# Convergence Conditions of Mixed States and their von Neumann Entropy in Continuous Quantum Measurements 

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#### Abstract

By carrying out appropriate continuous quantum measurements with a family of projection operators, a unitary channel can be approximated in an arbitrary precision in the trace norm sense. In particular, the quantum Zeno effect is described as an application. In the case of an infinite dimension, although the von Neumann entropy is not necessarily continuous, the difference of the entropies between the states, as mentioned above, can be arbitrarily made small under some conditions.


## 1 Introduction

The quantum Zeno effect (QZE) is a quantum effect which was shown by Misra and Sudarshan in [5]. This effect demonstrates that, in quantum mechanics, continuous measurements can freeze a state. Of course, this effect is peculiar to quantum mechanics. Such an effect is not observed in classical mechanics. The QZE has been extensively investigated by many researchers since its discovery.

Recently, some general mathematical aspects of quantum Zeno effect were investigated in [2]. In particular, continuous measurements of a state along a certain curve in a Hilbert space were considered. Roughly speaking,

[^0]continuous measurements made along a curve prescribed in advance change the initial state to the final state with probability 1 . This fact includes the QZE as a special case. However, in the paper [2], it is assumed that states under consideration are vector states.

In this paper, we show that a result similar to one in [2] holds with respect to mixed states too. By considering a mixed state, its von Neumann entropy can also be considered. In the case where the Hilbert space under consideration is infinite dimensional, the von Neumann entropy is not necessarily continuous with respect to the trace norm. Hence, by continuous measurements, even if the initial state converges to the final state in the trace norm sense, it does not always mean that the von Neumann entropy converges too. Moreover, the set of density operators with finite entropy is a first category [10]. Hence, it is meaningful to investigate convergence conditions of the von Neumann entropy in our continuous measurements.

In Section 2, we begin with defining the "continuous quantum measurements" as a certain type of quantum channel. We use two types of quantum channels and a combination of them. By doing so, a concept of "continuous quantum measurements" are defined clearly. We consider conditions for pointwise convergence and trace norm convergence. We apply obtained results to the QZE.

In Section 3, we consider the von Neumann entropy in infinite dimension. We show that the convergence conditions of the von Neumann entropy in continuous quantum measurement which considered in Section 2. Here, Simon's convergence theorem [4] plays a central role.

## 2 Continuous measurements for mixed states

### 2.1 Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space of state vectors of a quantum system $\mathcal{S}$. We denote the inner product and the norm of $\mathcal{H}$ by $\langle\cdot, \cdot\rangle$ (anti-linear in the first variable and linear in the second) and $\|\cdot\|$, respectively. Let $d(\leq \infty)$ be the dimension of $\mathcal{H}$. We denote all bounded linear operators, all compact operators, all trace-class operators, all density operators, and all unitary operators on $\mathcal{H}$ by $\mathfrak{B}(\mathcal{H}), \mathfrak{C}(\mathcal{H}), \mathfrak{T}(\mathcal{H}), \mathfrak{S}(\mathcal{H})$, and $\mathfrak{U}(\mathcal{H})$, respectively. A mixed state of $\mathcal{S}$ is represented as an element of $\mathfrak{S}(\mathcal{H})$. We denote the trace norm by $\|\cdot\|_{1}:=\operatorname{Tr}|\cdot|$. The Hamiltonian of the quantum system $\mathcal{S}$ is given by a self-adjoint operator $H$ which is time independent. The domain of $H$ is denoted as $D(H)$.

Let us consider the following two maps on $\mathfrak{S}(\mathcal{H})$ :

## 1. (Unitary channel)

Let $U$ be a unitary operator on $\mathcal{H}$ and $\mathcal{E}_{U}$ be a map on $\mathfrak{S}(\mathcal{H})$ which is given by

$$
\mathcal{E}_{U} \rho:=U \rho U^{*}, \forall \rho \in \mathfrak{S}(\mathcal{H})
$$

In particular, in the case $U=e^{-i t H}(t \in \mathbb{R})$, we denote $\mathcal{E}_{e^{-i t H}}$ by $\mathcal{E}_{t}$.

## 2. (Projection channel)

Let $\mathfrak{P}:=\left\{P_{n}\right\}_{n}$ be a family of projection operators on $\mathcal{H}$ with $P_{m} \perp$ $P_{n}(m \neq n), I=\sum_{n} P_{n}$, and $\mathcal{E}_{\mathfrak{P}}$ be a map on $\mathfrak{S}(\mathcal{H})$ which is given by

$$
\mathcal{E}_{\mathfrak{P}} \rho:=\sum_{n} P_{n} \rho P_{n}, \quad \forall \rho \in \mathfrak{S}(\mathcal{H})
$$

Now, consider a state $\rho \in \mathfrak{S}(\mathcal{H})$ fixed and suppose that one of the Schatten decompositions is given by

$$
\begin{equation*}
\rho=\sum_{n=1}^{d} \lambda_{n}\left|\Psi_{n}\right\rangle\left\langle\Psi_{n}\right| \tag{2.1}
\end{equation*}
$$

where, for all $\Psi, \Phi \in \mathcal{H}$, we denote the operator $\langle\Psi, \cdot\rangle \Phi$ by $|\Phi\rangle\langle\Psi|$. In (2.1), we allow $\lambda_{n}=0$ to take $\Psi_{n}$ such that $\left\{\Psi_{n}\right\}_{n=1}^{d}$ is a complete orthonormal system (CONS). We remark that it is not necessarily $\lambda_{n} \geq \lambda_{n+1}$ in this representation.

Let us consider a time interval $[0, \tau]$ with $\tau>0$. For the decomposition (2.1), consider a CONS of $\mathcal{H}$ denoted by $\left\{\Psi_{n}(t)\right\}_{n=1}^{d}$ which is parametrized by $t \in[0, \tau]$ with $\Psi_{n}(0)=\Psi_{n}(1 \leq \forall n \leq d)$. If $n \in \mathbb{N}$ is fixed, then $\Psi_{n}(\cdot)$ is a map from $[0, \tau]$ to $\mathcal{H}$.

We define

$$
\begin{equation*}
\mathfrak{P}(t):=\left\{\left|\Psi_{n}(t)\right\rangle\left\langle\Psi_{n}(t)\right|\right\}_{n=1}^{d}, \quad(t \in[0, \tau]) \tag{2.2}
\end{equation*}
$$

Let $\Delta: t_{0}, t_{1}, \cdots, t_{N}\left(t_{j} \in[0, \tau], j=0, \cdots, N\right)$ be an arbitrary partition of the interval $[0, \tau]$ :

$$
0=t_{0}<t_{1}<\cdots<t_{N-1}<t_{N}=\tau
$$

We set

$$
\Delta_{k}:=t_{k}-t_{k-1}, \quad(k=1, \cdots, N), \quad|\Delta|:=\max _{1 \leq k \leq N} \Delta_{k}
$$

and define

$$
\begin{equation*}
\rho_{\Delta}(\tau):=\varepsilon_{\mathfrak{P}\left(t_{N}\right)} \circ \varepsilon_{\Delta_{N}} \circ \mathcal{E}_{\mathfrak{P}\left(t_{N-1}\right)} \circ \mathcal{E}_{\Delta_{N-1}} \circ \cdots \circ \mathcal{E}_{\mathfrak{P}\left(t_{1}\right)} \circ \mathcal{E}_{\Delta_{1}} \rho . \tag{2.3}
\end{equation*}
$$

In the context of quantum mechanics where $\rho_{\Delta}(\tau)$ is interpreted as the posterior state that, in the successive measurements at time $t_{1}, \cdots, t_{N}$ by using the family of projection operators $\mathfrak{P}\left(t_{1}\right), \cdots, \mathfrak{P}\left(t_{N}\right)$, respectively. We remark that $\rho_{\Delta}(\tau)$ is dependent on the form of decomposition (2.1).

If $\rho_{\Delta}(\tau)$ converges with respect to $|\Delta| \rightarrow 0$ in a certain sense, we call such a measurements of a series "continuous quantum measurements".

By direct computations, we have

$$
\begin{equation*}
\rho_{\Delta}(\tau)=\sum_{k} \lambda_{\Delta, k}\left|\Psi_{k}(\tau)\right\rangle\left\langle\Psi_{k}(\tau)\right| \tag{2.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\lambda_{\Delta, k}:=\sum_{k_{0}, \cdots, k_{N-1}} \lambda_{k_{0}} \prod_{j=1}^{N}\left|\left\langle\Psi_{k_{j}}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k_{j-1}}\left(t_{j-1}\right)\right\rangle\right|^{2}, \quad\left(k_{N}=k\right) . \tag{2.5}
\end{equation*}
$$

### 2.2 Pointwise convergence

Let us consider a convergence condition of $\lambda_{\Delta, k}$ in the case $|\Delta| \rightarrow 0$.
Let

$$
\begin{align*}
& \gamma_{\Delta, k}:= \prod_{j=1}^{N}\left|\left\langle\Psi_{k}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k}\left(t_{j-1}\right)\right\rangle\right|^{2},  \tag{2.6}\\
& \epsilon_{\Delta, k}:=\sum_{\substack{k_{0}, \cdots, k_{N-1} \\
\exists l \in\{0, \cdots, N-1\}, k_{l} \neq k}} \lambda_{k_{0}} \prod_{j=1}^{N}\left|\left\langle\Psi_{k_{j}}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k_{j-1}}\left(t_{j-1}\right)\right\rangle\right|^{2}, \tag{2.7}
\end{align*}
$$

so that

$$
\begin{equation*}
\lambda_{\Delta, k}=\lambda_{k} \gamma_{\Delta, k}+\epsilon_{\Delta, k} . \tag{2.8}
\end{equation*}
$$

Theorem 2.1 Assume that there exists $k \in \mathbb{N}$ such that the following conditions hold:

$$
\begin{align*}
& \forall \lambda \in[0, \tau], \quad \Psi_{k}(\lambda) \in D(H),  \tag{2.9}\\
& \xi_{k}:=\sup _{0 \leq \lambda \leq \tau}\left\|H \Psi_{k}(\lambda)\right\|<\infty,  \tag{2.10}\\
& \eta_{k}:=\sup _{\substack{\lambda, \nu \in[0, \tau] \\
\lambda \neq \nu}} \frac{\left\|\Psi_{k}(\lambda)-\Psi_{k}(\nu)\right\|}{|\lambda-\nu|}<\infty,  \tag{2.11}\\
& \lim _{|\Delta| \rightarrow 0} \sum_{j=1}^{N} \operatorname{Re}\left\langle\Psi_{k}\left(t_{j}\right)-\Psi_{k}\left(t_{j-1}\right), \Psi_{k}\left(t_{j-1}\right)\right\rangle=0 . \tag{2.12}
\end{align*}
$$

Then we have

$$
\begin{equation*}
\lim _{|\Delta| \rightarrow 0} \lambda_{\Delta, k}=\lambda_{k} . \tag{2.13}
\end{equation*}
$$

Remark 2.2 Condition (2.11) implies that $\left\|\Psi_{k}(\lambda)-\Psi_{k}(\nu)\right\| \leq \eta_{k} \mid \lambda-$ $\mu \mid, \forall \lambda, \mu \in[0, \tau]$ (Lipschitz continuity). In particular, $\Psi_{k}(\cdot)$ is strongly continuous, so that the mapping $\Psi_{k}(\cdot):[0, t] \rightarrow \mathcal{H}$ is a curve in $\mathcal{H}$.

Proof. By using [2, THEOREM 4.2], the assumptions (2.9)-(2.12) imply that

$$
\begin{equation*}
\lim _{|\Delta| \rightarrow 0} \gamma_{\Delta, k}=1 \tag{2.14}
\end{equation*}
$$

On the other hand, we can estimate $\epsilon_{\Delta, k}$ as follows.

$$
\begin{align*}
\epsilon_{\Delta, k}= & \sum_{l=0}^{N-1} \sum_{\substack{k_{0}, \ldots, k_{N-1} \\
\forall i>l, k_{i}=k, k_{l} \neq k}} \lambda_{k_{0}} \prod_{j=1}^{N}\left|\left\langle\Psi_{k_{j}}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k_{j-1}}\left(t_{j-1}\right)\right\rangle\right|^{2}  \tag{2.15}\\
= & \sum_{l=0}^{N-1}\left\{\prod_{j=l+2}^{N}\left|\left\langle\Psi_{k}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k}\left(t_{j-1}\right)\right\rangle\right|^{2} \sum_{k_{l}, k_{l} \neq k}\left|\left\langle\Psi_{k}\left(t_{l+1}\right), e^{-i \Delta_{l+1} H} \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2}\right. \\
& \left.\times \sum_{k_{l-1}}\left|\left\langle\Psi_{k_{l}}\left(t_{l}\right), e^{-i \Delta_{l} H} \Psi_{k_{l-1}}\left(t_{l-1}\right)\right\rangle\right|^{2} \times \cdots \times \sum_{k_{0}}\left|\left\langle\Psi_{k_{1}}\left(t_{1}\right), e^{-i \Delta_{1} H} \Psi_{k_{0}}\left(t_{0}\right)\right\rangle\right|^{2} \lambda_{k_{0}}\right\}, \tag{2.16}
\end{align*}
$$

in the case where $l=0, N-1,\{\cdots\}$ in (2.16) is given by

$$
\begin{align*}
& \prod_{j=2}^{N}\left|\left\langle\Psi_{k}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k}\left(t_{j-1}\right)\right\rangle\right|^{2} \sum_{k_{0}, k_{0} \neq k}\left|\left\langle\Psi_{k_{1}}\left(t_{1}\right), e^{-i \Delta_{1} H} \Psi_{k_{0}}\left(t_{0}\right)\right\rangle\right|^{2} \lambda_{k_{0}}  \tag{2.17}\\
& \quad \sum_{k_{N-1}, k_{N-1} \neq k}\left|\left\langle\Psi_{k}\left(t_{l+1}\right), e^{-i \Delta_{l+1} H} \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2} \sum_{k_{N-2}}\left|\left\langle\Psi_{k_{N-1}}\left(t_{N-1}\right), e^{-i \Delta_{N-1} H} \Psi_{k_{l-2}}\left(t_{l-2}\right)\right\rangle\right|^{2} \\
& \cdots \times \sum_{k_{0}}\left|\left\langle\Psi_{k_{1}}\left(t_{1}\right), e^{-i \Delta_{1} H} \Psi_{k_{0}}\left(t_{0}\right)\right\rangle\right|^{2} \lambda_{k_{0}} \tag{2.18}
\end{align*}
$$

respectively.
By the Schwarz inequality, we have

$$
\begin{aligned}
\prod_{j=l+2}^{N}\left|\left\langle\Psi_{k}\left(t_{j}\right), e^{-i \Delta_{j} H} \Psi_{k}\left(t_{j-1}\right)\right\rangle\right|^{2} & \leq \prod_{j=l+2}^{N}\left\|\Psi_{k}\left(t_{j}\right)\right\|^{2} \cdot\left\|e^{-i \Delta_{j} H} \Psi_{k}\left(t_{j-1}\right)\right\|^{2} \\
& \leq 1, \quad \forall l \in\{0, \cdots, N-2\}
\end{aligned}
$$

For all $l \geq 1$,

$$
\begin{aligned}
& \sum_{k_{l-1}}\left|\left\langle\Psi_{k_{l}}\left(t_{l}\right), e^{-i \Delta_{l} H} \Psi_{k_{l-1}}\left(t_{l-1}\right)\right\rangle\right|^{2} \times \cdots \times \sum_{k_{0}}\left|\left\langle\Psi_{k_{1}}\left(t_{1}\right), e^{-i \Delta_{1} H} \Psi_{k_{0}}\left(t_{0}\right)\right\rangle\right|^{2} \lambda_{k_{0}} \\
\leq & \sum_{k_{l-1}}\left|\left\langle\Psi_{k_{l}}\left(t_{l}\right), e^{-i \Delta_{l} H} \Psi_{k_{l-1}}\left(t_{l-1}\right)\right\rangle\right|^{2} \times \cdots \times \sum_{k_{0}}\left|\left\langle e^{i \Delta_{1} H} \Psi_{k_{1}}\left(t_{1}\right), \Psi_{k_{0}}\left(t_{0}\right)\right\rangle\right|^{2} \\
\leq & \sum_{k_{l-1}}\left|\left\langle\Psi_{k_{l}}\left(t_{l}\right), e^{-i \Delta_{l} H} \Psi_{k_{l-1}}\left(t_{l-1}\right)\right\rangle\right|^{2} \times \cdots \times\left\|e^{i \Delta_{1} H} \Psi_{k_{1}}\left(t_{1}\right)\right\|^{2} \\
\leq & \cdots \leq 1
\end{aligned}
$$

Thus (2.16) implies that

$$
\begin{equation*}
\epsilon_{\Delta, k} \leq \sum_{l=0}^{N-1} \sum_{k_{l}, k_{l} \neq k}\left|\left\langle\Psi_{k}\left(t_{l+1}\right), e^{-i \Delta_{l+1} H} \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2} \tag{2.19}
\end{equation*}
$$

In the case where $k_{l} \neq k$, we have $\left\langle\Psi_{k}\left(t_{l}\right), \Psi_{k_{l}}\left(t_{l}\right)\right\rangle=0$. Hence

$$
\begin{align*}
& \sum_{k_{l}, k_{l} \neq k}\left|\left\langle\Psi_{k}\left(t_{l+1}\right), e^{-i \Delta_{l+1} H} \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2} \\
= & \sum_{k_{l}, k_{l} \neq k}\left|\left\langle\Psi_{k}\left(t_{l+1}\right),\left(e^{-i \Delta_{l+1} H}-1\right) \Psi_{k_{l}}\left(t_{l}\right)\right\rangle+\left\langle\Psi_{k}\left(t_{l+1}\right)-\Psi_{k}\left(t_{l}\right), \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2} \\
\leq & 2 \sum_{k_{l}, k_{l} \neq k}\left\{\left|\left\langle\left(e^{i \Delta_{l+1} H}-1\right) \Psi_{k}\left(t_{l+1}\right), \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2}+\left|\left\langle\Psi_{k}\left(t_{l+1}\right)-\Psi_{k}\left(t_{l}\right), \Psi_{k_{l}}\left(t_{l}\right)\right\rangle\right|^{2}\right\} \\
\leq & 2\left\{\left\|\left(e^{i \Delta_{l+1} H}-1\right) \Psi_{k}\left(t_{l+1}\right)\right\|^{2}+\left\|\Psi_{k}\left(t_{l+1}\right)-\Psi_{k}\left(t_{l}\right)\right\|^{2}\right\} . \tag{2.20}
\end{align*}
$$

Let $E_{H}(\cdot)$ be the spectral measure of Hamiltonian $H$. By the spectral theorem, we have

$$
\begin{align*}
\left\|\left(e^{i \Delta_{l+1} H}-1\right) \Psi_{k}\left(t_{l+1}\right)\right\|^{2} & =\int_{\mathbb{R}}\left|e^{i \Delta_{l+1} x}-1\right|^{2} d\left\|E_{H}(x) \Psi_{k}\left(t_{l+1}\right)\right\|^{2} \\
& \leq \int_{\mathbb{R}} \Delta_{l+1}^{2} x^{2} d\left\|E_{H}(x) \Psi_{k}\left(t_{l+1}\right)\right\|^{2} \\
& \leq \Delta_{l+1}^{2}\left\|H \Psi_{k}\left(t_{l+1}\right)\right\|^{2} \tag{2.21}
\end{align*}
$$

The assumptions (2.9)-(2.11) imply that

$$
\begin{equation*}
\left\|H \Psi_{k}\left(t_{l+1}\right)\right\|^{2} \leq \xi_{k}^{2}, \quad\left\|\Psi_{k}\left(t_{l+1}\right)-\Psi_{k}\left(t_{l}\right)\right\|^{2} \leq \Delta_{l+1}^{2} \eta_{k}^{2} \tag{2.22}
\end{equation*}
$$

Therefore, (2.19), (2.20), (2.21) and (2.22) implies that

$$
\begin{align*}
\epsilon_{\Delta, k} & \leq 2 \sum_{l=0}^{N-1}\left\{\left\|\left(e^{i \Delta_{l+1} H}-1\right) \Psi_{k}\left(t_{l+1}\right)\right\|^{2}+\left\|\Psi_{k}\left(t_{l+1}\right)-\Psi_{k}\left(t_{l}\right)\right\|^{2}\right\} \\
& \leq 2 \sum_{l=0}^{N-1}\left\{\Delta_{l+1}^{2}\left\|H \Psi_{k}\left(t_{l+1}\right)\right\|^{2}+\left\|\Psi_{k}\left(t_{l+1}\right)-\Psi_{k}\left(t_{l}\right)\right\|^{2}\right\} \\
& \leq 2\left(\xi_{k}^{2}+\eta_{k}^{2}\right) \sum_{l=1}^{N} \Delta_{l}^{2} \tag{2.23}
\end{align*}
$$

By [2, LEMMA 2.2],

$$
\lim _{|\Delta| \rightarrow 0} \sum_{l=1}^{N} \Delta_{l}^{2}=0
$$

Thus (2.23) implies that $\lim _{|\Delta| \rightarrow 0} \epsilon_{\Delta, k}=0$. Hence, by (2.8) and (2.14), we obtain (2.13)

Remark 2.3 Assume that the conditions of Theorem 2.1 hold. Let $a>1$ be a constant and take $|\Delta|$ such that

$$
\begin{equation*}
\left(\xi_{k}^{2}+2 \xi_{k} \eta_{k}\right)|\Delta|^{2}+2 \eta_{k}|\Delta| \leq \frac{\log a}{a} \tag{2.24}
\end{equation*}
$$

By the proof of [2, THEOREM 4.2],

$$
\begin{align*}
& \exp \left[-a\left\{\left(\xi_{k}^{2}+2 \xi_{k} \eta_{k}\right) \sum_{l=1}^{N} \Delta_{l}^{2}-2 \sum_{l=1}^{N} \operatorname{Re}\left\langle\Psi_{k}\left(t_{l}\right)-\Psi_{k}\left(t_{l-1}\right), \Psi_{k}\left(t_{l-1}\right)\right\rangle\right\}\right] \\
& \leq \gamma_{\Delta, k} \leq 1 \tag{2.25}
\end{align*}
$$

Then, by (2.23) and (2.25), we have

$$
\begin{align*}
& \left|\lambda_{\Delta, k}-\lambda_{k}\right|=\left|\lambda_{k}\left(\gamma_{\Delta, k}-1\right)+\epsilon_{\Delta, k}\right| \leq \lambda_{k}\left(1-\gamma_{\Delta, k}\right)+\epsilon_{\Delta, k} \\
\leq & \lambda_{k}\left(1-\exp \left[-a\left\{\left(\xi_{k}^{2}+2 \xi_{k} \eta_{k}\right) \sum_{l=1}^{N} \Delta_{l}^{2}-2 \sum_{l=1}^{N} \operatorname{Re}\left\langle\Psi_{k}\left(t_{l}\right)-\Psi_{k}\left(t_{l-1}\right), \Psi_{k}\left(t_{l-1}\right)\right\rangle\right\}\right]\right) \\
& +2\left(\xi_{k}^{2}+\eta_{k}^{2}\right) \sum_{l=1}^{N} \Delta_{l}^{2} . \tag{2.26}
\end{align*}
$$

The following corollary can be easily proven by using [2, COROLLARY 4.4].

Corollary 2.4 Assume that there exists $k \in \mathbb{N}$ such that the following conditions hold:

$$
\begin{align*}
& \Psi_{k}(\cdot):[0, \tau] \rightarrow \mathcal{H} \quad \text { is a strongly differentiable mapping, }  \tag{2.27}\\
& \forall \lambda \in[0, \tau], \quad \Psi_{k}(\lambda) \in D(H)  \tag{2.28}\\
& \xi_{k}<\infty,  \tag{2.29}\\
& \sup _{0 \leq \lambda \leq \tau}\left\|\Psi_{k}^{\prime}(\lambda)\right\|<\infty,  \tag{2.30}\\
& \text { where } \Psi_{k}^{\prime}(\cdot) \text { denotes the strong derivative of } \Psi_{k}(\cdot)
\end{align*}
$$

Then (2.9)-(2.12) hold. Therefore, by Theorem [2.1, (2.13) holds.
Example 2.5 Let $A$ be a self-adjoint operator on $\mathcal{H}$. Assume that there exists $k \in \mathbb{N}$ such that the following conditions hold:

$$
\begin{align*}
& \Psi_{k} \in D(A) \cap \bigcap_{0 \leq \lambda \leq \tau} D\left(H e^{-i \lambda A}\right),  \tag{2.31}\\
& \sup _{0 \leq \lambda \leq \tau}\left\|H e^{-i \lambda A} \Psi_{k}\right\|<\infty  \tag{2.32}\\
& \forall \lambda \in[o, \tau], \Psi_{k}(\lambda)=e^{-i \lambda A} \Psi_{k} . \tag{2.33}
\end{align*}
$$

In this case, by [2, EXAMPLE 4.5], (2.27)-(2.30) hold. Then by using Corollary 2.4, (2.9)-(2.13) hold.

### 2.3 Trace norm convergence

For the decomposition (2.1), we define

$$
\begin{equation*}
\rho(t):=\sum_{n} \lambda_{n}\left|\Psi_{n}(t)\right\rangle\left\langle\Psi_{n}(t)\right|, \quad \forall t \in[0, \tau] . \tag{2.34}
\end{equation*}
$$

Let us consider conditions of convergence from $\rho_{\Delta}(\tau)$ to $\rho(\tau)$ in the trace norm sense.

Theorem 2.6 Assume that the conditions (2.9)-(2.12) hold for all $k \in \mathbb{N}$ satisfying $\lambda_{k}>0$.

Then we have

$$
\begin{equation*}
\lim _{|\Delta| \rightarrow 0}\left\|\rho_{\Delta}(\tau)-\rho(\tau)\right\|_{1}=0 \tag{2.35}
\end{equation*}
$$

Proof. By definition of $\rho_{\Delta}(\tau), \rho(\tau)$, and equation (2.8), we have

$$
\begin{align*}
\left\|\rho_{\Delta}(\tau)-\rho(\tau)\right\|_{1} & =\sum_{k}\left\langle\Psi_{k}(\tau),\right| \rho_{\Delta}(\tau)-\rho(\tau)\left|\Psi_{k}(\tau)\right\rangle \\
& =\sum_{k}\left|\lambda_{\Delta, k}-\lambda_{k}\right| \\
& =\sum_{k}\left|\lambda_{k}\left(\gamma_{\Delta, k}-1\right)+\epsilon_{\Delta, k}\right| \\
& \leq \sum_{k} \lambda_{k}\left(1-\gamma_{\Delta, k}\right)+\sum_{k} \epsilon_{\Delta, k} \\
& =\sum_{k} \lambda_{k}\left(1-\gamma_{\Delta, k}\right)+\sum_{k}\left(\lambda_{\Delta, k}-\lambda_{k} \gamma_{\Delta, k}\right) \\
& =2-2 \sum_{k} \lambda_{k} \gamma_{\Delta, k} . \tag{2.36}
\end{align*}
$$

Note that

$$
\left|\lambda_{k} \gamma_{\Delta, k}\right| \leq \lambda_{k}(\forall k \in \mathbb{N}), \quad \sum_{k} \lambda_{k}=1 .
$$

The assumptions (2.9)-(2.12) imply that

$$
\lim _{|\Delta| \rightarrow 0} \lambda_{k} \gamma_{\Delta, k}=\lambda_{k}(\forall k \in \mathbb{N}) .
$$

Hence, by using Lebesgue's dominated convergence theorem, we have

$$
\lim _{|\Delta| \rightarrow 0} \sum_{k} \lambda_{k} \gamma_{\Delta, k}=1 .
$$

Therefore, by (2.36), we obtain (2.35).

Remark 2.7 Assume that the conditions of Theorem 2.6 hold and that $\sup _{k, \lambda_{k} \neq 0} \xi_{k}<\infty$ and $\sup _{k, \lambda_{k} \neq 0} \eta_{k}<\infty$ hold. Then, for $a>1$, we can take $|\Delta|$ such that (2.24) holds for all $k$ with $\lambda_{k} \neq 0$. Then we have (2.25) for all $k \in \mathbb{N}$ with $\lambda_{k} \neq 0$. Hence, by (2.36), for all $k \in \mathbb{N}$, we obtain the following estimation:

$$
\begin{align*}
& \left|\lambda_{\Delta, k}-\lambda_{k}\right| \leq\left\|\rho_{\Delta}(\tau)-\rho(\tau)\right\|_{1} \\
\leq & 2-2 \sum_{k} \lambda_{k} \exp \left[-a\left\{\left(\xi_{k}^{2}+2 \xi_{k} \eta_{k}\right) \sum_{l=1}^{N} \Delta_{l}^{2}-2 \sum_{l=1}^{N} \operatorname{Re}\left\langle\Psi_{k}\left(t_{l}\right)-\Psi_{k}\left(t_{l-1}\right), \Psi_{k}\left(t_{l-1}\right)\right\rangle\right\}\right] \tag{2.37}
\end{align*}
$$

The following corollary and example can be easily proven by using Corollary 2.4. Example 2.5, and Theorem 2.6.

Corollary 2.8 Assume that the conditions (2.27)-(2.30) hold for all $k \in \mathbb{N}$ with $\lambda_{k}>0$. Then we have (2.35).

Example 2.9 Let $A$ be a self-adjoint operator on $\mathcal{H}$. Assume that the conditions (2.31)- (2.33) hold for all $k \in \mathbb{N}$ with $\lambda_{k}>0$. Then we have (2.35).

In Example 2.9, let us consider the case of $d<\infty$. It is easy to see that the assumptions $(2.31)-(2.32)$ are satisfied. On the other hand, by Stone's theorem, for all $U \in \mathfrak{U}(\mathcal{H})$, there exists a self-adjoint operator $A$ such that $U=e^{-i \tau A}$. Since $\rho(\tau)=U \rho U^{*}$, we have $\lim _{|\Delta| \rightarrow 0}\left\|\rho_{\Delta}(\tau)-U \rho U^{*}\right\|_{1}=0$. This fact shows that, in the case $d<\infty$, an arbitrary state in $\left\{U \rho U^{*} \mid U \in\right.$ $\mathfrak{U}(\mathcal{H})\}$ can be approximated (in the trace norm sense) by states obtained after an appropriate continuous measurements. In other words, in this case, we can approximate an arbitrary unitary channel by continuous quantum measurements.

### 2.4 Application to quantum Zeno effect for mixed states

Let $\Psi_{k} \in D(H)$ and $\Psi_{k}(\lambda)=\Psi_{k}(\forall \lambda \in[0, \tau])$ holds for all $k \in \mathbb{N}$ with $\lambda_{k}>0$.

This is the case where $A=0$ in Example 2.9, Then (2.9) $-(2.12)$ hold for all $k \in \mathbb{N}$ with $\lambda_{k}>0$. Hence, we have (2.35).

This means that, by the series of measurement with respect to the family of the projection operators $\left\{\left|\Psi_{k}\right\rangle\left\langle\Psi_{k}\right|\right\}_{k}$, transitions to states different from the initial state are hindered. This can be interpreted as a quantum Zeno effect for mixed states.

## 3 Convergence condition of the von Neumann entropy

Let $\varphi:[0, \infty) \ni \lambda \mapsto-\lambda \log \lambda \in[0, \infty)$, where $\varphi(0):=0$. Then $\varphi$ is continuous, concave, and subadditive. Let $S(\rho)$ be the von Neumann entropy of $\rho \in \mathfrak{S}(\mathcal{H})$. i.e.

$$
S(\rho):=\operatorname{Tr} \varphi(\rho) .
$$

In the case $d<\infty$, by Fannes' inequality, we have for all $\rho_{1}, \rho_{2} \in \mathfrak{S}(\mathcal{H})$

$$
\left\|\rho_{1}-\rho_{2}\right\|_{1} \leq 1 / e \Longrightarrow\left|S\left(\rho_{1}\right)-S\left(\rho_{2}\right)\right| \leq\left\|\rho_{1}-\rho_{2}\right\|_{1} \log d+\varphi\left(\left\|\rho_{1}-\rho_{2}\right\|_{1}\right)
$$

Therefore the von Neumann entropy is continuous with respect to the trace norm.

On the other handin the case $d=\infty$ although the von Neumann entropy is lower semi-continuous with respect to the trace norm i.e. $\lim _{n \rightarrow \infty} \| \rho_{n}-$ $\rho \|_{1}=0 \Rightarrow S(\rho) \leq \liminf _{n \rightarrow \infty} S\left(\rho_{n}\right)$ ), it is not necessarily continuous. Moreover, it is known that the set $\{\rho \in \mathfrak{S}(\mathcal{H}) \mid S(\rho)<\infty\}$ is of the first category [10].

In what follows, we deal with the case where $d=\infty$ only.
For $\rho_{\Delta}(\tau)$ and $\rho$ considered in the section 2 , conditions of the convergence $S\left(\rho_{\Delta}(\tau)\right) \rightarrow S(\rho)$ are given by the following theorem.

Theorem 3.1 Assume that the conditions (2.9)-(2.11) hold for all $k \in \mathbb{N}$, and that the condition (2.12) holds for all $k \in \mathbb{N}$ with $\lambda_{k}>0$. Suppose that the following conditions hold:

$$
\begin{align*}
& \xi_{k} \rightarrow 0, \quad \eta_{k} \rightarrow 0(k \rightarrow 0),  \tag{3.1}\\
& S(\rho)<\infty,  \tag{3.2}\\
& \sum_{k} \varphi\left(\xi_{k}^{2}\right)<\infty, \quad \sum_{k} \varphi\left(\eta_{k}^{2}\right)<\infty . \tag{3.3}
\end{align*}
$$

Then

$$
\begin{equation*}
\lim _{|\Delta| \rightarrow 0} S\left(\rho_{\Delta}(\tau)\right)=S(\rho(\tau))=S(\rho) . \tag{3.4}
\end{equation*}
$$

Remark 3.2 The function $\varphi$ is monotone increasing on $[0,1 / e]$ and

$$
\xi_{k}^{2}=\sup _{0 \leq \lambda \leq \tau}\left\|H \Psi_{k}(\lambda)\right\|^{2}=\sup _{0 \leq \lambda \leq \tau} \int_{\mathbb{R}} x^{2} d\left\|E_{H}(x) \Psi_{k}(\lambda)\right\|^{2}
$$

Hence, $\xi_{k} \rightarrow 0(k \rightarrow \infty)$ implies that there exists $N_{0} \in \mathbb{N}$ such that, for all $k>N_{0}$,

$$
\varphi\left(\xi_{k}^{2}\right) \geq \sup _{0 \leq \lambda \leq \tau} \varphi\left(\int_{\mathbb{R}} x^{2} d\left\|E_{H}(x) \Psi_{k}(\lambda)\right\|^{2}\right)
$$

By Jensen's inequality, we have

$$
\varphi\left(\int_{\mathbb{R}} x^{2} d\left\|E_{H}(x) \Psi_{k}(\lambda)\right\|^{2}\right) \geq \int_{\mathbb{R}} \varphi\left(x^{2}\right) d\left\|E_{H}(x) \Psi_{k}(\lambda)\right\|^{2}
$$

Hence, for all $k>N_{0}$,

$$
\varphi\left(\xi_{k}^{2}\right) \geq \sup _{0 \leq \lambda \leq \tau} \int_{\mathbb{R}} \varphi\left(x^{2}\right) d\left\|E_{H}(x) \Psi_{k}(\lambda)\right\|^{2}
$$

Then, we have
$\forall k>N_{0}, \forall \lambda \in[0, \tau], \Psi_{k}(\lambda) \in D\left(\sqrt{\varphi\left(H^{2}\right)}\right), \varphi\left(\xi_{k}^{2}\right) \geq \sup _{0 \leq \lambda \leq \tau}\left\|\sqrt{\varphi\left(H^{2}\right)} \Psi_{k}(\lambda)\right\|^{2}$.
Moreover, using the estimate that

$$
\begin{aligned}
\sum_{k} \varphi\left(\xi_{k}^{2}\right)=\sum_{k=1}^{N_{0}} \varphi\left(\xi_{k}^{2}\right)+\sum_{k=N_{0}+1}^{\infty} \varphi\left(\xi_{k}^{2}\right) & \geq \sum_{k=1}^{N_{0}} \varphi\left(\xi_{k}^{2}\right)+\sum_{k=N_{0}+1}^{\infty} \sup _{0 \leq \lambda \leq \tau}\left\|\sqrt{\varphi\left(H^{2}\right)} \Psi_{k}(\lambda)\right\|^{2} \\
& \geq \sum_{k=1}^{N_{0}} \varphi\left(\xi_{k}^{2}\right)+\sup _{0 \leq \lambda \leq \tau} \sum_{k=N_{0}+1}^{\infty}\left\|\sqrt{\varphi\left(H^{2}\right)} \Psi_{k}(\lambda)\right\|^{2}
\end{aligned}
$$

we obtain
$\xi_{k} \rightarrow 0(k \rightarrow \infty), \sum_{k} \varphi\left(\xi_{k}^{2}\right)<\infty \Longrightarrow \exists N_{0} \in \mathbb{N}, \sup _{0 \leq \lambda \leq \tau} \sum_{k=N_{0}+1}^{\infty}\left\|\sqrt{\varphi\left(H^{2}\right)} \Psi_{k}(\lambda)\right\|^{2}<\infty$.
Particularly, in the case $H \in \mathfrak{B}(\mathcal{H})$, we have, for all $\Phi \in \mathcal{H}$,
$\int_{\mathbb{R}} \varphi\left(x^{2}\right) d\left\|E_{H}(x) \Phi\right\|^{2} \leq \sup _{x \in \sigma(H)} \varphi\left(x^{2}\right) \int_{\mathbb{R}} d\left\|E_{H}(x) \Phi\right\|^{2}=\sup _{x \in \sigma(H)} \varphi\left(x^{2}\right) \cdot\|\Phi\|<\infty$.

Hence, we obtain $\sqrt{\varphi\left(H^{2}\right)} \in \mathfrak{B}(\mathcal{H})$. Therefore, by (3.5), we have

$$
\begin{equation*}
\xi_{k} \rightarrow 0(k \rightarrow \infty), \sum_{k} \varphi\left(\xi_{k}^{2}\right)<\infty \Longrightarrow \varphi\left(H^{2}\right) \in \mathfrak{T}(\mathcal{H}) \tag{3.6}
\end{equation*}
$$

We remark that, in this case, if Hamiltonian $H$ is represented as a density operator, then $\varphi\left(H^{2}\right) \in \mathfrak{T}(\mathcal{H})$ means $S\left(H^{2}\right)<\infty$.

Proof. The assumption of this theorem and Theorem 2.6 imply that $\lim _{|\Delta| \rightarrow 0}\left\|\rho_{\Delta}(\tau)-\rho(\tau)\right\|_{1}=0$. Hence we have w- $\lim _{|\Delta| \rightarrow 0} \rho_{\Delta}(\tau)=\rho(\tau)$, where w- lim means weak limit.

By (2.8), (2.23) and $\gamma_{\Delta, k} \leq 1$, we have

$$
\lambda_{\Delta, k} \leq \lambda_{k}+2\left(\xi_{k}^{2}+\eta_{k}^{2}\right) \sum_{l=1}^{N} \Delta_{l}^{2}
$$

By $\lim _{|\Delta| \rightarrow 0} \sum_{l=1}^{N} \Delta_{l}^{2}=0$, there exists $\delta>0$ such that, for all $\Delta,|\Delta|<\delta \Rightarrow$ $\sum_{l=1}^{N} \Delta_{l}^{2}<1 / 2$. Thus

$$
\begin{equation*}
\lambda_{\Delta, k} \leq \lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2} \quad(|\Delta|<\delta) . \tag{3.7}
\end{equation*}
$$

We set

$$
\begin{equation*}
\sigma:=\sum_{k}\left(\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}\right)\left|\Psi_{k}(\tau)\right\rangle\left\langle\Psi_{k}(\tau)\right| . \tag{3.8}
\end{equation*}
$$

By the assumption of this theorem, $\sigma \in \mathfrak{C}(\mathcal{H})$. On the other hand, (3.7) implies that

$$
\begin{equation*}
\rho_{\Delta}(\tau) \leq \sigma \quad(|\Delta|<\delta) \tag{3.9}
\end{equation*}
$$

Moreover, by the assumption of this theorem and subadditivity of $\varphi$, we have

$$
\begin{align*}
S(\sigma) & =\sum_{k} \varphi\left(\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}\right)  \tag{3.10}\\
& \leq S(\rho)+\sum_{k} \varphi\left(\xi_{k}^{2}\right)+\sum_{k} \varphi\left(\eta_{k}^{2}\right)<\infty \tag{3.11}
\end{align*}
$$

Hence, by Simon's dominated convergence theorem for entropy 4. THEOREM A.3], we have

$$
\lim _{|\Delta| \rightarrow 0} S\left(\rho_{\Delta}(\tau)\right)=S(\rho(\tau))
$$

It is obvious that $S(\rho(\tau))=S(\rho)$ holds.

Remark 3.3 In the proof of Theorem 3.1, we used that

$$
\begin{equation*}
S(\rho)<\infty, \sum_{k} \varphi\left(\xi_{k}^{2}\right)<\infty, \sum_{k} \varphi\left(\eta_{k}^{2}\right)<\infty \Longrightarrow \sum_{k} \varphi\left(\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}\right)<\infty \tag{3.12}
\end{equation*}
$$

Conversely, we can show that, under condition (3.1),

$$
\begin{equation*}
\sum_{k} \varphi\left(\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}\right)<\infty \Longrightarrow S(\rho), \sum_{k} \varphi\left(\xi_{k}^{2}\right), \sum_{k} \varphi\left(\eta_{k}^{2}\right)<\infty \tag{3.13}
\end{equation*}
$$

as follows. By $\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2} \rightarrow 0(k \rightarrow \infty)$, we have

$$
\exists N_{0} \in \mathbb{N}, \forall k>N_{0}, \max \left\{\lambda_{k}, \xi_{k}^{2}, \eta_{k}^{2}\right\} \leq \lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}<1 / e
$$

Hence, by the fact that $\varphi$ is a monotone increasing function on $[0,1 / e]$, we obtain
$\max \left\{\sum_{k=N_{0}+1}^{\infty} \varphi\left(\lambda_{k}\right), \sum_{k=N_{0}+1}^{\infty} \varphi\left(\xi_{k}^{2}\right), \sum_{k=N_{0}+1}^{\infty} \varphi\left(\eta_{k}^{2}\right)\right\} \leq \sum_{k=N_{0}+1}^{\infty} \varphi\left(\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}\right)$.
Therefore, we have (3.13). Thus, in Theorem 3.1, we can replace the condition (3.2) and (3.3) with $\sum_{k} \varphi\left(\lambda_{k}+\xi_{k}^{2}+\eta_{k}^{2}\right)<\infty$.

Example 3.4 Let $A$ be a self-adjoint operator on $\mathcal{H}$. Assume that $A, H \in$ $\mathfrak{C}(\mathcal{H})$, and that $A$ and $H$ are strongly commuting. Moreover, we assume that

$$
\begin{align*}
& \forall k \in \mathbb{N}, \forall \lambda \in[0, \tau], \Psi_{k}(\lambda)=e^{-i \lambda A} \Psi_{k}  \tag{3.14}\\
& S(\rho)<\infty, \sum_{k} \varphi\left(\left\|H \Psi_{k}\right\|^{2}\right)<\infty, \sum_{k} \varphi\left(\left\|A \Psi_{k}\right\|^{2}\right)<\infty \tag{3.15}
\end{align*}
$$

Then, the compactness, the strong commutativity of $A$ and $H$, and (3.14) imply that $\xi_{k}=\left\|H \Psi_{k}\right\| \rightarrow 0, \eta_{k}=\left\|A \Psi_{k}\right\| \rightarrow 0(k \rightarrow \infty)$. Hence, the assumption of Theorem 3.1 is satisfied. Hence, we have $S\left(\rho_{\Delta}(\tau)\right) \rightarrow S(\rho)(|\Delta| \rightarrow$ $0)$.

In Example 3.4, let us consider the case of $A=0$. The following fact can be easily seen:

$$
\begin{align*}
& H \in \mathfrak{C}(\mathcal{H}), \Psi_{k}(\lambda)=\Psi_{k}(\forall k \in \mathbb{N}, \forall \lambda \in[0, \tau]), S(\rho)<\infty, \sum_{k} \varphi\left(\left\|H \Psi_{k}\right\|^{2}\right)<\infty \\
& \Longrightarrow \lim _{|\Delta| \rightarrow 0} S\left(\rho_{\Delta}(\tau)\right)=S(\rho) \tag{3.16}
\end{align*}
$$

This is the case of QZE. We remark that, if $\left\{\Psi_{k}\right\}_{k}$ is a sequence of eigenvectors of $H$, we have $\sum_{k} \varphi\left(\left\|H \Psi_{k}\right\|^{2}\right)=\operatorname{Tr} \varphi\left(H^{2}\right)<\infty$. Then, in (3.16), we can replace the condition $\sum_{k} \varphi\left(\left\|H \Psi_{k}\right\|^{2}\right)<\infty$ with $\varphi\left(H^{2}\right) \in \mathfrak{T}(\mathcal{H})$.

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