Average of uncertainty-product for bounded observables

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Abstract

The goal of this paper is to calculate exactly the average of uncertainty-product of two bounded observables and to establish its typicality over the whole set of finite dimensional quantum pure states. Here we use the uniform ensembles of pure and isospectral states as well as the states distributed uniformly according to the measure induced by the Hilbert-Schmidt norm. Firstly, we investigate the average uncertainty of an observable over isospectral density matrices. By letting the isospectral density matrices be of rank-one, we get the average uncertainty of an observable restricted to pure quantum states. These results can help us check how large the gap is between the uncertainty-product and any obtained lower bounds about the uncertainty-product. Although our method in the present paper cannot give a tighter lower bound of uncertainty-product for bounded observables, it can help us drop any one that is not tighter than the known one substantially.

Keywords: uncertainty relation; random quantum state; observable

1 Introduction

Uncertainty principle (aka Heisenberg's uncertainty relation) is one of basic constraints in quantum mechanics. It means that we cannot principally obtain precise measurement outcomes simultaneously when we measure two incomparable observables at the same time. The mathematical formulation of uncertainty relation is in terms of any of a variety of inequalities, where a fundamental limit to the precision with which certain pairs of physical properties of a particle, i.e. complementary variables, such as position $\hat{x}$ and momentum $\hat{p}$, can be known simultaneously. The uncertainty relation [1], introduced by Heisenberg in 1927, relates the standard deviation of momentum $\Delta\hat{p}$ and the standard deviation of position $\Delta\hat{x}$, it indicates that the more precisely the momentum of some particle is determined, the less precisely its position can be known, and vice versa. Specifically, the quantitative relation of such two standard deviations was derived by Kennard [2] later that year:

$$\Delta\hat{x} \cdot \Delta\hat{p} \geqslant \frac{\hbar}{2}, \tag{1.1}$$

where $\Delta \hat{p} = \sqrt{\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2}$ and $\Delta \hat{x} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$.

The most common general form of the uncertainty principle is the Robertson-Schrödinger uncertainty relations [3, 4]. In order to state it explicitly, we need some notions. The precision to which the value of an observable A can be known is quantified by its uncertainty function

$$\Delta A(\rho) := \sqrt{\langle A^2 \rangle_\rho - \langle A \rangle_\rho^2} \tag{1.2}$$

where $\langle O \rangle_\rho := \mathrm{Tr}\,(O\rho)$ for any observable O. Furthermore, the precision to which the values of two observables A and B can be known simultaneously is limited by the Robertson-Schrödinger uncertainty relation

$$(\Delta A(\rho) \cdot \Delta B(\rho))^2 \geqslant (\langle \{A, B\} \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho)^2 + \langle [A, B] \rangle_\rho^2, \tag{1.3}$$

where $\{A, B\} := \frac{1}{2}(AB + BA)$ and $[A, B] := \frac{1}{2\sqrt{-1}}(AB - BA)$. We see from Robertson-Schrödinger uncertainty relation that this uncertainty relation depends on the state under consideration. There are a lot of literatures devoting to improve the right hand side (rhs) of the above inequality [5, 6]. Moreover, recently many researchers proposed new perspective, instead of description of uncertainty-product, they used the sum of uncertainty [7, 8], and its various generalizations [9, 10], etc. Besides, many researchers generalize the uncertainty relation from pure state to isospectral mixed states by employing symplectic geometric tools [11]. Many contributions are given to another reformulation of uncertainty relation, for instance entropic uncertainty relation [12, 13] and its applications [14]. A connection is also established between entropic uncertainty and wave-particle duality [15]. There are literatures devoted to study the connection among uncertainty, and entanglement [16, 17, 18, 7], and the reversibility of measurement [19].

The purpose of this paper is to give a new perspective to state-independent uncertainty relation in terms of representation theory of unitary group and random matrix theory. Caution: because observables may be unbounded, for instance, the position operator $\hat{x}$, in physical regime, an unbounded observable may take infinity at some state. Throughout this paper, we will focus on bounded observables. Consider the following particular statistical ensembles: The used distribution of random state is uniform distribution induced by Hilbert-Schmidt measure defined over the set of all density matrices. By using tools from representation theory of unitary group and random matrix theory, we can give an exact calculation of such average value (in the pure state case or mixed state case, respectively) and consider its typicality under some restriction. Theoretically, as the typicality suggests that without measuring such bounded observables, we may claim that at most sampled states, one can get their uncertainty-product is close to their average value with overwhelming probability. Equivalently, their uncertainty-product deviates their average value with exponentially small probability. Our method proposed here in fact can help check how large the gap is between the uncertainty-product and any obtained lower bounds about the uncertainty-product. Specifically, except calculate the average of uncertainty-product, we also calculate the averages of the obtained lower bounds of uncertainty-product. Clearly the obtained lower bounds are state-dependent.

This paper is organized as follows. In Sect. 2, we will introduce various measures on state space. Specifically, there is a unique probability which is unitarily invariant on the pure state space. But, however, there is no unique unitarily-invariant probability measure over the mixed state space because of the existence of environment. Sect. 3 discusses the motivation why we take the average over corresponding

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state ensembles. Sect. 4 deals with the isospectral average of uncertainty-product of two bounded observables over the set of isospectral quantum states. Furthermore, separately, we consider the average of uncertainty-product for a random pure state, and also for a random mixed state. In Sect. 5, we make a discussion about the concentration of measure phenomenon about the quantity, i.e., the uncertainty-product of two bounded observables over the set of mixed states. Finally, some necessary materials for reasoning of our results are provided in the Appendix, see Sect. 7, for example, two specific examples in lower dimensions are provided in Sect. 7.8.

2 Measures on the state spaces

Given a measure μ on the set of quantum states, one can calculate the corresponding averages over all states with respect to this measure [20]. We will consider the set of pure quantum states. For a d-dimensional Hilbert space $\mathcal{H}_d$, the set of pure states consists of all unit vectors in $\mathcal{H}_d$. On this set, there exists a unique measure which is unitarily invariant, i.e., uniform probability measure $\mathrm{d}\mu(\psi)$ or induced by normalized Haar measure $\mathrm{d}\mu_{\mathrm{Haar}}(U)$ over the unitary group $\mathrm{U}(d)$. Indeed, any random pure state $|\psi\rangle$ is generated by a random unitary matrix $U \in \mathrm{U}(d)$ on any fixed pure state $|\psi_0\rangle$ via $|\psi\rangle = U|\psi_0\rangle$. The uniform ensemble of pure quantum states of finite-dimensional Hilbert space studied extensively in the context of foundations of quantum statistical mechanics, entanglement theory or various protocols/features of quantum information theory. Related literatures are too numerous to mention. Here we mention our two works using such particular ensemble to investigate the typicality of quantum coherence and average entropy of isospectral quantum states, see [21, 22]. Then we can define the average value of some function f on the set of pure states as follows:

$$\langle f(\psi) \rangle := \int_{\mathbb{S}^k} f(\psi)\mathrm{d}\mu(\psi) = \int_{\mathrm{U}(d)} f(U\psi_0)\mathrm{d}\mu_{\mathrm{Haar}}(U).$$ (2.1)

Unlike the case of pure states, it is known that there exist various measures on the set of mixed states, $\mathrm{D}(\mathcal{H}_d)$, the set of all positive semidefinite matrices with unit trace. As a matter of fact, one assumes naturally the distributions of eigenvalues and eigenvectors of a quantum state ρ, via the spectral decomposition $\rho = U\Lambda U^\dagger$, are independent. Thus any probability measure μ on $\mathrm{D}(\mathcal{H}_d)$ will be of product form: $\mathrm{d}\mu(\rho) = \mathrm{d}\nu(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(U)$, where $\mathrm{d}\mu_{\mathrm{Haar}}(U)$ is the unique Haar measure [23] on the unitary group and ν defines the distribution of eigenvalues without unique choice for it. The utility of ν in the average entropy or average coherence can be found in [24, 25, 20].

The measures used frequently over the $\mathrm{D}(\mathcal{H}_d)$ can be obtained by partially tracing over the Haar-distributed pure states in the higher dimension Hilbert space $\mathcal{H}_d \otimes \mathcal{H}_k$, say $\mathbb{C}^d \otimes \mathbb{C}^k$. In order to be convenience we suppose that $d \leqslant k$. Following [20], the joint probability density function of spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_d\}$ of ρ is given by

$$\mathrm{d}\nu_{d,k}(\Lambda) = C_{d,k}\delta\left(1 - \sum_{j=1}^{d}\lambda_j\right)\prod_{1 \leqslant i < j \leqslant d}(\lambda_j - \lambda_i)^2\prod_{j=1}^{d}\lambda_j^{k-d}\theta(\lambda_j)\mathrm{d}\lambda_j,$$ (2.2)

where the theta function θ ensures that ρ is positive definite, $C_{d,k}$ is the normalization constant, given by

$$C_{d,k} = \frac{\Gamma(dk)}{\prod_{j=0}^{d-1}\Gamma(d-j+1)\Gamma(k-j)}.$$ (2.3)

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In particular, in the present paper we will consider a special case where $d = k$, which corresponds to the Hilbert-Schmidt measure, a flat metric over the $\mathrm{D}(\mathcal{H}_d)$, denoted by $\mathrm{d}\mu_{\mathrm{HS}}(\rho)$. We also denote $\mathrm{d}\nu_{d,k} = \mathrm{d}\nu$ and $C_{d,k} = C_{\mathrm{HS}}$ if $d = k$. Thus we have

$$\mathrm{d}\mu_{\mathrm{HS}}(\rho) = \mathrm{d}\nu(\Lambda) \times \mathrm{d}\mu_{\mathrm{Haar}}(U), \tag{2.4}$$

where $\rho = U\Lambda U^\dagger$.

For convenience, we let A, B be observables, $\rho = U\Lambda U^\dagger$, and introduce the following symbol for convenience:

$$t_k = \mathrm{Tr}\left(\Lambda^k\right) = \mathrm{Tr}\left(\rho^k\right), \quad \mathscr{E}_k(\Lambda) := \int (U\Lambda U^\dagger)^{\otimes k}\mathrm{d}\mu_{\mathrm{Haar}}(U). \tag{2.5}$$

3   Motivation

In order to explain why we take the average of uncertainty-product for bounded observables, some words are needed. Denote $L_0(A, B, \rho) := \left(\langle\{A, B\}\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho\right)^2 + \langle[A, B]\rangle_\rho^2$. Clearly Eq. (1.3) becomes $(\Delta A(\rho) \cdot \Delta B(\rho))^2 \geqslant L_0(A, B, \rho)$. If one is obtained another lower bound, say $L(A, B, \rho)$, via some mathematical methods, then $(\Delta A(\rho) \cdot \Delta B(\rho))^2 \geqslant L(A, B, \rho)$. Now we need to compare lower bounds L_0 and L. If $L(A, B, \rho) \geqslant L_0(A, B, \rho)$, then we can say the lower bounds of the uncertainty principle are improved, that is, we get a tighter lower bound. However, such improvement sometimes is not essential, it is possible that

$$\int L(A, B, \rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho) = \int L_0(A, B, \rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho). \tag{3.1}$$

This shows that $L(A, B, \rho) = L_0(A, B, \rho)$ is satisfied almost every except a zero measure in state space by the Measure Theory. This is not real improvement. In fact, there are two observables such that the lower bound of the uncertainty principle Eq. (1.3) cannot be improved, see below Eq. (4.58). This example tell us that getting a universal uncertainty principle for any observables in which the lower bound is really improved, compared with L_0, seems impossible. At least, the statement is applicable for Eq. (4.58). However, if $(\Delta A(\rho) \cdot \Delta B(\rho))^2 \geqslant L(A, B, \rho) \geqslant L_0(A, B, \rho)$ and

$$\int L(A, B, \rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho) > \int L_0(A, B, \rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho), \tag{3.2}$$

then we say that the uncertainty principle $(\Delta A(\rho) \cdot \Delta B(\rho))^2 \geqslant L(A, B, \rho)$ really improves the uncertainty principle $(\Delta A(\rho) \cdot \Delta B(\rho))^2 \geqslant L_0(A, B, \rho)$. Therefore $L(A, B, \rho)$ is tighter than $L_0(A, B, \rho)$ substantially. This is what we want. But there is another situation that appears. We maybe get a new one $\hat{L}(A, B, \rho)$ without knowing the relationship between $\hat{L}$ and L_0. But we can still determine wether or not

$$\int \hat{L}(A, B, \rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho) > \int L_0(A, B, \rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho). \tag{3.3}$$

If it were the case, then $\hat{L}(A, B, \rho) > L_0(A, B, \rho)$ would hold in a subset of the state space. Improvement of uncertainty principle is possible limited to local range.

4 Isospectral average of uncertainty-product

In this section, we focus on the ensemble of isospectral density matrices. This ensemble has been recently studied in various contexts of quantum information. In fact, we also do some work in this field [22].

Consider the set of all isospectral density matrices $\mathcal{U}_\Lambda := \{\rho : \rho = U\Lambda U^\dagger, U \in U(d)\}$ with a fixed spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_d\}$, where $\lambda_j \geqslant 0$ for each j and $\sum_{j=1}^d \lambda_j = 1$. Now we can explicitly compute the average (squared) uncertainty of observable A over the set of isospectral density matrices $\mathcal{U}_\Lambda$ as follows:

$$\int \Delta A(\rho)^2 d\mu_{\text{Haar}}(U) = \text{Tr}\left(A^2 \mathscr{E}_1(\Lambda)\right) - \text{Tr}\left(A^{\otimes 2} \mathscr{E}_2(\Lambda)\right), \tag{4.1}$$

where $\mathcal{E}_k(\Lambda)$ is from (2.5). The details of computation about $\mathcal{E}_k(\Lambda)$, where $k = 1, 2, 3, 4$, are gathered in the Appendix, i.e., Section 7.

From the relations (7.16) and (7.17), we see that

$$\int \Delta A(\rho)^2 d\mu_{\text{Haar}}(U) = \frac{d - \text{Tr}\left(\Lambda^2\right)}{d^2 - 1}\left[\text{Tr}\left(A^2\right) - \frac{1}{d}(\text{Tr}(A))^2\right]. \tag{4.2}$$

By (7.51), we have

$$\int_{D(\mathcal{H}_d)} \Delta A(\rho)^2 d\mu_{\text{HS}}(\rho) = \frac{d}{d^2 + 1}\left[\text{Tr}\left(A^2\right) - \frac{1}{d}(\text{Tr}(A))^2\right]. \tag{4.3}$$

On the other hand, for any state $\rho \in D(\mathcal{H}_d)$,

$$\begin{aligned}
\Delta A(\rho)^2 \cdot \Delta B(\rho)^2 &= \text{Tr}\left(\left[A^2 \otimes B^2\right]\rho^{\otimes 2}\right) + \text{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right]\rho^{\otimes 4}\right) \\
&\quad - \text{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\rho^{\otimes 3}\right) - \text{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right]\rho^{\otimes 3}\right).
\end{aligned} \tag{4.4}$$

Thus

$$\begin{aligned}
&\int \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 d\mu_{\text{Haar}}(U) \\
&= \text{Tr}\left(\left[A^2 \otimes B^2\right]\mathscr{E}_2(\Lambda)\right) + \text{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right]\mathscr{E}_4(\Lambda)\right) \\
&\quad - \text{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\mathscr{E}_3(\Lambda)\right) - \text{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right]\mathscr{E}_3(\Lambda)\right),
\end{aligned} \tag{4.5}$$

where $\rho \in \mathcal{U}_\Lambda$.

With these identities, we calculate the the averaged uncertainty-product over the isospectral density matrices. By the tedious but simple calculations, we have the following result:

Theorem 4.1. *For two observables A and B on $\mathcal{H}_d$, the average of uncertainty-product over the set of all isospectral density matrices ρ on $\mathcal{H}_d$ is given by a symmetric function in arguments A and B*

$$\int \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 d\mu_{\text{Haar}}(U) = \sum_{j=1}^8 \omega_j(\Lambda) \cdot \Omega_j(A, B), \tag{4.6}$$

where $\Omega_j(A, B)$ are symmetric in arguments A and B for each j: $\Omega_j(A, B) = \Omega_j(B, A)$ and

$$\Omega_1(A, B) = \operatorname{Tr}(A)^2 \operatorname{Tr}(B)^2, \tag{4.7}$$

$$\Omega_2(A, B) = \operatorname{Tr}\left(A^2\right) \operatorname{Tr}(B)^2 + \operatorname{Tr}(A)^2 \operatorname{Tr}\left(B^2\right), \tag{4.8}$$

$$\Omega_3(A, B) = \operatorname{Tr}(AB) \operatorname{Tr}(A) \operatorname{Tr}(B), \tag{4.9}$$

$$\Omega_4(A, B) = \operatorname{Tr}\left(A^2\right) \operatorname{Tr}\left(B^2\right), \tag{4.10}$$

$$\Omega_5(A, B) = \operatorname{Tr}(AB) \operatorname{Tr}(AB), \tag{4.11}$$

$$\Omega_6(A, B) = \operatorname{Tr}\left(A^2 B\right) \operatorname{Tr}(B) + \operatorname{Tr}(A) \operatorname{Tr}\left(AB^2\right), \tag{4.12}$$

$$\Omega_7(A, B) = \operatorname{Tr}\left(A^2 B^2\right), \tag{4.13}$$

$$\Omega_8(A, B) = \operatorname{Tr}(ABAB), \tag{4.14}$$

and $\omega_j(\Lambda)$ are given by the following:

$$\omega_1(\Lambda) = \frac{1}{24}\Delta_4^{(4)} + \frac{3}{8}\Delta_4^{(3,1)} + \frac{1}{6}\Delta_4^{(2,2)} + \frac{3}{8}\Delta_4^{(2,1,1)} + \frac{1}{24}\Delta_4^{(1,1,1,1)}, \tag{4.15}$$

$$\omega_2(\Lambda) = \left(\frac{1}{24}\Delta_4^{(4)} + \frac{1}{8}\Delta_4^{(3,1)} - \frac{1}{8}\Delta_4^{(2,1,1)} - \frac{1}{24}\Delta_4^{(1,1,1,1)}\right) - \left(\frac{1}{6}\Delta_3^{(3)} + \frac{2}{3}\Delta_3^{(2,1)} + \frac{1}{6}\Delta_3^{(1,1,1)}\right), \tag{4.16}$$

$$\omega_3(\Lambda) = \frac{1}{6}\Delta_4^{(4)} + \frac{1}{2}\Delta_4^{(3,1)} - \frac{1}{2}\Delta_4^{(2,1,1)} - \frac{1}{6}\Delta_4^{(1,1,1,1)}, \tag{4.17}$$

$$\omega_4(\Lambda) = \left(\frac{1}{24}\Delta_4^{(4)} - \frac{1}{8}\Delta_4^{(3,1)} + \frac{1}{6}\Delta_4^{(2,2)} - \frac{1}{8}\Delta_4^{(2,1,1)} + \frac{1}{24}\Delta_4^{(1,1,1,1)}\right)$$
$$+ \left(\frac{\Delta_2^{(2)}}{2} + \frac{\Delta_2^{(1,1)}}{2}\right) - \left(\frac{1}{3}\Delta_3^{(3)} - \frac{1}{3}\Delta_3^{(1,1,1)}\right), \tag{4.18}$$

$$\omega_5(\Lambda) = \frac{1}{12}\Delta_4^{(4)} - \frac{1}{4}\Delta_4^{(3,1)} + \frac{1}{3}\Delta_4^{(2,2)} - \frac{1}{4}\Delta_4^{(2,1,1)} + \frac{1}{12}\Delta_4^{(1,1,1,1)}, \tag{4.19}$$

$$\omega_6(\Lambda) = \left(\frac{1}{6}\Delta_4^{(4)} - \frac{1}{3}\Delta_4^{(2,2)} + \frac{1}{6}\Delta_4^{(1,1,1,1)}\right) - \left(\frac{1}{3}\Delta_3^{(3)} - \frac{1}{3}\Delta_3^{(1,1,1)}\right), \tag{4.20}$$

$$\omega_7(\Lambda) = \left(\frac{1}{6}\Delta_4^{(4)} - \frac{1}{2}\Delta_4^{(3,1)} + \frac{1}{2}\Delta_4^{(2,1,1)} - \frac{1}{6}\Delta_4^{(1,1,1,1)}\right)$$
$$+ \left(\frac{\Delta_2^{(2)}}{2} - \frac{\Delta_2^{(1,1)}}{2}\right) - 2\left(\frac{1}{3}\Delta_3^{(3)} - \frac{2}{3}\Delta_3^{(2,1)} + \frac{1}{3}\Delta_3^{(1,1,1)}\right), \tag{4.21}$$

$$\omega_8(\Lambda) = \frac{1}{12}\Delta_4^{(4)} - \frac{1}{4}\Delta_4^{(3,1)} + \frac{1}{4}\Delta_4^{(2,1,1)} - \frac{1}{12}\Delta_4^{(1,1,1,1)}. \tag{4.22}$$

Here the meanings of the notations $\Delta_4^{(4)}, \Delta_4^{(3,1)}, \Delta_4^{(2,1,1)}, \Delta_4^{(1,1,1,1)}$ can be found from (7.37) to (7.41).

The hard part of the proof centers around the calculations of $\mathcal{E}_k(\Lambda)$ by using Schur-Weyl duality. Among other things, the key ingredient here is the Weingarten function, defined over the permutation group S_k, see the definition (7.10) for the unitary group. There are many ways that can be used to define the Weingarten function, for instance, a sum over partitions or equivalently, Young tableaux of $k \in \mathbb{N}$ and the characters of the symmetric group. In the case where permutation groups of lower orders are considered (such as $k = 2, 3, 4$ in our paper), the Weingarten functions can be explicitly evaluated. When k becomes larger, the explicit evaluation of such function is considerably complicated, and naturally the asymptotics is concerned. The proof of Theorem 4.1 is placed in Section 7.5.

Remark 4.2. Let $N_d^{-1} = d^2(d^2-1)(d^2-4)(d^2-9)$. We can write down more specific expressions for $\omega_j(\Lambda)$, where $t_k = \text{Tr}\left(\Lambda^k\right)$ for natural number k.

$$\omega_1(\Lambda) = N_d\left((d^4-8d^2+6)-6d(d^2-4)t_2+3(d^2+6)t_2^2+8(2d^2-3)t_3-30dt_4\right), \quad (4.23)$$

$$\omega_2(\Lambda) = N_d\left(-d(d^4-10d^2+14)+2d^2(2d^2-13)t_2-d(d^2+6)t_2^2-8d(d^2-4)t_3+10d^2t_4\right), \quad (4.24)$$

$$\omega_3(\Lambda) = N_d\left(-4d(d^2-4)+4d^2(d^2+1)t_2-4d(d^2+6)t_2^2-16d(d^2+1)t_3+40d^2t_4\right), \quad (4.25)$$

$$\omega_4(\Lambda) = N_d\left((d^6-11d^4+19d^2+6)-d(3d^4-25d^2+12)t_2 \right. \quad (4.26)$$

$$\left. +(d^4-6d^2+18)t_2^2+4(d^4-5d^2-6)t_3-2d(2d^2-3)t_4\right), \quad (4.27)$$

$$\omega_5(\Lambda) = N_d\left(2(d^2+6)-4d(d^2+6)t_2+2(d^4-6d^2+18)t_2^2+16(2d^2-3)t_3-4d(2d^2-3)t_4\right) \quad (4.28)$$

$$\omega_6(\Lambda) = N_d\left(2(d^4-5d^2-6)-2d(d^4-d^2-12)t_2+12(2d^2-3)t_2^2 \right. \quad (4.29)$$

$$\left. +8(d^4-3d^2+6)t_3-12d(d^2+1)t_4\right), \quad (4.30)$$

$$\omega_7(\Lambda) = N_d\left(-d(d^3+4d^2-9d-16)+d^2(d^4-d^2-32)t_2 \right. \quad (4.31)$$

$$\left. -4d(2d^2-3)t_2^2-4d(d^4-5d^2+4)t_3+4d^2(d^2+1)t_4\right), \quad (4.32)$$

$$\omega_8(\Lambda) = N_d\left(-10d+20d^2t_2-2d(2d^2-3)t_2^2-8d(d^2+1)t_3+2d^2(d^2+1)t_4\right). \quad (4.33)$$

Because A and B are bounded observables, i.e., Hermitian operators, we see that $\text{Tr}(A)$, $\text{Tr}(B)$, and $\text{Tr}(AB)$ are real numbers and $\text{Tr}(A^2) \geqslant 0$, and $\text{Tr}(B^2) \geqslant 0$. Then $\text{Tr}(A)^2\text{Tr}(B)^2 \geqslant 0$, $\text{Tr}(A^2)\text{Tr}(B)^2 + \text{Tr}(A)^2\text{Tr}(B^2) \geqslant 0$, $\text{Tr}(A^2)\text{Tr}(B^2) \geqslant 0$, i.e., $\Omega_j(A,B) \geqslant 0$ for $j = 1,2,4,5$ by the definition. In addition, $\text{Tr}(A^2B^2) = \text{Tr}(BA^2B) \geqslant 0$ since $BA^2B \geqslant 0$. Thus $\Omega_7(A,B) \geqslant 0$. Consider the operator $X = AB + BA$. Clearly X is a Hermitian operator. Moreover $X^2 \geqslant 0$, thus $\text{Tr}(X^2) \geqslant 0$. Because $\text{Tr}(X^2) = 2(\text{Tr}(ABAB) - \text{Tr}(A^2B^2))$, we have that $\text{Tr}(ABAB) \geqslant \text{Tr}(A^2B^2) \geqslant 0$. Hence $\Omega_8(A,B) \geqslant 0$. In summary, $\Omega_j(A,B) \geqslant 0$ for $j = 1,2,4,5,7,8$. However, $\Omega_3(A,B)$ and $\Omega_6(A,B)$ are not always non-negative.

Remark 4.3. The rhs of (4.6) remind us of one of applications to random matrix theory from free probability theory, established by Voiculescu [26]. Specifically, we can consider two independent random observables A and B from Gaussian unitary ensemble (GUE), according to free probability theory, A and B are asymptotic free (see the meaning of freeness in [27]). Indeed, denote $\varphi(\cdot) = \frac{1}{d}\text{Tr}(\cdot)$, where $\text{Tr}(\cdot)$ means the trace of matrix, when d becomes large enough, we have

$$\varphi(ABAB) \simeq \varphi(A^2)\varphi(B)^2 + \varphi(A)^2\varphi(B^2) - \varphi(A)^2\varphi(B)^2, \quad (4.34)$$

that is,

$$\Omega_8(A,B) \simeq d^{-2}\Omega_2(A,B) - d^{-3}\Omega_1(A,B). \quad (4.35)$$

Similarly, we have

$$\Omega_3(A,B) \simeq d^{-1}\Omega_1(A,B), \quad (4.36)$$

$$\Omega_5(A,B) \simeq d^{-2}\Omega_1(A,B), \quad (4.37)$$

$$\Omega_6(A,B) \simeq d^{-1}\Omega_2(A,B), \quad (4.38)$$

$$\Omega_7(A,B) \simeq d^{-1}\Omega_4(A,B). \quad (4.39)$$

Furthermore, we obtain that

$$\int \Delta A(U\Lambda U^\dagger)^2 \cdot \Delta B(U\Lambda U^\dagger)^2 \mathrm{d}\mu_{\text{Haar}}(U) \simeq \left(\omega_1(\Lambda) + d^{-1}\omega_3(\Lambda) + d^{-2}\omega_5(\Lambda) - d^{-3}\omega_8(\Lambda)\right)\Omega_1(A,B)$$
$$+ \left(\omega_2(\Lambda) + d^{-1}\omega_6(\Lambda) + d^{-2}\omega_8(\Lambda)\right)\Omega_2(A,B)$$
$$+ \left(\omega_4(\Lambda) + d^{-1}\omega_7(\Lambda)\right)\Omega_4(A,B). \tag{4.40}$$

The calculation in Theorem 4.1, and the subsequent remark suggest us that there are three terms, i.e., $\Omega_1(A,B), \Omega_2(A,B)$, and $\Omega_4(A,B)$, as the dimension grows large, play a leading role in estimating the average of uncertainty-product within isospectral density matrices. This also tells us that if we want to get a better lower bound about uncertainty-product, then when we take average of any improved lower bound, we should get larger coefficients of such three terms.

Besides, for a fixed Λ, we may view the left hand side of (4.6) as a function of two random observables A and B, for instance, from GUE or Wishart ensemble. We can also consider the concentration of measure phenomenon about such two observables. We leave these questions in the future research.

4.1 Average of uncertainty-product on pure states

For the pure state case, the average of uncertainty-product is easier to calculate. What we have obtained is the following:

Theorem 4.4. *For two observables A and B on $\mathcal{H}_d$, the average of uncertainty-product taken over the whole set of all pure states in $\mathcal{H}_d$ is given by*

$$\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 \mathrm{d}\mu(\psi) = \sum_{j=1}^{8} u_j \Omega_j(A,B), \tag{4.41}$$

where $\Omega_j(A,B)$ is from Theorem 4.1, and for $K_d = (d(d+1)(d+2)(d+3))^{-1}$,

$$u_1 = K_d, \quad u_2 = -(d+2)K_d, \quad u_3 = 4K_d, \quad u_4 = (d^2+3d+1)K_d, \tag{4.42}$$
$$u_5 = 2K_d, \quad u_6 = -2(d+1)K_d, \quad u_7 = (d^2+d-2)K_d, \quad u_8 = 2K_d. \tag{4.43}$$

We also have that

$$\int \mathrm{d}\mu(\psi)\left[\left(\langle\{A,B\}\rangle_\psi - \langle A\rangle_\psi\langle B\rangle_\psi\right)^2 + \langle[A,B]\rangle_\psi^2\right] = \sum_{j=1}^{8} l_j \Omega_j(A,B), \tag{4.44}$$

where

$$l_1 = K_d, \quad l_2 = K_d, \quad l_3 = -2(d+1)K_d, \quad l_4 = K_d, \quad l_5 = (d+1)(d+2)K_d, \tag{4.45}$$
$$l_6 = -2(d+1)K_d, \quad l_7 = (d^2+d-2)K_d, \quad l_8 = -2(2d+5)K_d. \tag{4.46}$$

In the above theorem, we investigate average behavior of both sides of Heisenberg-Robertson-Kennard relations on uniform pure state ensemble. For the case of the average of product of uncertainties (or the corresponding lower bounds for this quantity) over pure Haar-distributed quantum states, the corresponding integrals are very easy to perform as the integral

$$\int \psi^{\otimes k}\mathrm{d}\mu(\psi)$$

involved in all the averages are proportional to the projectors on the symmetric powers of the relevant Hilbert space. This will be clear in the proof, see (7.75). The details of the proof of Theorem 4.4 can be found in Subsection 7.6.

Remark 4.5. In higher dimensional space, there are two terms playing major role in the average uncertainty-product relative to other terms, i.e., $\Omega_4(A, B) = \text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right)$ and $\Omega_7(A, B) = \text{Tr}\left(A^2 B^2\right)$. However, the terms which play major role in the average lower bound of uncertainty-product is $\Omega_5(A, B) = \text{Tr}\left(AB\right)\text{Tr}\left(AB\right)$, and $\Omega_7(A, B) = \text{Tr}\left(A^2 B^2\right)$. Furthermore, we can derive that

$$\int d\mu(\psi)\left\{\Delta A(\psi)^2 \cdot \Delta B(\psi)^2 - \left[\left(\langle\{A, B\}\rangle_\psi - \langle A\rangle_\psi\langle B\rangle_\psi\right)^2 + \langle[A, B]\rangle_\psi^2\right]\right\}$$
$$= -(d + 3)K_d\Omega_2(A, B) + 2(d + 3)K_d\Omega_3(A, B) + d(d + 3)K_d\Omega_4(A, B)$$
$$- d(d + 3)K_d\Omega_5(A, B) + 4(d + 3)K_d\Omega_8(A, B). \tag{4.47}$$

By the nonnegativity of the left hand side of (4.47), we get the following inequality:

$$2\Omega_3(A, B) + d\Omega_4(A, B) + 4\Omega_8(A, B) \geqslant \Omega_2(A, B) + d\Omega_5(A, B). \tag{4.48}$$

That is,

$$2\,\text{Tr}\left(AB\right)\text{Tr}\left(A\right)\text{Tr}\left(B\right) + d\,\text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 4\,\text{Tr}\left(ABAB\right)$$
$$\geqslant \text{Tr}\left(A^2\right)\text{Tr}\left(B\right)^2 + \text{Tr}\left(A\right)^2\text{Tr}\left(B^2\right) + d\,\text{Tr}\left(AB\right)\text{Tr}\left(AB\right). \tag{4.49}$$

It seems difficult to show the above matrix trace inequality (4.49) directly. This inequality about two observables is what we want to get, i.e., uncertainty relation which is independent of state.

Remark 4.6. Naturally, a pure state $|\psi\rangle$ is called the *average state* with respect to uncertainty product of observables (A, B) if it satisfies that

$$\Delta A(\psi)^2 \cdot \Delta B(\psi)^2 = \sum_{j=1}^{8} u_j\Omega_j(A, B). \tag{4.50}$$

What properties do such state have? Answering this question can reveal principally why we do not need to take any measurements, and we can guess the uncertainty about observables by taking average.

Corollary 4.7. *For two observables A and B on $\mathbb{C}^2$, the average of uncertainty-product taken over the whole set of all pure states is given by*

$$\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 d\mu(\psi) = \frac{1}{120}\Omega_1(A, B) - \frac{1}{30}\Omega_2(A, B) + \frac{1}{30}\Omega_3(A, B) + \frac{11}{120}\Omega_4(A, B)$$
$$+ \frac{1}{60}\Omega_5(A, B) - \frac{1}{20}\Omega_6(A, B) + \frac{1}{30}\Omega_7(A, B) + \frac{1}{60}\Omega_8(A, B). \tag{4.51}$$

We also have that

$$\int d\mu(\psi)\left[\left(\langle\{A, B\}\rangle_\psi - \langle A\rangle_\psi\langle B\rangle_\psi\right)^2 + \langle[A, B]\rangle_\psi^2\right]$$
$$= \frac{1}{120}\Omega_1(A, B) + \frac{1}{120}\Omega_2(A, B) - \frac{1}{20}\Omega_3(A, B) + \frac{1}{120}\Omega_4(A, B)$$
$$+ \frac{1}{10}\Omega_5(A, B) - \frac{1}{20}\Omega_6(A, B) + \frac{1}{30}\Omega_7(A, B) - \frac{3}{20}\Omega_8(A, B). \tag{4.52}$$

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Next, as an example, we take $A = \sigma_i$ and $B = \sigma_j$, where σ_i and σ_j are any two different matrices from three Pauli's matrices, using the above Corollary, then we get the average of uncertainty-product of A and B is given by

$$\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 \mathrm{d}\mu(\psi) = \frac{2}{5}. \tag{4.53}$$

Moreover,

$$\int \mathrm{d}\mu(\psi) \left[(\langle \{A, B\} \rangle_\psi - \langle A \rangle_\psi \langle B \rangle_\psi)^2 + \langle [A, B] \rangle_\psi^2 \right] = \frac{2}{5}. \tag{4.54}$$

This is surprising! As we have seen that the following inequality

$$\Delta A(\psi)^2 \cdot \Delta B(\psi)^2 \geqslant (\langle \{A, B\} \rangle_\psi - \langle A \rangle_\psi \langle B \rangle_\psi)^2 + \langle [A, B] \rangle_\psi^2 \tag{4.55}$$

holds for all pure state $|\psi\rangle$. From the above discussion, we see that

$$\int \mathrm{d}\mu(\psi) f(\psi) = 0, \tag{4.56}$$

where f is defined by

$$f(\psi) = \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 - \left[(\langle \{A, B\} \rangle_\psi - \langle A \rangle_\psi \langle B \rangle_\psi)^2 + \langle [A, B] \rangle_\psi^2 \right], \tag{4.57}$$

which is obviously a non-negative function of the pure state $|\psi\rangle$. By Lebesgue integration theory, we get that $f(\psi)$ vanishes almost everywhere except a zero-measure subset of all pure states. In other words,

$$\Delta A(\psi)^2 \cdot \Delta B(\psi)^2 = (\langle \{A, B\} \rangle_\psi - \langle A \rangle_\psi \langle B \rangle_\psi)^2 + \langle [A, B] \rangle_\psi^2, \quad \text{a.e.} \tag{4.58}$$

From the above observation, we see that any desire to improve universally the uncertainty-product seems impossible, at least in the qubit case for two observables σ_i and σ_j chosen from three Pauli's matrices.

4.2 Average of uncertainty-product on the mixed states

For the mixed state, comparing with the pure state, the calculation is more complicated, we have the following result.

Theorem 4.8. *For two observables A and B on $\mathcal{H}_d$, the average of uncertainty-product taken over the whole set of all density matrices $\mathrm{D}(\mathcal{H}_d)$ is given by*

$$\int \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 \mathrm{d}\mu_{\mathrm{HS}}(\rho) = \sum_{j=1}^{8} \overline{\omega}_j \cdot \Omega_j(A, B), \tag{4.59}$$

where $\overline{\omega}_j = \int \omega_j(\Lambda) \mathrm{d}\nu(\Lambda) (j = 1, \ldots, 8)$.

Proof. The proof follows directly from Theorem 4.1 by using Proposition 7.4 and Lemma 7.6. $\qquad\square$

Remark 4.9. In fact, we can give the final formulae for $\overline{\omega}_j$'s. We ignore the tedious but simple calculations.

$$\overline{\omega}_1 \;=\; N_d\left(d^4 - 20d^2 + 158 - \frac{50}{d^2+1} + \frac{792}{d^2+2} - \frac{1512}{d^2+3}\right), \tag{4.60}$$

$$\overline{\omega}_2 \;=\; N_d\left(-d^5 + 18d^3 - 118d - \frac{50d}{d^2+1} + \frac{504d}{d^2+3}\right), \tag{4.61}$$

$$\overline{\omega}_3 \;=\; N_d\left(4d^3 - 80d + \frac{200d}{d^2+1} - \frac{1584d}{d^2+2} + \frac{2016d}{d^2+3}\right), \tag{4.62}$$

$$\overline{\omega}_4 \;=\; N_d\left(d^6 - 17d^4 + 99d^2 - 316 - \frac{50}{d^2+1} + \frac{396}{d^2+2} + \frac{504}{d^2+3}\right), \tag{4.63}$$

$$\overline{\omega}_5 \;=\; N_d\left(2d^2 - 40 + \frac{100}{d^2+1} - \frac{792}{d^2+2} + \frac{1008}{d^2+3}\right), \tag{4.64}$$

$$\overline{\omega}_6 \;=\; N_d\left(-2d^4 + 38d^2 - 276 - \frac{792}{d^2+2} + \frac{2016}{d^2+3}\right), \tag{4.65}$$

$$\overline{\omega}_7 \;=\; N_d\left(2d^5 - d^4 - 28d^3 + 9d^2 + 136d + \frac{100d}{d^2+1} - \frac{672d}{d^2+3}\right), \tag{4.66}$$

$$\overline{\omega}_8 \;=\; N_d\left(2d - \frac{100d}{d^2+1} + \frac{396d}{d^2+2} - \frac{336d}{d^2+3}\right). \tag{4.67}$$

From the above formulae, we can see that in higher dimensional space, $\Omega_4(A,B) = \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right)$ plays a leading role relative to other terms. We also see from Remark 4.3 that, for the large enough dimension d, when observables A and B taken from GUE are independent,

$$\int \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 \mathrm{d}\mu_{\mathrm{HS}}(\rho) \simeq m_1\Omega_1(A,B) + m_2\Omega_2(A,B) + m_4\Omega_4(A,B). \tag{4.68}$$

where

$$m_1 \;=\; \overline{\omega}_1 + d^{-1}\overline{\omega}_3 + d^{-2}\overline{\omega}_5 - d^{-3}\overline{\omega}_8, \tag{4.69}$$

$$m_2 \;=\; \overline{\omega}_2 + d^{-1}\overline{\omega}_6 + d^{-2}\overline{\omega}_8, \tag{4.70}$$

$$m_4 \;=\; \overline{\omega}_4 + d^{-1}\overline{\omega}_7. \tag{4.71}$$

Similar to the pure state case (see (4.50)), a mixed state ρ is called the *average state* with respect to uncertainty product of observables (A,B) if it satisfies that

$$\Delta A(\rho)^2 \cdot \Delta B(\rho)^2 = \sum_{j=1}^{8} \overline{\omega}_j \cdot \Omega_j(A,B). \tag{4.72}$$

We can ask analogous problems parallel to the pure state case. But we are not concerned these problems in this paper.

4.3 Average lower bound of uncertainty-product

Here we also calculate the average of the lower bound of uncertainty-product in (1.3).

Theorem 4.10. *For two observables A and B on $\mathcal{H}_d$, it holds that*

$$\int_{\mathrm{D}(\mathcal{H}_d)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\left(\langle\{A,B\}\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho\right)^2 = \sum_{j=1}^{8} \beta_j\Omega_j(A,B), \tag{4.73}$$

where $N_d^{-1} = d^2(d^2 - 1)(d^2 - 4)(d^2 - 9)$ and

$$\beta_1 = N_d \left(d^4 - 18d^2 + 158 - \frac{50}{d^2 + 1} + \frac{792}{d^2 + 2} - \frac{1512}{d^2 + 3} \right), \tag{4.74}$$

$$\beta_2 = N_d \left(d^3 - 20d + \frac{50d}{d^2 + 1} - \frac{396d}{d^2 + 2} + \frac{504d}{d^2 + 3} \right), \tag{4.75}$$

$$\beta_3 = N_d \left(-2d^5 + 38d^3 - 276d - \frac{792d}{d^2 + 2} + \frac{2016d}{d^2 + 3} \right), \tag{4.76}$$

$$\beta_4 = N_d \left(-2d^2 - 20 + \frac{50}{d^2 + 1} - \frac{396}{d^2 + 2} + \frac{504}{d^2 + 3} \right), \tag{4.77}$$

$$\beta_5 = N_d \left(d^6 - 15d^4 - 2d^3 + 60d^2 + 34d - 140 + \frac{200d + 200}{d^2 + 1} - \frac{396d + 792}{d^2 + 2} + \frac{1008}{d^2 + 3} \right), \tag{4.78}$$

$$\beta_6 = N_d \left(-2d^3 + 4d^2 + 34d - 380 + \frac{200d + 500}{d^2 + 1} - \frac{396d + 1584}{d^2 + 2} + \frac{2016}{d^2 + 3} \right), \tag{4.79}$$

$$\beta_7 = N_d \left(-d^7 + \frac{27}{2}d^5 - \frac{91}{2}d^3 + 70d - \frac{50d}{d^2 + 1} + \frac{396d}{d^2 + 2} - \frac{672d}{d^2 + 3} \right), \tag{4.80}$$

$$\beta_8 = N_d \left(-d^7 + \frac{27}{2}d^5 - \frac{91}{2}d^3 + 68d + \frac{50d}{d^2 + 1} - \frac{336d}{d^2 + 3} \right). \tag{4.81}$$

Thus

$$\int_{D(\mathcal{H}_d)} d\mu_{\text{HS}}(\rho) \left[\left(\langle \{A, B\} \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho \right)^2 + \langle [A, B] \rangle_\rho^2 \right] = \sum_{j=1}^8 \beta_j' \Omega_j(A, B), \tag{4.82}$$

where

$$\beta_1' = \beta_1, \ \beta_2' = \beta_2, \ \beta_3' = \beta_3, \ \beta_4' = \beta_4, \ \beta_5' = \beta_5, \ \beta_6' = \beta_6, \tag{4.83}$$

$$\beta_7' = N_d \left(-d^7 + 14d^5 - 53d^3 + 102d - \frac{100d}{d^2 + 1} + \frac{396d}{d^2 + 2} - \frac{672d}{d^2 + 3} \right), \tag{4.84}$$

$$\beta_8' = N_d \left(-d^7 + 13d^5 - 38d^3 + 36d + \frac{100d}{d^2 + 1} - \frac{336d}{d^2 + 3} \right). \tag{4.85}$$

The average of the lower bound of uncertainty-product can be the reference value for improving the lower bound of uncertainty-product, as suggested in Section 3. The proof of Theorem 4.10 is put in Subsection 7.7.

Remark 4.11. From the above Theorem 4.10, we see that in higher dimensional space, $\Omega_7(A, B) = \text{Tr}\left(A^2 B^2 \right)$ and $\Omega_8(A, B) = \text{Tr}\left(ABAB \right)$ play a leading role relative to other terms.

Remark 4.12. We can still compare (4.59) and (4.82) in order to obtain another matrix trace inequality:

$$\sum_{j=1}^8 (\overline{\omega}_j - \beta_j') \cdot \Omega_j(A, B) \geqslant 0. \tag{4.86}$$

As a matter of fact, (4.48) and (4.86) are just two special cases of the following matrix trace inequalities:

$$\sum_{j=1}^8 f_j(d) \cdot \Omega_j(A, B) \geqslant 0, \tag{4.87}$$

where $f_j(d) (j = 1, \ldots, 8)$ are the dimension-dependent factors under some constraints.

5 Concentration of measure phenomenon

In order to discuss the concentration of measure phenomenon might being happened to the uncertainty-product, we will use the concentration of measure phenomenon on the special unitary group $\mathrm{SU}(\mathcal{H}_d)$, established recently by Oszmaniec in his thesis [28].

Proposition 5.1 (Concentration of measure on $\mathrm{SU}(\mathcal{H}_d)$). *Consider a special unitary group $\mathrm{SU}(\mathcal{H}_d)$ equipped with the Haar measure μ_{Haar} and a Riemann metric g_{HS}. Let $f : \mathrm{SU}(\mathcal{H}_d) \to \mathbb{R}$ be a smooth function on $\mathrm{SU}(\mathcal{H}_d)$ with the mean $\bar{f} = \int_{\mathrm{SU}(\mathcal{H}_d)} f(U) \mathrm{d}\mu_{\mathrm{Haar}}(U)$, let*

$$L = \sqrt{\max\left\{ g_{\mathrm{HS}}(\nabla f, \nabla f) : U \in \mathrm{SU}(\mathcal{H}_d) \right\}} \tag{5.1}$$

be the Lipschitz constant of f. Then, for every $\epsilon \geqslant 0$, the following concentration inequalities hold

$$\mu_{\mathrm{Haar}}\left\{ U \in \mathrm{SU}(\mathcal{H}_d) : f(U) - \bar{f} \geqslant \epsilon \right\} \quad \leqslant \quad \exp\left(-\frac{d\epsilon^2}{4L^2} \right), \tag{5.2}$$

$$\mu_{\mathrm{Haar}}\left\{ U \in \mathrm{SU}(\mathcal{H}_d) : f(U) - \bar{f} \leqslant -\epsilon \right\} \quad \leqslant \quad \exp\left(-\frac{d\epsilon^2}{4L^2} \right). \tag{5.3}$$

Denote

$$\Phi(U) = \Delta A(U\rho U^\dagger)^2 \cdot \Delta B(U\rho U^\dagger)^2. \tag{5.4}$$

From (4.4), we see that

$$\Phi(U) = \mathrm{Tr}\left(U^{\otimes 4} \rho^{\otimes 4} U^{\dagger, \otimes 4} \left[A^2 \otimes B^2 \otimes \mathbb{1}^{\otimes 2} + A^{\otimes 2} \otimes B^{\otimes 2} - A^2 \otimes B^{\otimes 2} \otimes \mathbb{1} - A^{\otimes 2} \otimes B^2 \otimes \mathbb{1} \right] \right). \tag{5.5}$$

By using the result in [28, Lemma 6.1], we see that the Lipschitz constant L_Φ of the function Φ, with respect to the metric tensor g_{HS}, satisfies

$$\begin{aligned}
L_\Phi &\leqslant 8 \left\| A^2 \otimes B^2 \otimes \mathbb{1}^{\otimes 2} + A^{\otimes 2} \otimes B^{\otimes 2} - A^2 \otimes B^{\otimes 2} \otimes \mathbb{1} - A^{\otimes 2} \otimes B^2 \otimes \mathbb{1} \right\|_\infty \\
&\leqslant 32 \left\| A \right\|_\infty^2 \left\| B \right\|_\infty^2.
\end{aligned} \tag{5.6}$$

Thus we have the following result:

Theorem 5.2 (Concentration of measure within isospectral density matrices). *For every $\epsilon \geqslant 0$, the following concentration inequalities hold*

$$\mu_{\mathrm{Haar}}\left\{ U \in \mathrm{SU}(\mathcal{H}_d) : \Phi(U) - \overline{\Phi} \geqslant \epsilon \right\} \quad \leqslant \quad \exp\left(-\frac{d\epsilon^2}{4096 \left\| A \right\|_\infty^4 \left\| B \right\|_\infty^4} \right), \tag{5.7}$$

$$\mu_{\mathrm{Haar}}\left\{ U \in \mathrm{SU}(\mathcal{H}_d) : \Phi(U) - \overline{\Phi} \leqslant -\epsilon \right\} \quad \leqslant \quad \exp\left(-\frac{d\epsilon^2}{4096 \left\| A \right\|_\infty^4 \left\| B \right\|_\infty^4} \right). \tag{5.8}$$

This result shows that when we consider the uncertainty-product for two bounded observables A and B over the set of isospectral density matrices, the uncertainty-product around its average, in (4.6)

$$\overline{\Phi} = \sum_{j=1}^8 \omega_j(\Lambda) \cdot \Omega_j(A, B), \tag{5.9}$$

has an overwhelming probability.

Lemma 5.3 (Lévy's lemma). *Let $f : \mathbb{S}^k \to \mathbb{R}$ be a Lipschitz function from k-sphere to real line with the Lipschitz constant L (with respect to the Euclidean norm) and a point $u \in \mathbb{S}^k$ be chosen uniformly at random. Then, for all $\epsilon > 0$,*

$$\mathbf{Pr}\left\{\left|f(u) - \bar{f}\right| > \epsilon\right\} \leqslant 2\exp\left(-\frac{(k+1)\epsilon^2}{9\pi^3 L^2 \ln 2}\right), \tag{5.10}$$

where $\bar{f} := \int_{\mathbb{S}^k} f(u)\mathrm{d}\mu(u)$ means the mean value of f with respect to uniform probability measure on the unit sphere $\mathbb{S}^k$.

Let $f(\rho) = \Delta A(\rho)^2 \cdot \Delta B(\rho)^2$. Then

$$
\begin{aligned}
f(\rho) - f(\sigma) &= \mathrm{Tr}\left(\left[A^2 \otimes B^2\right]\left[\rho^{\otimes 2} - \sigma^{\otimes 2}\right]\right) + \mathrm{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right]\left[\rho^{\otimes 4} - \sigma^{\otimes 4}\right]\right) \\
&\quad - \mathrm{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\left[\rho^{\otimes 3} - \sigma^{\otimes 3}\right]\right) - \mathrm{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right]\left[\rho^{\otimes 3} - \sigma^{\otimes 3}\right]\right).
\end{aligned}
\tag{5.11}
$$

Thus

$$
\begin{aligned}
|f(\rho) - f(\sigma)| &\leqslant \left|\mathrm{Tr}\left(\left[A^2 \otimes B^2\right]\left[\rho^{\otimes 2} - \sigma^{\otimes 2}\right]\right)\right| + \left|\mathrm{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right]\left[\rho^{\otimes 4} - \sigma^{\otimes 4}\right]\right)\right| \\
&\quad + \left|\mathrm{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\left[\rho^{\otimes 3} - \sigma^{\otimes 3}\right]\right)\right| + \left|\mathrm{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right]\left[\rho^{\otimes 3} - \sigma^{\otimes 3}\right]\right)\right| \\
&\leqslant \left\|A^2 \otimes B^2\right\|_{\infty}\left\|\rho^{\otimes 2} - \sigma^{\otimes 2}\right\|_1 + \left\|A^{\otimes 2} \otimes B^{\otimes 2}\right\|_{\infty}\left\|\rho^{\otimes 4} - \sigma^{\otimes 4}\right\|_1 \\
&\quad + \left\|A^2 \otimes B^{\otimes 2}\right\|_{\infty}\left\|\rho^{\otimes 3} - \sigma^{\otimes 3}\right\|_1 + \left\|B^2 \otimes A^{\otimes 2}\right\|_{\infty}\left\|\rho^{\otimes 3} - \sigma^{\otimes 3}\right\|_1.
\end{aligned}
\tag{5.12}
$$

Since

$$\left\|\rho^{\otimes k} - \sigma^{\otimes k}\right\|_1 \leqslant k\left\|\rho - \sigma\right\|_1, \tag{5.13}$$

it follows that

$$|f(\rho) - f(\sigma)| \leqslant \left(12\left\|A\right\|_{\infty}^2 \left\|B\right\|_{\infty}^2\right)\left\|\rho - \sigma\right\|_1 \tag{5.14}$$

For the pure states, that is, $\rho = |\psi\rangle\langle\psi|$ and $\sigma = |\phi\rangle\langle\phi|$, we have $\left\|\psi - \phi\right\|_1 \leqslant \sqrt{2}\left\|\psi - \phi\right\|_2$, implying

$$|f(\psi) - f(\phi)| \leqslant L \cdot \left\|\psi - \phi\right\|_2, \tag{5.15}$$

where $L := 12\sqrt{2}\left\|A\right\|_{\infty}^2 \left\|B\right\|_{\infty}^2$. Note here that $k = 2d - 1$ since pure states live in $\mathbb{C}^d$. Then

$$\mathbf{Pr}\left\{\left|f(\psi) - \bar{f}\right| > \epsilon\right\} \leqslant 2\exp\left(-\frac{d\epsilon^2}{1296\pi^3 \left\|A\right\|_{\infty}^4 \left\|B\right\|_{\infty}^4 \ln 2}\right). \tag{5.16}$$

When $\left\|A\right\|_{\infty}$ and $\left\|B\right\|_{\infty}$ are independent of the dimension d, it shows the concentration of measure phenomenon.

In fact, we can view ρ and σ in $\mathrm{D}\left(\mathcal{H}_d\right)$ as reduced states of Haar-distributed bipartite states $|\Psi_\rho\rangle$ and $|\Psi_\sigma\rangle$ in $\mathcal{H}_d \otimes \mathcal{H}_d$, then let $g(\Psi_\rho) = \Delta A(\rho)^2 \cdot \Delta B(\rho)^2$, where $\Psi_\rho = |\Psi_\rho\rangle\langle\Psi_\rho|$ and $\rho = \mathrm{Tr}_2\left(|\Psi_\rho\rangle\langle\Psi_\rho|\right)$. Thus

$$\left|g(\Psi_\rho) - g(\Psi_\sigma)\right| \leqslant (12\sqrt{2}\left\|A\right\|_{\infty}^2 \left\|B\right\|_{\infty}^2)\left\|\Psi_\rho - \Psi_\sigma\right\|_2. \tag{5.17}$$

Then

$$\mathbf{Pr}\left\{\left|f(\rho) - \bar{f}\right| > \epsilon\right\} \leqslant 2\exp\left(-\frac{d^2\epsilon^2}{1296\pi^3 \left\|A\right\|_{\infty}^4 \left\|B\right\|_{\infty}^4 \ln 2}\right). \tag{5.18}$$

Thus we have the following:

Theorem 5.4 (Concentration of measure). *Assume that $\|A\|_\infty$ and $\|B\|_\infty$ are independent of dimension, where A and B are bounded observables. It holds that*

$$\mathbf{Pr}\left\{\left|\Delta A(\psi)^2 \cdot \Delta B(\psi)^2 - \langle \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 \rangle \right| > \epsilon \right\} \leqslant 2\exp\left(-\frac{d\epsilon^2}{1296\pi^3 \|A\|_\infty^4 \|B\|_\infty^4 \ln 2}\right) \tag{5.19}$$

and

$$\mathbf{Pr}\left\{\left|\Delta A(\rho)^2 \cdot \Delta B(\rho)^2 - \langle \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 \rangle \right| > \epsilon \right\} \leqslant 2\exp\left(-\frac{d^2\epsilon^2}{1296\pi^3 \|A\|_\infty^4 \|B\|_\infty^4 \ln 2}\right). \tag{5.20}$$

Here $\langle f(\rho) \rangle = \int f(\rho)\mathrm{d}\mu_{\mathrm{HS}}(\rho)$.

Generally, observables A and B are dimension-dependent, thus we cannot obtain the concentration of measure phenomenon universally. But of course, even though A and B are dimension-dependent, we could still get the concentration of measure phenomenon, for instance, whenever their operator norms are uniformly bounded. Besides, inequalities presented above do not have to be tight, i.e., even if the right hand side is "large", the relevant left hand side might still be very small.

6 Concluding remarks

This paper deals with uncertainty relations in various random state ensembles. As suggested, taking a state at random also corresponds to assuming minimal prior knowledge about the system in question. We make an attempt in describing uncertainty relation using only observables by taking average of uncertainty-product of any two bounded observables in our random state ensemble (see (4.48)). We also establish the typicality of a random state with respect to any two bounded observables under restricted conditions. The concentration of measure phenomenon is a very important property for a random state since it predicates the bulk behavior of a large number of quantum particles without any practical detections. Theoretically, sampled states randomly will show up average behavior with respect to a pair of bounded observables as we increases the level of the quantum system under consideration. In addition, we have also present an interesting result: beyond the set of zero-measure of all pure qubit states, it holds that

$$\Delta A(\psi)^2 \cdot \Delta B(\psi)^2 = \left(\langle \{A, B\}\rangle_\psi - \langle A\rangle_\psi \langle B\rangle_\psi\right)^2 + \langle [A, B]\rangle_\psi^2. \tag{6.1}$$

This result indicates that any desire to improve the uncertainty-product universally seems impossible, at least in the qubit case for two distinct observables σ_i and σ_j chosen from three Pauli's matrices. Our calculations can help us check how large the gap is between the uncertainty-product and any obtained lower bounds about the uncertainty-product. We hope the results obtained in this paper will shed new light on quantum information processing tasks.

7   Appendix: the computation of $\mathscr{E}_k^\circ(\Lambda)$

Consider a system of k qudits, each with a standard local computational basis $\{|i\rangle, i = 1, \ldots, d\}$. The Schur-Weyl duality relates transforms on the system performed by local d-dimensional unitary operations to those performed by permutation of the qudits. Recall that the symmetric group S_k is the group of all permutations of k objects. This group is naturally represented in our system by

$$\mathbf{P}(\pi)|i_1 \cdots i_k\rangle := |i_{\pi^{-1}(1)} \cdots i_{\pi^{-1}(k)}\rangle, \tag{7.1}$$

where $\pi \in S_k$ is a permutation and $|i_1 \cdots i_k\rangle$ is shorthand for $|i_1\rangle \otimes \cdots \otimes |i_k\rangle$. Let $\mathrm{U}(d)$ be the group of $d \times d$ unitary operators. This group is naturally represented in our system by

$$\mathbf{Q}(U)|i_1 \cdots i_k\rangle := U|i_1\rangle \otimes \cdots \otimes U|i_k\rangle, \tag{7.2}$$

where $U \in \mathrm{U}(d)$. Thus we have the following famous result:

Theorem 7.1 (Schur). *Let $\mathcal{A} = \mathrm{span}\{\mathbf{P}(\pi) : \pi \in S_k\}$ and $\mathcal{B} = \mathrm{span}\{\mathbf{Q}(U) : U \in \mathrm{U}(d)\}$. Then:*

$$\mathcal{A}' = \mathcal{B} \quad and \quad \mathcal{A} = \mathcal{B}' \tag{7.3}$$

The following result concerns with a wonderful decomposition of the representations on k-fold tensor space $(\mathbb{C}^d)^{\otimes k}$ of $\mathrm{U}(d)$ and S_k, respectively, using their corresponding irreps accordingly.

Theorem 7.2 (Schur-Weyl duality). *There exist a basis, known as Schur basis, in which representation $\left(\mathbf{QP}, (\mathbb{C}^d)^{\otimes k}\right)$ of $\mathrm{U}(d) \times S_k$ decomposes into irreducible representations $\mathbf{Q}_\lambda$ and $\mathbf{P}_\lambda$ of $\mathrm{U}(d)$ and S_k, respectively:*

(i) $(\mathbb{C}^d)^{\otimes k} \cong \bigoplus_{\lambda \vdash (k,d)} \mathbf{Q}_\lambda \otimes \mathbf{P}_\lambda$;

(ii) $\mathbf{P}(\pi) \cong \bigoplus_{\lambda \vdash (k,d)} \mathbb{1}_{\mathbf{Q}_\lambda} \otimes \mathbf{P}_\lambda(\pi)$;

(iii) $\mathbf{Q}(U) \cong \bigoplus_{\lambda \vdash (k,d)} \mathbf{Q}_\lambda(U) \otimes \mathbb{1}_{\mathbf{P}_\lambda}$.

Since $\mathbf{Q}$ and $\mathbf{P}$ commute, we can define representation $\left(\mathbf{QP}, (\mathbb{C}^d)^{\otimes k}\right)$ of $\mathrm{U}(d) \times S_k$ as

$$\mathbf{QP}(U, \pi) = \mathbf{Q}(U)\mathbf{P}(\pi) = \mathbf{P}(\pi)\mathbf{Q}(U) \quad \forall (U, \pi) \in \mathrm{U}(d) \times S_k. \tag{7.4}$$

Then:

$$\mathbf{QP}(U, \pi) = U^{\otimes k} P_\pi = P_\pi U^{\otimes k} \cong \bigoplus_{\lambda \vdash (k,d)} \mathbf{Q}_\lambda(U) \otimes \mathbf{P}_\lambda(\pi). \tag{7.5}$$

The dimensions of pairing irreps for U (d) and S_k, respectively, in Schur-Weyl duality can be computed by so-called *hook length formulae*. The hook of box (i, j) in a Young diagram determined by a partition λ is given by the box itself, the boxes to its right and below. The hook length is the number of boxes in a hook. Specifically, we have the following result without its proof:

Theorem 7.3 (Hook length formulae). *The dimensions of pairing irreps for* U (d) *and* S_k, *respectively, in Schur-Weyl duality can be given as follows:*

$$
\dim(\mathbf{Q}_\lambda) \quad = \quad \prod_{(i,j) \in \lambda} \frac{d + j - i}{h(i, j)} = \prod_{1 \leqslant i < j \leqslant d} \frac{\lambda_i - \lambda_j + j - i}{j - i}, \tag{7.6}
$$

$$
\dim(\mathbf{P}_\lambda) \quad = \quad \frac{k!}{\prod_{(i,j) \in \lambda} h(i, j)}. \tag{7.7}
$$

In [29], Schur-Weyl duality is employed to give a computation about the integral of the following form:

$$
\int_{\mathrm{U}(d)} U^{\otimes k} M (U^{\otimes k})^\dagger \mathrm{d}\mu_{\mathrm{Haar}}(U). \tag{7.8}
$$

Moreover we have obtained that

$$
\int_{\mathrm{U}(d)} U^{\otimes k} M (U^{\otimes k})^\dagger \mathrm{d}\mu_{\mathrm{Haar}}(U) = \left(\sum_{\pi \in S_k} \mathrm{Tr}\left(MP(\pi)\right) P(\pi^{-1}) \right) \left(\sum_{\pi \in S_k} \mathrm{Wg}(\pi) P(\pi^{-1}) \right), \tag{7.9}
$$

where *Weingarten function* Wg is defined over S_k by

$$
\mathrm{Wg}(\pi) := \frac{1}{(k!)^2} \sum_{\lambda \vdash k} \frac{\dim(\mathbf{P}_\lambda)^2}{\dim(\mathbf{Q}_\lambda)} \chi_\lambda(\pi) \tag{7.10}
$$

for each $\pi \in S_k$ and $\chi_\lambda(\pi) = \mathrm{Tr}\left(P_\lambda(\pi)\right)$ is the value of the character of irrep $\mathbf{P}_\lambda$ at $\pi \in S_k$.

Here we consider a special case where the above-mentioned $M = \Lambda^{\otimes k}$ for a given spectrum Λ and any natural number k, thus we introduce a new symbol for convenience:

$$
\mathscr{E}_k(\Lambda) := \int (U\Lambda U^\dagger)^{\otimes k} \mathrm{d}\mu_{\mathrm{Haar}}(U). \tag{7.11}
$$

Throughout this paper, we frequently leave out the integral domain U (d) when we consider matrix integral taken over the whole unitary group U (d) unless stated otherwise. We see that

$$
\mathscr{E}_k(\Lambda) = \sum_{\lambda \vdash (k,d)} \frac{\mathrm{Tr}\left(\Lambda^{\otimes k} C_\lambda\right)}{\mathrm{Tr}\left(C_\lambda\right)} C_\lambda, \tag{7.12}
$$

where

$$
C_\lambda := \frac{\dim(\mathbf{P}_\lambda)}{k!} \sum_{\pi \in S_k} \chi_\lambda(\pi) P(\pi). \tag{7.13}
$$

7.1 The case where $k = 1, 2$.

It is already known in [29] that

$$
\int U X U^\dagger \mathrm{d}\mu_{\mathrm{Haar}}(U) = \frac{\mathrm{Tr}\left(X\right)}{d} \mathbb{1}_d \tag{7.14}
$$

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and

$$\int (U \otimes U) M (U \otimes U)^\dagger \mathrm{d}\mu_{\mathrm{Haar}}(U)$$
$$= \left(\frac{\mathrm{Tr}\,(M)}{d^2 - 1} - \frac{\mathrm{Tr}\,(MF)}{d(d^2 - 1)} \right) \mathbb{1}_d \otimes \mathbb{1}_d - \left(\frac{\mathrm{Tr}\,(M)}{d(d^2 - 1)} - \frac{\mathrm{Tr}\,(MF)}{d^2 - 1} \right) F, \tag{7.15}$$

where $F := \sum_{i,j=1}^d |ji\rangle\langle ij|$ is called a swap operator. Thus

$$\mathscr{E}_1(\Lambda) = \frac{\mathbb{1}_d}{d}, \tag{7.16}$$

$$\mathscr{E}_2(\Lambda) = \frac{1}{d^2 - 1} \left[\left(1 - \frac{\mathrm{Tr}\,(\Lambda^2)}{d} \right) \mathbb{1}_d \otimes \mathbb{1}_d - \left(\frac{1}{d} - \mathrm{Tr}\,(\Lambda^2) \right) F \right]$$

$$= \Delta_2^{(2)} C_{(2)} + \Delta_2^{(1,1)} C_{(1,1)}, \tag{7.17}$$

where

$$\Delta_2^{(2)} := \frac{1 + \mathrm{Tr}\,(\Lambda^2)}{d(d+1)}, \quad \Delta_2^{(1,1)} := \frac{1 - \mathrm{Tr}\,(\Lambda^2)}{(d-1)d} \tag{7.18}$$

and

$$C_\lambda = \begin{cases} \frac{1}{2}(P_{(1)} + P_{(12)}), & \text{if } \lambda = (2), \\ \frac{1}{2}(P_{(1)} - P_{(12)}), & \text{if } \lambda = (1,1). \end{cases} \tag{7.19}$$

7.2 The formula of $\mathscr{E}_3(\Lambda)$

In what follows, we compute $\mathscr{E}_3(\Lambda)$. Note that we get the following decomposition via Schur-Weyl duality

$$(\mathbb{C}^d)^{\otimes 3} \cong \mathbf{Q}_{(3)} \otimes \mathbf{P}_{(3)} \bigoplus \mathbf{Q}_{(2,1)} \otimes \mathbf{P}_{(2,1)} \bigoplus \mathbf{Q}_{(1,1,1)} \otimes \mathbf{P}_{(1,1,1)} \tag{7.20}$$

where

$$\dim(\mathbf{Q}_\lambda) = \begin{cases} \frac{d(d+1)(d+2)}{6}, & \text{if } \lambda = (3), \\ \frac{(d-1)d(d+1)}{3}, & \text{if } \lambda = (2,1), \\ \frac{(d-2)(d-1)d}{6}, & \text{if } \lambda = (1,1,1), \end{cases} \quad \text{and} \quad \dim(\mathbf{P}_\lambda) = \begin{cases} 1, & \text{if } \lambda = (3), \\ 2, & \text{if } \lambda = (2,1), \\ 1, & \text{if } \lambda = (1,1,1). \end{cases} \tag{7.21}$$

Hence

$$C_\lambda = \begin{cases} \frac{1}{6}\left(P_{(1)} + P_{(12)} + P_{(13)} + P_{(23)} + P_{(123)} + P_{(132)} \right), & \text{if } \lambda = (3), \\ \frac{1}{3}\left(2P_{(1)} - P_{(123)} - P_{(132)} \right), & \text{if } \lambda = (2,1), \\ \frac{1}{6}\left(P_{(1)} - P_{(12)} - P_{(13)} - P_{(23)} + P_{(123)} + P_{(132)} \right), & \text{if } \lambda = (1,1,1). \end{cases} \tag{7.22}$$

It follows that

$$\mathrm{Tr}\,(C_\lambda) = \begin{cases} \frac{d(d+1)(d+2)}{6}, & \text{if } \lambda = (3), \\ \frac{2(d-1)d(d+1)}{3}, & \text{if } \lambda = (2,1), \\ \frac{(d-2)(d-1)d}{6}, & \text{if } \lambda = (1,1,1) \end{cases} \tag{7.23}$$

and

$$
\mathrm{Tr}\left(\Lambda^{\otimes 3}C_\lambda\right) =
\begin{cases}
\frac{1}{6}\left[1 + 3\,\mathrm{Tr}\left(\Lambda^2\right) + 2\,\mathrm{Tr}\left(\Lambda^3\right)\right], & \text{if } \lambda = (3), \\
\frac{2}{3}\left[1 - \mathrm{Tr}\left(\Lambda^3\right)\right], & \text{if } \lambda = (2,1), \\
\frac{1}{6}\left[1 - 3\,\mathrm{Tr}\left(\Lambda^2\right) + 2\,\mathrm{Tr}\left(\Lambda^3\right)\right], & \text{if } \lambda = (1,1,1).
\end{cases}
\tag{7.24}
$$

Therefore

$$
\mathscr{E}_3(\rho) = \Delta_3^{(3)}C_{(3)} + \Delta_3^{(2,1)}C_{(2,1)} + \Delta_3^{(1,1,1)}C_{(1,1,1)},
\tag{7.25}
$$

where

$$
\Delta_3^{(3)} \quad := \quad \frac{1 + 3\,\mathrm{Tr}\left(\Lambda^2\right) + 2\,\mathrm{Tr}\left(\Lambda^3\right)}{d(d+1)(d+2)},
\tag{7.26}
$$

$$
\Delta_3^{(2,1)} \quad := \quad \frac{1 - \mathrm{Tr}\left(\Lambda^3\right)}{(d-1)d(d+1)},
\tag{7.27}
$$

$$
\Delta_3^{(1,1,1)} \quad := \quad \frac{1 - 3\,\mathrm{Tr}\left(\Lambda^2\right) + 2\,\mathrm{Tr}\left(\Lambda^3\right)}{(d-2)(d-1)d}.
\tag{7.28}
$$

7.3 The formula of $\mathscr{E}_4(\Lambda)$

Similar we get the following decomposition:

$$
\begin{aligned}
(\mathbb{C}^d)^{\otimes 4} \quad \cong \quad & \mathbf{Q}_{(4)} \otimes \mathbf{P}_{(4)} \bigoplus \mathbf{Q}_{(3,1)} \otimes \mathbf{P}_{(3,1)} \bigoplus \mathbf{Q}_{(2,2)} \otimes \mathbf{P}_{(2,2)} \\
& \bigoplus \mathbf{Q}_{(2,1,1)} \otimes \mathbf{P}_{(2,1,1)} \bigoplus \mathbf{Q}_{(1,1,1,1)} \otimes \mathbf{P}_{(1,1,1,1)},
\end{aligned}
\tag{7.29}
$$

where

$$
\dim(\mathbf{Q}_\lambda) =
\begin{cases}
\frac{d(d+1)(d+2)(d+3)}{24}, & \text{if } \lambda = (4), \\
\frac{(d-1)d(d+1)(d+2)}{8}, & \text{if } \lambda = (3,1), \\
\frac{(d-1)d^2(d+1)}{12}, & \text{if } \lambda = (2,2), \\
\frac{(d-2)(d-1)d(d+1)}{8}, & \text{if } \lambda = (2,1,1), \\
\frac{(d-3)(d-2)(d-1)d}{24}, & \text{if } \lambda = (1,1,1,1),
\end{cases}
\quad \text{and} \quad
\dim(\mathbf{P}_\lambda) =
\begin{cases}
1, & \text{if } \lambda = (4), \\
3, & \text{if } \lambda = (3,1), \\
2, & \text{if } \lambda = (2,2), \\
3, & \text{if } \lambda = (2,1,1), \\
1, & \text{if } \lambda = (1,1,1,1).
\end{cases}
\tag{7.30}
$$

Hence we have:

$$
\begin{aligned}
C_{(4)} \quad = \quad & \frac{1}{24}P_{(1)} + \frac{1}{24}\left(P_{(12)} + P_{(13)} + P_{(14)} + P_{(23)} + P_{(24)} + P_{(34)}\right) \\
& + \frac{1}{24}\left(P_{(12)(34)} + P_{(13)(24)} + P_{(14)(23)}\right) \\
& + \frac{1}{24}\left(P_{(123)} + P_{(132)} + P_{(124)} + P_{(142)} + P_{(134)} + P_{(143)} + P_{(234)} + P_{(243)}\right) \\
& + \frac{1}{24}\left(P_{(1234)} + P_{(1243)} + P_{(1324)} + P_{(1342)} + P_{(1423)} + P_{(1432)}\right),
\end{aligned}
\tag{7.31}
$$

$$
\begin{aligned}
C_{(3,1)} \quad = \quad & \frac{3}{8}P_{(1)} + \frac{1}{8}\left(P_{(12)} + P_{(13)} + P_{(14)} + P_{(23)} + P_{(24)} + P_{(34)}\right) \\
& - \frac{1}{8}\left(P_{(12)(34)} + P_{(13)(24)} + P_{(14)(23)}\right) \\
& - \frac{1}{8}\left(P_{(1234)} + P_{(1243)} + P_{(1324)} + P_{(1342)} + P_{(1423)} + P_{(1432)}\right),
\end{aligned}
\tag{7.32}
$$

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$$C_{(2,2)} = \frac{1}{6}P_{(1)} + \frac{1}{6}\left(P_{(12)(34)} + P_{(13)(24)} + P_{(14)(23)}\right)$$
$$-\frac{1}{12}\left(P_{(123)} + P_{(132)} + P_{(124)} + P_{(142)} + P_{(134)} + P_{(143)} + P_{(234)} + P_{(243)}\right), \tag{7.33}$$

$$C_{(2,1,1)} = \frac{3}{8}P_{(1)} - \frac{1}{8}\left(P_{(12)} + P_{(13)} + P_{(14)} + P_{(23)} + P_{(24)} + P_{(34)}\right)$$
$$-\frac{1}{8}\left(P_{(12)(34)} + P_{(13)(24)} + P_{(14)(23)}\right)$$
$$+\frac{1}{8}\left(P_{(1234)} + P_{(1243)} + P_{(1324)} + P_{(1342)} + P_{(1423)} + P_{(1432)}\right), \tag{7.34}$$

$$C_{(1,1,1,1)} = \frac{1}{24}P_{(1)} - \frac{1}{24}\left(P_{(12)} + P_{(13)} + P_{(14)} + P_{(23)} + P_{(24)} + P_{(34)}\right)$$
$$+\frac{1}{24}\left(P_{(12)(34)} + P_{(13)(24)} + P_{(14)(23)}\right)$$
$$+\frac{1}{24}\left(P_{(123)} + P_{(132)} + P_{(124)} + P_{(142)} + P_{(134)} + P_{(143)} + P_{(234)} + P_{(243)}\right)$$
$$-\frac{1}{24}\left(P_{(1234)} + P_{(1243)} + P_{(1324)} + P_{(1342)} + P_{(1423)} + P_{(1432)}\right), \tag{7.35}$$

$$\mathscr{E}_4(\rho) = \Delta_4^{(4)}C_{(4)} + \Delta_4^{(3,1)}C_{(3,1)} + \Delta_4^{(2,2)}C_{(2,2)} + \Delta_4^{(2,1,1)}C_{(2,1,1)} + \Delta_4^{(1,1,1,1)}C_{(1,1,1,1)}, \tag{7.36}$$

where

$$\Delta_4^{(4)} := \frac{1 + 6\,\mathrm{Tr}\left(\Lambda^2\right) + 3\,\mathrm{Tr}\left(\Lambda^2\right)^2 + 8\,\mathrm{Tr}\left(\Lambda^3\right) + 6\,\mathrm{Tr}\left(\Lambda^4\right)}{d(d+1)(d+2)(d+3)}, \tag{7.37}$$

$$\Delta_4^{(3,1)} := \frac{1 + 2\,\mathrm{Tr}\left(\Lambda^2\right) - \mathrm{Tr}\left(\Lambda^2\right)^2 - 2\,\mathrm{Tr}\left(\Lambda^4\right)}{(d-1)d(d+1)(d+2)}, \tag{7.38}$$

$$\Delta_4^{(2,2)} := \frac{1 + 3\,\mathrm{Tr}\left(\Lambda^2\right)^2 - 4\,\mathrm{Tr}\left(\Lambda^3\right)}{(d-1)d^2(d+1)}, \tag{7.39}$$

$$\Delta_4^{(2,1,1)} := \frac{1 - 2\,\mathrm{Tr}\left(\Lambda^2\right) - \mathrm{Tr}\left(\Lambda^2\right)^2 + 2\,\mathrm{Tr}\left(\Lambda^4\right)}{(d-2)(d-1)d(d+1)}, \tag{7.40}$$

$$\Delta_4^{(1,1,1,1)} := \frac{1 - 6\,\mathrm{Tr}\left(\Lambda^2\right) + 3\,\mathrm{Tr}\left(\Lambda^2\right)^2 + 8\,\mathrm{Tr}\left(\Lambda^3\right) - 6\,\mathrm{Tr}\left(\Lambda^4\right)}{(d-3)(d-2)(d-1)d}. \tag{7.41}$$

7.3.1 The $(k,d) = (2,2)$ case

We have

$$\Delta_2^{(2)} = \frac{1 + t_2}{6}, \quad \Delta_2^{(1,1)} = \frac{1 - t_2}{2}. \tag{7.42}$$

7.3.2 The $(k,d) = (3,2)$ case

We have

$$\Delta_3^{(3)} = \frac{1 + 3t_2 + 2t_3}{24}, \quad \Delta_3^{(2,1)} := \frac{1 - t_3}{6}, \quad \Delta_3^{(1,1,1)} = 0. \tag{7.43}$$

7.3.3 The $(k,d) = (4,2)$ case

We have

$$\Delta_4^{(4)} = \frac{1 + 6t_2 + 3t_2^2 + 8t_3 + 6t_4}{120}, \tag{7.44}$$

$$\Delta_4^{(3,1)} = \frac{1 + 2t_2 - t_2^2 - 2t_4}{24}, \tag{7.45}$$

$$\Delta_4^{(2,2)} = \frac{1 + 3t_2^2 - 4t_3}{12}, \tag{7.46}$$

$$\Delta_4^{(1,1,1,1)} = 0. \tag{7.47}$$

7.4 The moment of $\mathrm{Tr}\left(\rho^k\right) = t_k$

In fact, we have already known that

Proposition 7.4 ([20]). *We have:*

$$\langle t_2 \rangle = \int d\mu_{\mathrm{HS}}(\rho)\,\mathrm{Tr}\left(\rho^2\right) = \frac{2d}{d^2 + 1}, \tag{7.48}$$

$$\langle t_3 \rangle = \int d\mu_{\mathrm{HS}}(\rho)\,\mathrm{Tr}\left(\rho^3\right) = \frac{5d^2 + 1}{(d^2 + 1)(d^2 + 2)}, \tag{7.49}$$

$$\langle t_4 \rangle = \int d\mu_{\mathrm{HS}}(\rho)\,\mathrm{Tr}\left(\rho^4\right) = \frac{14d^3 + 10d}{(d^2 + 1)(d^2 + 2)(d^2 + 3)}. \tag{7.50}$$

Remark 7.5. It is obvious that

$$\langle t_2 \rangle = \int d\nu(\Lambda)\,\mathrm{Tr}\left(\Lambda^2\right) = \frac{2d}{d^2 + 1}, \tag{7.51}$$

$$\langle t_3 \rangle = \int d\nu(\Lambda)\,\mathrm{Tr}\left(\Lambda^3\right) = \frac{5d^2 + 1}{(d^2 + 1)(d^2 + 2)}, \tag{7.52}$$

$$\langle t_4 \rangle = \int d\nu(\Lambda)\,\mathrm{Tr}\left(\Lambda^4\right) = \frac{14d^3 + 10d}{(d^2 + 1)(d^2 + 2)(d^2 + 3)}. \tag{7.53}$$

Lemma 7.6. *It holds that*

$$\left\langle t_2^2 \right\rangle = \int d\mu_{\mathrm{HS}}(\rho)\left[\mathrm{Tr}\left(\rho^2\right)\right]^2 = \frac{4d^4 + 18d^2 + 2}{(d^2 + 1)(d^2 + 2)(d^2 + 3)}. \tag{7.54}$$

Proof. In what follows, we calculate the integral:

$$\int d\mu_{\mathrm{HS}}(\rho)\left[\mathrm{Tr}\left(\rho^2\right)\right]^2 = \int d\nu(\Lambda)\,\mathrm{Tr}\left(\Lambda^2\right)^2 = \int d\nu(\Lambda)\left(\mathrm{Tr}\left(\Lambda^4\right) + 2\sum_{i<j}\lambda_i^2\lambda_j^2\right)$$

$$= \int d\nu(\Lambda)\,\mathrm{Tr}\left(\Lambda^4\right) + 2C_{\mathrm{HS}}^d\int\left(\sum_{i<j}\lambda_i^2\lambda_j^2\right)\delta\left(1 - \sum_{j=1}^d\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^d d\lambda_j$$

$$= \int d\nu(\Lambda)\,\mathrm{Tr}\left(\Lambda^4\right) + 2C_{\mathrm{HS}}^d\binom{d}{2}\int\left(\lambda_1^2\lambda_2^2\right)\delta\left(1 - \sum_{j=1}^d\lambda_j\right)|\Delta(\lambda)|^2\prod_{j=1}^d d\lambda_j, \tag{7.55}$$

where C_{HS}^d is the normalization constant:

$$C_{\mathrm{HS}}^d = \frac{\Gamma\left(d^2\right)}{\Gamma(d+1)\prod_{j=1}^d\Gamma(j)^2}. \tag{7.56}$$

Next, we calculate the following integral:

$$\int \left(\lambda_1^2 \lambda_2^2 \right) \delta \left(1 - \sum_{j=1}^d \lambda_j \right) |\Delta(\lambda)|^2 \prod_{j=1}^d d\lambda_j. \tag{7.57}$$

Let

$$F(t) = \int \left(\lambda_1^2 \lambda_2^2 \right) \delta \left(t - \sum_{j=1}^d \lambda_j \right) |\Delta(\lambda)|^2 \prod_{j=1}^d d\lambda_j. \tag{7.58}$$

Performing Laplace transform $(t \to s)$ of $F(t)$ gives rise to

$$\begin{aligned}
\widetilde{F}(s) &= \int_0^\infty \left(\lambda_1^2 \lambda_2^2 \right) \exp\left(-s \sum_{j=1}^d \lambda_j \right) |\Delta(\lambda)|^2 \prod_{j=1}^d d\lambda_j \\
&= s^{-(d^2+4)} \int_0^\infty \left(\mu_1^2 \mu_2^2 \right) \exp\left(-\sum_{j=1}^d \mu_j \right) |\Delta(\mu)|^2 \prod_{j=1}^d d\mu_j.
\end{aligned} \tag{7.59}$$

Using the inverse Laplace transform result $(s \to t)$: $\mathscr{L}^{-1}(s^\alpha) = \frac{t^{-\alpha-1}}{\Gamma(-\alpha)}$, it follows that

$$F(t) = \frac{1}{\Gamma(d^2+4)} t^{d^2+3} \int_0^\infty \left(\mu_1^2 \mu_2^2 \right) \exp\left(-\sum_{j=1}^d \mu_j \right) |\Delta(\mu)|^2 \prod_{j=1}^d d\mu_j. \tag{7.60}$$

Then

$$\int \left(\lambda_1^2 \lambda_2^2 \right) \delta \left(1 - \sum_{j=1}^d \lambda_j \right) |\Delta(\lambda)|^2 \prod_{j=1}^d d\lambda_j = \frac{1}{\Gamma(d^2+4)} \int_0^\infty \left(\mu_1^2 \mu_2^2 \right) \exp\left(-\sum_{j=1}^d \mu_j \right) |\Delta(\mu)|^2 \prod_{j=1}^d d\mu_j. \tag{7.61}$$

Denote

$$\langle f(\mu) \rangle_q = \frac{\int_0^\infty \cdots \int_0^\infty f(\mu) q(\mu) d\mu}{\int_0^\infty \cdots \int_0^\infty q(\mu) d\mu}, \tag{7.62}$$

where

$$q(\mu) \equiv q(\mu_1, \ldots, \mu_d) = |\Delta(\mu)|^{2\gamma} \prod_{j=1}^d \mu_j^{\alpha-1} e^{-\mu_j} d\mu_j. \tag{7.63}$$

From Mehta's book [30, Eq. (17.8.3), pp. 324], we see that

$$\langle \mu_1^2 \cdots \mu_k^2 \mu_{k+1} \cdots \mu_m \rangle_q = \prod_{j=1}^k (\alpha + 1 + \gamma(2d - m - j)) \prod_{j=1}^m (\alpha + \gamma(d - j)). \tag{7.64}$$

Letting $k = m = 2$ and $(\alpha, \gamma) = (1, 1)$ in the above equation, we obtain that

$$\langle \mu_1^2 \mu_2^2 \rangle_q = \prod_{j=1}^2 (2d - j) \cdot \prod_{j=1}^2 (d - j + 1) = 2d(d-1)^2(2d-1). \tag{7.65}$$

This implies that

$$\int_0^\infty \left(\mu_1^2 \mu_2^2 \right) \exp\left(-\sum_{j=1}^d \mu_j \right) |\Delta(\mu)|^2 \prod_{j=1}^d d\mu_j = 2d(d-1)^2(2d-1) \int_0^\infty \exp\left(-\sum_{j=1}^d \mu_j \right) |\Delta(\mu)|^2 \prod_{j=1}^d d\mu_j. \tag{7.66}$$

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Therefore

$$\int \left(\lambda_1^2 \lambda_2^2\right) \delta \left(1 - \sum_{j=1}^{d} \lambda_j\right) |\Delta(\lambda)|^2 \prod_{j=1}^{d} d\lambda_j = \frac{2d(d-1)^2(2d-1)}{\Gamma(d^2+4)} \int_0^{\infty} \exp\left(-\sum_{j=1}^{d} \mu_j\right) |\Delta(\mu)|^2 \prod_{j=1}^{d} d\mu_j. \quad (7.67)$$

Since

$$\int_0^{\infty} \cdots \int_0^{\infty} q(\mu) d\mu = \prod_{j=1}^{d} \Gamma(j)\Gamma(j+1) = \Gamma(d+1) \prod_{j=1}^{d} \Gamma(j)^2. \quad (7.68)$$

Finally we get

$$\int \left(\lambda_1^2 \lambda_2^2\right) \delta \left(1 - \sum_{j=1}^{d} \lambda_j\right) |\Delta(\lambda)|^2 \prod_{j=1}^{d} d\lambda_j = 2d(d-1)^2(2d-1) \frac{\Gamma(d+1)}{\Gamma(d^2+4)} \prod_{j=1}^{d} \Gamma(j)^2. \quad (7.69)$$

Based on this computation, we finally obtain that

$$\begin{aligned}
\int d\mu_{\mathrm{HS}}(\rho) \left[\mathrm{Tr}\left(\rho^2\right)\right]^2 &= \frac{14d^3 + 10d}{(d^2+1)(d^2+2)(d^2+3)} \\
&\quad + 2\frac{\Gamma\left(d^2\right)}{\Gamma(d+1)\prod_{j=1}^{d}\Gamma(j)^2}\binom{d}{2} 2d(d-1)^2(2d-1)\frac{\Gamma(d+1)}{\Gamma(d^2+4)}\prod_{j=1}^{d}\Gamma(j)^2 \\
&= \frac{14d^3 + 10d}{(d^2+1)(d^2+2)(d^2+3)} + \frac{2(d-1)^3(2d-1)}{(d^2+1)(d^2+2)(d^2+3)}. \quad (7.70)
\end{aligned}$$

Therefore we completes the proof. □

7.5 The proof of Theorem 4.1

For the first term in the left hands (lhs) of the above equation:

$$\begin{aligned}
\mathrm{Tr}\left(\left[A^2 \otimes B^2\right]\mathscr{E}_2(\Lambda)\right) &= \frac{\Delta_2^{(2)}}{2}\left[\mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) + \mathrm{Tr}\left(A^2 B^2\right)\right] \\
&\quad + \frac{\Delta_2^{(1,1)}}{2}\left[\mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) - \mathrm{Tr}\left(A^2 B^2\right)\right]. \quad (7.71)
\end{aligned}$$

Then for the third and fourth terms:

$$\begin{aligned}
&\mathrm{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\mathscr{E}_3(\Lambda)\right) \\
&= \frac{\Delta_3^{(3)}}{6}\left[\mathrm{Tr}\left(A^2\right)\mathrm{Tr}(B)^2 + 2\,\mathrm{Tr}\left(A^2 B\right)\mathrm{Tr}(B) + \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) + 2\,\mathrm{Tr}\left(A^2 B^2\right)\right] \\
&\quad + \frac{2\Delta_3^{(2,1)}}{3}\left[\mathrm{Tr}\left(A^2\right)\mathrm{Tr}(B)^2 - \mathrm{Tr}\left(A^2 B^2\right)\right] \\
&\quad + \frac{\Delta_3^{(1,1,1)}}{6}\left[\mathrm{Tr}\left(A^2\right)\mathrm{Tr}(B)^2 - 2\,\mathrm{Tr}\left(A^2 B\right)\mathrm{Tr}(B) - \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) + 2\,\mathrm{Tr}\left(A^2 B^2\right)\right] \quad (7.72)
\end{aligned}$$

and

$$\mathrm{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right] \mathscr{E}_3(\Lambda)\right)$$

$$= \frac{\Delta_3^{(3)}}{6}\left[\mathrm{Tr}\left(B^2\right)\mathrm{Tr}\left(A\right)^2 + 2\,\mathrm{Tr}\left(B^2 A\right)\mathrm{Tr}\left(A\right) + \mathrm{Tr}\left(B^2\right)\mathrm{Tr}\left(A^2\right) + 2\,\mathrm{Tr}\left(B^2 A^2\right)\right]$$

$$+ \frac{2\Delta_3^{(2,1)}}{3}\left[\mathrm{Tr}\left(B^2\right)\mathrm{Tr}\left(A\right)^2 - \mathrm{Tr}\left(B^2 A^2\right)\right]$$

$$+ \frac{\Delta_3^{(1,1,1)}}{6}\left[\mathrm{Tr}\left(B^2\right)\mathrm{Tr}\left(A\right)^2 - 2\,\mathrm{Tr}\left(B^2 A\right)\mathrm{Tr}\left(A\right) - \mathrm{Tr}\left(B^2\right)\mathrm{Tr}\left(A^2\right) + 2\,\mathrm{Tr}\left(B^2 A^2\right)\right]. \quad (7.73)$$

The second term is:

$$\mathrm{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right] \mathscr{E}_4(\Lambda)\right)$$

$$= \frac{\Delta_4^{(4)}}{24}\left[\mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B\right)^2 + \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B\right)^2 + 4\,\mathrm{Tr}\left(AB\right)\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right) + \mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B^2\right) + \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right)\right.$$

$$\left. + 2\,\mathrm{Tr}\left(AB\right)^2 + 4\,\mathrm{Tr}\left(A^2 B\right)\mathrm{Tr}\left(B\right) + 4\,\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(AB^2\right) + 4\,\mathrm{Tr}\left(A^2 B^2\right) + 2\,\mathrm{Tr}\left(ABAB\right)\right]$$

$$+ \frac{\Delta_4^{(3,1)}}{8}\left[3\,\mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B\right)^2 + \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B\right)^2 + 4\,\mathrm{Tr}\left(AB\right)\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right) + \mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B^2\right)\right.$$

$$\left. - \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) - 2\,\mathrm{Tr}\left(AB\right)^2 - 4\,\mathrm{Tr}\left(A^2 B^2\right) - 2\,\mathrm{Tr}\left(ABAB\right)\right]$$

$$+ \frac{\Delta_4^{(2,2)}}{12}\left[2\,\mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B\right)^2 + 2\,\mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) + 4\,\mathrm{Tr}\left(AB\right)^2 - 4\,\mathrm{Tr}\left(A^2 B\right)\mathrm{Tr}\left(B\right) - 4\,\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(AB^2\right)\right]$$

$$+ \frac{\Delta_4^{(2,1,1)}}{8}\left[3\,\mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B\right)^2 - \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B\right)^2 - 4\,\mathrm{Tr}\left(AB\right)\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right) - \mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B^2\right)\right.$$

$$\left. - \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) - 2\,\mathrm{Tr}\left(AB\right)^2 + 4\,\mathrm{Tr}\left(A^2 B^2\right) + 2\,\mathrm{Tr}\left(ABAB\right)\right]$$

$$+ \frac{\Delta_4^{(1,1,1,1)}}{24}\left[\mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B\right)^2 - \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B\right)^2 - 4\,\mathrm{Tr}\left(AB\right)\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(B\right) - \mathrm{Tr}\left(A\right)^2\mathrm{Tr}\left(B^2\right) + \mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right)\right.$$

$$\left. + 2\,\mathrm{Tr}\left(AB\right)^2 + 4\,\mathrm{Tr}\left(A^2 B\right)\mathrm{Tr}\left(B\right) + 4\,\mathrm{Tr}\left(A\right)\mathrm{Tr}\left(AB^2\right) - 4\,\mathrm{Tr}\left(A^2 B^2\right) - 2\,\mathrm{Tr}\left(ABAB\right)\right]. \quad (7.74)$$

Therefore, we get the conclusion.

7.6 The proof of Theorem 4.4

Clearly, for $k = 2, 3, 4$, we know that

$$\int \mathrm{d}\mu(\psi)|\psi\rangle\langle\psi|^{\otimes k} = \Delta_k^{(k)} C_{(k)}. \quad (7.75)$$

Then

$$\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 \mathrm{d}\mu(\psi)$$

$$= \int \mathrm{d}\mu(\psi)\,\mathrm{Tr}\left(\left[A^2 \otimes B^2\right]\psi^{\otimes 2}\right) + \int \mathrm{d}\mu(\psi)\,\mathrm{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right]\psi^{\otimes 4}\right)$$

$$- \int \mathrm{d}\mu(\psi)\,\mathrm{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\psi^{\otimes 3}\right) - \int \mathrm{d}\mu(\psi)\,\mathrm{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right]\psi^{\otimes 3}\right), \quad (7.76)$$

Now,

$$\int \mathrm{Tr}\left(\left[A^2 \otimes B^2\right]\psi^{\otimes 2}\right)\mathrm{d}\mu(\psi) = \frac{1}{d(d+1)}\left[\mathrm{Tr}\left(A^2\right)\mathrm{Tr}\left(B^2\right) + \mathrm{Tr}\left(A^2 B^2\right)\right] \quad (7.77)$$

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$$\int \text{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right] \psi^{\otimes 4}\right) d\mu(\psi) = \frac{1}{d(d+1)(d+2)(d+3)}\Omega(A,B), \tag{7.78}$$

where

$$
\begin{aligned}
\Omega(A,B) &= \text{Tr}(A)^2 \text{Tr}(B)^2 + \text{Tr}\left(A^2\right)\text{Tr}(B)^2 + 4\,\text{Tr}(AB)\,\text{Tr}(A)\,\text{Tr}(B) + \text{Tr}(A)^2 \text{Tr}\left(B^2\right) \\
&\quad + \text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 2\,\text{Tr}(AB)^2 + 4\,\text{Tr}\left(A^2 B\right)\text{Tr}(B) + 4\,\text{Tr}(A)\,\text{Tr}\left(AB^2\right) \\
&\quad + 4\,\text{Tr}\left(A^2 B^2\right) + 2\,\text{Tr}(ABAB).
\end{aligned} \tag{7.79}
$$

Moreover

$$
\begin{aligned}
&\int \text{Tr}\left(\left[A^2 \otimes B^{\otimes 2}\right]\psi^{\otimes 3}\right) d\mu(\psi) \\
&= \frac{1}{d(d+1)(d+2)}\left[\text{Tr}\left(A^2\right)\text{Tr}(B)^2 + 2\,\text{Tr}\left(A^2 B\right)\text{Tr}(B) + \text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 2\,\text{Tr}\left(A^2 B^2\right)\right]
\end{aligned} \tag{7.80}
$$

and

$$
\begin{aligned}
&\int \text{Tr}\left(\left[B^2 \otimes A^{\otimes 2}\right]\psi^{\otimes 3}\right) d\mu(\psi) \\
&= \frac{1}{d(d+1)(d+2)}\left[\text{Tr}\left(B^2\right)\text{Tr}(A)^2 + 2\,\text{Tr}\left(B^2 A\right)\text{Tr}(A) + \text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 2\,\text{Tr}\left(A^2 B^2\right)\right]
\end{aligned} \tag{7.81}
$$

Thus

$$
\begin{aligned}
&d(d+1)(d+2)(d+3)\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 d\mu(\psi) \\
&= \Omega(A,B) + (d+2)(d+3)\left[\text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + \text{Tr}\left(A^2 B^2\right)\right] \\
&\quad - (d+3)\left[\text{Tr}\left(A^2\right)\text{Tr}(B)^2 + 2\,\text{Tr}\left(A^2 B\right)\text{Tr}(B) + \text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 2\,\text{Tr}\left(A^2 B^2\right)\right] \\
&\quad - (d+3)\left[\text{Tr}\left(B^2\right)\text{Tr}(A)^2 + 2\,\text{Tr}\left(B^2 A\right)\text{Tr}(A) + \text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 2\,\text{Tr}\left(A^2 B^2\right)\right]
\end{aligned} \tag{7.82}
$$

$$
\begin{aligned}
&d(d+1)(d+2)(d+3)\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 d\mu(\psi) \\
&= \text{Tr}(A)^2 \text{Tr}(B)^2 - (d+2)\,\text{Tr}\left(A^2\right)\text{Tr}(B)^2 + 4\,\text{Tr}(AB)\,\text{Tr}(A)\,\text{Tr}(B) \\
&\quad - (d+2)\,\text{Tr}(A)^2 \text{Tr}\left(B^2\right) + (d^2 + 3d + 1)\,\text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + 2\,\text{Tr}(AB)^2 \\
&\quad - 2(d+1)\,\text{Tr}\left(A^2 B\right)\text{Tr}(B) - 2(d+1)\,\text{Tr}(A)\,\text{Tr}\left(AB^2\right) \\
&\quad + (d^2 + d - 2)\,\text{Tr}\left(A^2 B^2\right) + 2\,\text{Tr}(ABAB).
\end{aligned} \tag{7.83}
$$

Therefore

$$
\begin{aligned}
&d(d+1)(d+2)(d+3)\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 d\mu(\psi) \\
&= \text{Tr}(A)^2 \text{Tr}(B)^2 + 4\,\text{Tr}(AB)\,\text{Tr}(A)\,\text{Tr}(B) + 2\,\text{Tr}(AB)^2 + 2\,\text{Tr}(ABAB) \\
&\quad - (d+2)\left[\text{Tr}\left(A^2\right)\text{Tr}(B)^2 + \text{Tr}(A)^2 \text{Tr}\left(B^2\right)\right] \\
&\quad - 2(d+1)\left[\text{Tr}\left(A^2 B\right)\text{Tr}(B) + \text{Tr}(A)\,\text{Tr}\left(AB^2\right)\right] \\
&\quad + (d^2 + 3d + 1)\,\text{Tr}\left(A^2\right)\text{Tr}\left(B^2\right) + (d^2 + d - 2)\,\text{Tr}\left(A^2 B^2\right),
\end{aligned} \tag{7.84}
$$

that is,

$$\int \Delta A(\psi)^2 \cdot \Delta B(\psi)^2 \mathrm{d}\mu(\psi) = \sum_{j=1}^{8} u_j \Omega_j(A, B), \tag{7.85}$$

where $\Omega_j(A, B)$ is from Theorem 4.1, and for $K_d = (d(d+1)(d+2)(d+3))^{-1}$,

$$u_1 = K_d, \quad u_2 = -(d+2)K_d, \quad u_3 = 4K_d, \quad u_4 = (d^2+3d+1)K_d, \tag{7.86}$$

$$u_5 = 2K_d, \quad u_6 = -2(d+1)K_d, \quad u_7 = (d^2+d-2)K_d, \quad u_8 = 2K_d. \tag{7.87}$$

In the following we calculate the average lower bound,

$$\begin{aligned}
&\int \mathrm{d}\mu(\psi) \left(\langle \{A, B\} \rangle_\psi - \langle A \rangle_\psi \langle B \rangle_\psi \right)^2 = \int \mathrm{d}\mu(\psi) \left[\langle \{A, B\} \rangle_\psi^2 + \langle A \rangle_\psi^2 \langle B \rangle_\psi^2 - 2\langle \{A, B\} \rangle_\psi \langle A \rangle_\psi \langle B \rangle_\psi \right] \\
&= \frac{1}{d(d+1)} \left[\mathrm{Tr}\,(\{A, B\})^2 + \mathrm{Tr}\left(\{A, B\}^2\right) \right] + \frac{1}{d(d+1)(d+2)(d+3)} \Omega(A, B) \\
&\quad - \frac{2}{d(d+1)(d+2)} [\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(\{A, B\}) + \mathrm{Tr}\,(AB)\,\mathrm{Tr}\,(\{A, B\}) + \mathrm{Tr}\,(A\{A, B\})\,\mathrm{Tr}\,(B) \\
&\quad + \mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B\{A, B\}) + \mathrm{Tr}\,(AB\{A, B\}) + \mathrm{Tr}\,(A\{A, B\}B)] \\
&= \frac{1}{d(d+1)} \left[\mathrm{Tr}\,(AB)^2 + \frac{1}{2}\mathrm{Tr}\,(ABAB) + \frac{1}{2}\mathrm{Tr}\left(A^2B^2\right) \right] + \frac{1}{d(d+1)(d+2)(d+3)} \Omega(A, B) \\
&\quad - \frac{2}{d(d+1)(d+2)} [\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B)\,\mathrm{Tr}\,(AB) + \mathrm{Tr}\,(AB)^2 + \mathrm{Tr}\left(A^2B\right)\,\mathrm{Tr}\,(B) \\
&\quad + \mathrm{Tr}\left(B^2A\right)\,\mathrm{Tr}\,(A) + 2\,\mathrm{Tr}\,(ABAB) + 2\,\mathrm{Tr}\left(A^2B^2\right)]
\end{aligned} \tag{7.88}$$

and

$$\begin{aligned}
\int \langle [A, B] \rangle_\psi^2 \mathrm{d}\mu(\psi) &= \int \mathrm{d}\mu(\psi) \left[\langle \psi | AB | \psi \rangle^2 + \langle \psi | BA | \psi \rangle^2 - 2 \langle \psi | AB | \psi \rangle \langle \psi | BA | \psi \rangle \right] \\
&= \frac{1}{2d(d+1)} \left[\mathrm{Tr}\left(A^2B^2\right) - \mathrm{Tr}\,(ABAB) \right].
\end{aligned} \tag{7.89}$$

Therefore we have that

$$\int \mathrm{d}\mu(\psi) \left[\left(\langle \{A, B\} \rangle_\psi - \langle A \rangle_\psi \langle B \rangle_\psi \right)^2 + \langle [A, B] \rangle_\psi^2 \right] = \sum_{j=1}^{8} l_j \Omega_j(A, B), \tag{7.90}$$

where

$$l_1 = K_d, \quad l_2 = K_d, \quad l_3 = -2(d+1)K_d, \quad l_4 = K_d, \quad l_5 = (d+1)(d+2)K_d, \tag{7.91}$$

$$l_6 = -2(d+1)K_d, \quad l_7 = (d^2+d-2)K_d, \quad l_8 = -2(2d+5)K_d. \tag{7.92}$$

7.7 The proof of Theorem 4.10

Since the first term in the rhs of (1.3) can be rewritten as

$$\begin{aligned}
\left(\langle \{A, B\} \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho \right)^2 &= \langle \{A, B\} \rangle_\rho^2 + \langle A \rangle_\rho^2 \langle B \rangle_\rho^2 - 2\langle \{A, B\} \rangle_\rho \langle A \rangle_\rho \langle B \rangle_\rho \\
&= \mathrm{Tr}\left(\rho^{\otimes 2} \{A, B\}^{\otimes 2}\right) + \mathrm{Tr}\left(\rho^{\otimes 4} \left[A^{\otimes 2} \otimes B^{\otimes 2} \right]\right) - 2\,\mathrm{Tr}\left(\rho^{\otimes 3} \left[\{A, B\} \otimes A \otimes B \right]\right),
\end{aligned} \tag{7.93}$$

it follows that

$$\int_{\mathrm{D}(\mathcal{H}_d)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\langle\{A,B\}\rangle_\rho^2 = \int \mathrm{d}\nu(\Lambda)\,\mathrm{Tr}\left(\mathscr{E}_2(\Lambda)\{A,B\}^{\otimes 2}\right)$$

$$= \frac{1}{d^2+1}\mathrm{Tr}\,(AB)^2 + \frac{1}{2d(d^2+1)}\left[\mathrm{Tr}\left(A^2B^2\right) + \mathrm{Tr}\,(ABAB)\right] \quad (7.94)$$

and

$$\int_{\mathrm{D}(\mathcal{H}_d)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\langle A\rangle_\rho^2\langle B\rangle_\rho^2 = \int \mathrm{d}\nu(\Lambda)\,\mathrm{Tr}\left(\mathscr{E}_4(\Lambda)\left[A^{\otimes 2}\otimes B^{\otimes 2}\right]\right)$$
$$= \alpha_1\Omega_1(A,B) + \alpha_2\left(\Omega_2(A,B) + 4\Omega_3(A,B)\right) + \alpha_3\left(\Omega_4(A,B) + 2\Omega_5(A,B)\right)$$
$$+ \alpha_4\Omega_6(A,B) + \alpha_5\left(2\Omega_7(A,B) + \Omega_8(A,B)\right), \quad (7.95)$$

where

$$\alpha_1 = N_d\left(d^4 - 18d^2 + 158 - \frac{50}{d^2+1} + \frac{792}{d^2+2} - \frac{1512}{d^2+3}\right) \quad (7.96)$$

$$\alpha_2 = N_d\left(d^3 - 20d + \frac{50d}{d^2+1} - \frac{396d}{d^2+2} + \frac{504d}{d^2+3}\right), \quad (7.97)$$

$$\alpha_3 = N_d\left(-2d^2 - 20 + \frac{50}{d^2+1} - \frac{396}{d^2+2} + \frac{504}{d^2+3}\right), \quad (7.98)$$

$$\alpha_4 = N_d\left(4d^2 - 380 + \frac{500}{d^2+1} - \frac{1584}{d^2+2} + \frac{2016}{d^2+3}\right), \quad (7.99)$$

$$\alpha_5 = N_d\left(2d - \frac{100d}{d^2+1} + \frac{396d}{d^2+2} - \frac{336d}{d^2+3}\right). \quad (7.100)$$

Moreover

$$\mathrm{Tr}\left(\mathscr{E}_3(\Lambda)\left[\{A,B\}\otimes A\otimes B\right]\right) = \frac{1}{6}(\Delta_3^{(3)} + 4\Delta_3^{(2,1)} + \Delta_3^{(1,1,1)})\left[\mathrm{Tr}\,(AB)\,\mathrm{Tr}\,(A)\,\mathrm{Tr}\,(B)\right]$$
$$+ \frac{1}{6}(\Delta_3^{(3)} - \Delta_3^{(1,1,1)})\left[\mathrm{Tr}\left(A^2B\right)\mathrm{Tr}\,(B) + \mathrm{Tr}\left(AB^2\right)\mathrm{Tr}\,(A) + (\mathrm{Tr}\,(AB))^2\right]$$
$$+ \frac{1}{6}(\Delta_3^{(3)} - 2\Delta_3^{(2,1)} + \Delta_3^{(1,1,1)})\left[\mathrm{Tr}\,(ABAB) + \mathrm{Tr}\left(A^2B^2\right)\right]. \quad (7.101)$$

Therefore

$$\int_{\mathrm{D}(\mathcal{H}_d)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\langle\{A,B\}\rangle_\rho\langle A\rangle_\rho\langle B\rangle_\rho = \int \mathrm{d}\nu(\Lambda)\,\mathrm{Tr}\left(\mathscr{E}_3(\Lambda)\left[\{A,B\}\otimes A\otimes B\right]\right)$$

$$= L_d\left(d^2 - 8 - \frac{10}{d^2+1} + \frac{36}{d^2+2}\right)\Omega_3(A,B)$$

$$+ L_d\left(1 + \frac{10}{d^2+1} - \frac{18}{d^2+2}\right)\left[\Omega_5(A,B) + \Omega_6(A,B)\right]$$

$$+ L_d\left(\frac{1}{2}d^4 - 2d^2 + 1 + \frac{10}{d^2+1} - \frac{18}{d^2+2}\right)\left[\Omega_7(A,B) + \Omega_8(A,B)\right] \quad (7.102)$$

where $L_d = d(d^2-1)(d^2-4)$.

Since

$$\int_{\mathrm{D}(\mathcal{H}_d)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\left(\langle\{A,B\}\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho\right)^2$$
$$= \int_{\mathrm{D}(\mathcal{H}_d)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\left(\langle\{A,B\}\rangle_\rho^2 - 2\langle\{A,B\}\rangle_\rho\langle A\rangle_\rho\langle B\rangle_\rho + \langle A\rangle_\rho^2\langle B\rangle_\rho^2\right), \quad (7.103)$$

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and by (7.94), (7.95) and (7.102), we obtain the equality (4.73).

Moreover,

$$
\begin{aligned}
\int_{D(\mathcal{H}_d)} d\mu_{\mathrm{HS}}(\rho)\langle [A,B]\rangle_\rho^2 &= \int d\nu(\Lambda)\,\mathrm{Tr}\left(\mathscr{E}_2(\Lambda)[A,B]^{\otimes 2}\right) \\
&= \frac{1}{2d(d^2+1)}\left[\mathrm{Tr}\left(A^2 B^2\right) - \mathrm{Tr}\left(ABAB\right)\right], \\
&= \frac{1}{2d(d^2+1)}\left[\Omega_7(A,B) - \Omega_8(A,B)\right]. \tag{7.104}
\end{aligned}
$$

so we get (4.82).

7.8 Two examples in lower dimensions

In this section, we will present two examples in lower dimensions. Note that the results obtained previously are live in the space of the dimension being larger than three, as examples, we will deal with the same problem in the 2-dimensional and 3-dimensional spaces, respectively.

Theorem 7.7. *For two observables A and B on $\mathbb{C}^2$, the average of uncertainty-product taken over the whole set of all density matrices $D\left(\mathbb{C}^2\right)$ is given by*

$$
\begin{aligned}
&\int \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 d\mu_{\mathrm{HS}}(\rho) \\
&= \frac{2}{105}\Omega_1 - \frac{2}{35}\Omega_2 + \frac{4}{105}\Omega_3 + \frac{29}{210}\Omega_4 + \frac{1}{105}\Omega_5 - \frac{1}{21}\Omega_6 + \frac{3}{70}\Omega_7 + \frac{1}{210}\Omega_8. \tag{7.105}
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&\int d\mu_{\mathrm{HS}}(\rho)\left[\left(\langle\{A,B\}\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho\right)^2 + \langle [A,B]\rangle_\rho^2\right] \\
&= \frac{2}{105}\Omega_1 + \frac{1}{105}\Omega_2 - \frac{2}{21}\Omega_3 + \frac{1}{210}\Omega_4 + \frac{1}{7}\Omega_5 - \frac{1}{21}\Omega_6 + \frac{8}{105}\Omega_7 - \frac{1}{35}\Omega_8. \tag{7.106}
\end{aligned}
$$

Proof. For $d=2$, we have

$$
\langle t_2\rangle_2 = \frac{4}{5}, \quad \langle t_3\rangle_2 = \frac{7}{10}, \quad \langle t_4\rangle_2 = \frac{22}{35}, \quad \langle t_2^2\rangle_2 = \frac{23}{35}. \tag{7.107}
$$

Hence,

$$
\langle \Delta_2^{(2)}\rangle_2 = \frac{3}{10}, \quad \langle \Delta_2^{(1,1)}\rangle_2 = \frac{1}{10}, \tag{7.108}
$$

$$
\langle \Delta_3^{(3)}\rangle_2 = \frac{1}{5}, \quad \langle \Delta_3^{(2,1)}\rangle_2 = \frac{1}{20}, \tag{7.109}
$$

$$
\langle \Delta_4^{(4)}\rangle_2 = \frac{1}{7}, \quad \langle \Delta_4^{(3,1)}\rangle_2 = \frac{1}{35}, \quad \langle \Delta_4^{(2,2)}\rangle_2 = \frac{1}{70}. \tag{7.110}
$$

Since

$$
\begin{aligned}
\int_{D(\mathcal{H}_2)} \mathrm{Tr}\left(\left[A^2\otimes B^2\right]\mathscr{E}_2(\Lambda)\right) d\nu(\Lambda) &= \frac{3}{20}(\Omega_4 + \Omega_7) + \frac{1}{20}(\Omega_4 - \Omega_7) \\
&= \frac{1}{5}\Omega_4 + \frac{1}{10}\Omega_7, \tag{7.111}
\end{aligned}
$$

it follows that

$$\int_{D(\mathcal{H}_2)} \text{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2}\right] \mathscr{E}_4(\Lambda)\right) d\nu(\Lambda)$$

$$= \frac{1}{7} \times \frac{1}{24} \left(\Omega_1 + \Omega_2 + 4\Omega_3 + \Omega_4 + 2\Omega_5 + 4\Omega_6 + 4\Omega_7 + 2\Omega_8\right)$$

$$+ \frac{1}{35} \times \frac{1}{8} \left(3\Omega_1 + \Omega_2 + 4\Omega_3 - \Omega_4 - 2\Omega_5 - 4\Omega_7 - 2\Omega_8\right)$$

$$+ \frac{1}{70} \times \frac{1}{12} \left(2\Omega_1 + 2\Omega_4 + 4\Omega_5 - 4\Omega_6\right)$$

$$= \frac{2}{105}\Omega_1 + \frac{1}{105}\Omega_2 + \frac{4}{105}\Omega_3 + \frac{1}{210}\Omega_4 + \frac{1}{105}\Omega_5 + \frac{2}{105}\Omega_6 + \frac{1}{105}\Omega_7 + \frac{1}{210}\Omega_8, \tag{7.112}$$

and

$$\int_{D(\mathcal{H}_2)} \text{Tr}\left(\left[A^2 \otimes B^{\otimes 2} + B^2 \otimes A^{\otimes 2}\right] \mathscr{E}_3(\Lambda)\right) d\nu(\Lambda)$$

$$= \frac{1}{5} \times \frac{1}{6} \left(\Omega_2 + 2\Omega_4 + 2\Omega_6 + 4\Omega_7\right) + \frac{1}{20} \times \frac{2}{3} \left(\Omega_2 - 2\Omega_7\right)$$

$$= \frac{1}{15}\Omega_2 + \frac{1}{15}\Omega_4 + \frac{1}{15}\Omega_6 + \frac{1}{15}\Omega_7, \tag{7.113}$$

then by (4.5), we get

$$\int_{D(\mathcal{H}_2)} \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 d\mu_{\text{HS}}(\rho)$$

$$= \frac{2}{105}\Omega_1 - \frac{2}{35}\Omega_2 + \frac{4}{105}\Omega_3 + \frac{29}{210}\Omega_4 + \frac{1}{105}\Omega_5 - \frac{1}{21}\Omega_6 + \frac{3}{70}\Omega_7 + \frac{1}{210}\Omega_8. \tag{7.114}$$

Since

$$\int_{D(\mathcal{H}_2)} \text{Tr}\left(\{A,B\}^{\otimes 2} \mathscr{E}_2(\Lambda)\right) d\nu(\Lambda) = \frac{1}{5}\Omega_5 + \frac{1}{20}\left(\Omega_7 + \Omega_8\right), \tag{7.115}$$

$$\int_{D(\mathcal{H}_2)} \text{Tr}\left(\left[\{A,B\} \otimes A \otimes B\right] \mathscr{E}_3(\Lambda)\right) d\nu(\Lambda) = \frac{1}{15}\Omega_3 + \frac{1}{30}\left(\Omega_5 + \Omega_6\right) + \frac{1}{60}\left(\Omega_7 + \Omega_8\right) \tag{7.116}$$

and

$$\int_{D(\mathcal{H}_2)} \text{Tr}\left([A,B]^{\otimes 2} \mathscr{E}_2(\Lambda)\right) d\nu(\Lambda) = \frac{1}{20}\left(\Omega_7 - \Omega_8\right), \tag{7.117}$$

then by (7.93), we get

$$\int_{D(\mathcal{H}_2)} d\mu_{\text{HS}}(\rho) \left[\left(\langle\{A,B\}\rangle_\rho - \langle A\rangle_\rho \langle B\rangle_\rho\right)^2 + \langle[A,B]\rangle_\rho^2\right]$$

$$= \frac{2}{105}\Omega_1 + \frac{1}{105}\Omega_2 - \frac{2}{21}\Omega_3 + \frac{1}{210}\Omega_4 + \frac{1}{7}\Omega_5 - \frac{1}{21}\Omega_6 + \frac{8}{105}\Omega_7 - \frac{1}{35}\Omega_8. \tag{7.118}$$

We are done. $\qquad\square$

Theorem 7.8. *For two observables A and B on $\mathbb{C}^3$, the average of uncertainty-product taken over the whole set of all density matrices $D\left(\mathbb{C}^3\right)$ is given by*

$$\int \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 d\mu_{\text{HS}}(\rho)$$

$$= \frac{3}{440}\Omega_1 - \frac{1}{40}\Omega_2 + \frac{1}{110}\Omega_3 + \frac{109}{1032}\Omega_4 + \frac{1}{660}\Omega_5 - \frac{1}{66}\Omega_6 + \frac{1}{45}\Omega_7 + \frac{1}{1980}\Omega_8. \tag{7.119}$$

Moreover, we have

$$\int d\mu_{\mathrm{HS}}(\rho) \left[\left(\langle \{A, B\} \rangle_\rho - \langle A \rangle_\rho \langle B \rangle_\rho \right)^2 + \langle [A, B] \rangle_\rho^2 \right]$$

$$= \frac{3}{440}\Omega_1 + \frac{1}{440}\Omega_2 - \frac{1}{22}\Omega_3 + \frac{1}{1320}\Omega_4 + \frac{1}{12}\Omega_5 - \frac{1}{66}\Omega_6 + \frac{14}{495}\Omega_7 - \frac{1}{180}\Omega_8. \qquad (7.120)$$

Proof. For $d = 3$, we have

$$\langle t_2 \rangle_3 = \frac{3}{5}, \quad \langle t_3 \rangle_3 = \frac{23}{55}, \quad \langle t_4 \rangle_3 = \frac{17}{55}, \quad \langle t_2^2 \rangle_3 = \frac{61}{165}. \qquad (7.121)$$

Hence,

$$\langle \Delta_2^{(2)} \rangle_3 = \frac{2}{15}, \ \langle \Delta_2^{(1,1)} \rangle_3 = \frac{1}{15}, \qquad (7.122)$$

$$\langle \Delta_3^{(3)} \rangle_3 = \frac{2}{33}, \ \langle \Delta_3^{(2,1)} \rangle_3 = \frac{4}{165}, \ \langle \Delta_3^{(1,1,1)} \rangle_3 = \frac{1}{165} \qquad (7.123)$$

$$\langle \Delta_4^{(4)} \rangle_3 = \frac{1}{33}, \ \langle \Delta_4^{(3,1)} \rangle_3 = \frac{1}{99}, \ \langle \Delta_4^{(2,2)} \rangle_3 = \frac{1}{165} \ \langle \Delta_4^{(2,1,1)} \rangle_3 = \frac{1}{495}. \qquad (7.124)$$

Since

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{Tr}\left(\left[A^2 \otimes B^2 \right] \mathscr{E}_2(\Lambda) \right) d\nu(\Lambda) \ = \ \frac{1}{15}(\Omega_4 + \Omega_7) + \frac{1}{30}(\Omega_4 - \Omega_7) \qquad (7.125)$$

$$= \ \frac{1}{10}\Omega_4 + \frac{1}{30}\Omega_7, \qquad (7.126)$$

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{Tr}\left(\left[A^{\otimes 2} \otimes B^{\otimes 2} \right] \mathscr{E}_4(\Lambda) \right) d\nu(\Lambda)$$

$$= \frac{1}{33} \times \frac{1}{24} \left(\Omega_1 + \Omega_2 + 4\Omega_3 + \Omega_4 + 2\Omega_5 + 4\Omega_6 + 4\Omega_7 + 2\Omega_8 \right)$$

$$+ \frac{1}{99} \times \frac{1}{8} \left(3\Omega_1 + \Omega_2 + 4\Omega_3 - \Omega_4 - 2\Omega_5 - 4\Omega_7 - 2\Omega_8 \right)$$

$$+ \frac{1}{165} \times \frac{1}{12} \left(2\Omega_1 + 2\Omega_4 + 4\Omega_5 - 4\Omega_6 \right)$$

$$+ \frac{1}{495} \times \frac{1}{8} \left(3\Omega_1 - \Omega_2 - 4\Omega_3 - \Omega_4 - 2\Omega_5 + 4\Omega_7 + 2\Omega_8 \right)$$

$$= \frac{3}{440}\Omega_1 + \frac{1}{440}\Omega_2 + \frac{1}{110}\Omega_3 + \frac{1}{1320}\Omega_4 + \frac{1}{660}\Omega_5 + \frac{1}{330}\Omega_6 + \frac{1}{990}\Omega_7 + \frac{1}{1980}\Omega_8 \qquad (7.127)$$

and

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{Tr}\left(\left[A^2 \otimes B^{\otimes 2} + B^2 \otimes A^{\otimes 2} \right] \mathscr{E}_3(\Lambda) \right) d\nu(\Lambda)$$

$$= \frac{2}{33} \times \frac{1}{6} \left(\Omega_2 + 2\Omega_4 + 2\Omega_6 + 4\Omega_7 \right) + \frac{4}{165} \times \frac{2}{3} \left(\Omega_2 - 2\Omega_7 \right)$$

$$+ \frac{1}{165} \times \frac{1}{6} \left(\Omega_2 - 2\Omega_4 - 2\Omega_6 + 4\Omega_7 \right)$$

$$= \frac{3}{110}\Omega_2 + \frac{1}{55}\Omega_4 + \frac{1}{55}\Omega_6 + \frac{2}{165}\Omega_7, \qquad (7.128)$$

then by (4.5), we get

$$\int_{\mathrm{D}(\mathcal{H}_3)} \Delta A(\rho)^2 \cdot \Delta B(\rho)^2 d\mu_{\mathrm{HS}}(\rho)$$

$$= \frac{3}{440}\Omega_1 - \frac{1}{40}\Omega_2 + \frac{1}{110}\Omega_3 + \frac{109}{1032}\Omega_4 + \frac{1}{660}\Omega_5 - \frac{1}{66}\Omega_6 + \frac{1}{45}\Omega_7 + \frac{1}{1980}\Omega_8. \qquad (7.129)$$

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Since

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{Tr}\left(\{A,B\}^{\otimes 2}\mathscr{E}_2(\Lambda)\right) \mathrm{d}\nu(\Lambda) = \frac{1}{10}\Omega_5 + \frac{1}{60}\left(\Omega_7 + \Omega_8\right), \tag{7.130}$$

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{Tr}\left([\{A,B\}\otimes A \otimes B]\,\mathscr{E}_3(\Lambda)\right) \mathrm{d}\nu(\Lambda) = \frac{3}{110}\Omega_3 + \frac{1}{110}\left(\Omega_5 + \Omega_6\right) + \frac{1}{330}\left(\Omega_7 + \Omega_8\right) \tag{7.131}$$

and

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{Tr}\left([A,B]^{\otimes 2}\mathscr{E}_2(\Lambda)\right) \mathrm{d}\nu(\Lambda) = \frac{1}{60}\left(\Omega_7 - \Omega_8\right), \tag{7.132}$$

then by (7.93), we get

$$\int_{\mathrm{D}(\mathcal{H}_3)} \mathrm{d}\mu_{\mathrm{HS}}(\rho)\left[\left(\langle\{A,B\}\rangle_\rho - \langle A\rangle_\rho\langle B\rangle_\rho\right)^2 + \langle[A,B]\rangle_\rho^2\right]$$

$$= \frac{3}{440}\Omega_1 + \frac{1}{440}\Omega_2 - \frac{1}{22}\Omega_3 + \frac{1}{1320}\Omega_4 + \frac{1}{12}\Omega_5 - \frac{1}{66}\Omega_6 + \frac{14}{495}\Omega_7 - \frac{1}{180}\Omega_8. \tag{7.133}$$

This completes the proof. $\qquad\qquad\square$

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