GLOBAL GEOMETRIC DIFFERENCE BETWEEN SEPARABLE AND POSITIVE PARTIAL TRANSPOSE STATES

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ABSTRACT. In the convex set of all $3 \otimes 3$ states with positive partial transposes, we show that one can take two extreme points whose convex combinations belong to the interior of the convex set. Their convex combinations may be even in the interior of the convex set of all separable states. In general, we need at least mn extreme points to get an interior point by their convex combination, for the case of the convex set of all $m \otimes n$ separable states. This shows a sharp distinction between PPT states and separable states. We also consider the same questions for positive maps and decomposable maps.

1. INTRODUCTION

Distinguishing entanglement from separability is one of the most important question in the theory of quantum entanglement, and the positive partial transpose (PPT) criterion [8, 30] gives a simple but strong necessary condition for separability. The PPT condition is actually equivalent to the separability if the rank of a given PPT state is sufficiently low by [19]. In the case that the rank is not so high, it turns out that the local geometry is quite useful to distinguish and construct entanglement among PPT states. Basic idea is to consider the smallest faces determined by a given separable state in the convex sets of all separable states and all PPT states respectively, and compare those. See [14, 16] for recent progresses in this direction.

In this paper, we turn our attention to the global geometries for separable and PPT states, and look for the differences. We denote by $\mathbb{S}_{m,n}$ the convex set of all $m \otimes n$ separable states, and by $\mathbb{T}_{m,n}$ the convex set of all $m \otimes n$ PPT states. For the convex set $\mathbb{S}_{m,n}$, it is easy to see that a convex combination of two extreme points is always on the boundary of the convex set. Actually, the line segment between two extreme points of $\mathbb{S}_{2,2}$ is already a nontrivial face of the convex set, in most cases. See [25] for more details for the convex geometry of $\mathbb{S}_{2,2}$. More generally, the convex hull of max $\{m, n\}$ extreme points of $\mathbb{S}_{m,n}$ is a face of the convex set, in most cases by [1, 22]. Therefore, it is natural to ask how these properties are retained for the convex set $\mathbb{T}_{m,n}$.

¹⁹⁹¹ Mathematics Subject Classification. 81P15, 15A30, 46L05.

Key words and phrases. states with positive partial transposes, separable states, extreme points, boundary, positive maps, decomposable maps.

KCH is partially supported by NRFK 2013-020897. SHK is partially supported by NRFK 2009-0083521.

For a convex compact set C in a finite dimensional real vector space, we introduce the number $\nu(C)$ the smallest natural number k such that the convex combination of k extreme points of C may be an interior point of C. We recall that the interior of a convex set is defined by the interior with respect to the relative topology induced by the affine manifold generated by itself. Sometimes, it is more convenient to consider the convex cone \tilde{C} generated by the convex compact set C. For example, $\tilde{\mathbb{S}}_{m,n}$ ($\tilde{\mathbb{T}}_{m,n}$, respectively) is the convex cone of all $m \otimes n$ unnormalized separable (PPT, respectively) states. In this case, we may replace extreme points by extreme rays to get the same number $\nu(C)$. Recall that a point x of a convex compact set C is an extreme point of C if and only if x generates an extreme ray of the convex cone \tilde{C} , whenever the hyperplane generated by C does not contain the origin.

It is easy to see that

$$\nu(\mathbb{S}_{m,n}) = mn,$$

for every $m, n = 2, 3, \ldots$ The main purpose of this note is to show that

$$\nu(\mathbb{T}_{3,3}) = 2,$$

to see the geometric difference between $S_{3,3}$ and $T_{3,3}$. In other words, we can take just two extreme points of $T_{3,3}$ whose convex combinations belong to the interior of $T_{3,3}$.

To do this, we consider the following $3 \otimes 3$ states

$$\varrho_{b,\theta} = \begin{pmatrix}
p_{\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & -e^{-i\theta} \\
\cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & b & \cdot & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\
\cdot & -e^{i\theta} & \cdot & b & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot \\
-e^{-i\theta} & \cdot & \cdot & p_{\theta} & \cdot & \cdot & -e^{-i\theta} & \cdot \\
\cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & \frac{1}{b} & \cdot & -e^{-i\theta} & \cdot \\
\cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & -e^{i\theta} & \cdot & b & \cdot \\
-e^{i\theta} & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & p_{\theta}
\end{pmatrix}$$

for a given positive number b > 0 and a real number θ , where \cdot denotes zero and

$$p_{\theta} = \max\{e^{i(\theta - \frac{2}{3}\pi)} + e^{-i(\theta - \frac{2}{3}\pi)}, e^{i\theta} + e^{-i\theta}, e^{i(\theta + \frac{2}{3}\pi)} + e^{-i(\theta + \frac{2}{3}\pi)}\}$$

is the smallest positive number a so that the following 3×3 matrix

$$\begin{pmatrix} a & -e^{i\theta} & -e^{-i\theta} \\ -e^{-i\theta} & a & -e^{i\theta} \\ -e^{i\theta} & -e^{-i\theta} & a \end{pmatrix}$$

is positive, as it was discussed in Section 2 of [13]. We note that $1 \le p_{\theta} \le 2$. Therefore, it is immediate to see that $\rho_{b,\theta}$ is a PPT state. These PPT states have been constructed in [26] for $-\pi/3 < \theta < \pi/3$. The main point is to extend this construction for the full range of θ . We check that they are extreme points of $\mathbb{T}_{3,3}$ in most cases, with a few exceptions. Note that the case of b = 2 and $\theta = \pi/6$ has been checked to be extreme in [4]. If we divide the parameter $e^{i\theta}$ into three arcs and take any two extreme points from different arcs then their convex combinations lie in the interior of $\mathbb{T}_{3,3}$. We see that some of them turn out to be even in the interior of $\mathbb{S}_{3,3}$. It had been asked in [6] whether the sum of two PPT entangled extreme states can be separable, and the authors [14] gave an affirmative answer. More precisely, it was shown that sum of two extreme PPT entangled states with rank four may be separable. Our construction in this paper shows that sum of two extreme PPT entangled states with rank five may be even diagonal matrices with positive diagonal entries.

Let M_n be the C^* -algebra consisting of all $n \times n$ matrices over the complex field. We also consider the same question for the convex cone $\mathbb{P}_{m,n}$ (respectively $\mathbb{D}_{m,n}$) of all positive maps (respectively decomposable positive maps) from M_m into M_n , to show that $\nu(\mathbb{D}_{m,n}) \geq m + n - 2$. In the case of n = m = 3, we have $\nu(\mathbb{D}_{3,3}) = 4$ and $\nu(\mathbb{P}_{3,3}) = 2$. By the Jamiołkowski-Choi isomorphism [7, 20], the cones $\mathbb{P}_{m,n}$ and $\mathbb{D}_{m,n}$ are considered as subsets of $M_m \otimes M_n$, and we have the relation

$$\tilde{\mathbb{S}}_{m,n} \subset \tilde{\mathbb{T}}_{m,n} \subset \mathbb{D}_{m,n} \subset \mathbb{P}_{m,n}.$$

We also recall [9] that $\mathbb{S}_{m,n}$ and $\mathbb{P}_{m,n}$ (respectively $\mathbb{T}_{m,n}$ and $\mathbb{D}_{m,n}$) are dual to each others with respect to the bilinear pairing

(1)
$$\langle \rho, \phi \rangle = \operatorname{Tr}(\rho C_{\phi}^{\mathsf{t}}), \qquad \rho \in \tilde{\mathbb{S}}_{m,n}, \ \phi \in \mathbb{P}_{m,n},$$

where C_{ϕ} is the Choi matrix of ϕ defined by $\sum |i\rangle\langle j| \otimes \phi(|i\rangle\langle j|)$, and interior points of these convex sets can be characterized by the above duality:

- ρ is an interior point of $\mathbb{S}_{m,n}$ if and only if $\langle \rho, \phi \rangle > 0$ for all nonzero $\phi \in \mathbb{P}_{m,n}$.
- ϕ is an interior point of $\mathbb{P}_{m,n}$ if and only if $\langle \rho, \phi \rangle > 0$ for all $\rho \in \mathbb{S}_{m,n}$.

In this characterization of interior points of convex sets, we note that it suffices to check the positivity of the pairing only for extreme points (rays) of the dual convex set (convex cone). See Proposition 5.1 and 5.4 of [24].

In the next section, we examine the properties of the states $\rho_{b,\theta}$ and how to choose two of them to get an interior point by their convex combination. In Section 3, we show that they are extreme points in $\mathbb{T}_{3,3}$, in most cases. We consider the convex cones $\mathbb{P}_{3,3}$ and $\mathbb{D}_{m,n}$ in Section 4, and close this note with discussions in Section 5.

2. Separable states and PPT states

The facial structures for the convex set $\mathbb{T}_{m,n}$ are well understood [11]. Every face of $\mathbb{T}_{m,n}$ is of the form

$$\tau(D,E) = \{ \varrho \in \mathbb{T}_{m,n} : \mathcal{R}\varrho \subset D, \ \mathcal{R}\varrho^{\Gamma} \subset E \},\$$

for subspaces D and E of $\mathbb{C}^m \otimes \mathbb{C}^n$, and its interior is given by

int
$$\tau(D, E) = \{ \varrho \in \mathbb{T}_{m,n} : \mathcal{R}\varrho = D, \ \mathcal{R}\varrho^{\Gamma} = E \},\$$

where $\mathcal{R}\varrho$ denotes the range space of ϱ , and ϱ^{Γ} is the partial transpose of ϱ . Especially, a PPT state ϱ is an interior point of $\mathbb{T}_{m,n}$ if and only if the ranges of ϱ and ϱ^{Γ} are full spaces.

Extreme points of the convex set $\mathbb{S}_{m,n}$ are nothing but product states by the definition of separability. If we take k product states with k < mn and form a separable state $\varrho \in \mathbb{S}_{m,n}$ with their convex combination then the range space of ϱ is never the full space, and so ϱ is on the boundary of $\mathbb{T}_{m,n}$. By the relation $\mathbb{S}_{m,n} \subset \mathbb{T}_{m,n}$, we conclude that ϱ is also on the boundary of $\mathbb{S}_{m,n}$. Therefore, we have $\nu(\mathbb{S}_{m,n}) \ge mn$. Since the identity matrix is in the interior of $\mathbb{S}_{m,n}$, we conclude that $\nu(\mathbb{S}_{m,n}) = mn$. In fact, it is easy to see that every diagonal matrix with nonzero positive diagonal entries is an interior point of the convex set $\mathbb{S}_{m,n}$, by the duality between separable states and positive maps.

Now, we proceed to examine the properties of the states $\rho_{b,\theta}$. We also consider the PPT states defined by

$$\sigma_{b,\theta} = \begin{pmatrix} p_{\theta} & \cdot & \cdot & -e^{i\theta} & \cdot & \cdot & -e^{-i\theta} \\ \cdot & \frac{1}{b} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & b & \cdot & \cdot & \cdot & \cdot & \cdot \\ -e^{-i\theta} & \cdot & \cdot & \cdot & p_{\theta} & \cdot & \cdot & \cdot & -e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{b} & \cdot & \cdot \\ -e^{i\theta} & \cdot & \cdot & -e^{-i\theta} & \cdot & \cdot & p_{\theta} \end{pmatrix}$$

If $0 < |\theta| < \frac{\pi}{3}$ then $\sigma_{b,\theta}$ is nothing but PPT entangled edge states of type (8,6) constructed in [26]. We recall that a PPT state ρ is said to be of type (p,q) if the ranks of ρ and ρ^{Γ} are p and q, respectively. We note that the state $\rho_{b,\theta}$ defined in Introduction is given by

$$\varrho_{b,\theta} = \sigma_{b,\theta} + \sigma_{b,\theta}^{\Gamma} - \operatorname{Diag} \sigma_{b,\theta},$$

which is block-wise symmetric.

If $\theta = 0$ then $\sigma_{b,0}$ was shown to be separable for each b > 0 in [26]. On the other hand, if $\theta = \pi$ then $\sigma_{b,\pi}$ was shown [12] to be separable if and only if b = 1. When $b \neq 1$, we note that $\sigma_{b,\pi}$ is nothing but PPT entangled state given by Størmer [31] in the early eighties. We also know that both $\sigma_{b,\theta}$ and $\rho_{b,\theta}$ are PPT entangled edge states for $0 < |\theta| < \frac{\pi}{3}$ by [26]. Now, we turn our attention to the state $\rho_{b,\pi}$. We note that $\rho_{1,\pi}$ is the separable state given by the following four product vectors

$$(1,1,1)^{t} \otimes (1,1,1)^{t}, \qquad (1,1,-1)^{t} \otimes (1,1,-1)^{t}, \\ (1,-1,1)^{t} \otimes (1,-1,1)^{t}, \quad (-1,1,1)^{t} \otimes (-1,1,1)^{t},$$

as it was shown in [14]. The states $\rho_{b,\pi}$ with $b \neq 1$ appear in the construction [17] of PPT entangled states of type (4, 4) using the duality between positive linear maps and separable states. The special case $\rho_{2,\pi}$ is just the first example of $3 \otimes 3$ PPT entangled state given by Choi [8]. In short, we see that $\rho_{b,\pi}$ is separable if and only if b = 1.

If we take the diagonal unitary $U = \text{Diag}(1, e^{-\frac{2}{3}\pi i}, e^{\frac{2}{3}\pi i})$, then we have

$$U^{-1} \begin{pmatrix} p_{\theta} & -e^{i\theta} & -e^{-i\theta} \\ -e^{-i\theta} & p_{\theta} & -e^{i\theta} \\ -e^{i\theta} & -e^{-i\theta} & p_{\theta} \end{pmatrix} U = \begin{pmatrix} p_{\theta} & -e^{i(\theta-\frac{2}{3}\pi)} & -e^{-i(\theta-\frac{2}{3}\pi)} \\ -e^{-i(\theta-\frac{2}{3}\pi)} & p_{\theta} & -e^{i(\theta-\frac{2}{3}\pi)} \\ -e^{i(\theta-\frac{2}{3}\pi)} & -e^{-i(\theta-\frac{2}{3}\pi)} & p_{\theta} \end{pmatrix},$$

and so it follows that

(2)
$$(I \otimes U)^{-1} \varrho_{b,\theta}(I \otimes U) = \varrho_{b,\theta-\frac{2}{3}\pi}, \qquad (I \otimes U)^{-1} \sigma_{b,\theta}(I \otimes U) = \sigma_{b,\theta-\frac{2}{3}\pi}$$

Therefore, the separability and PPT properties of $\rho_{b,\theta}$ and $\sigma_{b,\theta}$ are invariant under the translation of θ by $\frac{2}{3}\pi$, as well as the types of the states. Therefore, we have the following:

Theorem 2.1. For states σ_{θ} and ϱ_{θ} , we have the following:

- (i) If $\theta \neq \frac{n}{3}\pi$ for any integer n, then $\sigma_{b,\theta}$ and $\varrho_{b,\theta}$ are PPT entangled edge states of type (8,6) and (5,5), respectively.
- (ii) If $\theta = \frac{n}{3}\pi$ for an even integer n, then $\sigma_{b,\theta}$ and $\varrho_{b,\theta}$ are separable states of type (8,6) and (5,5), respectively.
- (iii) If $\theta = \frac{n}{3}\pi$ for an odd integer n and b = 1, then $\sigma_{b,\theta}$ and $\varrho_{b,\theta}$ are separable states of type (7,6) and (4,4), respectively.
- (iv) If $\theta = \frac{n}{3}\pi$ for an odd integer n and $b \neq 1$, then $\sigma_{b,\theta}$ and $\varrho_{b,\theta}$ are PPT entangled states of type (7,6) and (4,4), respectively.

The separability and entangledness of the states $\rho_{b,\theta}$ and $\sigma_{b,\theta}$ are summarized in Figure 1. We note that the circle $\{e^{i\theta} : \theta \in \mathbb{R}\}$ is divided by three arcs by the range of the variable θ :

$$\left(-\pi, -\frac{\pi}{3}\right), \quad \left(-\frac{\pi}{3}, \frac{\pi}{3}\right), \quad \left(\frac{\pi}{3}, \pi\right).$$

We also note that the following three vectors

$$w_1(\theta) = (0, b, 0; e^{i\theta}, 0, 0; 0, 0, 0),$$

$$w_2(\theta) = (0, 0, 0; 0, 0, b; 0, e^{i\theta}, 0),$$

$$w_3(\theta) = (0, 0, e^{i\theta}; 0, 0, 0; b, 0, 0),$$



FIGURE 1. The points on the arcs represent PPT entangled states, and the small circles represent separable states.

belong to the kernel of $\rho_{b,\theta}$, regardless of the values of b and θ . There are extra kernel vectors:

$$w_{-} = (1, 0, 0; 0, e^{\frac{2}{3}\pi i}, 0; 0, 0, e^{-\frac{2}{3}\pi i}), \qquad -\pi < \theta < -\frac{\pi}{3}, w_{0} = (1, 0, 0; 0, 1, 0; 0, 0, 1), \qquad -\frac{\pi}{3} < \theta < +\frac{\pi}{3}, w_{+} = (1, 0, 0; 0, e^{-\frac{2}{3}\pi i}, 0; 0, 0, e^{\frac{2}{3}\pi i}), \qquad \frac{\pi}{3} < \theta < \pi.$$

If we take (b, θ) and (c, τ) so that $e^{i\theta}$ and $e^{i\tau}$ belong to the different arcs, then it is clear that the kernels of $\rho_{b,\theta}$ and $\rho_{c,\tau}$ have the trivial intersection. This means that the nontrivial convex combination ρ of these two states has the full range space, as well as the partial conjugate. Therefore, we conclude that this PPT state ρ belongs to the interior of the convex set $\mathbb{T}_{3,3}$. In the next section, we will show that each state $\rho_{b,\theta}$ is an extreme point in the convex set $\mathbb{T}_{3,3}$ consisting of all $3 \otimes 3$ PPT states, whenever $\theta \neq \frac{n}{3}\pi$ for an integer n. From this, we conclude that $\nu(\mathbb{T}_{3,3}) = 2$. If we take (b, θ) and (c, τ) so that $e^{i\theta}$ and $e^{i\tau}$ belong to the same arc, then we note that the convex combinations of $\rho_{b,\theta}$ and $\rho_{c,\tau}$ are on the boundary.

For a given $e^{i\theta}$, we take the antipodal point $e^{i(\theta+\pi)} = -e^{i\theta}$ then we see that

$$\frac{1}{2}(\varrho_{b,\theta} + \varrho_{c,\theta+\pi})$$

is a diagonal matrix, and so it is separable. Actually, it is an interior point of $S_{3,3}$, since there is no zero entry in the diagonal. This shows that the convex combination of two extreme PPT states may be in the interior of the convex set $S_{3,3}$ of all separable states. We note that $p_{\theta} + p_{\theta+\pi} > 2$ for each θ , and so we may take b > 0 so that $b + \frac{1}{b} = p_{\theta} + p_{\theta+\pi}$. Then we see that the sum $\rho_{b,\theta} + \rho_{1/b,\theta+\pi}$ of two extreme PPT states is a scalar multiple of the identity matrix.

3. Extremeness

First, we briefly explain the method [27, 10, 2] to check if a given face $\tau(D, E)$ is an extreme point or not, where D and E are subspaces of $\mathbb{C}^m \otimes \mathbb{C}^n$. Let $(M_m \otimes M_n)_h$ be the real Hilbert space of all $mn \times mn$ hermitian matrices in $M_m \otimes M_n$ with the inner product $\langle X, Y \rangle = \text{Tr}(YX^{\text{t}})$, and orthogonal projections P_D and P_E in $(M_m \otimes M_n)_h$ onto D and E, respectively. We define real linear maps ϕ_D and ϕ_E between $(M_m \otimes M_n)_h$ by

$$\phi_D(X) = P_D X P_D - X, \quad \phi_E(X) = (P_E X^{\Gamma} P_E)^{\Gamma} - X, \quad X \in (M_m \otimes M_n)_h.$$

Then we see that $\tau(D, E) \subset \operatorname{Ker} \phi_D \cap \operatorname{Ker} \phi_E$, where $\operatorname{Ker} \phi_D$ denotes the kernel space of ϕ_D . Therefore, if $\operatorname{Ker} \phi_D \cap \operatorname{Ker} \phi_E$ is one-dimensional then $\tau(D, E)$ must be an extreme point. It is not so difficult to see that the converse does hold. Thus, we can conclude that $\tau(D, E)$ is an extreme point if and only if the condition

$$\dim(\operatorname{Ker}\phi_D \cap \operatorname{Ker}\phi_E) = 1$$

holds.

Now, we proceed to show that $\rho_{b,\theta}$ is an extreme point in the convex set $\mathbb{T}_{3,3}$, whenever $0 < |\theta| < \frac{\pi}{3}$. Let $D = \mathcal{R}\rho_{b,\theta}$ and $E = \mathcal{R}\rho_{b,\theta}^{\Gamma}$. We note that $\rho_{b,\theta} = \rho_{b,\theta}^{\Gamma}$, so we see that $P_D = P_E$. Applying the Gram-Schmidt process to linearly independent vectors of $\mathcal{R}\rho_{b,\theta}$, we can compute the orthogonal projection P_D as follows:

$$P_D = P_E = \begin{pmatrix} \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & \frac{1}{1+b^2} & 0 & -\frac{be^{-i\theta}}{1+b^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{b^2}{1+b^2} & 0 & 0 & 0 & -\frac{be^{i\theta}}{1+b^2} & 0 & 0 \\ 0 & -\frac{be^{i\theta}}{1+b^2} & 0 & \frac{b^2}{1+b^2} & 0 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & 0 & 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{1+b^2} & 0 & -\frac{be^{-i\theta}}{1+b^2} & 0 \\ 0 & 0 & -\frac{be^{-i\theta}}{1+b^2} & 0 & 0 & 0 & \frac{1}{1+b^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{be^{i\theta}}{1+b^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{be^{i\theta}}{1+b^2} & 0 & \frac{b^2}{1+b^2} & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} \end{pmatrix}.$$

By a direct computation, we can show that both $\operatorname{Ker} \phi_D$ and $\operatorname{Ker} \phi_E$ are twenty-five dimensional real linear subspaces. Let $\{E_{ij}\}$ be the usual matrix units in M_9 . Then, we can find a basis $\{X_i : 1 \leq i \leq 25\}$ of real linear space $\operatorname{Ker} \phi_D$, which consists of hermitian matrices including the following vectors:

$$X_{1} = E_{11} + E_{55} - E_{15} - E_{51}$$

$$X_{2} = E_{11} + E_{99} - E_{19} - E_{91}$$

$$X_{3} = E_{55} + E_{99} - E_{59} - E_{95}$$

$$X_{4} = i(E_{19} - E_{15} - E_{59}) - i(E_{91} - E_{51} - E_{95})$$

$$X_{5} = e^{-i\theta}E_{24} + e^{i\theta}E_{42} - bE_{44} - \frac{1}{b}E_{22},$$

$$X_{6} = e^{-i\theta}E_{68} + e^{i\theta}E_{86} - bE_{88} - \frac{1}{b}E_{66},$$

$$X_{7} = e^{-i\theta}E_{73} + e^{i\theta}E_{37} - bE_{33} - \frac{1}{b}E_{77}.$$

We also see that Ker $\phi_E = \text{span}\{Y_i : 1 \le i \le 25\}$ with hermitian matrices Y_i 's. Here, we just write down the list of Y_i for $i = 1, 2, \dots, 7$, as follows:

$$\begin{split} Y_1 &= E_{11} + E_{55} - E_{24} - E_{42}, \\ Y_2 &= E_{11} + E_{99} - E_{37} - E_{73}, \\ Y_3 &= E_{55} + E_{99} - E_{68} - E_{86}, \\ Y_4 &= i(E_{37} + E_{42} + E_{86}) - i(E_{73} + E_{24} + E_{68}), \\ Y_5 &= e^{-i\theta}E_{19} + e^{i\theta}E_{91} - bE_{33} - \frac{1}{b}E_{77}, \\ Y_6 &= e^{-i\theta}E_{51} + e^{i\theta}E_{15} - bE_{44} - \frac{1}{b}E_{22}, \\ Y_7 &= e^{-i\theta}E_{95} + e^{i\theta}E_{59} - bE_{88} - \frac{1}{b}E_{66}. \end{split}$$

For the full list of vectors X_i and Y_i for $8 \le i \le 25$, see the appendix. By solving the linear equation $\sum_{i=1}^{25} x_i X_i = \sum_{j=1}^{25} y_j Y_j$ with respect to x_i 's and y_j 's, we see that the subspace Ker $\Phi_D \cap$ Ker Φ_E is generated $\varrho_{b,\theta}$. In fact, we have

$$\varrho_{b,\theta} = \cos \theta (X_1 + X_2 + X_3) + \sin \theta X_4 - X_5 - X_6 - X_7$$
$$= \cos \theta (Y_1 + Y_2 + Y_3) - \sin \theta Y_4 - Y_5 - Y_6 - X_7.$$

Therefore, we see that $\rho_{b,\theta}$ is an extreme point in $\mathbb{T}_{3,3}$ for $0 < |\theta| < \frac{\pi}{3}$.

We note that $\rho_{b,\theta-\frac{2}{3}\pi}$ is extreme if and only if $\rho_{b,\theta}$ is so by the relation (2). Consequently, we may conclude that $\rho_{b,\theta}$ is extreme whenever $\theta \neq \frac{n}{3}\pi$ for an integer n.

4. Decomposable and positive maps

In order to see that $\nu(\mathbb{P}_{3,3}) = 2$, we recall the positive linear map $\Phi_{\theta}(t)$ considered in [15], which maps a 3×3 matrix $X = (x_{ij})$ to the following 3×3 matrix

$$\begin{pmatrix} a(t)x_{11} + b(t)x_{22} + c(t)x_{33} & -e^{i\theta}x_{12} & -e^{-i\theta}x_{13} \\ -e^{-i\theta}x_{21} & c(t)x_{11} + a(t)x_{22} + b(t)x_{33} & -e^{i\theta}x_{23} \\ -e^{i\theta}x_{31} & -e^{-i\theta}x_{32} & b(t)x_{11} + c(t)x_{22} + a(t)x_{33} \end{pmatrix},$$

where

$$a(t) = 1 - \frac{(p_{\theta} - 1)t}{1 - t + t^2}, \quad b(t) = \frac{(p_{\theta} - 1)t^2}{1 - t + t^2}, \quad c(t) = \frac{(p_{\theta} - 1)t}{1 - t + t^2}$$

with $0 < t < \infty$. It was shown that $\Phi_{\theta}(t)$ generates an exposed ray of the convex cone $\mathbb{P}_{3,3}$, and so generates an extreme ray of $\mathbb{P}_{3,3}$, whenever the condition

$$\theta \neq \frac{2n-1}{3}\pi, \qquad (\theta,t) \neq \left(\frac{2n}{3}\pi, 1\right)$$

holds. It is now clear that if we take the convex combination of two antipodal maps $\Phi_{\theta}(t)$ and $\Phi_{\theta+\pi}(s)$ then we get a positive map whose Choi matrix is a diagonal matrix with positive diagonal entries, and so we see that this map is an interior point of $\mathbb{P}_{3,3}$ by duality.

It remains to consider the convex cone $\mathbb{D}_{m,n}$ consisting of all decomposable maps from M_m into M_n . We first note that every decomposable map is the convex combination of the maps

$$\phi_V : X \mapsto V^* X V, \qquad \phi^W : X \mapsto W^* X^{\mathsf{t}} W, \qquad X \in M_m,$$

for $m \times n$ matrices V and W, where X^{t} denotes the transpose of X. Therefore, every decomposable map from M_{m} into M_{n} is of the form

(3)
$$\phi_{\mathcal{V}} + \phi^{\mathcal{W}} = \sum_{i} \phi_{V_i} + \sum_{j} \phi^{W_j},$$

for a finite sets $\mathcal{V} = \{V_i\}$ and $\mathcal{W} = \{W_j\}$ of $m \times n$ matrices. We also note that if the map (3) is on the boundary of the cone $\mathbb{P}_{m,n}$ then it is also on the boundary of the cone $\mathbb{D}_{m,n}$. For a product vector $|z\rangle = |\xi\rangle \otimes |\eta\rangle$, the pairing in (1) is given by

$$\langle |z\rangle\langle z|, \phi_{\mathcal{V}} + \phi^{\mathcal{W}}\rangle = \sum_{i} |\langle \xi|V_{i}|\bar{\eta}\rangle|^{2} + \sum_{j} |\langle \bar{\xi}|W_{j}|\bar{\eta}\rangle|^{2}$$

Therefore, the map (3) is on the boundary of $\mathbb{P}_{m,n}$ if and only if the equation

$$\langle \xi | V_i | \bar{\eta} \rangle = 0, \qquad \langle \bar{\xi} | W_j | \bar{\eta} \rangle = 0$$

has a common solution $|\xi\rangle \otimes |\eta\rangle \in \mathbb{C}^m \otimes \mathbb{C}^n$. If we put $k = \dim \operatorname{span} \mathcal{V}$ and $\ell = \dim \operatorname{span} \mathcal{W}$ then it was shown in [21] that

(i) If $k + \ell < m + n - 2$, then there exists a solution

(ii) If $k + \ell = m + n - 2$ and

$$\sum_{r+s=m-1} (-1)^r \binom{k}{r} \binom{\ell}{s} \neq 0.$$

then there exists a solution.

(iii) If $k + \ell > m + n - 2$, then the existence of solutions is not guaranteed.

Therefore, we have the following:

Theorem 4.1. For given natural numbers $m, n = 2, 3, \ldots$, consider the equation

(4)
$$k + \ell = m + n - 2, \qquad \sum_{r+s=m-1} (-1)^r \binom{k}{r} \binom{\ell}{s} = 0$$

with unknowns k and ℓ . Then, we have the following:

- (i) We have $\nu(\mathbb{D}_{m,n}) \ge m + n 2$ in general.
- (ii) If the equation (4) has no solution then $\nu(\mathbb{D}_{m,n}) \ge m + n 1$.

The polynomial in the Diophantine equation (4) is called the Krawtchouk polynomial which plays an important role in coding theory. See [28] and [32]. The equation (4) has not yet completely solved.

In order to get an upper bound for $\nu(\mathbb{D}_{m,n})$, we have to construct decomposable maps in the interior of the cone $\mathbb{D}_{m,n}$. By the duality between decomposable maps and PPT states with respect to the pairing (1), we see that the map in (3) lies on the boundary of the cone $\mathbb{D}_{m,n}$ if and only if there exists a PPT states σ such that the ranges of σ and the partial transpose σ^{Γ} coincide with \mathcal{V}^{\perp} and \mathcal{W}^{\perp} , respectively.

We consider the case with m = 2, to take an n - 1 dimensional subspace D of $\mathbb{C}^2 \otimes \mathbb{C}^n$ with no product vectors [33, 29]. Then it is clear that there is no PPT state σ such that $\mathcal{R}\sigma = D$ and $\mathcal{R}\sigma^{\Gamma} = \mathbb{C}^2 \otimes \mathbb{C}^n$. Indeed, if we assume that there is such a state σ then σ must be separable by [23], but this state violates the range criterion [18]. Therefore, if we take a basis \mathcal{V} in D^{\perp} then the map $\phi_{\mathcal{V}}$ is an interior point of the cone $\mathbb{D}_{2,n}$. This shows that $\nu(\mathbb{D}_{2,n}) \leq n+1$. In the case of m = 2, it was shown in [21] that the equation (4) has a solution if and only if n is an even number. This proves the odd case of the following:

(5)
$$\nu(\mathbb{D}_{2,n}) = \begin{cases} n+1, & n \text{ is odd,} \\ n, & n \text{ is even.} \end{cases}$$

When $n = 2\mu$ is an even integer than the equation (4) has the unique solution $(k, \ell) = (\mu, \mu)$, and one can construct $\mathcal{V} = \{V_1, \ldots, V_\mu\}$ and $\mathcal{W} = \{W_1, \ldots, W_\mu\}$ so that the decomposable map (3) is an interior point of $\mathbb{D}_{2,n}$, following the argument in [21]. To do this, we consider the $2 \times 2\mu$ matrix V_i whose *i*-th 2×2 block is the identity matrix and other entries are all zero. We also consider the $2 \times 2\mu$ matrix W_i whose *i*-th block is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and other entries are all zero. Then we see that

$$\sum_{i=1}^{\mu} \phi_{V_i} + \sum_{i=1}^{\mu} \phi^{W_i}$$

is just the trace map sending $X \in M_2$ to $\text{Tr}(X)I \in M_{2\mu}$, which is an interior point of $\mathbb{D}_{2,2\mu}$. This shows the above equality (5) when *n* is even. For the 2 \otimes 2 system, the whole facial structures of the cone $\mathbb{D}_{2,2}$ have been characterized in [3].

In the case of m = 3, we know that the equation (4) has a solution if and only if n is of the form $n = \mu(\mu + 2)$, with the solution $(k, \ell) = (\binom{\mu+1}{2}, \binom{\mu+2}{2})$. Especially, in the $3 \otimes 3$ case, we have the solution $(k, \ell) = (1, 3)$. In this case, we see that the map

$$\phi_I + \phi^{E_{12} - E_{21}} + \phi^{E_{23} - E_{32}} + \phi^{E_{31} - E_{13}}$$

is exactly the trace map, which is an interior point of $\mathbb{D}_{3,3}$. Therefore, we have

$$\nu(\mathbb{D}_{3,3}) = 4.$$

5. Discussion

For a given convex set, we have considered the smallest number of extreme points with which we may get an interior point by their convex combinations. For $3 \otimes 3$ PPT states and positive maps, these numbers turned out to be just 2. This means that there exist 'antipodal' extreme points. Poor knowledge on extreme PPT states and extreme positive maps prevent the authors to extend these results to higher dimensions.

For the cases of separable states and decomposable maps, these numbers exceed 2. This means that there exist no 'antipodal' extreme points, and might reflect the facts that the notions of separability and decomposability are defined by convex hulls of prescribed given extreme points, and that there are no easy intrinsic characterizations for these notions.

The equality $\nu(\mathbb{S}_{m,n}) = mn$ tells us that the number mn is the minimum of the lengths of interior points of $\mathbb{S}_{m,n}$. Recall that the length of a separable state ϱ is given by the minimum number of product states with which ϱ can be expressed as a convex combination. It seems to be an interesting question to ask if every interior point of $\mathbb{S}_{m,n}$ has the length mn. The authors [14, 16] have recently constructed separable states in $\mathbb{S}_{m,n}$ whose lengths exceed the number mn, for the cases (m, n) = (3, 3) and (2, 4). All of those are boundary points of the convex set $\mathbb{S}_{m,n}$. See also [5]. For some faces of $\mathbb{S}_{m,n}$, it is possible to characterize the interior by lengths. For example, this is clearly the case if a face of $\mathbb{S}_{m,n}$ is affinely isomorphic to a simplex. See [14, 16] for constructions of such faces in the $3 \otimes 3$ or $2 \otimes n$ cases. This is also the case [25] for a face of $\mathbb{S}_{2,n}$ which is affinely isomorphic to the convex set generated by trigonometric moment curve.

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6. Appendix

In this appendix we list up the remaining vectors X_i and Y_i constituting bases of real spaces $\text{Ker}(\phi_D)$ and $\text{Ker}(\phi_E)$, respectively, as follows:

$$\begin{split} X_8 &= e^{-i\theta}(E_{29} - E_{21}) + e^{i\theta}(E_{92} - E_{12}) + b(E_{14} + E_{41} - E_{49} - E_{94}), \\ X_9 &= e^{-i\theta}(E_{71} - E_{75}) + e^{i\theta}(E_{17} - E_{57}) + b(E_{35} + E_{53} - E_{13} - E_{31}), \\ X_{10} &= e^{-i\theta}(E_{79} - E_{71}) + e^{i\theta}(E_{97} - E_{17}) + b(E_{13} + E_{31} - E_{39} - E_{93}), \\ X_{11} &= e^{-i\theta}(E_{61} - E_{65}) + e^{i\theta}(E_{16} - E_{56}) + b(E_{58} + E_{85} - E_{18} - E_{81}), \\ X_{12} &= e^{-i\theta}(E_{69} - E_{61}) + e^{i\theta}(E_{96} - E_{16}) + b(E_{18} + E_{81} - E_{89} - E_{98}), \\ X_{13} &= -e^{-i\theta}(E_{21} + E_{25}) - e^{i\theta}(E_{12} + E_{52}) + b(E_{14} + E_{41} - E_{45} - E_{54}), \\ X_{14} &= e^{-i\theta}(E_{13} - E_{53}) + e^{i\theta}(E_{31} - E_{39}) + \frac{1}{b}(E_{97} + E_{79} - E_{17} - E_{71}), \\ X_{15} &= e^{-i\theta}(E_{13} - E_{93}) + e^{i\theta}(E_{41} - E_{49}) + \frac{1}{b}(E_{29} + E_{92} - E_{12} - E_{21}), \\ X_{16} &= e^{-i\theta}(E_{14} - E_{54}) + e^{i\theta}(E_{41} - E_{49}) + \frac{1}{b}(E_{29} + E_{92} - E_{12} - E_{21}), \\ X_{17} &= e^{-i\theta}(E_{18} - E_{98}) + e^{i\theta}(E_{81} - E_{89}) + \frac{1}{b}(E_{69} + E_{96} - E_{16} - E_{61}), \\ X_{19} &= e^{-i\theta}(E_{58} - E_{18}) + e^{i\theta}(E_{36} + E_{87}) - b(E_{38} + E_{83}) - \frac{1}{b}(E_{67} + E_{76}), \\ X_{20} &= e^{-i\theta}(E_{63} + E_{78}) + e^{i\theta}(E_{32} + E_{47}) + b(E_{34} + E_{43}) + \frac{1}{b}(E_{26} + E_{62}), \\ X_{22} &= -e^{-i\theta}(E_{23} + E_{74}) - e^{i\theta}(E_{32} + E_{46}) + b(E_{48} + E_{84}) + \frac{1}{b}(E_{26} + E_{62}), \\ X_{23} &= e^{-i\theta}(E_{67} + b^2 E_{83} - be^{-i\theta} E_{63}) + e^{i\theta}(E_{84} + \frac{1}{b^2} E_{62} - \frac{1}{b}e^{i\theta} E_{82}) - \frac{1}{b}(E_{46} + E_{64}), \\ X_{24} &= e^{-i\theta}(E_{48} + \frac{1}{b^2} E_{26} - \frac{1}{b}e^{-i\theta} E_{23}) - e^{i\theta}(E_{34} + \frac{1}{b^2} E_{72} - \frac{1}{b}e^{i\theta} E_{32}) + \frac{1}{b}(E_{47} + E_{74}), \\ y_{13} &= e^{-i\theta}(E_{43} + \frac{1}{b^2} E_{27} - \frac{1}{b}e^{-i\theta} E_{23}) - e^{i\theta}(E_{34} + \frac{1}{b^2} E_{72} - \frac{1}{b}e^{i\theta} E_{32}) + \frac{1}{b}(E_{47} + E_{74}), \\ x_{25} &= -e^{-i\theta}(E_{43} + \frac{1}{b^2} E_{27} - \frac{1}{b}e^{-i\theta} E_{23}) - e^{i\theta}(E_{34} + \frac{1}{b^2} E_{72} - \frac{1}{b}e^{i\theta} E_{32}) + \frac{1}{b}(E_{47} + E_{74}), \\ y_{13} &= e^{-i\theta}(E_{14} + \frac{1}{b^2$$

$$\begin{split} &Y_8 = e^{-i\theta}(E_{21} - E_{33}) + e^{i\theta}(E_{12} - E_{38}) + b(E_{67} + E_{76} - E_{14} - E_{41}), \\ &Y_9 = e^{-i\theta}(E_{21} - E_{52}) + e^{i\theta}(E_{12} - E_{25}) + b(E_{45} + E_{54} - E_{14} - E_{41}), \\ &Y_{10} = e^{-i\theta}(E_{34} - E_{65}) + e^{i\theta}(E_{43} - E_{66}) + b(E_{58} + E_{85} - E_{27} - E_{72}), \\ &Y_{11} = e^{-i\theta}(E_{34} - E_{96}) + e^{i\theta}(E_{84} - E_{71}) + b(E_{13} + E_{31} - E_{26} - E_{62}), \\ &Y_{13} = e^{-i\theta}(E_{79} - E_{17}) + e^{i\theta}(E_{97} - E_{71}) + b(E_{13} + E_{31} - E_{39} - E_{39}), \\ &Y_{14} = e^{-i\theta}(E_{26} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{48} - E_{84}), \\ &Y_{15} = e^{-i\theta}(E_{39} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{48} - E_{84}), \\ &Y_{15} = e^{-i\theta}(E_{39} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{79} - E_{97}), \\ &Y_{14} = e^{-i\theta}(E_{26} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{48} - E_{84}), \\ &Y_{15} = e^{-i\theta}(E_{39} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{79} - E_{97}), \\ &Y_{14} = e^{-i\theta}(E_{26} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{79} - E_{97}), \\ &Y_{15} = e^{-i\theta}(E_{39} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{79} - E_{97}), \\ &Y_{16} = e^{-i\theta}(E_{39} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{79} - E_{97}), \\ &Y_{16} = e^{-i\theta}(E_{39} - E_{13}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{17} + E_{71} - E_{79} - E_{97}), \\ &Y_{16} = e^{-i\theta}(E_{72} - E_{85}) + e^{i\theta}(E_{62} - E_{31}) + \frac{1}{b}(E_{55} + E_{52} - E_{12} - E_{21}), \\ &Y_{17} = e^{-i\theta}(E_{67} - E_{41}) + e^{i\theta}(E_{76} - E_{14}) + \frac{1}{b}(E_{59} + E_{96} - E_{34} - E_{43}), \\ &Y_{18} = e^{-i\theta}(E_{72} - E_{89}) + e^{i\theta}(E_{27} - E_{89}) + \frac{1}{b}(E_{69} + E_{96} - E_{34} - E_{43}), \\ &Y_{20} = e^{-i\theta}(E_{64} + E_{82}) - e^{i\theta}(E_{63} + E_{87}) - b(E_{29} + E_{92}) - \frac{1}{b}(E_{49} + E_{94}), \\ &Y_{21} = -e^{-i\theta}(E_{64} + E_{82}) - e^{i\theta}(E_{64} + E_{28}) + b(E_{57} + E_{75}) + \frac{1}{b}(E_{46} + E_{64}), \\$$

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