

A simple comparative analysis of exact and approximate quantum error correction

Carlo Cafaro^{1,2} and Peter van Loock²

¹*Max-Planck Institute for the Science of Light, 91058 Erlangen, Germany and*

²*Institute of Physics, Johannes-Gutenberg University Mainz, 55128 Mainz, Germany*

We present a comparative analysis of exact and approximate quantum error correction by means of simple unabridged analytical computations. For the sake of clarity, using primitive quantum codes, we study the exact and approximate error correction of the two simplest unital (Pauli errors) and nonunital (non-Pauli errors) noise models, respectively. The similarities and differences between the two scenarios are stressed. In addition, the performances of quantum codes quantified by means of the entanglement fidelity for different recovery schemes are taken into consideration in the approximate case. Finally, the role of self-complementarity in approximate quantum error correction is briefly addressed.

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I. INTRODUCTION

It is known that decoherence is one of the most important obstacles in quantum information processing, since it causes a quantum computer to lose its quantum properties destroying its performance advantages over a classical computer. There are different methods for preserving quantum coherence. One possible technique exploits redundancy in encoding information. As pointed out in [1], one might think that redundancy cannot be of any use in quantum computing, since quantum states cannot be cloned [2]. However, using the property of quantum entanglement, Shor and Steane discovered a clever scheme for exploiting redundancy [3, 4]. This scheme is known as *quantum error correcting codes* (QECCs). For a comprehensive introduction to QECCs, we refer to [5]. Within such scheme, information is encoded in linear subspaces (codes) of the total Hilbert space in such a way that errors induced by the interaction with the environment can be detected and corrected. The QECCs approach may be interpreted as an active stabilization of a quantum state in which, by monitoring the system and conditionally carrying on suitable operations, one prevents the loss of information. In detail, the errors occur on a qubit when its evolution differs from the ideal one. This happens by interaction of the qubit with an environment.

Among the first and most famous QECCs, there are the Shor nine-qubit code [3], the Calderbank-Shor-Steane seven-qubit code [4, 6] and the perfect 1-error correcting five-qubit code [7, 8] with transmission rate equal to $1/5$. In general, scientists aim at searching for new quantum codes capable of combatting very general error models and correcting for arbitrary errors at unknown positions in the codeword. These results, in general, have more theoretical than practical importance, since they assume the existence of a fairly sophisticated quantum computer which has not been built yet. Fortunately, in many realistic situations, additional information on possible errors is available. Ideally, this knowledge should be taken into consideration in order to construct the simplest code with the highest transmission rate that can have a good chance to be implemented in a real laboratory. We stress *ideally* since, in general, it is a difficult problem to design quantum codes for any particular noise model. For instance, there are quantum systems for which the noise leads to dephasing errors only or bit-flip errors only. It has been shown that for such restricted types of decoherence, it is possible to perform error correction of one arbitrary dephasing/bit-flip error by encoding a single logical qubit into a minimum of three physical qubits [9]. Furthermore, uncovering efficient codes for restricted error models may be important for proof of principle demonstrations of quantum error correction [10]. Consider an error model where the position of the erroneous qubits is known. Such errors at known positions are denoted as *erasures* [10]. A t -error correcting code is a $2t$ -erasure correcting code. Also, while $1/5$ is the highest rate for a 1-error correcting code, four qubits are sufficient for a code to correct one arbitrary erasure. Omitting suitable normalization factors, the perfect four-qubit code for the correction of one erasure reads [10],

$$|0_L\rangle \stackrel{\text{def}}{=} |0000\rangle + |1111\rangle \quad \text{and} \quad |1_L\rangle \stackrel{\text{def}}{=} |1001\rangle + |0110\rangle. \quad (1)$$

In general, when no knowledge on the noise model is assumed, the errors to be corrected are completely random. This scenario may lead to the error correction of Pauli-type errors X , Y , Z (with $X \stackrel{\text{def}}{=} \sigma_x$, $Y \stackrel{\text{def}}{=} \sigma_y$, $Z \stackrel{\text{def}}{=} \sigma_z$) that occur with equal probability $p_X = p_Y = p_Z = \frac{1}{3}$ (symmetric depolarizing channel, [11]). However, if further information about an error process is available, more efficient codes can be designed as pointed out earlier. As a matter of fact, in many physical systems, the types of noise are likely to be unbalanced between amplitude (X -type) and phase (Z -type) errors. These are asymmetric error models which are still described by Kraus operators that are (unitary) Pauli matrices (Pauli Kraus operators). However, there are types of noise models seen in realistic settings that are

not described by Pauli Kraus operators and, for these cases as well, the task of constructing good error correcting codes is very challenging. The amplitude damping (AD) channel is the simplest nonunital channel whose Kraus operators cannot be described by (unitary) Pauli operations [11]. The two Kraus operators for AD noise are given by $A_0 \stackrel{\text{def}}{=} I - \mathcal{O}(\gamma)$ and $A_1 \stackrel{\text{def}}{=} \sqrt{\gamma}|0\rangle\langle 1|$ where γ denotes the damping rate (or, damping probability parameter). As we may observe, there is no simple way of reducing A_1 to one Pauli error operator since $|0\rangle\langle 1|$ is not normal. Observe that $A_1^\dagger \propto \sigma_x - i\sigma_y$, therefore the linear span of A_1 and A_1^\dagger equals the linear span of σ_x and σ_y . If the quantum system interacts with an environment at finite temperature, the Kraus operator A_1^\dagger will appear in the noise model (as stressed in [12], the error space to be corrected is a subspace of that spanned by the interaction operators, selected by the initial state of the environment) [11]. Therefore, if a code is capable of correcting t σ_x - and t σ_y -errors, it can also correct t A_1 and t A_1^\dagger errors. For the AD channel, we only need to deal with the error A_1 but not with A_1^\dagger . For such a reason, requiring to be capable of correcting both σ_x - and σ_y -errors is a less efficient way for constructing quantum codes for the AD channel. The first quantum code correcting single-AD errors was a $[[4, 1]]$ code presented by Leung et al. in [13]. Omitting proper normalization factors, the Leung et al. $[[4, 1]]$ code reads,

$$|0_L\rangle \stackrel{\text{def}}{=} |0000\rangle + |1111\rangle \quad \text{and} \quad |1_L\rangle \stackrel{\text{def}}{=} |0011\rangle + |1100\rangle. \quad (2)$$

The four-qubit code spanned by the codewords $|0_L\rangle$ and $|1_L\rangle$ in Eq. (2) represents a departure from standard QECCs that seek to *perfectly* correct up to t arbitrary errors on the system. The key-point advanced in [13] is that exact correctability is too strong a restriction. Relaxing the Knill-Laflamme QEC conditions (KL-conditions) [14] in such a manner that they are only *approximately* satisfied and allowing for a negligible error in the recovery scheme, better codes with higher transmission rates can be uncovered. The adaptation of the code to the noise model, an idea remarked later also in [15], is a crucial factor behind the success of the Leung et al. four-qubit code. Following the lead of [13], other works concerning the error correction of amplitude damping errors have appeared into the literature [15–20]. In [16, 17], it is emphasized that the concept of self-complementarity is crucial for error correcting amplitude damping errors, although the self-complementarity of the Leung et al. code is not specifically mentioned. In [15], the performance of various quantum error correcting codes for AD errors are numerically analyzed. In particular, a numerical analysis of the performance (quantified by means of Schumacher’s entanglement fidelity, [21]) of the four-qubit code for two recovery schemes can be found. The recovery schemes employed are the so-called code projected and the optimal channel adapted recovery schemes. The latter scheme was computed via semidefinite programming methods in [19]. However, no explicit analytical investigation (similar, for instance, to the investigations presented in [22] and [23] for depolarizing and Weyl unitary errors, respectively) is available. Furthermore, as pointed out in [20], numerically computed recovery maps are difficult to describe and understand analytically.

Inspired by [16, 17], we uncover that among the possible 28-pairs of orthonormal self-complementary codewords in \mathcal{H}_2^4 , only three pairs are indeed good (locally permutation equivalent, [25]) single AD-error correcting codes. In particular, two of these pairs define the perfect 1-erasure correcting code in Eq. (1) and the well-known four-qubit Leung et al. code in Eq. (2). Moreover, motivated by the numerical analysis presented in [15], we provide a fully analytical investigation of the performances, quantified in terms of Schumacher’s entanglement fidelity, of the Leung et al. four-qubit code. For the four-qubit code, the performance is evaluated for three different recovery schemes: the standard QEC recovery operation, the code-projected recovery operation and, finally, an analytically-optimized Fletcher’s-type channel-adapted recovery operation [15].

The layout of this article is as follows. In Section II, we describe necessary conditions for approximate quantum error correction together with necessary and sufficient conditions for exact quantum error correction. In Section III, we present a detailed study for the exact error correction of the bit-flip (or, similarly, phase-flip) noise errors by means of the three-qubit repetition code. We also present an analytical investigation of amplitude errors by means of the Leung et al. four-qubit code. The performance of each code is quantified by means of the entanglement fidelity. In particular, we compare three different recovery schemes in the approximate case. The concluding remarks appear in Section IV. A number of appendices with technical details of calculations are also provided.

II. FROM EXACT TO APPROXIMATE QUANTUM ERROR CORRECTION CONDITIONS

The very first *sufficient conditions* for approximate quantum error correction were introduced by Leung et al. in [13]. They showed that quantum codes can be effective in the error correction procedure even though they violated the standard KL-conditions. However, these violations characterized by small deviations from the standard error-correction conditions are allowed provided that they do not affect the desired fidelity order.

A. Exact quantum error correction conditions

For the sake of reasoning, let us consider a quantum stabilizer code \mathcal{C} with code parameters $[[n, k, d]]$ encoding k -logical qubits in the Hilbert space \mathcal{H}_2^k into n -physical qubits in the Hilbert space \mathcal{H}_2^n and distance d . Assume that the noise model after the encoding procedure is $\Lambda(\rho)$ and can be described by an operator-sum representation,

$$\Lambda(\rho) \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K}} A_k \rho A_k^\dagger, \quad (3)$$

where \mathcal{K} is the index set of all the enlarged Kraus operators A_k that appear in the sum. The noise channel Λ is a CPTP (completely positive and trace preserving) map. The codespace of \mathcal{C} is a k -dimensional subspace of \mathcal{H}_2^n where some error operators that characterize the error model Λ being considered can be reversed. Denote with $\mathcal{A}_{\text{reversible}} \subset \mathcal{A} \stackrel{\text{def}}{=} \{A_k\}$ with $k \in \mathcal{K}$ the set of reversible enlarged errors A_k on \mathcal{C} such that $\mathcal{K}_{\text{reversible}} \stackrel{\text{def}}{=} \{k : A_k \in \mathcal{A}_{\text{reversible}}\}$ is the index set of $\mathcal{A}_{\text{reversible}}$. Therefore, the noise model $\Lambda'(\rho)$ given by,

$$\Lambda'(\rho) \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K}_{\text{reversible}}} A_k \rho A_k^\dagger, \quad (4)$$

is reversible on $\mathcal{C} \subset \mathcal{H}_2^n$. The noise channel Λ' denotes a CP but non-TP map. The enlarged error operators A_k in $\mathcal{A}_{\text{reversible}}$ satisfy the standard error correction conditions [11],

$$P_{\mathcal{C}} A_l^\dagger A_m P_{\mathcal{C}} = \alpha_{lm} P_{\mathcal{C}}, \quad (5)$$

for any $l, m \in \mathcal{K}_{\text{reversible}}$, $P_{\mathcal{C}}$ denotes the projector on the codespace and α_{lm} are entries of a positive Hermitian matrix. Furthermore, a subset of error operators A_k in $\mathcal{A}_{\text{reversible}}$ is detectable if it satisfies the following detectability conditions,

$$P_{\mathcal{C}} A_k P_{\mathcal{C}} = \lambda_{A_k} P_{\mathcal{C}}, \quad (6)$$

where λ_{A_k} denotes a proportionality constant between $P_{\mathcal{C}} A_k P_{\mathcal{C}}$ and $P_{\mathcal{C}}$. The fulfillment of Eq. (5) for some subset of enlarged error operators A_k that characterize the operator sum representation of the noise model Λ implies that there exists an new operator-sum decomposition of Λ such that $\Lambda'(\rho)$ in Eq. (4) becomes,

$$\Lambda'(\rho) \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K}'_{\text{reversible}}} A'_k \rho A'_k{}^\dagger, \quad (7)$$

where (5) is replaced by

$$P_{\mathcal{C}} A'_l{}^\dagger A'_m P_{\mathcal{C}} = p_m \delta_{lm} P_{\mathcal{C}}, \quad (8)$$

for any $l, m \in \mathcal{K}'_{\text{reversible}}$ with the error detection probabilities p_m non-negative c -numbers. We remark that Eq. (8) is equivalent to the usual (exact) orthogonality and non-deformation conditions for a nondegenerate code,

$$\langle i_L | A'_l{}^\dagger A'_m | j_L \rangle = \delta_{ij} \delta_{lm} p_m \quad (9)$$

for any i, j labelling the logical states and $l, m \in \mathcal{K}_{\text{reversible}}$.

Observe that for any linear operator A'_k on a vector space V there exists a unitary U_k and a positive operator $J \stackrel{\text{def}}{=} \sqrt{A'_k{}^\dagger A'_k}$ such that [11],

$$A'_k = U_k J = U_k \sqrt{A'_k{}^\dagger A'_k}. \quad (10)$$

We stress that J is the unique positive operator that satisfies Eq. (10). As a matter of fact, multiplying $A'_k = U_k J$ on the left by the adjoint equation $A'_k{}^\dagger = J U_k{}^\dagger$ gives,

$$A'_k{}^\dagger A'_k = J U_k{}^\dagger U_k J = J^2 \Rightarrow J = \sqrt{A'_k{}^\dagger A'_k}. \quad (11)$$

Furthermore, if A'_k is invertible (that is, $\det A'_k \neq 0$), U_k is unique and reads,

$$U_k \stackrel{\text{def}}{=} A'_k J^{-1} = A'_k \left(\sqrt{A'_k{}^\dagger A'_k} \right)^{-1}. \quad (12)$$

How do we choose the unitary U_k when A'_k is not invertible? The operator J is a positive operator and belongs to a special subclass of Hermitian operators such that for any vector $|v\rangle \in V$, $\langle v|J|v\rangle$ is a *real* and non-negative number. Therefore, J has a spectral decomposition

$$J \stackrel{\text{def}}{=} \sqrt{A'_k{}^\dagger A'_k} = \sum_l \lambda_l |l\rangle \langle l|, \quad (13)$$

where $\lambda_l \geq 0$ and $\{|l\rangle\}$ denotes an orthonormal basis for the vector space V . Define the vectors $|\psi_l\rangle \stackrel{\text{def}}{=} A'_k |l\rangle$ and notice that,

$$\langle \psi_l | \psi_l \rangle = \langle l | A'_k{}^\dagger A'_k | l \rangle = \langle l | J^2 | l \rangle = \lambda_l^2. \quad (14)$$

For the time being, consider only those l for which $\lambda_l \neq 0$. For those l , consider the vectors $|e_l\rangle$ defined as

$$|e_l\rangle \stackrel{\text{def}}{=} \frac{|\psi_l\rangle}{\lambda_l} = \frac{A'_k |l\rangle}{\lambda_l}, \quad (15)$$

with $\langle e_l | e_{l'} \rangle = \delta_{ll'}$. For those l for which $\lambda_l = 0$, extend the orthonormal set $\{|e_l\rangle\}$ in such a manner that it forms an orthonormal basis $\{|E_l\rangle\}$. Then, a suitable choice for the unitary operator U_k such that

$$A'_k |l\rangle = U_k J |l\rangle, \quad (16)$$

with $\{|l\rangle\}$ an orthonormal basis for V reads,

$$U_k \stackrel{\text{def}}{=} \sum_l |E_l\rangle \langle l|. \quad (17)$$

In summary, the unitary U_k is uniquely determined by Eq. (12) when A'_k is invertible or Eq. (17) when A'_k is not necessarily invertible. We finally stress that the non-uniqueness of U_k when $\det A'_k = 0$ is due to the freedom in choosing the orthonormal basis $\{|l\rangle\}$ for the vector space V .

In the scenario being considered, when Eq. (8) is satisfied, the enlarged error operators A'_m admit polar decompositions,

$$A'_m P_C = \sqrt{p_m} U_m P_C, \quad (18)$$

with $k \in \mathcal{K}_{\text{reversible}}$. From Eqs. (8) and (18), we get

$$p_m \delta_{lm} P_C = P_C A_l{}^\dagger A'_m P_C = \sqrt{p_l p_m} P_C U_l{}^\dagger U_m P_C, \quad (19)$$

that is,

$$P_C U_l{}^\dagger U_m P_C = \delta_{lm} P_C. \quad (20)$$

We stress that Eq. (20) is needed for an unambiguous syndrome detection since, as a consequence of the orthogonality of different $R_m \stackrel{\text{def}}{=} U_m P_C$, the recovery operation $\mathcal{R} \stackrel{\text{def}}{=} \{R_m\}$ is trace preserving. This can be shown as follows.

Let \mathcal{V}^{iL} be the subspace of \mathcal{H}_2^n spanned by the corrupted images $\{A'_k |i_L\rangle\}$ of the codewords $|i_L\rangle$. Let $\{|v_r^{iL}\rangle\}$ be an orthonormal basis for \mathcal{V}^{iL} . We define such a subspace \mathcal{V}^{iL} for each of the codewords. Because of the KL-conditions [14],

$$\begin{aligned} \langle i_L | A_k{}^\dagger A_{k'} | i_L \rangle &= \langle j_L | A_k{}^\dagger A_{k'} | j_L \rangle, \quad \forall i, j \\ \langle i_L | A_k{}^\dagger A_{k'} | j_L \rangle &= 0, \quad \forall i \neq j, \end{aligned} \quad (21)$$

the subspaces \mathcal{V}^{iL} and \mathcal{V}^{jL} with $i \neq j$ are orthogonal subspaces. If $\mathcal{V}^{iL} \oplus \mathcal{V}^{jL}$ is a proper subset of \mathcal{H}_2^n with $\mathcal{V}^{iL} \oplus \mathcal{V}^{jL} \neq \mathcal{H}_2^n$, we denote its orthogonal complement by \mathcal{O} . We then have,

$$\mathcal{H}_2^n \stackrel{\text{def}}{=} (\mathcal{V}^{iL} \oplus \mathcal{V}^{jL}) \oplus (\mathcal{V}^{iL} \oplus \mathcal{V}^{jL})^\perp = (\mathcal{V}^{iL} \oplus \mathcal{V}^{jL}) \oplus \mathcal{O}, \quad (22)$$

where,

$$\mathcal{O} \stackrel{\text{def}}{=} (\mathcal{V}^{i_L} \oplus \mathcal{V}^{j_L})^\perp. \quad (23)$$

Let $\{|o_k\rangle\}$ be an orthonormal basis for \mathcal{O} . Then, the set of states $\{|v_r^{i_L}\rangle, |o_k\rangle\}$ constitutes an orthonormal basis for \mathcal{H}_2^n . We introduce the quantum recovery operation \mathcal{R} with operation elements

$$\mathcal{R} \stackrel{\text{def}}{=} \{R_1, \dots, R_r, \dots, \hat{O}\}, \quad (24)$$

with,

$$\mathcal{R}(\rho) \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K}'_{\text{reversible}}} R_k \rho R_k^\dagger + \hat{O} \rho \hat{O}^\dagger, \quad (25)$$

where,

$$R_r \stackrel{\text{def}}{=} \sum_i |i_L\rangle \langle v_r^{i_L}|, \quad (26)$$

and \hat{O} (with $\hat{O} = \hat{O}^\dagger = \hat{O}^\dagger \hat{O}$) is a projector onto the subspace \mathcal{O} in Eq. (23),

$$\hat{O} \stackrel{\text{def}}{=} \sum_k |o_k\rangle \langle o_k|. \quad (27)$$

We remark that the recovery operation \mathcal{R} is a trace preserving quantum operation by construction because,

$$\begin{aligned} \sum_r R_r^\dagger R_r + \hat{O}^\dagger \hat{O} &= \sum_r \left[\left(\sum_i |i_L\rangle \langle v_r^{i_L}| \right)^\dagger \left(\sum_j |j_L\rangle \langle v_r^{j_L}| \right) \right] + \left(\sum_k |o_k\rangle \langle o_k| \right)^\dagger \left(\sum_{k'} |o_{k'}\rangle \langle o_{k'}| \right) \\ &= \sum_{r, i, j} |v_r^{i_L}\rangle \langle i_L | j_L \rangle \langle v_r^{j_L}| + \sum_{k, k'} |o_k\rangle \langle o_k | o_{k'} \rangle \langle o_{k'}| \\ &= \sum_{r, i, j} |v_r^{i_L}\rangle \langle v_r^{j_L}| \delta_{ij} + \sum_{k, k'} |o_k\rangle \langle o_{k'}| \delta_{kk'} \\ &= \sum_{r, i} |v_r^{i_L}\rangle \langle v_r^{i_L}| + \sum_k |o_k\rangle \langle o_k| \\ &= \mathcal{I}_{2^n \times 2^n}, \end{aligned} \quad (28)$$

since $\mathcal{B}_{\mathcal{H}_2^n} \stackrel{\text{def}}{=} \{|v_r^{j_L}\rangle, |o_k\rangle\}$ is an orthonormal basis for \mathcal{H}_2^n . For more details, we refer to [14].

B. Approximate quantum error correction conditions

In general, approximate quantum error correction becomes useful when the operator-sum representation of the noise model is defined by errors parametrized by a certain number of small parameters such as the coupling strength between the environment and the quantum system. For the sake of simplicity, suppose the error model is characterized by a single small parameter δ and assume the goal is to uncover a quantum code for the noise model Λ' with fidelity,

$$\mathcal{F} \geq 1 - O(\delta^{\beta+1}), \quad (29)$$

for some $\beta \geq 0$. How strong can be the violation of the standard (exact) KL-conditions in order to preserve the desired fidelity order in Eq. (29)? In other words, how relaxed can the approximate error correction conditions be so that the inequality in (29) is satisfied? The answer to this important question was provided by Leung *et al.* in [13].

It turns out that for both exact and approximate quantum error correction conditions, it is necessary that

$$P_{\text{detection}} \stackrel{\text{def}}{=} \sum_{k \in \mathcal{K}'_{\text{reversible}}} p_k \geq \mathcal{F}, \quad (30)$$

where $P_{\text{detection}}$ denotes the total error detection probability. Eq. (30) requires that all the enlarged error operators A'_l with maximum detection probability must be included in $\mathcal{A}'_{\text{reversible}}$,

$$\max_{|\psi_{in}\rangle \in \mathcal{C}} \text{Tr} \left(|\psi_{in}\rangle \langle \psi_{in}| A'_l{}^\dagger A'_l \right) \approx O(\delta^\alpha) \text{ with } \alpha \leq \beta. \quad (31)$$

The important point is that a good overlap between the input and output states is needed while it is not necessary to recover the exact input state $|\psi_{in}\rangle \langle \psi_{in}|$, since we do not require $\mathcal{F} = 1$. In terms of the enlarged error operators restricted to the codespace, this means that such errors need to be only approximately unitary and mutually orthogonal. These considerations lead to the relaxed sufficient error correction conditions.

In analogy to (18), assume that the polar decomposition for A'_l is given by,

$$A'_l P_{\mathcal{C}} = U_l \sqrt{P_{\mathcal{C}} A'_l{}^\dagger A'_l P_{\mathcal{C}}}. \quad (32)$$

Since $P_{\mathcal{C}} A'_l{}^\dagger A'_l P_{\mathcal{C}}$ restricted to the codespace \mathcal{C} have different eigenvalues, the exact error correction conditions are not fulfilled. Let us say that p_l and $\lambda_l p_l$ are the largest and the smallest eigenvalues, respectively, where both p_l and λ_l are c -numbers. Furthermore, let us define the so-called residue operator π_l as [13],

$$\pi_l \stackrel{\text{def}}{=} \sqrt{P_{\mathcal{C}} A'_l{}^\dagger A'_l P_{\mathcal{C}}} - \sqrt{\lambda_l p_l} P_{\mathcal{C}}, \quad (33)$$

where,

$$0 \leq |\pi_l| \stackrel{\text{def}}{=} \left(\pi_l^\dagger \pi_l \right)^{\frac{1}{2}} \leq \sqrt{p_l} - \sqrt{\lambda_l p_l}. \quad (34)$$

Substituting (33) into (32), we get

$$A'_l P_{\mathcal{C}} = U_l \left(\sqrt{\lambda_l p_l} I + \pi_l \right) P_{\mathcal{C}}. \quad (35)$$

From Eq. (35) and imposing that $P_{\mathcal{C}} U_l{}^\dagger U_m P_{\mathcal{C}} = \delta_{lm} P_{\mathcal{C}}$, the analog of Eq. (8) becomes

$$P_{\mathcal{C}} A'_l{}^\dagger A'_m P_{\mathcal{C}} = \left(\sqrt{\lambda_l p_l} I + \pi_l^\dagger \right) \left(\sqrt{\lambda_m p_m} I + \pi_m \right) P_{\mathcal{C}} \delta_{lm}, \quad (36)$$

where,

$$p_l (1 - \lambda_l) \leq O(\delta^{\beta+1}), \forall l \in \mathcal{K}'_{\text{reversible}}. \quad (37)$$

We stress that when the exact error correction conditions are satisfied, $\lambda_l = 1$ and $\pi_l = 0$ (the null operator). Thus, in that scenario, Eqs. (8) and (36) coincide. Finally, we point out that an approximate recovery operation $\mathcal{R} \stackrel{\text{def}}{=} \{R_1, \dots, R_r, \dots, \hat{O}\}$ with R_k defined in Eq. (26) and \hat{O} formally defined just as in the exact case can be employed in this new scenario as well. However, assuming to consider $R_m^\dagger \stackrel{\text{def}}{=} U_m P_{\mathcal{C}}$, extra care in the explicit computation of the unitary operators U_m is needed in view of the fact that the polar decomposition (18) is replaced by the one in (32). For an explicit unabridged and correct computation of recovery operators as originally proposed by Leung et al., see Appendix A. For further theoretical details, we refer to reference [13].

III. FROM EXACT TO APPROXIMATE QEC: TWO SIMPLE NOISE MODELS

The objective in this section is to discuss in detail the exact and approximate error correction conditions for the simplest unital and nonunital channels, respectively. Specifically, we consider the bit-flip and the amplitude damping noise models. Error correction is performed by means of the three-qubit bit-flip repetition code for the unital channel while we employ the four-qubit Leung et al. code for the nonunital noise model. We acknowledge that the bit-flip noise model is certainly not the prototype of a truly quantum noise model. However, we believe its consideration is suitable for our purposes, since we wish to essentially stress the similarities and differences between exact and approximate error correction schemes avoiding unnecessary complications. The exact error correction analysis for more realistic and truly quantum error models along the lines presented here could be found in previous works of one of the Authors [22–24].

A. The simplest unital channel with Pauli errors

We consider a bit-flip noisy quantum channel and QEC is performed via the three-qubit bit-flip repetition code [11]. We remark that the bit-flip and the phase-flip (or, dephasing) channels are unitarily equivalent. This means that there exists a unitary operator U such that the action of one channel is the same as the other, provided the first channel is preceded by U and followed by U^\dagger . In the case being considered, it follows that

$$\Lambda_{\text{bit}}(\rho) \stackrel{\text{def}}{=} (H \circ \Lambda_{\text{phase}} \circ H^\dagger)(\rho),$$

where H denotes the Hadamard single-qubit gate. Error correction of dephasing errors by means of the three-qubit phase flip repetition code works very much like the error correction of bit-flip errors via the three-qubit bit flip repetition code. That said, we admit that a pure dephasing channel, with no other sources of noise at all, is physically improbable. However, in many physical systems, dephasing is indeed the dominant error source [26].

The performance of the error correcting code is quantified by means of the entanglement fidelity as function of the error probability. The bit flip noisy channel $\Lambda_{\text{bit}}^{(1)}$ (single use of the channel) is defined as follows,

$$\Lambda_{\text{bit}}^{(1)}(\rho) \stackrel{\text{def}}{=} (1-p)\rho + pX\rho X^\dagger, \quad (38)$$

where the matrix representation in the 1-qubit computational basis $\mathcal{B}_{\text{computational}} = \{|0\rangle, |1\rangle\}$ of the X -Pauli operator is given by,

$$[X]_{\mathcal{B}_{\text{computational}}} \stackrel{\text{def}}{=} \begin{pmatrix} \langle 0|X|0\rangle & \langle 0|X|1\rangle \\ \langle 1|X|0\rangle & \langle 1|X|1\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (39)$$

Observe that the bit-flip channel is a unital channel since $\Lambda_{\text{bit}}^{(1)}(I) = I$. Consider the three-qubit bit flip encoding defined as,

$$|0\rangle \rightarrow |0_L\rangle \stackrel{\text{def}}{=} |000\rangle, \quad |1\rangle \rightarrow |1_L\rangle \stackrel{\text{def}}{=} |111\rangle. \quad (40)$$

The action of three uses of the bit flip channel $\Lambda_{\text{bit}}^{(3)}(\rho)$ on 3-qubits quantum states reads,

$$\Lambda_{\text{bit}}^{(3)}(\rho) \stackrel{\text{def}}{=} \sum_{i_1, i_2, i_3=0}^1 p_{i_3} p_{i_2} p_{i_1} (A_{i_3} \otimes A_{i_2} \otimes A_{i_1}) \rho (A_{i_3} \otimes A_{i_2} \otimes A_{i_1})^\dagger, \quad (41)$$

where $A_0 \stackrel{\text{def}}{=} I$, $A_1 \stackrel{\text{def}}{=} X$ are Pauli operators. Furthermore,

$$p_0 \stackrel{\text{def}}{=} 1-p, \quad p_1 \stackrel{\text{def}}{=} p, \quad (42)$$

with,

$$\sum_{i_1, i_2, i_3=0}^1 p_{i_3} p_{i_2} p_{i_1} = p_0^3 + 3p_1 p_0^2 + 3p_1^2 p_0 + p_1^3 = 1. \quad (43)$$

To simplify our notation, we may assume that $A_{i_n} \otimes \dots \otimes A_{i_1} \equiv A_{i_n \dots i_1}$. The channel $\Lambda_{\text{bit}}^{(3)}(\rho)$ can be written as,

$$\Lambda_{\text{bit}}^{(3)}(\rho) \stackrel{\text{def}}{=} \sum_{k=0}^7 A'_k \rho A'_k{}^\dagger \quad \text{and} \quad \sum_{k=0}^7 A'_k{}^\dagger A'_k = I_{8 \times 8}, \quad (44)$$

where we denote with \mathcal{A} the superoperator defined in terms of the enlarged error operators $\{A'_0, \dots, A'_7\}$. In an explicit way, the error operators $\{A'_0, \dots, A'_7\}$ read,

$$\begin{aligned} A'_0 &\stackrel{\text{def}}{=} \sqrt{p_0^3} I^1 \otimes I^2 \otimes I^3, \quad A'_1 \stackrel{\text{def}}{=} \sqrt{p_1 p_0^2} X^1 \otimes I^2 \otimes I^3, \quad A'_2 \stackrel{\text{def}}{=} \sqrt{p_1^2 p_0} I^1 \otimes X^2 \otimes I^3, \\ A'_3 &\stackrel{\text{def}}{=} \sqrt{p_1 p_0^2} I^1 \otimes I^2 \otimes X^3, \quad A'_4 \stackrel{\text{def}}{=} \sqrt{p_1^2 p_0} X^1 \otimes X^2 \otimes I^3, \quad A'_5 \stackrel{\text{def}}{=} \sqrt{p_1^2 p_0} X^1 \otimes I^2 \otimes X^3, \\ A'_6 &\stackrel{\text{def}}{=} \sqrt{p_1^2 p_0} I^1 \otimes X^2 \otimes X^3, \quad A'_7 \stackrel{\text{def}}{=} \sqrt{p_1^3} X^1 \otimes X^2 \otimes X^3. \end{aligned} \quad (45)$$

The set of error operators satisfying the detectability condition, $P_C A'_k P_C = \lambda_{A'_k} P_C$, where $P_C \stackrel{\text{def}}{=} |0_L\rangle\langle 0_L| + |1_L\rangle\langle 1_L|$ is the projector operator on the code subspace $\mathcal{C} \stackrel{\text{def}}{=} \text{Span}\{|0_L\rangle, |1_L\rangle\}$ is given by,

$$\mathcal{A}_{\text{detectable}} \stackrel{\text{def}}{=} \{A'_0, A'_1, A'_2, A'_3, A'_4, A'_5, A'_6\} \subseteq \mathcal{A}. \quad (46)$$

The only non detectable error is A'_7 . Furthermore, since all the detectable errors are invertible, the set of correctable errors is such that $\mathcal{A}_{\text{correctable}}^\dagger \mathcal{A}_{\text{correctable}}$ is detectable [27]. It follows that,

$$\mathcal{A}_{\text{correctable}} \stackrel{\text{def}}{=} \{A'_0, A'_1, A'_2, A'_3\} \subseteq \mathcal{A}_{\text{detectable}} \subseteq \mathcal{A}. \quad (47)$$

To be more explicit, the set of enlarged error operators $\{A'_k\}$ with $k \in \{0, \dots, \bar{k}\}$ is correctable provided that,

$$P_C A'_l{}^\dagger A'_m P_C \propto P_C, \quad (48)$$

for any pair of (l, m) with $l, m \in \{0, \dots, \bar{k}\}$. Eq. (48) is satisfied if and only if,

$$\langle 0_L | A'_l{}^\dagger A'_m | 0_L \rangle = \langle 1_L | A'_l{}^\dagger A'_m | 1_L \rangle \quad \text{and} \quad \langle 0_L | A'_l{}^\dagger A'_m | 1_L \rangle = \langle 1_L | A'_l{}^\dagger A'_m | 0_L \rangle = 0, \quad (49)$$

for any pair of (l, m) with $l, m \in \{0, \dots, \bar{k}\}$. The enlarged error operators $\{A'_l\}$ in (45) can be rewritten as,

$$\begin{aligned} A'_0 &\stackrel{\text{def}}{=} \sqrt{(1-p)^3} I, \quad A'_1 \stackrel{\text{def}}{=} \sqrt{p(1-p)^2} X^1, \quad A'_2 \stackrel{\text{def}}{=} \sqrt{p(1-p)^2} X^2, \\ A'_3 &\stackrel{\text{def}}{=} \sqrt{p(1-p)^2} X^3, \quad A'_4 \stackrel{\text{def}}{=} \sqrt{p^2(1-p)} X^1 X^2, \quad A'_5 \stackrel{\text{def}}{=} \sqrt{p^2(1-p)} X^1 X^3, \\ A'_6 &\stackrel{\text{def}}{=} \sqrt{p^2(1-p)} X^2 X^3, \quad A'_7 \stackrel{\text{def}}{=} \sqrt{p^3} X^1 X^2 X^3. \end{aligned} \quad (50)$$

The action of the correctable error operators $\mathcal{A}_{\text{correctable}}$ on the codewords $|0_L\rangle$ and $|1_L\rangle$ is given by,

$$\begin{aligned} |0_L\rangle &\rightarrow A'_0 |0_L\rangle = \sqrt{p_0^3} |000\rangle, \quad A'_1 |0_L\rangle = \sqrt{p_1 p_0^2} |100\rangle, \quad A'_2 |0_L\rangle = \sqrt{p_1 p_0^2} |010\rangle, \quad A'_3 |0_L\rangle = \sqrt{p_1 p_0^2} |001\rangle \\ |1_L\rangle &\rightarrow A'_0 |1_L\rangle = \sqrt{p_0^3} |111\rangle, \quad A'_1 |1_L\rangle = \sqrt{p_1 p_0^2} |011\rangle, \quad A'_2 |1_L\rangle = \sqrt{p_1 p_0^2} |101\rangle, \quad A'_3 |1_L\rangle = \sqrt{p_1 p_0^2} |110\rangle. \end{aligned} \quad (51)$$

The two four-dimensional orthogonal subspaces \mathcal{V}^{0L} and \mathcal{V}^{1L} of \mathcal{H}_2^3 generated by the action of $\mathcal{A}_{\text{correctable}}$ on $|0_L\rangle$ and $|1_L\rangle$ are defined as,

$$\mathcal{V}^{0L} \stackrel{\text{def}}{=} \text{Span} \{ |v_1^{0L}\rangle = |000\rangle, |v_2^{0L}\rangle = |100\rangle, |v_3^{0L}\rangle = |010\rangle, |v_4^{0L}\rangle = |001\rangle \}, \quad (52)$$

and,

$$\mathcal{V}^{1L} \stackrel{\text{def}}{=} \text{Span} \{ |v_1^{1L}\rangle = |111\rangle, |v_2^{1L}\rangle = |011\rangle, |v_3^{1L}\rangle = |101\rangle, |v_4^{1L}\rangle = |110\rangle \}, \quad (53)$$

respectively. Notice that $\mathcal{V}^{0L} \oplus \mathcal{V}^{1L} = \mathcal{H}_2^3$. The recovery superoperator $\mathcal{R} \leftrightarrow \{R_l\}$ with $l = 1, \dots, 4$ is defined as [14],

$$R_l \stackrel{\text{def}}{=} V_l \sum_{i=0}^1 |v_l^{iL}\rangle \langle v_l^{iL}|, \quad (54)$$

where the unitary operator V_l is such that $V_l |v_l^{iL}\rangle \stackrel{\text{def}}{=} |iL\rangle$ for $i \in \{0, 1\}$. Substituting (52) and (53) into (54), it follows that the four recovery operators $\{R_0, R_1, R_2, R_3\}$ are given by,

$$\begin{aligned} R_0 &\stackrel{\text{def}}{=} \frac{1}{\sqrt{(1-p)^3}} P_C A'_0{}^\dagger = |000\rangle\langle 000| + |111\rangle\langle 111|, \quad R_1 \stackrel{\text{def}}{=} \frac{1}{\sqrt{p(1-p)^2}} P_C A'_1{}^\dagger = |000\rangle\langle 100| + |111\rangle\langle 011|, \\ R_2 &\stackrel{\text{def}}{=} \frac{1}{\sqrt{p(1-p)^2}} P_C A'_2{}^\dagger = |000\rangle\langle 010| + |111\rangle\langle 101|, \quad R_3 \stackrel{\text{def}}{=} \frac{1}{\sqrt{p^3}} P_C A'_3{}^\dagger = |000\rangle\langle 001| + |111\rangle\langle 110|, \end{aligned} \quad (55)$$

with,

$$\sum_{l=0}^3 R_l^\dagger R_l = I_{2^3 \times 2^3} \quad (56)$$

We observe that the four recovery operators R_j associated with the four correctable errors A_j' with $j \in \{0, 1, 2, 3\}$ are formally defined as,

$$R_j \stackrel{\text{def}}{=} \frac{|0_L\rangle \langle 0_L| A_j'^\dagger}{\sqrt{\langle 0_L | A_j'^\dagger A_j' | 0_L \rangle}} + \frac{|1_L\rangle \langle 1_L| A_j'^\dagger}{\sqrt{\langle 1_L | A_j'^\dagger A_j' | 1_L \rangle}}. \quad (57)$$

When considering exact quantum error correction, $R_j \propto P_C A_j'^\dagger$ where the coefficient of proportionality must be determined in such a manner that its product with A_j' leads to a unitary operator. This coefficient equals the square root of $\langle 0_L | A_j'^\dagger A_j' | 0_L \rangle$ (or, $\langle 1_L | A_j'^\dagger A_j' | 1_L \rangle$). We emphasize three features of exact-QEC, two of which concern the standard QEC recovery superoperator $\mathcal{R} \stackrel{\text{def}}{=} \{R_0, R_1, R_2, R_3\}$:

- The two eigenvalues λ_{\max} and λ_{\min} of the (2×2) -matrix associated with the operators $P_C A_l'^\dagger A_m' P_C$ (with A_k' correctable errors) on the codespace \mathcal{C} coincide. For example, for $P_C A_0'^\dagger A_0' P_C$ we have $\lambda_{\max} = \lambda_{\min} = (1-p)^3$;
- The projector on the codespace P_C belongs to \mathcal{R} , the standard QEC recovery;
- All the four recovery operators in \mathcal{R} are p -independent, where p denotes the error probability and is the single parameter that characterizes the noise model being considered.

For the simple bit-flip noise model with error correction performed by means of the three-qubit bit-flip code, the entanglement fidelity reads [21, 28],

$$\mathcal{F}_{[[3,1,1]]}(p) \stackrel{\text{def}}{=} \frac{1}{(\dim_{\mathbb{C}} \mathcal{C})^2} \sum_{l=0}^7 \sum_{k=0}^3 |\text{Tr}(R_k A_l')_{\mathcal{C}}|^2 = \frac{1}{4} \sum_{l=0}^7 \sum_{k=0}^3 |\langle 0_L | R_k A_l' | 0_L \rangle + \langle 1_L | R_k A_l' | 1_L \rangle|^2. \quad (58)$$

Substituting (57) into the RHS of (58), we obtain

$$\mathcal{F}_{[[3,1,1]]}(p) = \frac{1}{4} \sum_{l=0}^7 \sum_{k=0}^3 \left| \frac{\langle 0_L | A_k'^\dagger A_l' | 0_L \rangle}{\sqrt{\langle 0_L | A_k'^\dagger A_k' | 0_L \rangle}} + \frac{\langle 1_L | A_k'^\dagger A_l' | 1_L \rangle}{\sqrt{\langle 1_L | A_k'^\dagger A_k' | 1_L \rangle}} \right|^2. \quad (59)$$

We observe that $\mathcal{F}_{[[3,1,1]]}(p)$ is, in principle, the sum of $4 \times 8 = 32$ -terms that arise by considering all the possible pairs (k, l) with $k \in \{0, 1, 2, 3\}$ and $l \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. However, it turns out that only 4-terms are nonvanishing and contribute to the computation of the entanglement fidelity. They are $\{(k, l)\} = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$. Thus, only the four correctable errors $\{A_0', A_1', A_2', A_3'\}$ are recoverable. Indeed, they are fully recovered and,

$$(R_j A_j')_{\mathcal{C}} = \langle 0_L | A_j'^\dagger A_j' | 0_L \rangle I = \langle 1_L | A_j'^\dagger A_j' | 1_L \rangle I \propto I. \quad (60)$$

In the exact-QEC scenario, we stress:

- Only the correctable errors are recoverable. Indeed, they are fully recoverable. No off-diagonal contribution arises.

In summary, $\mathcal{F}_{[[3,1,1]]}(p)$ reads

$$\begin{aligned} \mathcal{F}_{[[3,1,1]]}(p) &= \frac{1}{4} \left[\left| \frac{\langle 0_L | A_0'^\dagger A_0' | 0_L \rangle}{\sqrt{\langle 0_L | A_0'^\dagger A_0' | 0_L \rangle}} + \frac{\langle 1_L | A_0'^\dagger A_0' | 1_L \rangle}{\sqrt{\langle 1_L | A_0'^\dagger A_0' | 1_L \rangle}} \right|^2 + \left| \frac{\langle 0_L | A_1'^\dagger A_1' | 0_L \rangle}{\sqrt{\langle 0_L | A_1'^\dagger A_1' | 0_L \rangle}} + \frac{\langle 1_L | A_1'^\dagger A_1' | 1_L \rangle}{\sqrt{\langle 1_L | A_1'^\dagger A_1' | 1_L \rangle}} \right|^2 + \right. \\ &\quad \left. \left| \frac{\langle 0_L | A_2'^\dagger A_2' | 0_L \rangle}{\sqrt{\langle 0_L | A_2'^\dagger A_2' | 0_L \rangle}} + \frac{\langle 1_L | A_2'^\dagger A_2' | 1_L \rangle}{\sqrt{\langle 1_L | A_2'^\dagger A_2' | 1_L \rangle}} \right|^2 + \left| \frac{\langle 0_L | A_3'^\dagger A_3' | 0_L \rangle}{\sqrt{\langle 0_L | A_3'^\dagger A_3' | 0_L \rangle}} + \frac{\langle 1_L | A_3'^\dagger A_3' | 1_L \rangle}{\sqrt{\langle 1_L | A_3'^\dagger A_3' | 1_L \rangle}} \right|^2 \right] \\ &= \frac{1}{4} \left[4 \langle 0_L | A_0'^\dagger A_0' | 0_L \rangle + 4 \langle 0_L | A_1'^\dagger A_1' | 0_L \rangle + 4 \langle 0_L | A_2'^\dagger A_2' | 0_L \rangle + 4 \langle 0_L | A_3'^\dagger A_3' | 0_L \rangle \right], \quad (61) \end{aligned}$$

that is,

$$\mathcal{F}_{[[3,1,1]]}(p) = \langle 0_L | A_0'^{\dagger} A_0' | 0_L \rangle + \langle 0_L | A_1'^{\dagger} A_1' | 0_L \rangle + \langle 0_L | A_2'^{\dagger} A_2' | 0_L \rangle + \langle 0_L | A_3'^{\dagger} A_3' | 0_L \rangle. \quad (62)$$

Substituting (50) into (62), we get

$$\mathcal{F}_{[[3,1,1]]}(p) = 1 - 3p^2 + 2p^3. \quad (63)$$

When the error probability p increases beyond a certain threshold \bar{p} , quantum error correction does more harm than good. To uncover this point \bar{p} , we have to check two conditions. First, we compare $\mathcal{F}_{[[3,1,1]]}(p)$ with the fidelity without coding and error correction (the so-called single-qubit baseline performance),

$$\mathcal{F}_{\text{no-QEC}}(p) \stackrel{\text{def}}{=} \frac{1}{4} \sum_{k=0}^1 |\text{Tr} A_k|^2 = 1 - 2p + p^2. \quad (64)$$

For this error correction scheme to be useful, it must be

$$\mathcal{F}_{\text{no-QEC}}(p) \leq \mathcal{F}_{[[3,1,1]]}(p). \quad (65)$$

In the case being considered, this inequality holds true for any $0 \leq p \leq 1$. Second, the error correction scheme is effective provided that the failure probability $P_{\text{failure}}(p) \stackrel{\text{def}}{=} 1 - \mathcal{F}_{[[3,1,1]]}(p)$ is smaller than the error probability p ,

$$P_{\text{failure}}(p) \leq p. \quad (66)$$

This second inequality holds true if and only if $0 \leq p \leq \frac{1}{2}$.

B. The simplest nonunital channel with non-Pauli errors

An approximate QEC framework turns out to be of great use when combatting non-Pauli errors. Within such framework, we allow for a negligible but non-vanishing error in the recovery so that errors need not be exactly orthogonal to be unambiguously detected and perfectly recovered. Indeed, we allow for slight non-orthogonalities between approximately correctable error operators. This way, correctable errors have to satisfy the KL-conditions only approximately. As a consequence, in such a scenario, the composite operation $\mathcal{R}\Lambda\mathcal{E}$ is necessarily only approximately close to the identity on the codespace. Observe that in such approximate QEC framework, for a given noise model, more codes satisfying the approximate KL-conditions can be constructed. Furthermore, it is not unusual to uncover codes of shorter block lengths which, although of less general applicability, may indeed be more efficient for the specific error model considered. For instance, when considering amplitude damping errors in the standard A_0, A_1 non-Pauli error basis on a n -qubits state, to the first order in γ , $(n+1)$ -errors may occur. Thus, in order to be correctable by this nondegenerate non-Pauli basis code, such errors must map the codeword space to orthogonal spaces if the syndrome is to be detected unambiguously. Thus, it must be $2^n \geq 2(n+1)$, that is $n \geq 3$. Instead, considering the error correction of the same decoherence model by means of nondegenerate standard Pauli basis codes, it must be $2^n \geq 2(2n+1)$, that is $n \geq 5$. The former scenario arises when considering the amplitude damping channel and using the Leung et al. $[[4, 1]]$ code.

In the case of amplitude damping, we model the environment as starting in the $|0\rangle$ state as it were at zero temperature. The AD quantum noisy channel is defined as [11],

$$\Lambda_{\text{AD}}(\rho) \stackrel{\text{def}}{=} \sum_{k=0}^1 A_k \rho A_k^{\dagger}, \quad (67)$$

where the Kraus error operators A_k read,

$$A_0 \stackrel{\text{def}}{=} \frac{1}{2} \left[\left(1 + \sqrt{1-\gamma}\right) I + \left(1 - \sqrt{1-\gamma}\right) \sigma_z \right], \quad A_1 \stackrel{\text{def}}{=} \frac{\sqrt{\gamma}}{2} (\sigma_x + i\sigma_y). \quad (68)$$

Observe that the AD channel is nonunital since $\Lambda_{\text{AD}}(I) \neq I$. The (2×2) -matrix representation of the A_k operators is given by,

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \quad \text{and} \quad A_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}. \quad (69)$$

The action of the A_k with $k \in \{0, 1\}$ operators on the computational basis vectors $|0\rangle$ and $|1\rangle$ reads,

$$A_0 |0\rangle = |0\rangle, A_0 |1\rangle = \sqrt{1-\gamma} |1\rangle, \quad (70)$$

and,

$$A_1 |0\rangle \equiv 0, A_1 |1\rangle = \sqrt{\gamma} |0\rangle, \quad (71)$$

respectively. The codewords of the Leung et al. $[[4, 1]]$ quantum code are given by [13],

$$|0_L\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (|0000\rangle + |1111\rangle) \text{ and } |1_L\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (|0011\rangle + |1100\rangle). \quad (72)$$

We underline that this code is a two-dimensional subspace of the 16-dimensional *complex* Hilbert space \mathcal{H}_2^4 and is spanned by self-complementary codewords. Recall that a code \mathcal{C} is called self-complementary if its codespace is spanned by codewords $\{|c_a\rangle\}$ defined as,

$$|c_a\rangle \stackrel{\text{def}}{=} \frac{|a\rangle + |\bar{a}\rangle}{\sqrt{2}}, \quad (73)$$

where a is a binary string of length n and $\bar{a} \stackrel{\text{def}}{=} \mathbf{1} \oplus a$ is the complement of a . In Appendix B, we show that, in addition to the Leung et al. four-qubit code, there are only two additional two-dimensional subspaces spanned by self-complementary codewords in \mathcal{H}_2^4 capable of error-correcting single AD-errors.

After the encoding operation, the total set of enlarged error operators is given by the following 16 enlarged error operators,

$$\mathcal{A}_{\text{total}} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} A_{0000}, A_{1000}, A_{0100}, A_{0010}, A_{0001}, A_{1100}, A_{1010}, A_{1001}, A_{0110}, A_{0101}, A_{0011}, \\ A_{1110}, A_{1011}, A_{0111}, A_{1101}, A_{1111} \end{array} \right\}, \quad (74)$$

where,

$$16 = 2^4 = \sum_{k=0}^4 \binom{4}{k} = \binom{4}{0} + \binom{4}{1} + \binom{4}{2} + \binom{4}{3} + \binom{4}{4}. \quad (75)$$

Consider the following quantum state $|\psi\rangle$,

$$|\psi\rangle \stackrel{\text{def}}{=} \alpha |0_L\rangle + \beta |1_L\rangle, \quad (76)$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$. Then, the action of the weight-0 enlarged error operator A_{0000} on $|\psi\rangle$ reads,

$$A_{0000} |\psi\rangle = \alpha \left[\frac{|0000\rangle + (1-\gamma)^2 |1111\rangle}{\sqrt{2}} \right] + \beta \left[(1-\gamma) \frac{(|0011\rangle + |1100\rangle)}{\sqrt{2}} \right]. \quad (77)$$

The action of the four weight-1 enlarged error operators is given by,

$$\begin{aligned} A_{1000} |\psi\rangle &= \sqrt{\frac{\gamma(1-\gamma)}{2}} [\alpha(1-\gamma) |0111\rangle + \beta |0100\rangle], A_{0100} |\psi\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}} [\alpha(1-\gamma) |1011\rangle + \beta |1000\rangle], \\ A_{0010} |\psi\rangle &= \sqrt{\frac{\gamma(1-\gamma)}{2}} [\alpha(1-\gamma) |1101\rangle + \beta |0001\rangle], A_{0001} |\psi\rangle = \sqrt{\frac{\gamma(1-\gamma)}{2}} [\alpha(1-\gamma) |1110\rangle + \beta |0010\rangle]. \end{aligned} \quad (78)$$

The action of the six weight-2 enlarged error operators reads,

$$\begin{aligned} A_{1100} |\psi\rangle &= \frac{\gamma}{\sqrt{2}} [\alpha(1-\gamma) |0011\rangle + \beta |0000\rangle], A_{1010} |\psi\rangle = \frac{\alpha}{\sqrt{2}} \gamma(1-\gamma) |0101\rangle, \\ A_{1001} |\psi\rangle &= \frac{\alpha}{\sqrt{2}} \gamma(1-\gamma) |0110\rangle, A_{0110} |\psi\rangle = \frac{\alpha}{\sqrt{2}} \gamma(1-\gamma) |1001\rangle, \\ A_{0101} |\psi\rangle &= \frac{\alpha}{\sqrt{2}} \gamma(1-\gamma) |1010\rangle, A_{0011} |\psi\rangle = \frac{\gamma}{\sqrt{2}} [\alpha(1-\gamma) |1100\rangle + \beta |0000\rangle]. \end{aligned} \quad (79)$$

The action of the four weight-3 enlarged error operators is given by,

$$\begin{aligned} A_{1110} |\psi\rangle &= \frac{\alpha}{\sqrt{2}} \gamma^{\frac{3}{2}} \sqrt{1-\gamma} |0001\rangle, & A_{1011} |\psi\rangle &= \frac{\alpha}{\sqrt{2}} \gamma^{\frac{3}{2}} \sqrt{1-\gamma} |0100\rangle, \\ A_{0111} |\psi\rangle &= \frac{\alpha}{\sqrt{2}} \gamma^{\frac{3}{2}} \sqrt{1-\gamma} |1000\rangle, & A_{1101} |\psi\rangle &= \frac{\alpha}{\sqrt{2}} \gamma^{\frac{3}{2}} \sqrt{1-\gamma} |0010\rangle. \end{aligned} \quad (80)$$

Finally, the action of the weight-4 enlarged error operator reads,

$$A_{1111} |\psi\rangle = \frac{\alpha}{\sqrt{2}} \gamma^2 |0000\rangle. \quad (81)$$

For the sake of completeness, observe that

$$\begin{aligned} \sum_{i_1, i_2, i_3, i_4=0}^1 P(A_{i_1 i_2 i_3 i_4}) &= \sum_{i_1, i_2, i_3, i_4=0}^1 \text{Tr} \left(A_{i_1 i_2 i_3 i_4} |\psi\rangle \langle \psi| A_{i_1 i_2 i_3 i_4}^\dagger \right) = \sum_{i_1, i_2, i_3, i_4=0}^1 \langle \psi | A_{i_1 i_2 i_3 i_4}^\dagger A_{i_1 i_2 i_3 i_4} | \psi \rangle \\ &= \left(\frac{\alpha^2}{2} + \frac{(1-\gamma)^4}{2} \alpha^2 + \beta^2 (1-\gamma)^2 \right) + 4 \left(\frac{\gamma(1-\gamma)}{2} (\alpha^2 (1-\gamma)^2 + \beta^2) \right) + \\ &\quad + \gamma^2 (\alpha^2 (1-\gamma)^2 + \beta^2) + 2\alpha^2 \gamma^2 (1-\gamma)^2 + 2\alpha^2 \gamma^3 (1-\gamma) + \frac{\alpha^2}{2} \gamma^4 \\ &= \alpha^2 + \beta^2 \equiv |\alpha|^2 + |\beta|^2 = 1, \end{aligned} \quad (82)$$

where $P(A_{i_1 i_2 i_3 i_4})$ denotes the probability that $A_{i_1 i_2 i_3 i_4}$ occurs. We recall that for a given code \mathcal{C} , the set of detectable errors is closed under linear combinations. That is, if E_1 and E_2 are both detectable, then so is $\alpha E_1 + \beta E_2$. This useful property implies that to check detectability, one has to consider only the elements of a linear basis for the space of errors of interest. An enlarged error operator A_k is detectable if and only if,

$$P_{\mathcal{C}} A_k P_{\mathcal{C}} \propto P_{\mathcal{C}}, \quad (83)$$

where $P_{\mathcal{C}} \stackrel{\text{def}}{=} |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|$ is the projector on the codespace \mathcal{C} . Condition (83) requires that for detectable errors it must be,

$$\langle 0_L | A_k | 0_L \rangle = \langle 1_L | A_k | 1_L \rangle \quad \text{and} \quad \langle 0_L | A_k | 1_L \rangle = 0 = \langle 1_L | A_k | 0_L \rangle. \quad (84)$$

For the weight-0 enlarged error operator A_{0000} , we have

$$\begin{aligned} \langle 0_L | A_{0000} | 0_L \rangle &= 1 - \gamma + \frac{1}{2} \gamma^2 = 1 - \gamma + \mathcal{O}(\gamma^2), & \langle 1_L | A_{0000} | 1_L \rangle &= 1 - \gamma, \\ \langle 0_L | A_{0000} | 1_L \rangle &= 0 = \langle 1_L | A_{0000} | 0_L \rangle. \end{aligned} \quad (85)$$

Therefore, A_{0000} is detectable. Similarly, it turns out that the four weight-1 enlarged error operators A_{1000} , A_{0100} , A_{0010} and A_{0001} are detectable. For instance, for A_{0100} we obtain

$$\langle 0_L | A_{0100} | 0_L \rangle = 0 = \langle 1_L | A_{0100} | 1_L \rangle \quad \text{and} \quad \langle 0_L | A_{0100} | 1_L \rangle = 0 = \langle 1_L | A_{0100} | 0_L \rangle. \quad (86)$$

Considering the weight-2 enlarged error operators, it follows that A_{1100} and A_{0011} are not detectable since

$$\langle 0_L | A_{1100} | 0_L \rangle = 0 = \langle 1_L | A_{1100} | 1_L \rangle \quad \text{and} \quad \langle 0_L | A_{1100} | 1_L \rangle = \frac{\gamma}{2} \neq 0, \quad \langle 1_L | A_{1100} | 0_L \rangle = \frac{\gamma(1-\gamma)}{2} \neq 0, \quad (87)$$

and,

$$\langle 0_L | A_{0011} | 0_L \rangle = 0 = \langle 1_L | A_{0011} | 1_L \rangle \quad \text{and} \quad \langle 0_L | A_{0011} | 1_L \rangle = \frac{\gamma}{2} \neq 0, \quad \langle 1_L | A_{0011} | 0_L \rangle = \frac{\gamma(1-\gamma)}{2} \neq 0. \quad (88)$$

On the contrary the weight-2 enlarged error operators A_{1010} , A_{1001} , A_{0110} and A_{0101} are detectable. For instance, for A_{1010} we have

$$\langle 0_L | A_{1010} | 0_L \rangle = 0 = \langle 1_L | A_{1010} | 1_L \rangle \text{ and, } \langle 0_L | A_{1010} | 1_L \rangle = 0 = \langle 1_L | A_{1010} | 0_L \rangle. \quad (89)$$

The four weight-3 enlarged error operators A_{1110} , A_{1011} , A_{0111} and A_{1101} are all detectable. For each one of them we get the same type of relations which hold true for A_{1110} . For instance,

$$\langle 0_L | A_{1110} | 0_L \rangle = 0 = \langle 1_L | A_{1110} | 1_L \rangle \text{ and, } \langle 0_L | A_{1110} | 1_L \rangle = 0 = \langle 1_L | A_{1110} | 0_L \rangle. \quad (90)$$

Finally, the weight-4 enlarged error operator A_{1111} is not detectable since

$$\langle 0_L | A_{1111} | 0_L \rangle = \frac{\gamma^2}{2} \neq 0 = \langle 1_L | A_{1111} | 1_L \rangle \text{ and, } \langle 0_L | A_{1111} | 1_L \rangle = 0 = \langle 1_L | A_{1111} | 0_L \rangle. \quad (91)$$

We point out that the physical reason why A_{0000} is detectable and A_{1111} is not detectable is as follows: there is a nonzero probability for the error A_{0000} to occur within the considered orders (up to linear orders in gamma), whereas the error A_{1111} would simply never occur in those allowed orders (0-th and 1-st in gamma); in other words, by ignoring terms proportional to γ^2 , the A_{1111} error (four photons get lost) simply does not exist and thus it cannot be referred to as detectable. In conclusion, we have

$$\mathcal{A}_{\text{detectable}} \stackrel{\text{def}}{=} \mathcal{A}_{\text{total}} / \{A_{0011}, A_{1100}, A_{1111}\} \subseteq \mathcal{A}_{\text{total}}. \quad (92)$$

We also recall that the notion of correctability depends on all the errors in the set under consideration and, unlike detectability, cannot be applied to individual errors [27]. Furthermore, it is important to note that a linear combination of correctable errors is also a correctable error. A set of enlarged error operators $\{A_k\}$ is correctable iff,

$$P_C A_l^\dagger A_m P_C \propto P_C. \quad (93)$$

Condition (93) requires that for correctable errors it must be,

$$\langle 0_L | A_l^\dagger A_m | 0_L \rangle = \langle 1_L | A_l^\dagger A_m | 1_L \rangle \text{ and, } \langle 0_L | A_l^\dagger A_m | 1_L \rangle = 0 = \langle 1_L | A_l^\dagger A_m | 0_L \rangle. \quad (94)$$

In the case under investigation, it turns out that the following set of enlarged error operators is correctable

$$\mathcal{A}_{\text{correctable}} \stackrel{\text{def}}{=} \{A_{0000}, A_{1000}, A_{0100}, A_{0010}, A_{0001}\} \subseteq \mathcal{A}_{\text{detectable}} \subseteq \mathcal{A}_{\text{total}}. \quad (95)$$

When $l \neq m$, error operators in $\mathcal{A}_{\text{correctable}}$ perfectly (arbitrary order in γ) satisfy the conditions in (94). When $l = m$, we have

$$\begin{aligned} \langle 0_L | A_{0000}^\dagger A_{0000} | 1_L \rangle &= 0 = \langle 1_L | A_{0000}^\dagger A_{0000} | 0_L \rangle, \quad \langle 0_L | A_{0000}^\dagger A_{0000} | 0_L \rangle = 1 - 2\gamma + \mathcal{O}(\gamma^2) = \langle 1_L | A_{0000}^\dagger A_{0000} | 1_L \rangle, \\ \langle 0_L | A_{1000}^\dagger A_{1000} | 1_L \rangle &= 0 = \langle 1_L | A_{1000}^\dagger A_{1000} | 0_L \rangle, \quad \langle 0_L | A_{1000}^\dagger A_{1000} | 0_L \rangle = \frac{\gamma}{2} + \mathcal{O}(\gamma^2) = \langle 1_L | A_{1000}^\dagger A_{1000} | 1_L \rangle, \\ \langle 0_L | A_{0100}^\dagger A_{0100} | 1_L \rangle &= 0 = \langle 1_L | A_{0100}^\dagger A_{0100} | 0_L \rangle, \quad \langle 0_L | A_{0100}^\dagger A_{0100} | 0_L \rangle = \frac{\gamma}{2} + \mathcal{O}(\gamma^2) = \langle 1_L | A_{0100}^\dagger A_{0100} | 1_L \rangle, \\ \langle 0_L | A_{0010}^\dagger A_{0010} | 1_L \rangle &= 0 = \langle 1_L | A_{0010}^\dagger A_{0010} | 0_L \rangle, \quad \langle 0_L | A_{0010}^\dagger A_{0010} | 0_L \rangle = \frac{\gamma}{2} + \mathcal{O}(\gamma^2) = \langle 1_L | A_{0010}^\dagger A_{0010} | 1_L \rangle, \\ \langle 0_L | A_{0001}^\dagger A_{0001} | 1_L \rangle &= 0 = \langle 1_L | A_{0001}^\dagger A_{0001} | 0_L \rangle, \quad \langle 0_L | A_{0001}^\dagger A_{0001} | 0_L \rangle = \frac{\gamma}{2} + \mathcal{O}(\gamma^2) = \langle 1_L | A_{0001}^\dagger A_{0001} | 1_L \rangle, \end{aligned} \quad (96)$$

therefore, the KL-conditions in (94) are only approximately fulfilled to the first order in the damping parameter γ .

1. The standard QEC recovery operation

Let us construct now the standard QEC recovery operators. Following our remarks in Section II, we observe that the suitable orthonormal basis $\mathcal{B}_{\mathcal{H}_2^4}$ for the 16-dimensional *complex* Hilbert space \mathcal{H}_2^4 reads,

$$\mathcal{B}_{\mathcal{H}_2^4} \stackrel{\text{def}}{=} \{|v_r^{jL}\rangle, |o_k\rangle\}, \quad (97)$$

with,

$$\begin{aligned} |v_0\rangle &\stackrel{\text{def}}{=} \frac{|0000\rangle + (1-\gamma)^2 |1111\rangle}{\sqrt{1+(1-\gamma)^4}}, \quad |v_1\rangle \stackrel{\text{def}}{=} \frac{|0011\rangle + |1100\rangle}{\sqrt{2}}, \quad |v_2\rangle \stackrel{\text{def}}{=} |0111\rangle, \quad |v_3\rangle \stackrel{\text{def}}{=} |0100\rangle, \\ |v_4\rangle &\stackrel{\text{def}}{=} |1011\rangle, \quad |v_5\rangle \stackrel{\text{def}}{=} |1000\rangle, \quad |v_6\rangle \stackrel{\text{def}}{=} |1101\rangle, \quad |v_7\rangle \stackrel{\text{def}}{=} |0001\rangle, \quad |v_8\rangle \stackrel{\text{def}}{=} |1110\rangle, \\ |v_9\rangle &\stackrel{\text{def}}{=} |0010\rangle, \end{aligned} \quad (98)$$

and,

$$\begin{aligned} |o_1\rangle &\equiv |v_{10}\rangle \stackrel{\text{def}}{=} |0101\rangle, \quad |o_2\rangle \equiv |v_{11}\rangle \stackrel{\text{def}}{=} |0110\rangle, \quad |o_3\rangle \equiv |v_{12}\rangle \stackrel{\text{def}}{=} |1001\rangle, \quad |o_4\rangle \equiv |v_{13}\rangle \stackrel{\text{def}}{=} |1010\rangle, \\ |o_5\rangle &\equiv |v_{14}\rangle \stackrel{\text{def}}{=} \frac{(1-\gamma)^2 |0000\rangle - |1111\rangle}{\sqrt{1+(1-\gamma)^4}}, \quad |o_6\rangle \equiv |v_{15}\rangle \stackrel{\text{def}}{=} \frac{|0011\rangle - |1100\rangle}{\sqrt{2}}. \end{aligned} \quad (99)$$

The standard QEC recovery superoperator \mathcal{R} is given by,

$$\mathcal{R} \stackrel{\text{def}}{=} \{R_0, R_1, R_2, R_3, R_4, \hat{O}\}, \quad (100)$$

where,

$$\begin{aligned} R_0 &\stackrel{\text{def}}{=} |0_L\rangle \langle v_0| + |1_L\rangle \langle v_1|, \quad R_1 \stackrel{\text{def}}{=} |0_L\rangle \langle v_2| + |1_L\rangle \langle v_3|, \quad R_2 \stackrel{\text{def}}{=} |0_L\rangle \langle v_4| + |1_L\rangle \langle v_5|, \\ R_3 &\stackrel{\text{def}}{=} |0_L\rangle \langle v_6| + |1_L\rangle \langle v_7|, \quad R_4 \stackrel{\text{def}}{=} |0_L\rangle \langle v_8| + |1_L\rangle \langle v_9|, \quad \hat{O} \stackrel{\text{def}}{=} \sum_{k=1}^6 |o_k\rangle \langle o_k|. \end{aligned} \quad (101)$$

We remark that, unlike the exact case, we have now:

- The two eigenvalues λ_{\max} and λ_{\min} of the (2×2) -matrix associated with the operators $P_{\mathcal{C}} A_l^\dagger A_m P_{\mathcal{C}}$ (with A_k correctable errors) on the codespace \mathcal{C} do not coincide. For example, for $P_{\mathcal{C}} A_0^\dagger A_0 P_{\mathcal{C}}$ we have $\lambda_{\max} = \frac{1+(1-\gamma)^4}{2}$ and $\lambda_{\min} = (1-\gamma)^2$. The discrepancy between the two eigenvalues is a fingerprint of the non-unitarity of $A_k P_{\mathcal{C}}$ where A_k is correctable;
- The non-tracelessness of the operators $P_{\mathcal{C}} A_l^\dagger A_m P_{\mathcal{C}}$ with $l \neq m$ is an indicator of the non-orthogonality between $A_m P_{\mathcal{C}}$ and $A_l P_{\mathcal{C}}$;
- The projector on the codespace $P_{\mathcal{C}}$ does not belong to \mathcal{R} , the standard QEC recovery;
- There exist recovery operators in \mathcal{R} that are γ -dependent, where γ denotes the damping probability and is the single parameter that characterizes the noise model being considered.

In this case, it turns out that the entanglement fidelity becomes,

$$\begin{aligned}
\mathcal{F}_{[[4,1]]}^{\text{QEC-recovery}}(\gamma) &\stackrel{\text{def}}{=} \frac{1}{(2)^2} \sum_{l,k} |\text{Tr}(R_k A_l)_C|^2 \\
&= \frac{1}{4} \left(\sqrt{\frac{1+(1-\gamma)^4}{2}} + \sqrt{\frac{2(1-\gamma)^2}{2}} \right)^2 + \left(\sqrt{\frac{\gamma(1-\gamma)^3}{2}} + \sqrt{\frac{\gamma(1-\gamma)}{2}} \right)^2 + \left(\frac{1}{4} \frac{2}{1+(1-\gamma)^4} \left(\frac{\gamma^2}{2} \right)^2 \right) + \\
&\quad + \left(\frac{1}{4} \left(\frac{\gamma^2(1-\gamma)^2((1-\gamma)^2-1)}{2(1+(1-\gamma)^4)} \right)^2 \right) \\
&\approx 1 - 2\gamma^2 + O(\gamma^3), \tag{102}
\end{aligned}$$

that is,

$$\mathcal{F}_{[[4,1]]}^{\text{QEC-recovery}}(\gamma) \approx 1 - 2\gamma^2 + O(\gamma^3). \tag{103}$$

We stress that $\mathcal{F}_{[[4,1]]}^{\text{QEC-recovery}}(\gamma)$ is, in principle, the sum of $5 \times 16 = 80$ -terms that arise by considering all the possible pairs (k, l) with $k \in \{0, \dots, 4\}$ and $l \in \{0, \dots, 16\}$. However, it turns out that only 6-terms are nonvanishing and contribute to the computation of the entanglement fidelity. They are $\{(k, l)\} = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (0, 15)\}$. Thus, not only the five correctable errors $\{A_0, A_1, A_2, A_3, A_4\}$ are partially recoverable since

$$\langle 0_L | A_j^\dagger A'_j | 0_L \rangle \neq \langle 1_L | A_j^\dagger A'_j | 1_L \rangle \quad (\text{for arbitrary orders in } \gamma), \tag{104}$$

but there is also the emergence of an off-diagonal contribution $(0, 15)$. Thus, unlike the exact scenario, we have for the approximate case that:

- Not only the correctable errors are recoverable. Indeed, they are not fully recoverable. Off-diagonal contributions do arise.

2. The code-projected recovery operation

As we have noticed in Eq. (101), the standard QEC recovery does not contain the projector on the codespace as possible recovery operator. In this new case, the chosen (orthonormal) basis vectors spanning \mathcal{H}_2^4 are given by,

$$\begin{aligned}
|v_0\rangle &\stackrel{\text{def}}{=} \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}, \quad |v_1\rangle \stackrel{\text{def}}{=} \frac{|0011\rangle + |1100\rangle}{\sqrt{2}}, \quad |v_2\rangle \stackrel{\text{def}}{=} \frac{|0000\rangle - |1111\rangle}{\sqrt{2}}, \quad |v_3\rangle \stackrel{\text{def}}{=} \frac{|0011\rangle - |1100\rangle}{\sqrt{2}}, \\
|v_4\rangle &\stackrel{\text{def}}{=} |0111\rangle, \quad |v_5\rangle \stackrel{\text{def}}{=} |0100\rangle, \quad |v_6\rangle \stackrel{\text{def}}{=} |1011\rangle, \quad |v_7\rangle \stackrel{\text{def}}{=} |1000\rangle, \quad |v_8\rangle \stackrel{\text{def}}{=} |1101\rangle, \quad |v_9\rangle \stackrel{\text{def}}{=} |0001\rangle, \\
|v_{10}\rangle &\stackrel{\text{def}}{=} |1110\rangle, \quad |v_{11}\rangle \stackrel{\text{def}}{=} |0010\rangle, \quad |v_{12}\rangle \stackrel{\text{def}}{=} |1001\rangle, \quad |v_{13}\rangle \stackrel{\text{def}}{=} |1010\rangle, \quad |v_{14}\rangle \stackrel{\text{def}}{=} |0101\rangle, \quad |v_{15}\rangle \stackrel{\text{def}}{=} |0110\rangle. \tag{105}
\end{aligned}$$

The code-projected recovery (CP recovery) becomes,

$$\mathcal{R} \stackrel{\text{def}}{=} \{R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_9, R_{10}\}, \tag{106}$$

where,

$$R_1 \stackrel{\text{def}}{=} |0_L\rangle \langle 0_L| + |1_L\rangle \langle 1_L|,$$

$$R_2 \stackrel{\text{def}}{=} |0_L\rangle \left(\frac{1}{\sqrt{2}} \langle 0000| - \frac{1}{\sqrt{2}} \langle 1111| \right) + |1_L\rangle \left(\frac{1}{\sqrt{2}} \langle 0011| - \frac{1}{\sqrt{2}} \langle 1100| \right),$$

$$R_3 \stackrel{\text{def}}{=} |0_L\rangle \langle 0111| + |1_L\rangle \langle 0100| = R_{A_{1000}}, R_4 \stackrel{\text{def}}{=} |0_L\rangle \langle 1011| + |1_L\rangle \langle 1000| = R_{A_{0100}}, R_5 \stackrel{\text{def}}{=} |0_L\rangle \langle 1101| + |1_L\rangle \langle 0001| = R_{A_{0010}},$$

$$R_6 \stackrel{\text{def}}{=} |0_L\rangle \langle 1110| + |1_L\rangle \langle 0010| = R_{A_{0001}}, R_7 \stackrel{\text{def}}{=} |0_L\rangle \langle 1001| = R_{A_{0110}}, R_8 \stackrel{\text{def}}{=} |0_L\rangle \langle 1010| = R_{A_{0101}},$$

$$R_9 \stackrel{\text{def}}{=} |0_L\rangle \langle 0101| = R_{A_{1010}}, R_{10} \stackrel{\text{def}}{=} |0_L\rangle \langle 0110| = R_{A_{1001}}. \quad (107)$$

Observe that,

$$\sum_{k=1}^{10} P_k \stackrel{\text{def}}{=} \sum_{k=1}^{10} R_k^\dagger R_k = R_1^\dagger R_1 + R_2^\dagger R_2 + R_3^\dagger R_3 + R_4^\dagger R_4 + R_5^\dagger R_5 + R_6^\dagger R_6 + R_7^\dagger R_7 + R_8^\dagger R_8 + R_9^\dagger R_9 + R_{10}^\dagger R_{10}$$

$$= |0000\rangle \langle 0000| + \dots + |1111\rangle \langle 1111| = I_{2^4 \times 2^4}. \quad (108)$$

The entanglement fidelity becomes,

$$\mathcal{F}_{[[4,1]]}^{\text{CP-recovery}}(\gamma) \stackrel{\text{def}}{=} \frac{1}{(2)^2} \sum_{k=0}^{15} \sum_{l=1}^{10} \left| \text{Tr}(R_l A'_k)_{|C} \right|^2, \quad (109)$$

where $R_l \in \mathcal{R}$ and $A'_0 \stackrel{\text{def}}{=} A_{0000}$, $A'_1 \stackrel{\text{def}}{=} A_{1000}, \dots, A'_{15} = A_{1111}$. We point out that both the recovery operators R_1 and R_2 contribute to the entanglement fidelity in Eq. (109) since,

$$\langle 0_L | R_1 A_{0000} | 0_L \rangle = 1 - \gamma + \frac{\gamma^2}{2}, \quad \langle 1_L | R_1 A_{0000} | 1_L \rangle = 1 - \gamma,$$

$$\langle 0_L | R_2 A_{0000} | 0_L \rangle = \gamma - \frac{\gamma^2}{2}, \quad \langle 1_L | R_2 A_{0000} | 1_L \rangle = 0, \quad (110)$$

and,

$$\langle 0_L | R_1 A_{1111} | 0_L \rangle + \langle 1_L | R_1 A_{1111} | 1_L \rangle = \frac{\gamma^2}{2} + 0 = \frac{\gamma^2}{2},$$

$$\langle 0_L | R_2 A_{1111} | 0_L \rangle + \langle 1_L | R_2 A_{1111} | 1_L \rangle = \frac{\gamma^2}{2} + 0 = \frac{\gamma^2}{2}. \quad (111)$$

Furthermore, the contributions of recovery operators R_3, R_4, R_5, R_6 are given by,

$$\langle 0_L | R_3 A_{1000} | 0_L \rangle + \langle 1_L | R_3 A_{1000} | 1_L \rangle = (1 - \gamma) \sqrt{\frac{\gamma(1 - \gamma)}{2}} + \sqrt{\frac{\gamma(1 - \gamma)}{2}},$$

$$\langle 0_L | R_4 A_{0100} | 0_L \rangle + \langle 1_L | R_4 A_{0100} | 1_L \rangle = (1 - \gamma) \sqrt{\frac{\gamma(1 - \gamma)}{2}} + \sqrt{\frac{\gamma(1 - \gamma)}{2}}, \quad (112)$$

and,

$$\langle 0_L | R_5 A_{0010} | 0_L \rangle + \langle 1_L | R_5 A_{0010} | 1_L \rangle = (1 - \gamma) \sqrt{\frac{\gamma(1 - \gamma)}{2}} + \sqrt{\frac{\gamma(1 - \gamma)}{2}},$$

$$\langle 0_L | R_6 A_{0001} | 0_L \rangle + \langle 1_L | R_6 A_{0001} | 1_L \rangle = (1 - \gamma) \sqrt{\frac{\gamma(1 - \gamma)}{2}} + \sqrt{\frac{\gamma(1 - \gamma)}{2}}, \quad (113)$$

respectively. Finally, the contribution arising from the recovery operators R_7, R_8, R_9, R_{10} becomes transparent once we consider the following relations,

$$\begin{aligned}
\langle 0_L | R_7 A_{0110} | 0_L \rangle &= \frac{\gamma(1-\gamma)}{\sqrt{2}}, \quad \langle 1_L | R_7 A_{0110} | 1_L \rangle = 0, \\
\langle 0_L | R_8 A_{0101} | 0_L \rangle + \langle 1_L | R_8 A_{0101} | 1_L \rangle &= \frac{\gamma(1-\gamma)}{\sqrt{2}} + 0 = \frac{\gamma(1-\gamma)}{\sqrt{2}}, \\
\langle 0_L | R_9 A_{1010} | 0_L \rangle + \langle 1_L | R_9 A_{1010} | 1_L \rangle &= \frac{\gamma(1-\gamma)}{\sqrt{2}} + 0 = \frac{\gamma(1-\gamma)}{\sqrt{2}}, \\
\langle 0_L | R_{10} A_{1001} | 0_L \rangle + \langle 1_L | R_{10} A_{1001} | 1_L \rangle &= \frac{\gamma(1-\gamma)}{\sqrt{2}} + 0 = \frac{\gamma(1-\gamma)}{\sqrt{2}}. \tag{114}
\end{aligned}$$

We notice that the six enlarged error operators $A_{1100}, A_{0011}, A_{1110}, A_{1011}, A_{0111}, A_{1101}$ do not contribute to the computation of the entanglement fidelity $\mathcal{F}_{[[4,1]]}^{\text{CP-recovery}}(\gamma)$. Finally, we obtain

$$\begin{aligned}
\mathcal{F}_{[[4,1]]}^{\text{CP-recovery}}(\gamma) &= \frac{1}{4} \left\{ \left[\left(1 - \gamma + \frac{\gamma^2}{2}\right) + (1 - \gamma) \right]^2 + \left(\gamma - \frac{\gamma^2}{2} \right)^2 + 2 \left(\frac{\gamma^2}{2} \right)^2 + 4 \left[(2 - \gamma) \sqrt{\frac{\gamma(1-\gamma)}{2}} \right]^2 + \right. \\
&\quad \left. + 4 \left[\frac{\gamma(1-\gamma)}{\sqrt{2}} \right]^2 \right\}, \\
&\approx 1 - \frac{7}{4} \gamma^2 + \mathcal{O}(\gamma^3), \tag{115}
\end{aligned}$$

that is,

$$\mathcal{F}_{[[4,1]]}^{\text{CP-recovery}}(\gamma) \approx 1 - \frac{7}{4} \gamma^2 + \mathcal{O}(\gamma^3). \tag{116}$$

From Eqs. (103) and (116), it turns out that as far as the entanglement fidelity concerns, the CP recovery scheme is more successful than the standard QEC recovery scheme.

3. An analytically-optimized Fletcher's-type channel-adapted recovery operation

In addition to the standard QEC and CP recovery schemes, it is possible to consider additional recovery schemes such as an analytically-optimized version of a channel-adapted recovery scheme as proposed by Fletcher *et al.* in [15].

Consider the CP recovery scheme in Eq. (106) where, however, the recovery operators R_1 and R_2 are defined as [15],

$$\begin{aligned}
R_1 &\stackrel{\text{def}}{=} |0_L\rangle (a \langle 0000| + b \langle 1111|) + |1_L\rangle \left(\frac{1}{\sqrt{2}} \langle 0011| + \frac{1}{\sqrt{2}} \langle 1100| \right) \\
&= |0_L\rangle (a \langle 0000| + b \langle 1111|) + |1_L\rangle \langle 1_L|, \tag{117}
\end{aligned}$$

and,

$$R_2 \stackrel{\text{def}}{=} |0_L\rangle (b^* \langle 0000| - a^* \langle 1111|) + |1_L\rangle \left(\frac{1}{\sqrt{2}} \langle 0011| - \frac{1}{\sqrt{2}} \langle 1100| \right), \tag{118}$$

respectively, where $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. The remaining eight recovery operators are defined just as in Eq. (106). The set of all ten recovery operators forms the Fletcher *et al.* recovery operation $\mathcal{R}_{\text{Fletcher}}$. Note that,

$$R_1^\dagger = a^* |0000\rangle \langle 0_L| + b^* |1111\rangle \langle 0_L| + |1_L\rangle \langle 1_L|, \tag{119}$$

and,

$$R_2^\dagger = b |0000\rangle \langle 0_L| - a |1111\rangle \langle 0_L| + \frac{1}{\sqrt{2}} |0011\rangle \langle 1_L| - \frac{1}{\sqrt{2}} |1100\rangle \langle 1_L|. \quad (120)$$

Using (119) and (120), we get

$$R_1^\dagger R_1 + R_2^\dagger R_2 = |0000\rangle \langle 0000| + |0011\rangle \langle 0011| + |1100\rangle \langle 1100| + |1111\rangle \langle 1111|. \quad (121)$$

It turns out that for the Fletcher *et al.* recovery (F-recovery) operations,

$$\sum_k P_k = \sum_k R_k^\dagger R_k = |0000\rangle \langle 0000| + \dots + |1111\rangle \langle 1111| = I_{2^4 \times 2^4}. \quad (122)$$

In this case, the entanglement fidelity $\mathcal{F}_{[[4,1]]}^{\text{F-recovery}}$ reads,

$$\mathcal{F}_{[[4,1]]}^{\text{F-recovery}} \stackrel{\text{def}}{=} \frac{1}{(2)^2} \sum_{k=0}^{15} \sum_{l=1}^{10} \left| \text{Tr} (R_l A'_k)_{|C} \right|^2, \quad (123)$$

where $R_l \in \mathcal{R}_{\text{Fletcher}}$ and $A'_0 \stackrel{\text{def}}{=} A_{0000}$, $A'_1 \stackrel{\text{def}}{=} A_{1000}, \dots, A'_{15} = A_{1111}$. Notice that,

$$\begin{aligned} \langle 0_L | R_1 A_{0000} | 0_L \rangle &= \frac{a + b(1-\gamma)^2}{\sqrt{2}}, \quad \langle 1_L | R_1 A_{0000} | 1_L \rangle = 1 - \gamma, \\ \langle 0_L | R_2 A_{0000} | 0_L \rangle &= \frac{b^* - a^*(1-\gamma)^2}{\sqrt{2}}, \quad \langle 1_L | R_2 A_{0000} | 1_L \rangle = 0, \\ \langle 0_L | R_1 A_{1111} | 0_L \rangle &= a \frac{\gamma^2}{\sqrt{2}}, \quad \langle 1_L | R_1 A_{1111} | 1_L \rangle = 0, \\ \langle 0_L | R_2 A_{1111} | 0_L \rangle &= b^* \frac{\gamma^2}{\sqrt{2}}, \quad \langle 1_L | R_2 A_{1111} | 1_L \rangle = 0, \end{aligned} \quad (124)$$

while the remaining terms are the same as obtained in the previous analysis performed with the traditional QEC recovery scheme. Following the line of reasoning provide in the former computations, we get $\mathcal{F}_{[[4,1]]}^{\text{F-recovery}}(\gamma) \equiv \mathcal{F}_{[[4,1]]}(a, b, \gamma)$ with,

$$\mathcal{F}_{[[4,1]]}(a, b, \gamma) = \frac{1}{4} \left\{ \left| \frac{a + b(1-\gamma)^2}{\sqrt{2}} + (1-\gamma) \right|^2 + \left| \frac{b^* - a^*(1-\gamma)^2}{\sqrt{2}} \right|^2 + 2\gamma(1-\gamma)(2-\gamma)^2 + 2\gamma^2(1-\gamma)^2 + \frac{\gamma^4}{2} \right\}. \quad (125)$$

We wish to maximize $\mathcal{F}_{[[4,1]]}(a, b, \gamma)$. The problem is to find \bar{a} and \bar{b} (perhaps γ -dependent quantities) such that $\mathcal{F}_{[[4,1]]}(\bar{a}, \bar{b}, \gamma)$ denotes the searched maximum,

$$\mathcal{F}_{[[4,1]]}(\bar{a}, \bar{b}, \gamma) = \max_{|a|^2 + |b|^2 = 1} \mathcal{F}(a, b, \gamma). \quad (126)$$

It can be shown that (for details, see Appendix C),

$$\mathcal{F}_{[[4,1]]}(\bar{a}, \bar{b}, \gamma) = 1 - \frac{3}{2}\gamma^2 + \mathcal{O}(\gamma^3), \quad (127)$$

with

$$\bar{a}(\gamma) \stackrel{\text{def}}{=} \frac{1}{\sqrt{1 + (1-\gamma)^4}} \quad \text{and} \quad \bar{b}(\gamma) \stackrel{\text{def}}{=} \frac{(1-\gamma)^2}{\sqrt{1 + (1-\gamma)^4}}. \quad (128)$$

Finally,

$$\mathcal{F}_{[[4,1]]}^{\text{F-recovery}}(\gamma) \approx 1 - \frac{3}{2}\gamma^2 + \mathcal{O}(\gamma^3). \quad (129)$$

From Eqs. (103), (116) and (129), it turns out that as far as the entanglement fidelity concerns, the analytically-optimized F-recovery scheme is better than both the standard QEC and CP recovery schemes. The comparison of the three recovery schemes employed can be visualized in Fig. 1.

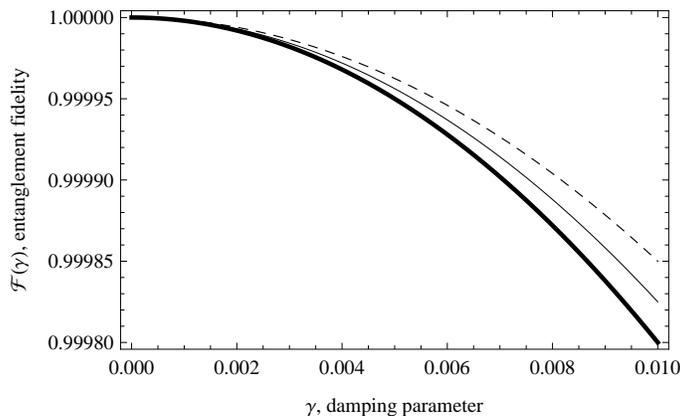


FIG. 1: The truncated series expansion of the entanglement fidelity $\mathcal{F}(\gamma)$ vs. the amplitude damping parameter γ with $0 \leq \gamma \leq 10^{-2}$ for the Leung et al. four-qubit code for amplitude damping errors; the Fletcher-type recovery (dashed line), the code-projected recovery (thin solid line) and, the standard QEC-recovery (thick solid line).

IV. FINAL REMARKS

In this article, we presented a comparative analysis of exact and approximate quantum error correction by means of simple unabridged analytical computations. For the sake of clarity, using primitive quantum codes, we showed a detailed study of exact and approximate error correction for the two simplest unital (Pauli errors) and nonunital (non-Pauli errors) noise models, respectively. The similarities and differences between the two scenarios were stressed. In addition, the performances of quantum codes quantified by means of the entanglement fidelity for different recovery schemes were taken into consideration in the approximate case.

Our main findings, some of which appear in the appendices to ease the readability of the article, can be outlined as follows:

1. We have explicitly constructed one of the recovery operators as originally proposed by Leung et al. in [13]. As a by-product, we also found the correct version of Eq. (41) in [13]. Our version is represented by Eq. (A10) in Appendix A.
2. We have explicitly discussed the similarities and differences between exact and approximate-QEC schemes for very simple noise models and very common stabilizer codes. Our analysis is purely analytical and no numerical consideration is required. Thus, it is straightforward to follow and, we believe, has considerable pedagogical and explanatory relevance. In particular, the points to be stressed in the *exact case* are:
 - The two eigenvalues λ_{\max} and λ_{\min} of the (2×2) -matrix associated with the operators $P_{\mathcal{C}}A_l^\dagger A_m P_{\mathcal{C}}$ (with A_k correctable errors) on the codespace \mathcal{C} coincide;
 - The projector on the codespace $P_{\mathcal{C}}$ belongs to the standard QEC recovery \mathcal{R} ;
 - All the recovery operators in \mathcal{R} are p -independent;
 - The correctable errors are fully recoverable. No off-diagonal contribution arises.

On the other side, the main points to be stressed in the *approximate case* are:

- The two eigenvalues λ_{\max} and λ_{\min} of the (2×2) -matrix associated with the operators $P_{\mathcal{C}}A_l^\dagger A_m P_{\mathcal{C}}$ (with A_k correctable errors) on the codespace \mathcal{C} do not coincide;
- The discrepancy between the two eigenvalues is a fingerprint of the non-unitarity of $A_k P_{\mathcal{C}}$ where A_k is a correctable error;
- The non-tracelessness of the operators $P_{\mathcal{C}}A_l^\dagger A_m P_{\mathcal{C}}$ with $l \neq m$ is an indicator of the non-orthogonality between $A_m P_{\mathcal{C}}$ and $A_l P_{\mathcal{C}}$;
- The projector on the codespace $P_{\mathcal{C}}$ does not belong to the standard QEC recovery \mathcal{R} ;

- There exist recovery operators in \mathcal{R} that are γ -dependent;
 - The correctable errors are not fully recoverable. Off-diagonal contributions do arise.
3. We have explicitly shown that there are only three possible self-complementary quantum codes characterized by a two-dimensional subspace of the sixteen-dimensional *complex* Hilbert space \mathcal{H}_2^4 capable of error-correcting single-AD errors. Thus, in this regard, the Leung *et al.* four-qubit code is not unique. Our three codes appear in Eqs. (B7), (B8) and (B9) in Appendix B.
 4. In the approximate-QEC case, we have explicitly computed the entanglement fidelity for three different recovery schemes. In particular, Eq. (103) for the standard QEC recovery has, to the best of our knowledge, never appeared in the literature (neither numerically nor analytically); furthermore, Eq. (116) for the code-projected recovery is the analytical counterpart of the numerical finding presented in [15]; finally, Eq. (129) represents our analytical contribution to the understanding of the numerical result presented in [19].

Although our investigation is limited to very simple noise models and very simple codes, we hope that it will inspire other researchers to pursue novel analytical studies of more realistic noise models and higher-dimensional quantum codes. After all, for such type of investigations, analytical computations can become considerably messy (as pointed out in [29]) and understanding in an analytical fashion recovery maps numerically computed can become quite a tricky task as well (as stressed in [20]).

In conclusion, also in view of these very last considerations, we are very confident about the relevance of the pedagogical nature of our analytical investigation carried out in this article.

Acknowledgments

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Appendix A: Unabridged computation of the recovery operation

For the sake of clarity, we limit our analysis to the explicit computation of the Leung *et al.* recovery operator for the enlarged error operator A_{0000} . In principle, the remaining recovery operators can be computed in the same manner. Observe that in general we should be dealing with operators acting on the 16-dimensional *complex* Hilbert space \mathcal{H}_2^4 . However, in what follows, we shall take into consideration only lower-dimensional matrix-representations of operators where the dimension is limited to nontrivial contributions. For instance, for A_{0000} and P_C , we consider their matrix-representation restricted to the four-dimensional subspace of \mathcal{H}_2^4 spanned by the orthonormal vectors,

$$\{|0000\rangle, |0011\rangle, |1100\rangle, |1111\rangle\}. \quad (\text{A1})$$

We obtain,

$$P_C \stackrel{\text{def}}{=} \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and,} \quad A_{0000} \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-\gamma & 0 & 0 \\ 0 & 0 & 1-\gamma & 0 \\ 0 & 0 & 0 & (1-\gamma)^2 \end{pmatrix}. \quad (\text{A2})$$

From Eq. (A2), it follows that the two eigenvalues of $P_C A_{0000}^\dagger A_{0000} P_C$ are given by,

$$\lambda_{\min} \stackrel{\text{def}}{=} (1-\gamma)^2 \quad \text{and,} \quad \lambda_{\max} \stackrel{\text{def}}{=} \frac{1+(1-\gamma)^4}{2}, \quad (\text{A3})$$

while $\sqrt{P_C A_{0000}^\dagger A_{0000} P_C}$ reads,

$$\sqrt{P_C A_{0000}^\dagger A_{0000} P_C} = \begin{pmatrix} \frac{(1-\gamma)^4+1}{4} & 0 & 0 & \frac{(1-\gamma)^4+1}{4} \\ 0 & \frac{(1-\gamma)^2}{2} & \frac{(1-\gamma)^2}{2} & 0 \\ 0 & \frac{(1-\gamma)^2}{2} & \frac{(1-\gamma)^2}{2} & 0 \\ \frac{(1-\gamma)^4+1}{4} & 0 & 0 & \frac{(1-\gamma)^4+1}{4} \end{pmatrix}^{\frac{1}{2}}. \quad (\text{A4})$$

After some algebra, we have

$$\sqrt{P_C A_{0000}^\dagger A_{0000} P_C} = \lambda_1 |v_1\rangle \langle v_1| + \lambda_2 |v_2\rangle \langle v_2| + \lambda_3 |v_3\rangle \langle v_3| + \lambda_4 |v_4\rangle \langle v_4|, \quad (\text{A5})$$

where,

$$\lambda_1 \stackrel{\text{def}}{=} 1 - \gamma, \lambda_2 \stackrel{\text{def}}{=} 0, \lambda_3 \stackrel{\text{def}}{=} 0, \lambda_4 \stackrel{\text{def}}{=} \sqrt{\frac{1 + (1 - \gamma)^4}{2}}, \quad (\text{A6})$$

and,

$$|v_1\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, |v_2\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, |v_3\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, |v_4\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (\text{A7})$$

Substituting (A7) and (A6) into (A5), we get

$$\sqrt{P_C A_{0000}^\dagger A_{0000} P_C} = \frac{1}{2} \begin{pmatrix} \sqrt{\frac{(1-\gamma)^4+1}{2}} & 0 & 0 & \sqrt{\frac{(1-\gamma)^4+1}{2}} \\ 0 & 1-\gamma & 1-\gamma & 0 \\ 0 & 1-\gamma & 1-\gamma & 0 \\ \sqrt{\frac{(1-\gamma)^4+1}{2}} & 0 & 0 & \sqrt{\frac{(1-\gamma)^4+1}{2}} \end{pmatrix}. \quad (\text{A8})$$

Recall that the residue operator π_{0000} is given by,

$$\pi_{0000} \stackrel{\text{def}}{=} \sqrt{P_C A_{0000}^\dagger A_{0000} P_C} - \sqrt{\lambda_{\min}} P_C, \quad (\text{A9})$$

that is,

$$\pi_{0000} = \begin{pmatrix} \frac{1}{2}\gamma + \frac{1}{2}\sqrt{\frac{1}{2}(\gamma-1)^4 + \frac{1}{2} - \frac{1}{2}} & 0 & 0 & \frac{1}{2}\gamma + \frac{1}{2}\sqrt{\frac{1}{2}(\gamma-1)^4 + \frac{1}{2} - \frac{1}{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}\gamma + \frac{1}{2}\sqrt{\frac{1}{2}(\gamma-1)^4 + \frac{1}{2} - \frac{1}{2}} & 0 & 0 & \frac{1}{2}\gamma + \frac{1}{2}\sqrt{\frac{1}{2}(\gamma-1)^4 + \frac{1}{2} - \frac{1}{2}} \end{pmatrix}. \quad (\text{A10})$$

Observe that for $\gamma \ll 1$, π_{0000} becomes

$$\pi_{0000} = \frac{1}{2}\gamma^2 \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(\gamma^4). \quad (\text{A11})$$

Let us focus now on the computation of the unitary operator U_{0000} . From Eqs. (A2) and (A7), we get

$$A_{0000} P_C |v_1\rangle \stackrel{\text{def}}{=} \frac{1-\gamma}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, A_{0000} P_C |v_2\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, A_{0000} P_C |v_3\rangle \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$A_{0000} P_C |v_4\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ (1-\gamma)^2 \end{pmatrix}. \quad (\text{A12})$$

Consider the following basis given by,

$$\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}, \quad (\text{A13})$$

with,

$$\begin{aligned}
|e_1\rangle &\stackrel{\text{def}}{=} \frac{A_{0000}P_{\mathcal{C}}|v_1\rangle}{\lambda_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |e_2\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad |e_3\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
|e_4\rangle &\stackrel{\text{def}}{=} \frac{A_{0000}P_{\mathcal{C}}|v_4\rangle}{\lambda_4} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\frac{1+(1-\gamma)^4}{2}}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ (1-\gamma)^2 \end{pmatrix}.
\end{aligned} \tag{A14}$$

Applying the Gram-Schmidt orthonormalization procedure to $\{|e_1\rangle, |e_2\rangle, |e_3\rangle, |e_4\rangle\}$, we get

$$\begin{aligned}
|E_1\rangle &\stackrel{\text{def}}{=} \frac{A_{0000}P_{\mathcal{C}}|v_1\rangle}{\lambda_1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad |E_2\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad |E_3\rangle \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\frac{1+(1-\gamma)^4}{2}}} \begin{pmatrix} -(1-\gamma)^2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \\
|E_4\rangle &\stackrel{\text{def}}{=} \frac{A_{0000}P_{\mathcal{C}}|v_4\rangle}{\lambda_4} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{\frac{1+(1-\gamma)^4}{2}}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ (1-\gamma)^2 \end{pmatrix},
\end{aligned} \tag{A15}$$

with $\langle E_i | E_k \rangle = \delta_{ik}$. Finally, the unitary operator U_{0000} reads,

$$U_{0000} \stackrel{\text{def}}{=} |E_1\rangle \langle v_1| + |E_2\rangle \langle v_2| + |E_3\rangle \langle v_3| + |E_4\rangle \langle v_4|, \tag{A16}$$

that is, using Eqs. (A7) and (A15),

$$U_{0000} \stackrel{\text{def}}{=} \begin{pmatrix} \frac{1}{\sqrt{2}} \frac{1+(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}} & 0 & 0 & \frac{1}{\sqrt{2}} \frac{1-(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} \frac{1-(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}} & 0 & 0 & \frac{1}{\sqrt{2}} \frac{1+(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}} \end{pmatrix}. \tag{A17}$$

Finally, the Leung *et al.* recovery operator associated with the enlarged error operator A_{0000} is given by,

$$R_{0000} = P_{\mathcal{C}} U_{0000}^\dagger, \tag{A18}$$

with $P_{\mathcal{C}}$ in Eq. (A2) and U_{0000} in Eq. (A17).

Appendix B: Self-complementary codes

1. Part 1

Let \mathcal{C} be a $[[n, k, d]]$ quantum stabilizer code that spans a 2^k -dimensional subspace of a 2^n -dimensional Hilbert space. Two quantum codes $\mathcal{C}^{(1)}$ and $\mathcal{C}^{(2)}$ are *locally permutation equivalent* if $\mathcal{C}^{(2)} = \tau \mathcal{C}^{(1)}$ with $\tau \stackrel{\text{def}}{=} \pi T$ where T is a local unitary transformation in $U(2)^{\otimes n}$ and π is a permutation of the qubits. When $\mathcal{C}^{(2)} = \tau \mathcal{C}^{(1)}$ with $\tau \stackrel{\text{def}}{=} T$, we say that the two quantum codes are *locally equivalent*. Finally, we say that the two codes are globally equivalent, or simply equivalent, if $\mathcal{C}^{(1)}$ is locally equivalent to a code obtained from $\mathcal{C}^{(2)}$ by a permutation on qubits.

Assuming single-qubit encoding, how many pairs $(|0_L\rangle, |1_L\rangle)$ [17],

$$\left(|0_L\rangle \stackrel{\text{def}}{=} \frac{|u\rangle + |\bar{u}\rangle}{\sqrt{2}}, \quad |1_L\rangle \stackrel{\text{def}}{=} \frac{|v\rangle + |\bar{v}\rangle}{\sqrt{2}} \right), \tag{B1}$$

of orthonormal self-complementary codewords can we construct in the *complex* Hilbert space \mathcal{H}_2^4 ? To be explicit, recall that the canonical computational basis of \mathcal{H}_2^4 reads,

$$\mathcal{B}_{\mathcal{H}_2^4} \stackrel{\text{def}}{=} \left\{ \begin{array}{l} |e_{0000}\rangle \stackrel{\text{def}}{=} |0000\rangle, |e_{1000}\rangle \stackrel{\text{def}}{=} |1000\rangle, |e_{0100}\rangle \stackrel{\text{def}}{=} |0100\rangle, |e_{0010}\rangle \stackrel{\text{def}}{=} |0010\rangle, \\ |e_{0001}\rangle \stackrel{\text{def}}{=} |0001\rangle, |e_{1100}\rangle \stackrel{\text{def}}{=} |1100\rangle, |e_{1010}\rangle \stackrel{\text{def}}{=} |1010\rangle, |e_{1001}\rangle \stackrel{\text{def}}{=} |1001\rangle, |e_{0110}\rangle \stackrel{\text{def}}{=} |0110\rangle, \\ |e_{0101}\rangle \stackrel{\text{def}}{=} |0101\rangle, |e_{0011}\rangle \stackrel{\text{def}}{=} |0011\rangle, |e_{1110}\rangle \stackrel{\text{def}}{=} |1110\rangle, |e_{1101}\rangle \stackrel{\text{def}}{=} |1101\rangle, |e_{0111}\rangle \stackrel{\text{def}}{=} |0111\rangle, \\ |e_{1011}\rangle \stackrel{\text{def}}{=} |1011\rangle, |e_{1111}\rangle \stackrel{\text{def}}{=} |1111\rangle \end{array} \right\}. \quad (\text{B2})$$

The number of possible pairs is,

$$\# \text{ pairs of orthonormal self-complementary codewords in } \mathcal{H}_2^4 = \frac{8^2 - 8}{2} = 28, \quad (\text{B3})$$

and they are given by,

$$\left(|v_i^{(+)}\rangle, |v_j^{(+)}\rangle \right), \quad i < j \in \{1, 2, \dots, 8\}, \quad (\text{B4})$$

where,

$$\begin{aligned} |v_1^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|0000\rangle + |1111\rangle}{\sqrt{2}}, & |v_2^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|1000\rangle + |0111\rangle}{\sqrt{2}}, & |v_3^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|0100\rangle + |1011\rangle}{\sqrt{2}}, & |v_4^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|0010\rangle + |1101\rangle}{\sqrt{2}}, \\ |v_5^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|0001\rangle + |1110\rangle}{\sqrt{2}}, & |v_6^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|1100\rangle + |0011\rangle}{\sqrt{2}}, & |v_7^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|1010\rangle + |0101\rangle}{\sqrt{2}}, & |v_8^{(+)}\rangle &\stackrel{\text{def}}{=} \frac{|1001\rangle + |0110\rangle}{\sqrt{2}}. \end{aligned} \quad (\text{B5})$$

Thus, the possible combinations are

$$\begin{aligned} &\left(|v_1^{(+)}\rangle, |v_2^{(+)}\rangle \right), \left(|v_1^{(+)}\rangle, |v_3^{(+)}\rangle \right), \left(|v_1^{(+)}\rangle, |v_4^{(+)}\rangle \right), \left(|v_1^{(+)}\rangle, |v_5^{(+)}\rangle \right), \left(|v_1^{(+)}\rangle, |v_6^{(+)}\rangle \right), \left(|v_1^{(+)}\rangle, |v_7^{(+)}\rangle \right), \\ &\left(|v_1^{(+)}\rangle, |v_8^{(+)}\rangle \right), \left(|v_2^{(+)}\rangle, |v_3^{(+)}\rangle \right), \left(|v_2^{(+)}\rangle, |v_4^{(+)}\rangle \right), \left(|v_2^{(+)}\rangle, |v_5^{(+)}\rangle \right), \left(|v_2^{(+)}\rangle, |v_6^{(+)}\rangle \right), \\ &\left(|v_2^{(+)}\rangle, |v_7^{(+)}\rangle \right), \left(|v_2^{(+)}\rangle, |v_8^{(+)}\rangle \right), \left(|v_3^{(+)}\rangle, |v_4^{(+)}\rangle \right), \left(|v_3^{(+)}\rangle, |v_5^{(+)}\rangle \right), \left(|v_3^{(+)}\rangle, |v_6^{(+)}\rangle \right), \left(|v_3^{(+)}\rangle, |v_7^{(+)}\rangle \right), \\ &\left(|v_3^{(+)}\rangle, |v_8^{(+)}\rangle \right), \left(|v_4^{(+)}\rangle, |v_5^{(+)}\rangle \right), \left(|v_4^{(+)}\rangle, |v_6^{(+)}\rangle \right), \left(|v_4^{(+)}\rangle, |v_7^{(+)}\rangle \right), \left(|v_4^{(+)}\rangle, |v_8^{(+)}\rangle \right), \left(|v_5^{(+)}\rangle, |v_6^{(+)}\rangle \right), \\ &\left(|v_5^{(+)}\rangle, |v_7^{(+)}\rangle \right), \left(|v_5^{(+)}\rangle, |v_8^{(+)}\rangle \right), \left(|v_6^{(+)}\rangle, |v_7^{(+)}\rangle \right), \left(|v_6^{(+)}\rangle, |v_8^{(+)}\rangle \right), \left(|v_7^{(+)}\rangle, |v_8^{(+)}\rangle \right). \end{aligned} \quad (\text{B6})$$

It turns out that among the $\binom{8}{2} = 28$ -pairs of possible self-complementary orthogonal codewords in \mathcal{H}_2^4 , only three pairs are indeed good single-AD error correcting codes. For more details, see Part 2 of Appendix C. They are given by:

- The $\left(|v_1^{(+)}\rangle, |v_6^{(+)}\rangle \right)$ -pair that represents the Leung et al. $[[4, 1]]$ -code. The non-normalized codewords read,
$$|0_L\rangle \stackrel{\text{def}}{=} |0000\rangle + |1111\rangle \quad \text{and} \quad |1_L\rangle \stackrel{\text{def}}{=} |0011\rangle + |1100\rangle. \quad (\text{B7})$$

- The $\left(|v_1^{(+)}\rangle, |v_8^{(+)}\rangle \right)$ -pair that represents the Grassl et al. perfect quantum erasure code. The non-normalized codewords read,

$$|0_L\rangle \stackrel{\text{def}}{=} |0000\rangle + |1111\rangle \quad \text{and} \quad |1_L\rangle \stackrel{\text{def}}{=} |1001\rangle + |0110\rangle. \quad (\text{B8})$$

- The $(|v_1^{(+)}\rangle, |v_7^{(+)}\rangle)$ -pair has no specific mention in the literature, to the best of our knowledge. The non-normalized codewords read,

$$|0_L\rangle \stackrel{\text{def}}{=} |0000\rangle + |1111\rangle \quad \text{and} \quad |1_L\rangle \stackrel{\text{def}}{=} |0101\rangle + |1010\rangle. \quad (\text{B9})$$

The three codes spanned by the codewords in (B7), (B8) and (B9) are indeed globally equivalent. We point out that the codeword $|0_L\rangle \stackrel{\text{def}}{=} |0000\rangle + |1111\rangle$ is the only codeword in these 28-pairs that is invariant under any cyclical permutation of qubits. This property of $|0_L\rangle$ turns out to be very useful when checking out the global equivalence among the three good single-AD error correcting codes. In particular, the Leung *et al.* $[[4, 1]]$ -code is globally equivalent to the Grassl *et al.* perfect quantum erasure code encoding one qubit and correcting one arbitrary erasure.

2. Part 2

Observe that for the Leung *et al.* four-qubit code (normalization factors are omitted), we have

$$\begin{aligned} A_{0000} |0_L\rangle &= |0000\rangle + (1 - \gamma)^2 |1111\rangle, \quad A_{0000} |1_L\rangle = (1 - \gamma) |0011\rangle + (1 - \gamma) |1100\rangle, \\ A_{1000} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |0111\rangle, \quad A_{1000} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0100\rangle, \\ A_{0100} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1011\rangle, \quad A_{0100} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |1000\rangle, \\ A_{0010} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1101\rangle, \quad A_{0010} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0001\rangle, \\ A_{0001} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1110\rangle, \quad A_{0001} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0010\rangle. \end{aligned} \quad (\text{B10})$$

For the Grassl *et al.* four-qubit code (normalization factors are omitted), we get

$$\begin{aligned} A_{0000} |0_L\rangle &= |0000\rangle + (1 - \gamma)^2 |1111\rangle, \quad A_{0000} |1_L\rangle = (1 - \gamma) |1001\rangle + (1 - \gamma) |0110\rangle, \\ A_{1000} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |0111\rangle, \quad A_{1000} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0001\rangle, \\ A_{0100} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1011\rangle, \quad A_{0100} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0010\rangle, \\ A_{0010} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1101\rangle, \quad A_{0010} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0100\rangle, \\ A_{0001} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1110\rangle, \quad A_{0001} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |1000\rangle. \end{aligned} \quad (\text{B11})$$

Finally, for the third four-qubit code defined in Eq. (B9) (normalization factors are omitted), we obtain

$$\begin{aligned} A_{0000} |0_L\rangle &= |0000\rangle + (1 - \gamma)^2 |1111\rangle, \quad A_{0000} |1_L\rangle = (1 - \gamma) |0101\rangle + (1 - \gamma) |1010\rangle, \\ A_{1000} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |0111\rangle, \quad A_{1000} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0010\rangle, \\ A_{0100} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1011\rangle, \quad A_{0100} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0001\rangle, \\ A_{0010} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1101\rangle, \quad A_{0010} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |1000\rangle, \\ A_{0001} |0_L\rangle &= \sqrt{\gamma} (1 - \gamma)^{\frac{3}{2}} |1110\rangle, \quad A_{0001} |1_L\rangle = \sqrt{\gamma(1 - \gamma)} |0100\rangle. \end{aligned} \quad (\text{B12})$$

From Eqs. (B10), (B11) and (B12) it is straightforward to show that the pairs $(|v_1^{(+)}\rangle, |v_6^{(+)}\rangle)$, $(|v_1^{(+)}\rangle, |v_8^{(+)}\rangle)$ and $(|v_1^{(+)}\rangle, |v_7^{(+)}\rangle)$, respectively, lead to good codes for the AD errors. Finally, it can be checked that,

$$\begin{aligned}
(|v_1^{(+)}\rangle, |v_2^{(+)}\rangle) & \text{ is not good because } \{A_{0000}, A_{1000}\} \text{ is not correctable;} \\
(|v_1^{(+)}\rangle, |v_3^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0100}\} \text{ is not correctable;} \\
(|v_1^{(+)}\rangle, |v_4^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0010}\} \text{ is not correctable;} \\
(|v_1^{(+)}\rangle, |v_5^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0001}\} \text{ is not correctable;} \\
(|v_2^{(+)}\rangle, |v_3^{(+)}\rangle) & \text{, is not good because } \{A_{1000}, A_{0100}\} \text{ is not correctable;} \\
(|v_2^{(+)}\rangle, |v_4^{(+)}\rangle) & \text{, is not good because } \{A_{1000}, A_{0010}\} \text{ is not correctable;} \\
(|v_2^{(+)}\rangle, |v_5^{(+)}\rangle) & \text{, is not good because } \{A_{1000}, A_{0001}\} \text{ is not correctable;} \\
(|v_2^{(+)}\rangle, |v_6^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0100}\} \text{ is not correctable;} \\
(|v_2^{(+)}\rangle, |v_7^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0010}\} \text{ is not correctable;} \\
(|v_2^{(+)}\rangle, |v_8^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0001}\} \text{ is not correctable,} \tag{B13}
\end{aligned}$$

and,

$$\begin{aligned}
(|v_3^{(+)}\rangle, |v_4^{(+)}\rangle) & \text{, is not good because } \{A_{0100}, A_{0010}\} \text{ is not correctable;} \\
(|v_3^{(+)}\rangle, |v_5^{(+)}\rangle) & \text{, is not good because } \{A_{0100}, A_{0001}\} \text{ is not correctable;} \\
(|v_3^{(+)}\rangle, |v_6^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{1000}\} \text{ is not correctable;} \\
(|v_3^{(+)}\rangle, |v_7^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0001}\} \text{ is not correctable;} \\
(|v_3^{(+)}\rangle, |v_8^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0010}\} \text{ is not correctable;} \\
(|v_4^{(+)}\rangle, |v_5^{(+)}\rangle) & \text{, is not good because } \{A_{0010}, A_{0001}\} \text{ is not correctable;} \\
(|v_4^{(+)}\rangle, |v_6^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0001}\} \text{ is not correctable;} \\
(|v_4^{(+)}\rangle, |v_7^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{1000}\} \text{ is not correctable;} \\
(|v_4^{(+)}\rangle, |v_8^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0100}\} \text{ is not correctable.} \tag{B14}
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
(|v_5^{(+)}\rangle, |v_6^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0010}\} \text{ is not correctable;} \\
(|v_5^{(+)}\rangle, |v_7^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{0100}\} \text{ is not correctable;} \\
(|v_5^{(+)}\rangle, |v_8^{(+)}\rangle) & \text{, is not good because } \{A_{0000}, A_{1000}\} \text{ is not correctable;} \\
(|v_6^{(+)}\rangle, |v_7^{(+)}\rangle) & \text{, is not good because } \{A_{0100}, A_{0010}\} \text{ is not correctable;} \\
(|v_6^{(+)}\rangle, |v_8^{(+)}\rangle) & \text{, is not good because } \{A_{0100}, A_{0001}\} \text{ is not correctable;} \\
(|v_7^{(+)}\rangle, |v_8^{(+)}\rangle) & \text{, is not good because } \{A_{0010}, A_{0001}\} \text{ is not correctable.} \tag{B15}
\end{aligned}$$

Appendix C: The complex optimization problem

In this Appendix, we use the notation $a \equiv \alpha$ and $b \equiv \beta$. The problem is to find $\bar{\alpha}$ and $\bar{\beta}$ (perhaps γ -dependent quantities) such that $\mathcal{F}(\bar{\alpha}, \bar{\beta}, \gamma)$ denotes the searched maximum,

$$\mathcal{F}(\bar{\alpha}, \bar{\beta}, \gamma) = \max_{|\alpha|^2 + |\beta|^2 = 1} \mathcal{F}(\alpha, \beta, \gamma). \quad (\text{C1})$$

Let us precede by brute force in an analytical fashion. Assume that,

$$\alpha = \text{Re } \alpha + i \text{Im } \alpha = \alpha_R + i\alpha_I \text{ and, } \beta = \text{Re } \beta + i \text{Im } \beta = \beta_R + i\beta_I. \quad (\text{C2})$$

Therefore, the two *complex*-variables complex optimization problem may be defined in terms of four *real*-variables optimization problem,

$$\mathcal{F}(\bar{\alpha}_R, \bar{\alpha}_I, \bar{\beta}_R, \bar{\beta}_I, \gamma) = \max_{\bar{\alpha}_R^2 + \bar{\alpha}_I^2 + \bar{\beta}_R^2 + \bar{\beta}_I^2 = 1} \mathcal{F}(\alpha_R, \alpha_I, \beta_R, \beta_I, \gamma). \quad (\text{C3})$$

Observe that $\mathcal{F}_{[[4,1]]}(\alpha, \beta, \gamma)$ can be rewritten as,

$$\mathcal{F}_{[[4,1]]}(\gamma)(\alpha, \beta, \gamma) = \frac{1}{4} \left\{ A + B + 2\gamma(1-\gamma)(2-\gamma)^2 + 2\gamma^2(1-\gamma)^2 + \frac{\gamma^4}{2} \right\}, \quad (\text{C4})$$

where,

$$\begin{aligned} A &\stackrel{\text{def}}{=} \left| \frac{\alpha + \beta(1-\gamma)^2}{\sqrt{2}} + (1-\gamma) \right|^2 = \left| \frac{(\alpha_R + i\alpha_I) + (\beta_R + i\beta_I)(1-\gamma)^2}{\sqrt{2}} + (1-\gamma) \right|^2 \\ &= \left(\frac{\alpha_R + \beta_R(1-\gamma)^2}{\sqrt{2}} + (1-\gamma) \right)^2 + \left(\frac{\alpha_I + \beta_I(1-\gamma)^2}{\sqrt{2}} \right)^2, \end{aligned} \quad (\text{C5})$$

and,

$$\begin{aligned} B &\stackrel{\text{def}}{=} \left| \frac{\beta^* - \alpha^*(1-\gamma)^2}{\sqrt{2}} \right|^2 = \left| \frac{(\beta_R - i\beta_I) - (\alpha_R - i\alpha_I)(1-\gamma)^2}{\sqrt{2}} \right|^2 = \left| \frac{\beta_R - \alpha_R(1-\gamma)^2}{\sqrt{2}} - i \frac{\beta_I - \alpha_I(1-\gamma)^2}{\sqrt{2}} \right|^2 \\ &= \left(\frac{\beta_R - \alpha_R(1-\gamma)^2}{\sqrt{2}} \right)^2 + \left(\frac{\beta_I - \alpha_I(1-\gamma)^2}{\sqrt{2}} \right)^2. \end{aligned} \quad (\text{C6})$$

After some algebraic manipulation of Eqs. (C6) and (C5), we get

$$\begin{aligned}
A + B &= \left(\frac{\alpha_R + \beta_R(1-\gamma)^2}{\sqrt{2}} + (1-\gamma) \right)^2 + \left(\frac{\alpha_I + \beta_I(1-\gamma)^2}{\sqrt{2}} \right)^2 + \left(\frac{\beta_R - \alpha_R(1-\gamma)^2}{\sqrt{2}} \right)^2 + \left(\frac{\beta_I - \alpha_I(1-\gamma)^2}{\sqrt{2}} \right)^2 \\
&= \left[\left(\frac{\alpha_R + \beta_R(1-\gamma)^2}{\sqrt{2}} + (1-\gamma) \right)^2 + \left(\frac{\beta_R - \alpha_R(1-\gamma)^2}{\sqrt{2}} \right)^2 \right] + \\
&\quad + \left[\left(\frac{\alpha_I + \beta_I(1-\gamma)^2}{\sqrt{2}} \right)^2 + \left(\frac{\beta_I - \alpha_I(1-\gamma)^2}{\sqrt{2}} \right)^2 \right] \\
&= \left[(\alpha_R^2 + \beta_R^2) \left(\frac{1+(1-\gamma)^4}{2} \right) + (1-\gamma)^2 + 2(1-\gamma) \frac{\alpha_R + \beta_R(1-\gamma)^2}{\sqrt{2}} \right] + \\
&\quad + \left[(\alpha_I^2 + \beta_I^2) \left(\frac{1+(1-\gamma)^4}{2} \right) \right] \\
&= \frac{1+(1-\gamma)^4}{2} + (1-\gamma)^2 + 2(1-\gamma) \frac{\alpha_R + \beta_R(1-\gamma)^2}{\sqrt{2}}. \tag{C7}
\end{aligned}$$

Therefore, substituting (C7) into (C4), $\mathcal{F}_{[[4,1]]}(\alpha, \beta, \gamma)$ becomes

$$\begin{aligned}
\mathcal{F}_{[[4,1]]}(\alpha_R, \alpha_I, \beta_R, \beta_I, \gamma) &= \frac{1}{4} \left\{ \frac{1+(1-\gamma)^4}{2} + (1-\gamma)^2 + 2(1-\gamma) \frac{\alpha_R + \beta_R(1-\gamma)^2}{\sqrt{2}} + 2\gamma(1-\gamma)(2-\gamma)^2 + \right. \\
&\quad \left. + 2\gamma^2(1-\gamma)^2 + \frac{\gamma^4}{2} \right\} \\
&= \mathcal{F}_0(\gamma) + \frac{2\alpha_R(1-\gamma) + 2\beta_R(1-\gamma)^3}{4\sqrt{2}}, \tag{C8}
\end{aligned}$$

where,

$$\mathcal{F}_0(\gamma) \stackrel{\text{def}}{=} \frac{1}{4} \left(\frac{1+(1-\gamma)^4}{2} + (1-\gamma)^2 + 2\gamma(1-\gamma)(2-\gamma)^2 + 2\gamma^2(1-\gamma)^2 + \frac{\gamma^4}{2} \right). \tag{C9}$$

Therefore, the *complex* optimization problem becomes

$$\begin{aligned}
\mathcal{F}_{[[4,1]]}(\bar{\alpha}_R, \bar{\alpha}_I, \bar{\beta}_R, \bar{\beta}_I, \gamma) &= \max_{\bar{\alpha}_R^2 + \bar{\alpha}_I^2 + \bar{\beta}_R^2 + \bar{\beta}_I^2 = 1} \mathcal{F}_{[[4,1]]}(\alpha_R, \alpha_I, \beta_R, \beta_I, \gamma) \\
&= \max_{\bar{\alpha}_R^2 + \bar{\alpha}_I^2 + \bar{\beta}_R^2 + \bar{\beta}_I^2 = 1} \left[\mathcal{F}_0(\gamma) + \frac{2\alpha_R(1-\gamma) + 2\beta_R(1-\gamma)^3}{4\sqrt{2}} \right]. \tag{C10}
\end{aligned}$$

We note that $\mathcal{F}_{[[4,1]]}(\alpha_R, \alpha_I, \beta_R, \beta_I, \gamma)$ does not depend on α_I and β_I . Setting $\alpha_I = \beta_I = 0$, the maximization problem becomes

$$\mathcal{F}_{[[4,1]]}(\bar{\alpha}_R, \bar{\beta}_R, \gamma) = \max_{\bar{\alpha}_R^2 + \bar{\beta}_R^2 = 1} \left[\mathcal{F}_0(\gamma) + \frac{2\alpha_R(1-\gamma) + 2\beta_R(1-\gamma)^3}{4\sqrt{2}} \right]. \tag{C11}$$

We observe that,

$$\frac{d}{d\alpha_R} \left(\mathcal{F}_0(\gamma) + \frac{2\alpha_R(1-\gamma) + 2(1-\alpha_R^2)^{\frac{1}{2}}(1-\gamma)^3}{4\sqrt{2}} \right) = 0, \quad (\text{C12})$$

implies that,

$$\alpha_R - 2\alpha_R\gamma - \sqrt{1-\alpha_R^2} + \alpha_R\gamma^2 = 0, \quad (\text{C13})$$

that is,

$$\bar{\alpha}_R(\gamma) \stackrel{\text{def}}{=} \frac{1}{\sqrt{1+(1-\gamma)^4}} \text{ and, } \bar{\beta}_R(\gamma) \stackrel{\text{def}}{=} \frac{(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}}. \quad (\text{C14})$$

Finally, we obtain

$$\mathcal{F}_{[[4,1]]}(\bar{\alpha}_R, \bar{\beta}_R, \gamma) \stackrel{\gamma \ll 1}{\approx} 1 - \frac{3}{2}\gamma^2 + \mathcal{O}(\gamma^3). \quad (\text{C15})$$

More precisely, we should set $\alpha_R^2 + \beta_R^2 \leq 1$ or, $\alpha_R^2 + \beta_R^2 = r^2$ with $r \leq 1$. In this case we have,

$$\frac{d}{d\alpha_R} \left(\frac{2\alpha_R(1-\gamma) + 2(r^2 - \alpha_R^2)^{\frac{1}{2}}(1-\gamma)^3}{4\sqrt{2}} \right) = \frac{1}{4}\sqrt{2} \frac{\gamma-1}{\sqrt{r^2 - \alpha_R^2}} \left(\alpha_R - \sqrt{r^2 - \alpha_R^2} - 2\gamma\alpha_R + \gamma^2\alpha_R \right) = 0, \quad (\text{C16})$$

that is,

$$\bar{\alpha}_R(\gamma) \stackrel{\text{def}}{=} \frac{r}{\sqrt{1+(1-\gamma)^4}} \text{ and, } \bar{\beta}_R(\gamma) \stackrel{\text{def}}{=} \frac{r(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}}. \quad (\text{C16})$$

Observe that,

$$\mathcal{F}_{[[4,1]]}(\bar{\alpha}_R, \bar{\beta}_R, \gamma) = \max_{\bar{\alpha}_R^2 + \bar{\beta}_R^2 = r^2} \left[\mathcal{F}_0(\gamma) + \frac{2\alpha_R(1-\gamma) + 2\beta_R(1-\gamma)^3}{4\sqrt{2}} \right], \quad (\text{C18})$$

that is,

$$\frac{2\alpha_R(1-\gamma) + 2\beta_R(1-\gamma)^3}{4\sqrt{2}} = \frac{2 \left(\frac{r}{\sqrt{1+(1-\gamma)^4}} \right) (1-\gamma) + 2 \left(\frac{r(1-\gamma)^2}{\sqrt{1+(1-\gamma)^4}} \right) (1-\gamma)^3}{4\sqrt{2}} \approx \frac{1}{2}r - r\gamma + r\gamma^2 + \mathcal{O}(\gamma^3), \quad (\text{C19})$$

thus,

$$\mathcal{F}_{[[4,1]]}(\bar{\alpha}_R, \bar{\beta}_R, \gamma) \approx \frac{1}{2}(1+r) + (1-r)\gamma - \left(\frac{5}{2} - r^2 \right) \gamma^2 + \mathcal{O}(\gamma^3). \quad (\text{C20})$$

It then turns out that for $r = 1$ we obtain the optimal fidelity,

$$\mathcal{F}_{[[4,1]]}(\bar{\alpha}_R, \bar{\beta}_R, \gamma) \approx 1 - \frac{3}{2}\gamma^2 + \mathcal{O}(\gamma^3). \quad (\text{C21})$$

In conclusion, setting $\alpha_I = \beta_I = 0$ and α_R, β_R given in Eq. (C16), $\mathcal{F}_{[[4,1]]}(\alpha, \beta, \gamma)$ in (C4) becomes

$$\mathcal{F}_{[[4,1]]}^{\text{F-recovery}}(\gamma) \approx 1 - \frac{3}{2}\gamma^2 + \mathcal{O}(\gamma^3). \quad (\text{C22})$$

The derivation of Eq. (C22) concludes our optimization problem.

We emphasize that after completing this work, we have become aware that Eq. (C22) has also appeared in [30]. However, the derivation presented in [30] is by no means as explicit as the one provided in our work.

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