On Functionally Commutative Quantum Systems

Takeo Kamizawa

Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Toruń, Poland

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Abstract

In this paper, a method to solve functionally commutative time-dependent linear homogeneous differential equation is discussed. We apply this technique to solve some dynamical quantum problems.

1 Introduction

The study of natural phenomena and their model analysis often involves differential equations, and it is well-known that once the initial condition is stated there is the unique solution for a linear differential equation. If we could obtain the solution of a differential equation in a closed form, we would be able to follow a trajectory completely, i.e. we can analyse the system well. However, the problem is that the class of integrable (or solvable) differential equations using the methods known today is quite small, and we do not have a uniform technique to solve for all differential equations. Thus, we look for the closed forms of the solutions of some particular types of differential equations using specific methods.

The dynamics in quantum mechanics is expressed by linear equations on the set of all bounded operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$:

$$\frac{d}{dt}\rho(t) = L(t)\rho(t),$$

but the general closed solution is not known if L is time-dependent. One of the well-studied class is the one such that the generator of the time evolution $L(t)$ is functionally commutative, i.e. $L(t)L(s) = L(s)L(t)$ for $t, s \in \mathcal{I}$ on

some open interval $\mathcal{I}$, and this class of open quantum systems is analysed in [4, 5], where the solution of functionally commutative quantum systems were obtained using the spectrum of $L(t)$. This technique is useful when the dimension is small (e.g. $\dim \mathcal{H} = 1, 2, 3$) because the spectrum can be calculated relatively easily, but it becomes difficult when the dimension increases, especially when $\dim \mathcal{H} \geq 5$ because of the Galois theory.

However, linear systems with functionally commutative generators have been studied and there are many important papers in this field [1, 3, 7, 20, 8, 18, 10, 14, 9, 22, 11]. Especially, Zhu's technique [21, 22] enables us to compute the solutions of these differential equations effectively. A benefit of this method is that this technique does not rely on the calculation of the eigenvalues of $L(t)$, so we may be able to obtain the solutions of high-dimensional differential equations, where the spectrum of the generator is seldom computable.

In this paper, we study functionally commutative differential equations. We study Zhu's technique in Sect. 2, and as examples this technique is applied in Sect. 3 to some 2-level quantum systems with commutative generators.

2 Preliminaries

2.1 Open Quantum Systems

First of all, let us introduce several notations and concepts in open quantum systems (for details, see [19, 12, 13]). Let $\mathcal{H}_S$ be an n-dimensional Hilbert space corresponding to a quantum system S, $\mathcal{L}(\mathcal{H}_S)$ be the set of all linear transformations on $\mathcal{H}_S$ and $\mathcal{B}_*(\mathcal{H}_S) \subset \mathcal{L}(\mathcal{H}_S)$ be the set of all self-adjoint bounded operators. The set of states on $\mathcal{H}_S$ is defined by

$$\mathcal{S}(\mathcal{H}_S) = \{\rho \in \mathcal{B}_*(\mathcal{H}_S) \mid \rho \geq 0, \ \mathrm{tr}\rho = 1\},$$

where $\rho \geq 0$ means that ρ is positive semidefinite, i.e. $\langle \varphi | \rho | \varphi \rangle \geq 0$ for all $|\varphi\rangle \in \mathcal{H}_S$. The dynamics on $\mathcal{S}(\mathcal{H}_S)$ is represented by a differential equation called a master equation as

$$\frac{d}{dt}\rho(t) = L(t)\rho(t), \tag{1}$$

where $L(t) : \mathcal{L}(\mathcal{H}_S) \to \mathcal{L}(\mathcal{H}_S)$ is a linear operator (often called a generator) on an open interval $\mathcal{I} \subset \mathbb{R}$ containing 0. The solution of (1) with the initial state $\rho_0 \in \mathcal{S}(\mathcal{H}_S)$ is a transformation $\rho : \mathcal{I} \to \mathcal{S}(\mathcal{H}_S)$ which satisfies (1)

and $\rho(0) = \rho_0$, and it is known that $\rho(t)$ for (1) is uniquely determined (cf. e.g. [19, 15]) and the solution is expressed using the linear transformation $\Phi : \mathcal{I} \times \mathcal{S}(\mathcal{H}_S) \to \mathcal{S}(\mathcal{H}_S)$ called the fundamental solution (or dynamical map, especially in quantum physics) as

$$\rho(t) = \Phi(t)\rho_0. \tag{2}$$

By putting (2) into (1), we also obtain the equation of this fundamental solution:

$$\frac{d}{dt}\Phi(t) = L(t)\Phi(t) \tag{3}$$

with $\Phi(0) = I$ (identity).

Geometrically, the set of states $\mathcal{S}(\mathcal{H}_S)$ forms a convex set, so Φ has to make this set invariant during its evolution, i.e. Φ maps from a positive element to a positive element (such a map is called a positive map). For making Φ a positive map, it is known that L has to be an operator called a Metzler operator and this is also a sufficient condition (see [12, 6]).

In the study of open quantum systems, we assume that our system $(\mathcal{H}_S, \Phi)$ is affected by its surroundings or environment, which is also supposed to form another system $(\mathcal{H}_E, \Gamma)$ on $\mathcal{I}$, and the system $(\mathcal{H}_S \otimes \mathcal{H}_E, \Phi \otimes \Gamma)$ is assumed to form the total system on $\mathcal{I}$. Indeed, the total system is considered to form a closed system, namely a system such that

$$L(t)\rho = -\frac{i}{\hbar}[H(t), \rho], \tag{4}$$

where $\hbar > 0$ is some constant, $[A, B] = AB - BA$ is the commutator, $H(t) : \mathcal{H}_S \otimes \mathcal{H}_E \to \mathcal{H}_S \otimes \mathcal{H}_E$ is some time-dependent self-adjoint linear operator called a Hamiltonian, and the equation (4) is called the von Neumann equation.

For the analysis of our system $(\mathcal{H}_S, \Phi)$, we are interested in the closed form of the solution Φ so that we will be able to follow all orbits in the system. However, because of the noise from the environment the structure of $L(t)$ can be complicated and it may be difficult to obtain the closed form of Φ. In this field, some models have been considered and analysed, and one of them is the functionally commutative class, where $[L(t), L(s)] = O$ for $t, s \in \mathcal{I}$ [4, 5]. The main topic in this paper is about open quantum systems of this class, and we discuss the integrability later.

2.2 Magnus Expansion

For a linear differential equation (1) with the variable coefficient $L(t)$, we are interested in the unique solution of this equation. If the generator is

time-independent, i.e. $L(t) = L$ for all $t \in \mathcal{I}$, the solution is given by

$$\rho(t) = \exp(Lt) \rho_0 \qquad (5)$$

for a given initial state $\rho_0 \in \mathcal{S}(\mathcal{H}_S)$, where

$$\exp(Lt) = \sum_{k=0}^{\infty} \frac{t^k}{k!} L^k = \sum_{k=0}^{m-1} \alpha_k(t) L^k,$$

m is the degree of the minimal polynomial of L and $\alpha_k : \mathcal{I} \to \mathbb{C}$ are some scalar functions.

We may expect that a similar theory should hold for a time-dependent generator $L(t)$, but in fact the solution of (1) with time-dependent generator $L(t)$ cannot be represented like (5) in general.

Magnus [16] gave a sufficient condition for this problem:

Theorem 1. *[2, Theorem 1]Let $\Phi(t)$ be a map satisfying (3) and $\Phi(0) = I$. If*

$$\frac{d}{dt} \Phi(t) = \sum_{k=0}^{\infty} \frac{B_k}{k!} \mathrm{ad}_{\Phi(t)}^k L(t)$$

converges for all t, where B_k is the k-th Bernoulli number and

$$\mathrm{ad}_A^0 B = B, \ \mathrm{ad}_A^1 B = [A, B], \ \mathrm{ad}_A^k B = \left[A, \mathrm{ad}_A^{k-1} B \right],$$

then $\Phi(t)$ is given by

$$\Phi(t) = \exp(L(t)).$$

Moreover, $\Phi(t)$ has an infinite series representation, which is now usually called the Magnus expansion:

$$\Phi(t) = \int_0^t L(t_1) dt_1 - \frac{1}{2} \int_0^t \left[\int_0^{t_1} L(t_2) dt_2, L(t_1) \right] dt_1 + \cdots. \qquad (6)$$

The Magnus expansion is a good formulation for the solution of a general linear differential equation of the form (1) with the time-dependent genera-tor. However, the Magnus expansion is an infinite series, which is difficult to be dealt with, so we are next interested in linear systems which has a finite series expression, which is discussed in the following sections.

2.3 Linear Differential Equation of Functionally Commutative Systems

In this section we study a particular type of differential equations; differential equations of the form (1) with the generators $L(t)$ which commute with its derivative on $\mathcal{I}$, i.e.

$$\left[L(t), \frac{d}{dt}L(t)\right] = O. \ (\forall t \in \mathcal{I}) \tag{7}$$

The study of this type of differential equations seems to have started before 1934 [11], and many discussions have been done on this problem [1, 3, 7, 20, 8, 18, 10, 14, 9, 22, 11].

One reason of the popularity of this type of systems is that the solution of (1) satisfying (7) is known to be [9, 15]:

$$\rho(t) = \exp\left(\int_0^t L(\tau)d\tau\right)\rho_0. \tag{8}$$

In addition, in some cases the generator has a finite-sum form

$$L(t) = \sum_{k=0}^{M} \alpha_k(t)L_k,$$

where L_k are linearly independent constant matrices and α_k are scalar functions [1, 14, 9], where we are able to calculate the closed form

$$\Phi(t) = \exp\left(\int_0^t L(\tau)d\tau\right)$$

easily.

It is important to remark that the condition (7) is equivalent to the commutativity of the generator $L(t)$ because of the following theorem:

Theorem 2. *[15, Theorem 7.4.1]. For integrable function $A(t)$ on $\mathcal{I}$ the following conditions are equivalent:*

- *$[A(t), A(s)] = O$ for all $t, s \in \mathcal{I}$;*

- *$\left[A(t), \int_s^t A(\tau)d\tau\right] = O$ for all $t, s \in \mathcal{I}$.*

Putting $A(t) = \frac{d}{dt}L(t)$, we obtain the equivalence of the functional commutativity and the commutivity of the generator with its derivative. Therefore, the functionally commutative systems have some ideal properties like the systems of the form (7).

2.4 Finite Form of the Exponential

Magnus expansion (6) is a useful formula to calculate the closed form of the solution $\Phi(t)$ for some systems. However, the Magnus expansion involves an infinite series, which is difficult to be dealt with. For applications and computational purposes, it is beneficial for us if the series can be also represented by a finite series.

The key starting point for this problem would be the result by Martin [18, Theorem 2]:

Theorem 3. *Suppose $L(t)$ is bounded and piecewise continuous for $t \geq 0$. Then $L(t)$ is functionally commutative on $t \geq 0$ iff there is a set of mutually commuting constant matrices $\{L_k = L(t_k)\}_{k=1}^{M}$ for some $M \leq n^2$ and bounded piecewise continuous scalar functions α_k such that*

$$L(t) = \sum_{k=1}^{M} \alpha_k(t) L_k.$$

A benefit of this theorem is that we are able to simply calculate

$$\Phi(t) = \exp\left(\int_0^t L(\tau)d\tau\right)$$

using the decomposition by Martin. However, as Zhu pointed out in [21], there are difficulties in finding the decomposition. In [21, 22] Zhu constructed a systematic method to obtain a finite decomposition of a functionally commutative generator $L(t)$, called a spatial decomposition. In this section his method and the application to calculate $\exp\left(\int_0^t L(\tau)d\tau\right)$ are introduced.

Let $\mathbb{L} = \mathbb{L}(\mathcal{I}, \mathbb{C})$ be a linear space of maps $f : \mathcal{I} \to \mathbb{C}$. A matrix function $F \in \mathcal{M}_n(\mathbb{L})$ is said to be proper if there are $G \in \mathcal{M}_n(\mathbb{C})$, $p \in \mathbb{N}$ and $\alpha_k \in \mathbb{L}$ $(k = 1, \ldots, p)$ such that

$$F(t) = \sum_{k=1}^{p} \alpha_k(t) G^{k-1},$$

and the associated function $f : \mathcal{I} \times \mathbb{C} \to \mathbb{C}$:

$$f(t, \lambda) = \sum_{k=1}^{p} \alpha_k(t) \lambda^{k-1}$$

is called a primitive function. Note that the primitive function need not be defined on the entire complex plane but is required that, for some $D \subset \mathbb{C}$, $f(\cdot, \lambda) \in \mathbb{L}$ for all $\lambda \in D$, $\lambda_i \in D$ and

$$\left\{ \frac{1}{k_i!} \cdot \frac{\partial^{k_i} f(t, \lambda)}{\partial \lambda^{k_i}} \bigg|_{\lambda = \lambda_i} \mid k_i = 0, 1, \ldots, m_i - 1, \; i = 1, 2, \ldots, r \right\} \subset \mathbb{L},$$

where $\sigma(G) = \{(\lambda_i, m_i)\}_{i=1}^{r}$ is the spectrum (i.e. λ_i is an eigenvalue and m_i is its multiplication) of G.

Recall that a matrix function $F(t)$ is said to be functionally commutative (or semiproper) on $\mathcal{I}$ if

$$F(t) F(s) = F(s) F(t) \; (\forall t, s \in \mathcal{I}).$$

An important discovery by Zhu was that a functionally commutative matrix function F can be decomposed into a finite sum of proper functions as follows:

Theorem 4. $F \in \mathcal{M}_n(\mathbb{L})$ *is functionally commutative on $\mathcal{I}$ iff F can be decomposed into a sum of proper functions (called a spatial decomposition):*

$$F(t) = \sum_{i=1}^{m} F_i(t) = \sum_{i=1}^{m} f_i(t, G_i),$$

where $[G_i, G_j] = O$ for all $i, j \leq m$.

Using the spatial decomposition of a functionally commutative matrix function, we are able to solve the linear differential equation (1) according to the following theorem:

Theorem 5. *[22, Theorem 4]. If $L \in \mathcal{M}_n(\mathbb{K})$ is semiproper on $\mathcal{I}$ having a decomposition*

$$L(t) = \sum_{i=1}^{m} L_i(t) = \sum_{i=1}^{m} f_i(t, G_i),$$

where $[G_i, G_j] = 0$ and m is the degree of the minimal polynomial of L, then the system (1) has the finite solution

$$\Phi(t) = \prod_{i=1}^{m} g_i(t, G_i),$$

where

$$g_i(t, \lambda) = \exp\left(\int_0^t f_i(\tau, \lambda) d\tau \right).$$

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3 Functionally Commutative Quantum Systems

As shown in the previous section, a functionally commutative linear homogeneous differential equation can be solved using the technique by Zhu. In this section we apply this method to some functionally commutative quantum systems.

Example 6. In [5], a functionally commutative system:

$$L(t)\rho = \sum_{k=1}^{n^2-1} \alpha_k(t) \left[U_k \rho U_k^* - \rho\right] \tag{9}$$

was analysed, where U_k are unitary operators such that $\{U_k\}_k \cup \{I\}$ forms a basis of $\mathcal{L}(\mathcal{H})$. In this case, the eigenvalues of L can be obtained relatively easily, and the solution was finally calculated using the spectrum.

Alternatively, as an example, we calculate the solution of (9) with $n = 2$ [5, Example 1] using the Zhu's technique. Let

$$\frac{d}{dt}\rho_t = L(t)\rho_t = \gamma \sum_{k=1}^{3} \alpha_k(t) \left(\sigma_k \rho_t \sigma_k^* - \rho_t\right),$$

where $\gamma \in \mathbb{C}$ is some constant, $\alpha_k : \mathbb{R} \to \mathbb{C}$ are some complex functions and σ_k are the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

According to the identity [17, pp.35]:

$$\mathrm{vec}\,(ABC) = \left(C^T \otimes A\right)\mathrm{vec}B,$$

the matrix form of $L(t)$ is given by

$$
\begin{aligned}
\tilde{L}(t) &= \gamma \sum_{k=1}^{3} \alpha_k(t)\left(\overline{\sigma_k} \otimes \sigma_k - I\right) \\
&= \gamma\left[\left(\alpha_1(t) + \alpha_2(t)\right)\mathrm{adiag}\,(1,0,0,1) + \left(\alpha_1(t) - \alpha_2(t)\right)\mathrm{adiag}\,(0,1,1,0)\right. \\
&\quad \left. - \left(\alpha_1(t) + \alpha_2(t)\right)I - 2\alpha_3(t)\,\mathrm{diag}\,(0,1,1,0)\right], \tag{10}
\end{aligned}
$$

where I is the identity matrix,

$$\mathrm{adiag}\,(a_1,\ldots,a_n) = \begin{pmatrix} O & & a_1 \\ & \cdot^{\cdot^{\cdot}} & \\ a_n & & O \end{pmatrix},$$

and

$$\text{vec} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a_{n2} \\ \vdots \\ a_{nn} \end{pmatrix}$$

is the vector form of the matrix, which is used to represent a superoperator in its matrix form. One can easily check L is commutative by a direct calculation.

In (10), put

$$\begin{aligned} G_1 &= \text{adiag}\,(1,0,0,1)\,, & f_1\,(t,\lambda) &= \gamma\left\{\alpha_1\,(t) + \alpha_2\,(t)\right\}\lambda, \\ G_2 &= \text{adiag}\,(0,1,1,0,) & f_2\,(t,\lambda) &= \gamma\left\{\alpha_1\,(t) - \alpha_2\,(t)\right\}\lambda, \\ G_3 &= I, & f_3\,(t,\lambda) &= -\gamma\left\{\alpha_1\,(t) + \alpha_2\,(t)\right\}\lambda, \\ G_4 &= \text{diag}\,(0,1,1,0)\,, & f_4 &= -2\gamma\alpha_3\,(t)\,\lambda, \end{aligned}$$

then, applying Theorem 5 we have

$$\begin{aligned} g_1\,(t,G_1) &= C_1 \sum_{\xi=0}^{\infty} \frac{\left(\int_0^t \gamma\left\{\alpha_1\,(\tau) + \alpha_2\,(\tau)\right\}d\tau\right)^{\xi}}{\xi!} G_1^{\xi} \\ &= C_1 \left[G_4 + \sum_{\xi=0}^{\infty} \frac{\left(\int_0^t \gamma\left\{\alpha_1\,(\tau) + \alpha_2\,(\tau)\right\}d\tau\right)^{2\xi+1}}{(2\xi+1)!} G_1 \right. \\ &\qquad \left. + \sum_{\xi=0}^{\infty} \frac{\left(\int_0^t \gamma\left\{\alpha_1\,(\tau) + \alpha_2\,(\tau)\right\}d\tau\right)^{2\xi}}{(2\xi)!} \text{diag}\,(1,0,0,1) \right] \\ &= C_1 \left[G_4 + \cosh\left(\int_0^t \gamma\left\{\alpha_1\,(\tau) + \alpha_2\,(\tau)\right\}d\tau\right) \text{diag}\,(1,0,0,1) \right. \\ &\qquad \left. + \sinh\left(\int_0^t \gamma\left\{\alpha_1\,(\tau) + \alpha_2\,(\tau)\right\}d\tau\right) \text{adiag}\,(1,0,0,1) \right] \end{aligned}$$

$$g_2\left(t, G_2\right) = C_2 \sum_{\xi=0}^{\infty} \frac{\left(\int_0^t \gamma \left\{\alpha_1\left(\tau\right) - \alpha_2\left(\tau\right)\right\} d\tau\right)}{\xi!} G_2^{\xi}$$

$$= C_2 \left[\operatorname{diag}(1,0,0,1) + \sum_{\xi=0}^{\infty} \frac{\left(\int_0^t \gamma \left\{\alpha_1\left(\tau\right) - \alpha_2\left(\tau\right)\right\} d\tau\right)^{2\xi+1}}{(2\xi+1)!} G_2 \right.$$

$$\left. + \sum_{\xi=0}^{\infty} \frac{\left(\int_0^t \gamma \left\{\alpha_1\left(\tau\right) - \alpha_2\left(\tau\right)\right\} d\tau\right)^{2\xi}}{(2\xi)!} G_4 \right]$$

$$= C_2 \left[\operatorname{diag}(1,0,0,1) + \sinh\left(\int_0^t \gamma \left\{\alpha_1\left(\tau\right) - \alpha_2\left(\tau\right)\right\} d\tau\right) G_2 \right.$$

$$\left. + \cosh\left(\int_0^t \gamma \left\{\alpha_1\left(\tau\right) - \alpha_2\left(\tau\right)\right\} d\tau\right) G_4 \right]$$

$$g_3\left(t, G_3\right) = C_3 \exp\left(-\gamma \int_0^t \left\{\alpha_1\left(\tau\right) + \alpha_2\left(\tau\right)\right\} d\tau\right) I$$

$$g_4\left(t, G_4\right) = C_4 \left[\operatorname{diag}(1,0,0,1) + \exp\left(-2\gamma \int_0^t \alpha_3\left(t\right) d\tau\right) G_4 \right],$$

where $C_1, \ldots, C_4 \in \mathbb{C}$ are arbitrary constants. Then we obtain the fundamental solution $\Phi\left(t\right)$ of (1)

$$\Phi\left(t\right) = \prod_{k=1}^{4} g_k\left(t, G_k\right)$$

$$= C \exp\left(-\gamma \int_0^t \left\{\alpha_1\left(\tau\right) + \alpha_2\left(\tau\right)\right\} d\tau\right)\left[\vphantom{\int_0^t} \right.$$

$$\cosh\left(\int_0^t \gamma \left\{\alpha_1(\tau) + \alpha_2(\tau)\right\} d\tau\right) \operatorname{diag}(1,0,0,1) + \sinh\left(\int_0^t \gamma \left\{\alpha_1(\tau) + \alpha_2(\tau)\right\} d\tau\right) \operatorname{adiag}(1,0,0,1)$$

$$+ \sinh\left(\int_0^t \gamma \left\{\alpha_1(\tau) - \alpha_2(\tau)\right\} d\tau\right) \exp\left(-2\gamma \int_0^t \alpha_3(\tau) d\tau\right) \operatorname{adiag}(0,1,1,0)$$

$$\left. + \cosh\left(\int_0^t \gamma \left\{\alpha_1(\tau) - \alpha_2(\tau)\right\} d\tau\right) \exp\left(-2\gamma \int_0^t \alpha_3(\tau) d\tau\right) \operatorname{diag}(0,1,1,0)\right]$$

$$= \frac{C}{2}\left[\left(1 + \exp\left(-2\gamma \int_0^t \left\{\alpha_1(\tau) + \alpha_2(\tau)\right\} d\tau\right)\right) \operatorname{diag}(1,0,0,1) \right.$$

$$+ \left(1 - \exp\left(-2\gamma \int_0^t \left\{\alpha_1(\tau) + \alpha_2(\tau)\right\} d\tau\right)\right) \operatorname{adiag}(1,0,0,1)$$

$$+ \left(\exp\left(-2\gamma \int_0^t \alpha_2(\tau) d\tau\right) - \exp\left(-2\gamma \int_0^t \alpha_1(\tau) d\tau\right)\right) \exp\left(-2\gamma \int_0^t \alpha_3(\tau) d\tau\right) \operatorname{adiag}(0,1,1,0)$$

$$\left. + \left(\exp\left(-2\gamma \int_0^t \alpha_2(\tau) d\tau\right) + \exp\left(-2\gamma \int_0^t \alpha_1(\tau) d\tau\right)\right) \exp\left(-2\gamma \int_0^t \alpha_3(\tau) d\tau\right) \operatorname{diag}(0,1,1,0)\right],$$

where $C = C_1C_2C_3C_4$ is some constant, or in another form

$$
\begin{aligned}
\Phi(t)\rho \;=\;& \frac{C}{2}\Bigg[\left(1+\exp\left(-2\gamma\int_0^t \{\alpha_1(\tau)+\alpha_2(\tau)\}\,d\tau\right)\right)(\sigma_3\rho\sigma_3+\rho) \\
&+\left(1-\exp\left(-2\gamma\int_0^t \{\alpha_1(\tau)+\alpha_2(\tau)\}\,d\tau\right)\right)(\sigma_1\rho\sigma_1+\sigma_2\rho\sigma_2) \\
&+\left(\exp\left(-2\gamma\int_0^t \alpha_2(\tau)d\tau\right)-\exp\left(-2\gamma\int_0^t \alpha_1(\tau)d\tau\right)\right)\exp\left(-2\gamma\int_0^t \alpha_3(\tau)d\tau\right)(\sigma_1\rho\sigma_1-\sigma_2\rho\sigma_2) \\
&+\left(\exp\left(-2\gamma\int_0^t \alpha_2(\tau)d\tau\right)+\exp\left(-2\gamma\int_0^t \alpha_1(\tau)d\tau\right)\right)\exp\left(-2\gamma\int_0^t \alpha_3(\tau)d\tau\right)(\rho-\sigma_3\rho\sigma_3)\Bigg].
\end{aligned}
$$

Since $\Phi(0)=I$, we have $C=1$.

Example 7. We consider another 2-level system

$$
\frac{d}{dt}\rho_t \;=\; L(t)\,\rho_t
$$

$$
L(t)\rho \;=\; -\frac{i}{2}\varepsilon(t)\,[\sigma_3,\rho] + \gamma(t)\,(\mu L_1 + (1-\mu)L_2)\,\rho
$$

$$
+\frac{1}{2}\sum_{\alpha,\beta=0}^{1} c_{\alpha,\beta}(t)\,([F_\alpha,\rho F_\beta]+[F_\alpha\rho,F_\beta]),
$$

where $\mu\in[0,1]$ is some constant,

$$
\begin{aligned}
L_1\rho &= \sigma^+\rho\sigma^- - \tfrac{1}{2}\{\sigma^-\sigma^+,\rho\}, \quad L_2\rho = \sigma^-\rho\sigma^+ - \tfrac{1}{2}\{\sigma^+\sigma^-,\rho\}, \\
F_0 &= \sigma^-\sigma^+, \qquad\qquad\qquad\qquad F_1 = \sigma^+\sigma^-, \\
\sigma^\pm &= \tfrac{1}{\sqrt{2}}(\sigma_1\pm i\sigma_2),
\end{aligned}
$$

$[A,B]=AB-BA$, $\{A,B\}=AB+BA$ and $\gamma,\varepsilon,c_{\alpha,\beta}$ are time-dependent complex functions. This example was considered in [4], but here we calculate the solution using the Zhu's method. By a simple calculation one finds the matrix form of $L(t)$ is given by

$$
\begin{aligned}
L \;=\;& \begin{pmatrix} 2\gamma(\mu-1) & 0 & 0 & 2\gamma\mu \\ 0 & -2c_{00}+4c_{01}-2c_{11}+i\varepsilon-\gamma & 0 & 0 \\ 0 & 0 & -2c_{00}+4c_{10}-2c_{11}-i\varepsilon-\gamma & 0 \\ 2\gamma(1-\mu) & 0 & 0 & -2\gamma\mu \end{pmatrix} \\
\;=\;& A_1(t)+A_2(t)+A_3(t),
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(t) &= 2\gamma(t)\,G_1 \\
A_2(t) &= \{4c_{01}(t)-2c_{00}(t)-2c_{11}(t)-\gamma(t)+i\varepsilon(t)\}\,G_2 \\
A_3(t) &= \{4c_{10}(t)-2c_{00}(t)-2c_{11}(t)-\gamma(t)-i\varepsilon(t)\}\,G_3
\end{aligned}
$$

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with

$$G_1 = \begin{pmatrix} \mu - 1 & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 - \mu & 0 & 0 & -\mu \end{pmatrix} \quad G_2 = \text{diag}\,(0, 1, 0, 0) \quad G_3 = \text{diag}\,(0, 0, 1, 0)$$

$$
\begin{aligned}
f_1(t, \lambda) &= 2\gamma(t)\lambda \\
f_2(t, \lambda) &= \{4c_{01}(t) - 2c_{00}(t) - 2c_{11}(t) - \gamma(t) + i\varepsilon(t)\}\lambda \\
f_3(t, \lambda) &= \{4c_{10}(t) - 2c_{00}(t) - 2c_{11}(t) - \gamma(t) - i\varepsilon(t)\}\lambda.
\end{aligned}
$$

Thus, by applying Theorem 5,

$$
\begin{aligned}
g_1(t, G_1) &= C_1 \exp\left(\int_0^t f_1(\tau, G_1)\, d\tau\right) \\
&= C_1 \left[\begin{pmatrix} \mu & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\mu & 0 & 0 & 1-\mu \end{pmatrix} + \cosh\left(\int_0^t 2\gamma(\tau)d\tau\right) \begin{pmatrix} 1-\mu & 0 & 0 & -\mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \mu-1 & 0 & 0 & \mu \end{pmatrix} \right. \\
&\quad \left. + \sinh\left(\int_0^t 2\gamma(\tau)d\tau\right) G_1 \right] \\
&= C_1 \begin{pmatrix} \mu & 0 & 0 & \mu \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1-\mu & 0 & 0 & 1-\mu \end{pmatrix} - C_1 \exp\left(-\int_0^t 2\gamma(\tau)d\tau\right) G_1
\end{aligned}
$$

$$
\begin{aligned}
g_2(t, G_2) &= C_2 \exp\left(\int_0^t f_2(\tau, G_2)\, d\tau\right) \\
&= C_2 \left[I - G_2 + \sum_{k=0}^{\infty} \frac{\left(\int_0^t \{4c_{01}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) + i\varepsilon(\tau)\} d\tau\right)^k}{k!} G_2 \right] \\
&= C_2 \text{diag}\,(1, 0, 1, 1) + C_2 \exp\left(\int_0^t \{4c_{01}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) + i\varepsilon(\tau)\} d\tau\right) G_2
\end{aligned}
$$

$$
\begin{aligned}
g_3(t, G_3) &= C_3 \exp\left(\int_0^t f_3(\tau, G_3)\, d\tau\right) \\
&= C_3 \left[I - G_3 + \sum_{k=0}^{\infty} \frac{\left(\int_0^t \{4c_{10}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) - i\varepsilon(\tau)\} d\tau\right)^k}{k!} G_3 \right] \\
&= C_3 \text{diag}\,(1, 1, 0, 1) + C_3 \exp\left(\int_0^t \{4c_{10}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) - i\varepsilon(\tau)\} d\tau\right) G_3,
\end{aligned}
$$

where $C_1, C_2, C_3 \in \mathbb{C}$ are arbitrary constants. Then, finally we obtain the solution

$$
\Phi(t) = C \begin{pmatrix} \mu & 0 & 0 & \mu \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1-\mu & 0 & 0 & 1-\mu \end{pmatrix}
$$

$$
+ C \exp\left(\int_0^t \{4c_{01}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) + i\varepsilon(\tau)\} \, d\tau \right) G_2
$$

$$
+ C \exp\left(\int_0^t \{4c_{10}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) - i\varepsilon(\tau)\} \, d\tau \right) G_3
$$

$$
- C \exp\left(-\int_0^t 2\gamma(\tau) d\tau \right) G_1
$$

where $C = C_1 C_2 C_3$ is a constant, or in another form

$$
\Phi(t)\rho = \frac{C}{4} \left[(2\mu - 1)(\rho\sigma_3 + \sigma_3\rho + i\sigma_1\rho\sigma_2 + i\sigma_2\rho\sigma_1) + \rho + \sigma_1\rho\sigma_1 - \sigma_2\rho\sigma_2 + \sigma_3\rho\sigma_3 \right]
$$

$$
+ \frac{C}{4} \exp\left(\int_0^t \{4c_{01}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) + i\varepsilon(\tau)\} \, d\tau \right) [\rho - \sigma_3\rho\sigma_3 + \rho\sigma_3 - \sigma_3\rho]
$$

$$
+ \frac{C}{4} \exp\left(\int_0^t \{4c_{10}(\tau) - 2c_{00}(\tau) - 2c_{11}(\tau) - \gamma(\tau) - i\varepsilon(\tau)\} \, d\tau \right) [\rho - \sigma_3\rho\sigma_3 - \rho\sigma_3 + \sigma_3\rho]
$$

$$
- \frac{C}{4} \exp\left(-\int_0^t 2\gamma(\tau) d\tau \right) [(2\mu - 1)(\rho\sigma_3 + \sigma_3\rho + i\sigma_1\rho\sigma_2 + i\sigma_2\rho\sigma_1) - \rho + \sigma_1\rho\sigma_1 - \sigma_2\rho\sigma_2 - \sigma_3\rho\sigma_3].
$$

The initial condition implies $C = 1$.

4   Concluding Remark

In this paper some results on ordinary linear differential equations with functionally commutative generators were introduced, and using the Zhu's result, a method to solve a certain type of differential equations effectively, some quantum master equations with functionally commutative generators were solved effectively, i.e. involving only finitely many terms. Indeed, this method does not rely on the spectrum of the generator explicitly, in other words we do not need to calculate the eigenvalues. This is an advantage from a computational point, and we may be able to solve higher dimensional and more complicated master equations, where it is difficult to calculate the eigenvalues.

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