# Open Quantum Random Walks and Quantum Markov chains on Trees I: Phase transitions

### FARRUKH MUKHAMEDOV

Department of Mathematical Sciences, College of Science,
United Arab Emirates University 15551, Al-Ain,
United Arab Emirates and
Institute of Mathematics named after V.I.Romanovski, 4,
University str., 100125, Tashkent, Uzbekistan
e-mail: far75m@gmail.com; farrukh.m@uaeu.ac.ae

#### Abdessatar Souissi

Department of Accounting, College of Business Management
Qassim University, Ar Rass, Saudi Arabia and
Preparatory institute for scientific and technical studies,
Carthage University, Amilcar 1054, Tunisia
e-mail: a.souaissi@qu.edu.sa; abdessattar.souissi@ipest.rnu.tn

### TAREK HAMDI

Department of Management Information Systems, College of Business Management Qassim University, Ar Rass, Saudi Arabia and Laboratoire d'Analyse Mathématiques et applications LR11ES11 Université de Tunis El-Manar, Tunisia e-mail: t.hamdi@qu.edu.sa

#### Abstract

In the present paper, we construct QMC (Quantum Markov Chains) associated with Open Quantum Random Walks such that the transition operator of the chain is defined by OQRW and the restriction of QMC to the commutative subalgebra coincides with the distribution  $\mathbb{P}_{\rho}$  of OQRW. However, we are going to look at the probability distribution as a Markov field over the Cayley tree. Such kind of consideration allows us to investigated phase transition phenomena associated for OQRW within QMC scheme. Furthermore, we first propose a new construction of QMC on trees, which is an extension of QMC considered in Ref. [10]. Using such a construction, we are able to construct QMCs on tress associated with OQRW. Our investigation leads to the detection of the phase transition phenomena within the proposed scheme. This kind of phenomena appears first time in this direction. Moreover, mean entropies of QMCs are calculated.

Mathematics Subject Classification: 46L53, 46L60, 82B10, 81Q10.

Key words: Open quantum random walks; Quantum Markov chain; Cayley tree; disordered phase; phase transition

### 1. Introduction

Discovering the aspects of quantum mechanics, such as superposition and interference, has lead to the idea of quantum walks; a generalization of classical random walks [22, 31, 32, 40]. Recently, in [20] a quantum phase transition has been explored by means of quantum walks in an optical lattice. On the other hand, in [33] it has been shown that discrete-time quantum walks (QW) can realise topological phases in 1D and 2D for all the symmetry classes of free-fermion systems. In particular,

they provide the QW protocols that simulate representatives of all topological phases, featured by the presence of robust symmetry- protected edge states [34]. In general, QW realisations are particularly useful, because, in addition to the simplicity of their mathematical description, the parameters that define them can be easily controlled in the lab.

Over the past decade, motivated largely by the prospect of superefficient algorithms, the theory of quantum Markov chains (QMC), especially in the guise of quantum walks, has generated a huge number of works, including many discoveries of fundamental importance [12, 25, 31, 36, 58]. In [27] a novel approach has been proposed to investigate quantum cryptography problems by means of QMC [30], where quantum effects are entirely encoded into super-operators labelling transitions, and the nodes of its transition graph carry only classical information and thus they are discrete. QMC have been applied [12, 23, 25, 24] to the investigations of so-called "open quantum random walks" (OQRW) [13, 18, 19, 35, 37]. We notice that OQRW are related to the study of asymptotic behavior of trace-preserving completely positive maps, which belong to fundamental topics of quantum information theory ( see for instance [17, 38, 54]).

For the sake of clarity, let us recall some necessary information about OQRW. Let  $\mathcal{K}$  denote a separable Hilbert space and let  $\{|i\rangle\}_{i\in\Lambda}$  be its orthonormal basis indexed by the vertices of some graph  $\Lambda$  (here the set  $\Lambda$  of vertices might be finite or countable). Let  $\mathcal{H}$  be another Hilbert space, which will describe the degrees of freedom given at each point of  $\Lambda$ . Then we will consider the space  $\mathcal{H} \otimes \mathcal{K}$ . For each pair i, j one associates a bounded linear operator  $B_j^i$  on  $\mathcal{H}$ . This operator describes the effect of passing from  $|j\rangle$  to  $|i\rangle$ . We will assume that for each j, one has

$$\sum_{i} B_j^{i*} B_j^i = \mathbf{1},$$

where, if infinite, such series is strongly convergent. This constraint means: the sum of all the effects leaving site j is  $\mathbb{1}$ . The operators  $B_j^i$  act on  $\mathcal{H}$  only, we dilate them as operators on  $\mathcal{H} \otimes \mathcal{K}$  by putting

$$M_i^i = B_i^i \otimes |i\rangle\langle j|$$
.

The operator  $M_j^i$  encodes exactly the idea that while passing from  $|j\rangle$  to  $|i\rangle$  on the lattice, the effect is the operator  $B_j^i$  on  $\mathcal{H}$ .

According to [13] one has

(2) 
$$\sum_{i,j} M_j^{i*} M_j^i = \mathbf{1}.$$

Therefore, the operators  $(M_i^i)_{i,j}$  define a completely positive mapping

(3) 
$$\mathcal{M}(\rho) = \sum_{i} \sum_{j} M_j^i \, \rho \, M_j^{i^*}$$

on  $\mathcal{H} \otimes \mathcal{K}$ .

In what follows, we consider density matrices on  $\mathcal{H} \otimes \mathcal{K}$  which take the form

(4) 
$$\rho = \sum_{i} \rho_{i} \otimes |i\rangle\langle i|,$$

assuming that  $\sum_{i} \operatorname{Tr}(\rho_i) = 1$ .

For a given initial state of such form, the *Open Quantum Random Walk (OQRW)* is defined by the mapping  $\mathcal{M}$ , which has the following form

(5) 
$$\mathcal{M}(\rho) = \sum_{i} \left( \sum_{j} B_{j}^{i} \rho_{j} B_{j}^{i*} \right) \otimes |i\rangle \langle i|.$$

By means of the map  $\mathcal{M}$  one defines a family of classical random process on  $\Omega = \Lambda^{\mathbb{Z}_+}$ . Namely, for any density operator  $\rho$  on  $\mathcal{H} \otimes \mathcal{K}$  (see (4)) the probability distribution is defined by

(6) 
$$\mathbb{P}_{\rho}(i_0, i_1, \dots, i_n) = \operatorname{Tr}(B_{i_{n-1}}^{i_n} \cdots B_{i_1}^{i_2} B_{i_0}^{i_1} \rho_{i_0} B_{i_0}^{i_1*} B_{i_1}^{i_2*} \cdots B_{i_{n-1}}^{i_n*}).$$

We point out that this distribution is not a Markov measure [14].

On the other hand, it is well-known [12, 53] that to each classical random walk one can associate a certain Markov chain and some properties of the walk can be explored by the constructed chain. Therefore, it is natural to construct a Quantum Markov chain (QMC) associated with OQRW and investigate its properties.

Recently, in [23, 25], we have found a quantum Markov chain <sup>1</sup> (or finitely correlated state [26])  $\varphi$  on the algebra  $\mathcal{A} = \bigotimes_{i \in \mathbb{Z}_+} \mathcal{A}_i$ , where  $\mathcal{A}_i$  is isomorphic to  $B(\mathcal{H}) \otimes B(\mathcal{K})$ ,  $i \in \mathbb{Z}_+$ , such that the transition operator P equals to the mapping  $\mathcal{M}^{*2}$  and the restriction of  $\varphi$  to the commutative subalgebra of  $\mathcal{A}$  coincides with the distribution  $\mathbb{P}_{\rho}$ , i.e.

(7) 
$$\varphi((\mathbb{1} \otimes |i_0 > < i_0|) \otimes \cdots \otimes (\mathbb{1} \otimes |i_n > < i_n|)) = \mathbb{P}_{\rho}(i_0, i_1, \dots, i_n).$$

Hence, this result allows us to interpret the distribution  $\mathbb{P}_{\rho}$  as a QMC, and to study further properties of  $\mathbb{P}_{\rho}$ .

In the present paper, we initiate to look at the probability distribution (6) as a Markov field over the Cayley tree  $\Gamma^k$  [22]. Roughly speaking,  $(i_0, i_1, \ldots, i_n)$  is considered as a configuration on  $\Omega = \Lambda^{\Gamma^k}$ . Such kind of consideration allows us to investigated phase transition phenomena associated for OQRW within QMC scheme [41, 43]. We stress that, in physics, a spacial classes of QMC, called "Matrix Product States" (MPS) and more generally "Tensor Network States" [21, 56] were used to investigate quantum phase transitions for several lattice models. This method uses the density matrix renormalization group (DMRG) algorithm which opened a new way of performing the renormalization procedure in 1D systems and gave extraordinary precise results. This is done by keeping the states of subsystems which are relevant to describe the whole wave-function, and not those that minimize the energy on that subsystems [59].

In [7, 8, 9, 10, 44, 45] a QMC approach has been used to investigate models defined over the Cayley trees. In this path the QMC scheme is based on the  $C^*$ -algebraic framework (see also [6, 46]). Furthermore, in [41, 42, 43, 47, 48] we have established that Gibbs measures of the Ising model with competing (Ising) interactions (with commuting interactions) on a Cayley trees, can be considered as QMC (see also [49]).

In this paper, we first propose a new construction of QMC on trees, which is an extension of QMC considered in [10, 25, 57]. Using such a construction, we are able to construct QMC on tress associated with OQRW. Furthermore, our investigation leads to the detection of the phase transition phenomena within the proposed scheme. This kind of phenomena appears first time in this direction. Moreover, mean entropies of QMCs are calculated (cp. [55, 60]). We point out that, recently, in [39] 1st and 2nd moments of the open quantum walk have been studied and found its standard deviation. A phase transition is observed by evaluating the standard deviation, i.e. whether the quantum walk has diffusive or ballistic behavior.

#### 2. Preliminaries

Let  $\Gamma_+^k = (V, E)$  be the semi-infinite Cayley tree of order k with root o. The Cayley tree of order k is characterized by being a tree for which every vertex has exactly k+1 nearest-neighbors (see [41]). Recall that, two vertices x and y are nearest neighbors (denoted  $x \sim y$ ) if they are joined through an edge (i.e.  $\langle x, y \rangle \in E$ ). A list  $x \sim x_1 \sim \cdots \sim x_{d-1} \sim y$  of vertices is called a path from x to y. The distance on the tree between two vertices x and y (denoted d(x,y)) is the length of the shortest edge-path joining them.

Define

$$W_n := \{ x \in V \mid d(x, o) = n \}$$

<sup>&</sup>lt;sup>1</sup>We note that a Quantum Markov Chain is a quantum generalization of a Classical Markov Chain where the state space is a Hilbert space, and the transition probability matrix of a Markov chain is replaced by a transition amplitude matrix, which describes the mathematical formalism of the discrete time evolution of open quantum systems, see [1]-[3],[26, 28, 51] for more details.

<sup>&</sup>lt;sup>2</sup>The dual of  $\mathcal{M}$  is defined by the equality  $\text{Tr}(\mathcal{M}(\rho)x) = \text{Tr}(\rho\mathcal{M}^*(x))$  for all density operators  $\rho$  and observables x.

$$\Lambda_n := \bigcup_{j \le n} W_j; \quad \Lambda_{[m,n]} = \bigcup_{j=m}^n W_j.$$

Recall a coordinate structure in  $\Gamma_+^k$ : every vertex x (except for  $x^0$ ) of  $\Gamma_+^k$  has coordinates  $(i_1, \ldots, i_n)$ , here  $i_m \in \{1, \ldots, k\}$ ,  $1 \le m \le n$  and for the vertex  $x^0$  we put (0). Namely, the symbol (0) constitutes level 0, and the sites  $(i_1, \ldots, i_n)$  form level n (i.e.  $d(x^0, x) = n$ ) of the lattice. Using this structure, vertices  $x_{W_n}^{(1)}, x_{W_n}^{(2)}, \cdots, x_{W_n}^{(|W_n|)}$  of  $W_n$  can be represented as follows:

(8) 
$$x_{W_n}^{(1)} = (1, 1, \dots, 1, 1), \quad x_{W_n}^{(2)} = (1, 1, \dots, 1, 2), \quad \dots \quad x_{W_n}^{(k)} = (1, 1, \dots, 1, k, ),$$
$$x_{W_n}^{(k+1)} = (1, 1, \dots, 2, 1), \quad x_{W_n}^{(2)} = (1, 1, \dots, 2, 2), \quad \dots \quad x_{W_n}^{(2k)} = (1, 1, \dots, 2, k),$$

:

$$x_{W_n}^{(|W_n|-k+1)} = (k, k, \dots, k, 1), \ x_{W_n}^{(|W_n|-k+2)} = (k, k, \dots, k, 2), \ \dots x_{W_n}^{|W_n|} = (k, k, \dots, k, k).$$

In the above notations, we write

$$W_n = \{(i_1, i_2, \cdots, i_n); i_j = 1, 2, \cdots, k\}$$

So one can see that  $|W_n| = k^n$ . The set of direct successors for a given vertex  $x \in V$  is defined by

(9) 
$$S(x) := \{ y \in V : x \sim y \text{ and } d(y, o) > d(x, o) \}.$$

The vertex x has exactly k direct successors denoted  $(x, i), i = 1, 2, \dots, k$ 

$$S(x) = \{(x,1), (x,2), \cdots, (x,k)\}.$$

To each vertex x, we associate a C\*-algebra of observable  $\mathcal{A}_x$  with identity  $\mathbf{1}_x$ . For a given bounded region  $V' \subset V$ , we consider the algebra  $\mathcal{A}_{V'} = \bigotimes_{x \in V'} \mathcal{A}_x$ . We have the following natural embedding

$$\mathcal{A}_{\Lambda_n} \equiv \mathcal{A}_{\Lambda_n} \otimes \mathbb{1}_{W_{n+1}} \subset \mathcal{A}_{\Lambda_{n+1}}.$$

The algebra  $\mathcal{A}_{\Lambda_n}$  is then a subalgebra of  $\mathcal{A}_{\Lambda_{n+1}}$ . It follows the local algebra

(10) 
$$\mathcal{A}_{V;loc} := \bigcup_{n \in \mathbb{N}} \mathcal{A}_{\Lambda_n}$$

and the quasi-local algebra

$$A_V := \overline{A_{V \cdot loc}}^{C^*}$$

The set of states on a C\*-algebra  $\mathcal{A}$  will be denoted  $\mathcal{S}(\mathcal{A})$ .

Consider a triplet  $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$  of C\*-algebras. A quasi-conditional expectation [4] is a completely positive identity preserving linear map  $E: \mathcal{A} \to \mathcal{B}$  such that E(ca) = cE(a), for all  $a \in \mathcal{A}$ ,  $c \in \mathcal{C}$ .

**Definition 2.1.** [4] Let  $\mathcal{B} \subseteq \mathcal{A}$  be two unitary  $C^*$ -algebra  $\mathbb{I}$ . A Markov transition expectation from  $\mathcal{A}$  into  $\mathcal{B}$  is a completely positive identity preserving map.

**Definition 2.2.** [6] A (backward) quantum Markov chain on  $A_V$  is a triplet  $(\phi_o, (E_{\Lambda_n})_{n\geq 0}, (h_n)_n)$  of initial state  $\phi_o \in \mathcal{S}(\mathcal{A}_o)$ , a sequence of quasi-conditional expectations  $(E_{\Lambda_n})_n$  w.r.t. the triple  $A_{\Lambda_{n-1}} \subseteq A_{\Lambda_n} \subseteq A_{\Lambda_{n+1}}$  and a sequence  $h_n \in A_{W_n,+}$  of boundary conditions such that for each  $a \in A_V$  the limit

(11) 
$$\varphi(a) := \lim_{n \to \infty} \phi_0 \circ E_{\Lambda_0} \circ E_{\Lambda_1} \circ \cdots \circ E_{\Lambda_n}(h_{n+1}^{1/2} a h_{n+1}^{1/2})$$

exists in the weak-\*-topology and defines a state. In this case the state  $\varphi$  defined by (11) is also called quantum Markov chain (QMC).

Remark 2.3. The above definition introduces quantum Markov chains on trees as a triplet generalizing the definitions considered in [5, 10, 11, 41] by adding the boundary conditions. On the other hand, it extends to trees the recent unifying definition for quantum Markov chains on the one-dimensional case [6].

## 3. QMC ASSOCIATED WITH OQRW ON TREES

Let  $\mathcal{H}$  and  $\mathcal{K}$  be two separable Hilbert spaces. Let  $\{|i\rangle\}_{i\in\Lambda}$  be an ortho-normal basis of  $\mathcal{K}$  indexed by a graph  $\Lambda$ . To each  $x\in V$  we associate the algebra  $\mathcal{A}_x\equiv\mathcal{A}:=\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ .

Let  $\mathcal{M}$  be a OQRW given by (5). In this section we will use notations from the previous sections. As before, for each  $(i,j) \in \Lambda^2$ , one associates an operator  $B_j^i \in \mathcal{B}(\mathcal{H})$  to describe the transition from the state  $|j\rangle$  to the state  $|i\rangle$  such that

(12) 
$$\sum_{i \in \Lambda} B_j^{i*} B_j^i = \mathbf{1}_{\mathcal{B}(\mathcal{H})}.$$

Consider the density operator  $\rho \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ , of the form

$$\rho = \sum_{i \in \Lambda} \rho_i \otimes |i\rangle\langle i|; \quad \rho_i \in \mathcal{B}(\mathcal{H})^+.$$

In what follows, for the sake of simplicity of calculations, we assume that  $\rho_i \neq 0$  for all  $i \in \Lambda$  (see [25, Remark 4.5] for other kind of initial states).

Let us consider

(13) 
$$M_j^i = B_j^i \otimes |i\rangle\langle j| \in \mathcal{B}(\mathcal{H} \otimes \mathcal{K}).$$

Put

(14) 
$$A_j^i := \frac{1}{\operatorname{Tr}(\rho_j)^{1/2}} \rho_j^{1/2} \otimes |i\rangle\langle j|, \quad i, j \in \Lambda.$$

For each  $u \in V$ , we set

(15) 
$$K_j^{i(\{u\} \cup S(u))} := M_j^{i*(u)} \otimes \bigotimes_{v \in S(u)} A_j^{i(v)} \in \mathcal{A}_{\{u\} \cup S(u)}.$$

Put

(16) 
$$\mathcal{E}_{u}(a) := \sum_{(i,j),(i',j')\in\Lambda^{2}} \operatorname{Tr}_{u}\left(K_{j}^{i(\{u\}\cup S(u))} a K_{j'}^{i'(\{u\}\cup S(u)),*}\right); \quad a \in \mathcal{A}_{\{u\}\cup S(u)}.$$

For the sake of shortness, if no confusion is caused, the operator  $K_j^{i(\{u\}\cup S(u))}$  will be denoted simply by  $K_j^i$ .

**Lemma 3.1.** For each  $u \in V$ , the map  $\mathcal{E}_u$  defines a Markov transition expectation from  $\mathcal{A}_{\{u\}\cup S(u)}$  into  $\mathcal{A}_u$ . Moreover, we have

(17) 
$$\mathcal{E}_{u}(a_{0}^{(u)} \otimes a_{1}^{(u,1)} \otimes \cdots \otimes a_{k}^{(u,k)}) = \sum_{(i,j,j') \in \Lambda^{3}} \left( \prod_{\ell=1}^{k} \varphi_{j,j'}(a_{\ell}^{(u,\ell)}) \right) M_{j}^{i*} a_{0}^{(u)} M_{j'}^{i}$$

where

(18) 
$$\varphi_{jj'}(b) := \frac{1}{\operatorname{Tr}(\rho_j)^{1/2}\operatorname{Tr}(\rho_{j'})^{1/2}}\operatorname{Tr}\left(\rho_j^{1/2}\rho_{j'}^{1/2}\otimes|j'\rangle\langle j|b\right); \quad \forall b \in \mathcal{B}(\mathcal{H})\otimes\mathcal{B}(\mathcal{K})$$

for every  $a_0, a_1, \cdots, a_k \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ 

*Proof.* Let  $a=a_0^{(u)}\otimes a_1^{(u,1)}\otimes \cdots \otimes a_k^{(u,k)},$  according to (16) one has

$$\mathcal{E}_{u}(a) = \sum_{(i,j),(i',j')\in\Lambda^{2}} \operatorname{Tr}_{u]} \left( K_{j}^{i} a_{0}^{(u,0)} \otimes a_{1}^{(u,1)} \cdots \otimes a_{k}^{(u,k)} K_{j'}^{i'*} \right)$$

$$= \operatorname{Tr}_{u]} \left( \left( \sum_{(i,j)\in\Lambda^{2}} K_{j}^{i} \right) a \left( \sum_{(i,j)\in\Lambda^{2}} K_{j}^{i} \right)^{*} \right)$$

Then  $\mathcal{E}_u$  has a Krauss form and it is completely positive. Taking into account (15) and (13) one gets

$$\mathcal{E}_{u}(a) = \sum_{(i,j),(i',j')\in\Lambda^{2}} \operatorname{Tr}_{u]} \left( K_{j}^{i(\{u\}\cup S(u))} a K_{j'}^{i'(\{u\}\cup S(u))*} \right)$$

$$= \sum_{(i,j),(i',j')\in\Lambda^{2}} M_{j}^{i*} a_{0}^{(u,0)} M_{j'}^{i'} \prod_{\ell=1}^{k} \operatorname{Tr}(A_{j}^{i} a_{\ell}^{(u,\ell)} A_{j'}^{i'*}).$$

From (14) for each  $\ell \in \{1, \dots, k\}$  one has

$$\operatorname{Tr}(A_{j}^{i}a_{\ell}^{(u,\ell)}A_{j'}^{i'*}) = \operatorname{Tr}(A_{j'}^{i'*}A_{j}^{i}a_{l}^{(u,\ell)}) \\
= \frac{1}{\operatorname{Tr}(\rho_{j})^{1/2}\operatorname{Tr}(\rho_{j'})^{1/2}}\operatorname{Tr}(\rho_{j'}^{1/2}\otimes|j'\rangle\langle i'|\rho_{j}^{1/2}\otimes|i\rangle\langle j|a_{\ell}^{(u,\ell)}) \\
= \frac{1}{\operatorname{Tr}(\rho_{i})^{1/2}\operatorname{Tr}(\rho_{i'})^{1/2}}\operatorname{Tr}(\rho_{j'}^{1/2}\rho_{j}^{1/2}\otimes|j'\rangle\langle j|a_{\ell}^{(u,\ell)})\delta_{i,i'}$$

where  $\delta_{i,i'}$  denotes the Kronecker symbol. This leads to (17) and finishes the proof.

**Lemma 3.2.** For each  $n \in \mathbb{N}$ , the map

(19) 
$$\mathcal{E}_{W_n} = \bigotimes_{u \in W_n} \mathcal{E}_u$$

defines a Markov transition expectation from  $\mathcal{A}_{\Lambda_{[n,n+1]}}$  into  $\mathcal{A}_{W_n}$ . Moreover, the map

$$(20) E_{\Lambda_n} = id_{\mathcal{A}_{\Lambda_{n-1}}} \otimes \mathcal{E}_{W_n}$$

is a quasi-conditional expectation w.r.t. the triplet  $\mathcal{A}_{\Lambda_{n-1}} \subset \mathcal{A}_{\Lambda_n} \subset \mathcal{A}_{\Lambda_{n+1}}$ .

*Proof.* Thanks to the Cayley tree structure  $W_{n+1} = \bigsqcup_{u \in W_n} S(u)$ , where  $\bigsqcup$  means the disjointedness of the union. One gets the result using Lemma 3.1.

**Remark 3.3.** In the notations of Definition 2.2 the triplet  $(\phi_o, (E_{\Lambda_n})_{n\geq 0}, (h_n)_n)$  defining a quantum Markov chain  $\varphi$  on  $\mathcal{A}_V$  through (11) reduces to a finer triplet  $(\phi_o, (\mathcal{E}_u)_{u\in V}, (h_u)_{u\in V})$  where  $\phi_o \in \mathcal{S}(\mathcal{A}_o)$ , the family of localized Markov transition expectations  $(\mathcal{E}_u)_{u\in V}$  relates to the sequence of quasiconditional expectations  $(E_{\Lambda_n})_n$  through (19) and (20) and  $h_n = \bigotimes_{u\in W_n} h_u$ .

**Theorem 3.4.** Let  $M_j^i$  and  $A_j^i$  be given by (14) and (13). In the notations of Lemma 3.1, if an initial density matrix  $\omega_o \in \mathcal{A}_{o;+}$  and a boundary condition  $(h_u)_{u \in V}$  satisfy

(21) 
$$\operatorname{Tr}(\omega_o h_o) = 1$$

and

(22) 
$$\sum_{i,j,j'\in\Lambda} M_j^{i*} M_{j'}^i \prod_{\ell=1}^k \varphi_{j,j'}(h_{(u,\ell)}) = h_u$$

Then the triplet  $(\omega_o, (\mathcal{E}_u)_{u \in V}, (h_u)_{u \in V})$  defines a quantum Markov chain  $\varphi$  on the algebra  $\mathcal{A}_V$ . Moreover, for each  $a = \bigotimes_{u \in \Lambda_n} a_u \in \mathcal{A}_{\Lambda_n}$  one has

(23) 
$$\varphi(a) = \sum_{j,j' \in \Lambda} \operatorname{Tr} \left( \mathcal{M}_{jj'}(\omega_o) a_o \right) \prod_{u \in \Lambda_{[1,n]}} \psi_{j,j'}(a_u) \prod_{v \in \Lambda_{n+1}} \varphi_{j,j'}(h^{(v)})$$

where  $\mathcal{E}_u$  is given by (17), the functional  $\varphi_{jj}$  is given by (18), and

(24) 
$$\mathcal{M}_{jj'}(\cdot) = \sum_{i \in \Lambda} M_{j'}^i \cdot M_j^{i*},$$

(25) 
$$\psi_{j,j'}(b) = \frac{1}{\text{Tr}(\rho_j)^{1/2} \text{Tr}(\rho_{j'})^{1/2}} \sum_{i \in \Lambda} \text{Tr}\left(B_{j'}^i \rho_{j'}^{1/2} \rho_j^{1/2} B_j^{i^*} \otimes |i\rangle\langle i| b\right).$$

*Proof.* Let us first prove the existence of the Markov chain  $\varphi$  by examining the limit (11) for  $E_{\Lambda_n}$  as in Lemma 3.2 and  $h_n = \bigotimes_{u \in W_n} h_u$ . From Lemma 3.1 the equality (22) is equivalent to

$$h_u = \mathcal{E}_u(\mathbf{1}_u \otimes h_{(u,1)} \otimes \cdots h_{(u,k)})$$

It follows that for each integer  $m \in \mathbb{N}$  one has

(26) 
$$\mathcal{E}_{W_m}(\mathbf{1}_{W_n} \otimes h_{m+1}) = \bigotimes_{u \in W_m} \mathcal{E}_u(\mathbf{1}_u \otimes h_{(u,1)} \otimes \cdots h_{(u,k)}) = h_m$$

Let  $a = \bigotimes_{u \in \Lambda_n} a_u$ , for each subset  $I \subseteq \Lambda_n$ , we denote  $a_I := \bigotimes_{u \in I} a_u \otimes \mathbf{1}_{W_n \setminus I}$ . For each  $m \ge n+1$  one has

$$\varphi_{m}(a) := \varphi_{o} \circ E_{\Lambda_{0}} \circ E_{\Lambda_{1}} \circ \dots \circ E_{\Lambda_{m}}(h_{m+1}^{1/2}ah_{m+1}^{1/2})$$

$$\stackrel{(20)}{=} \varphi_{o} \left( \mathcal{E}_{W_{0}} \left( a_{o} \otimes \mathcal{E}_{W_{1}} \left( a_{W_{n}} \cdots \mathcal{E}_{W_{n}} \left( a_{W_{n}} \otimes \mathcal{E}_{W_{n+1}} \left( \mathbf{1}_{W_{n+1}} \cdots \mathcal{E}_{W_{m}} \left( \mathbf{1}_{W_{m}} \otimes h_{m+1} \right) \right) \right) \right) \right) \right)$$

$$\stackrel{(26)}{=} \varphi_{o} \left( \mathcal{E}_{W_{0}} \left( a_{o} \otimes \mathcal{E}_{W_{1}} \left( a_{W_{n}} \cdots \mathcal{E}_{W_{n}} \left( a_{W_{n}} \otimes \mathcal{E}_{W_{n+1}} \left( \mathbf{1}_{W_{n+1}} \cdots \mathcal{E}_{W_{m-1}} \left( \mathbf{1}_{W_{m}} \otimes h_{m} \right) \right) \right) \right) \right) \right)$$

$$\vdots$$

$$= \varphi_{o} \left( \mathcal{E}_{W_{0}} \left( a_{W_{0}} \dots \left( \mathcal{E}_{W_{n-1}} \left( a_{W_{n-1}} \left( \mathcal{E}_{W_{n}} \left( a_{W_{n}} \otimes h_{n+1} \right) \right) \right) \right) \right) \right)$$

Then the limit (11) exists in the strongly finite sense and defines a positive functional  $\varphi$ . Thanks to (21)  $\varphi$  is a state on  $\mathcal{A}_V$ . Therefore, the triplet  $(\omega_o, (\mathcal{E}_u)_{u \in V}, (h_u)_{u \in V})$  defines a quantum Markov chain  $\varphi$  on the algebra  $\mathcal{A}_V$  given by

(27) 
$$\varphi(a) = \varphi_o\left(\mathcal{E}_{W_0}\left(a_{W_0}\dots\left(\mathcal{E}_{W_{n-1}}\left(a_{W_{n-1}}\left(\mathcal{E}_{W_n}\left(a_{W_n}\otimes h_{n+1}\right)\right)\right)\right)\right)\right)$$

Now let us determine the expression for  $\varphi$ . From the tree structure, we get

$$\mathcal{E}_{W_n}(a_{W_n} \otimes h_{n+1}) = \bigotimes_{v \in W_n} \mathcal{E}_v(a_v \otimes h_{(v,1)} \otimes \cdots \otimes h_{(v,k)})$$

and

$$\mathcal{E}_{W_{n-1}}\left(a_{W_{n-1}}\otimes\mathcal{E}_{W_n}\left(a_{W_n}\otimes h_{n+1}\right)\right) = \bigotimes_{u\in W_{n-1}}\mathcal{E}_u\left(a_u\otimes\bigotimes_{v\in S(u)}\mathcal{E}_v\left(a_v\otimes h_{(v,1)}\otimes\cdots\otimes h_{(v,k)}\right)\right)$$

For each  $u \in W_{n-1}$ , one finds

$$\mathcal{E}_{u}\left(a_{u} \otimes \bigotimes_{v \in S(u)} \mathcal{E}_{v}\left(a_{v} \otimes h_{(v,1)} \otimes \cdots \otimes h_{(v,k)}\right)\right) \\
\stackrel{(17)}{=} \mathcal{E}_{u}\left(a_{u} \otimes \bigotimes_{v \in W_{n}} \left(\sum_{(i_{v},j_{v},j'_{v}) \in \Lambda^{3}} \prod_{\ell=1}^{k} \varphi_{j_{v},j'_{v}}(h_{(v,\ell)}) M_{j_{v}}^{i_{v}*} a_{v} M_{j'_{v}}^{i_{v}}\right)\right) \\
= \sum_{\mathbf{i},\mathbf{j},\mathbf{j}' \in \Lambda^{S(u)}} \left(\prod_{v \in W_{n}} \prod_{\ell=1}^{k} \varphi_{j_{v},j'_{v}}(h_{(v,\ell)})\right) \mathcal{E}_{u}\left(a_{u} \otimes \bigotimes_{v \in S(u)} M_{j_{v}}^{i_{v}*} a_{v} M_{j'_{v}}^{i_{v}}\right) \\
= \sum_{\mathbf{i},\mathbf{j},\mathbf{j}' \in \Lambda^{S(u)}} \sum_{i_{u},j_{u},i'_{u}j'_{u} \in \Lambda} \left(\prod_{v \in W_{n}} \prod_{\ell=1}^{k} \varphi_{j_{v},j'_{v}}(h_{(v,\ell)})\right) \operatorname{Tr}\left(A_{j_{u}}^{i_{u}} M_{j_{v}}^{i_{v}*} a_{v} M_{j'_{u}}^{i_{v}*} A_{j'_{u}}^{i'_{u}*}\right) M_{j_{u}}^{i_{u}*} a_{u} M_{j'_{u}}^{i'_{u}}$$

where  $\mathbf{i}' = (i_v)_{v \in S(u)}$ ,  $\mathbf{j} = (j_v)_{v \in S(u)}$ ,  $\mathbf{j}' = (j_v')_{v \in S(u)}$  are sequences of elements of  $\Lambda$  induced by S(u).

According to (14) and (13), for each  $v \in S(u)$ , we obtain

$$\sum_{i_{v} \in \Lambda} \operatorname{Tr} \left( A_{j_{u}}^{i_{u}} M_{j_{v}}^{i_{v}}^{*} a_{v} M_{j_{v}'}^{i_{v}} A_{j_{u}'}^{i_{u}}^{*} \right) = \operatorname{Tr} \left( M_{j_{v}'}^{i_{v}} A_{j_{u}'}^{i_{u}}^{*} A_{j_{u}}^{i_{u}} M_{j_{v}}^{i_{v}}^{*} a_{v} \right) \\
= \sum_{i_{v} \in \Lambda} \frac{1}{\sqrt{\operatorname{Tr}(\rho_{j_{u}}) \operatorname{Tr}(\rho_{j_{u}'})}} \operatorname{Tr} \left( B_{j_{v}'}^{i_{v}} \rho_{j_{u}'}^{1/2} \rho_{j_{u}}^{1/2} B_{j_{v}}^{i_{v}} \otimes |i_{v}\rangle \langle j_{v}'||j_{u}'\rangle \langle i_{u}'||i_{u}\rangle \langle j_{u}||j_{v}\rangle \langle i_{v}|a_{v} \right) \\
= \frac{1}{\sqrt{\operatorname{Tr}(\rho_{j_{u}}) \operatorname{Tr}(\rho_{j_{u}'})}} \sum_{i_{v} \in \Lambda} \operatorname{Tr} \left( B_{j_{v}'}^{i_{v}} \rho_{j_{u}'}^{1/2} \rho_{j_{u}}^{1/2} B_{j_{v}}^{i_{v}} \otimes |i_{v}\rangle \langle i_{v}|a_{v} \right) \delta_{j_{v},j_{u}} \delta_{j_{v}',j_{u}'} \delta_{i_{u}',i_{u}} \\
= \frac{1}{\sqrt{\operatorname{Tr}(\rho_{j_{v}}) \operatorname{Tr}(\rho_{j_{v}'})}} \sum_{i_{v} \in \Lambda} \operatorname{Tr} \left( B_{j_{v}'}^{i_{v}} \rho_{j_{v}'}^{1/2} \rho_{j_{v}}^{1/2} B_{j_{v}}^{i_{v}} \otimes |i_{v}\rangle \langle i_{v}|a_{v} \right) \delta_{j_{v},j_{u}} \delta_{j_{v}',j_{u}'} \delta_{i_{u}',i_{u}} \\
= \psi_{j_{v}j_{v}'}(a_{v}) \delta_{j_{v},j_{u}} \delta_{j_{v}',j_{u}'} \delta_{i_{u}',i_{u}}.$$

here, as before,  $\psi_{j_u,j'_u}$  is given by (25).

This implies that, for any  $u \in W_{n-1}$  among the configurations  $\mathbf{j}, \mathbf{j}' \in \Lambda^{\{u\} \cup S(u)}$  only ones satisfying  $(j_u, j_u') = (j_v, j_v')$  for all  $v \in S(u)$ , appear on the sum of the right hand side of (28). It follows that (28) becomes

$$\sum_{i,j,j'\in\Lambda} \left( \prod_{v\in S(u)} \psi_{j,j'\in\Lambda}(a_v) \right) \left( \prod_{v\in S(u)} \prod_{\ell=1}^k \varphi_{j,j'}(h_{(v,\ell)}) \right) M_j^{i*} a_u M_{j'}^{i}$$

Iterating the above procedure, for each  $m \leq n$ , one finds

$$\mathcal{E}_{W_m}(a_{W_m} \otimes \mathcal{E}_{W_{m+1}}(a_{W_{m+1}} \otimes \cdots \mathcal{E}_{W_n}(a_{W_n} \otimes h_{n+1})))$$

$$= \sum_{j,j' \in \Lambda^{W_m}} \left( \prod_{v \in \Lambda_{[m,n]}} \psi_{j,j' \in \Lambda}(a_v) \right) \left( \prod_{w \in W_{n+1}} \prod_{\ell=1}^k \varphi_{j,j'}(h_w) \right) \bigotimes_{u \in W_m} \sum_{i_u} {M_j^{i_u}}^* a_u M_{j'}^{i_u}$$

Since for m=0, one has  $W_0=\{o\}$  then

$$\varphi(a) = \sum_{j,j' \in \Lambda} \left( \prod_{v \in \Lambda_n} \psi_{j,j' \in \Lambda}(a_v) \right) \left( \prod_{w \in W_{n+1}} \prod_{\ell=1}^k \varphi_{j,j'}(h_w) \right) \varphi_o\left( \sum_{i \in \Lambda} {M_j^{i*}} a_u M_{j'}^i \right)$$

hence

$$\varphi_o\left(\sum_{i\in\Lambda} M_j^{i*} a_u M_{j'}^i\right) = \operatorname{Tr}\left(\omega_o \sum_{i\in\Lambda} M_j^{i*} a_o M_{j'}^i\right)$$
$$= \sum_{i\in\Lambda} \operatorname{Tr}(\omega_o M_j^{i*} a_o M_{j'}^i) = \sum_{i\in\Lambda} \operatorname{Tr}(M_{j'}^i \omega_o M_j^{i*} a_o) = \operatorname{Tr}(\mathcal{M}_{jj'}(\omega_o) a_o)$$

where  $\mathcal{M}_{ii'}$  is given by (24). This completes the proof.

**Remark 3.5.** The maps  $\varphi_{jj'}$  and  $\psi_{jj'}$  are linear functionals. If the particular case j = j', the linear functionals  $\varphi_{jj}$  and  $\psi_{jj}$  define two states, and we have

(29) 
$$\varphi_{jj}(b) = \frac{1}{\operatorname{Tr}(\rho_j)} \operatorname{Tr}(\rho_j \otimes |j\rangle\langle j|b).$$

and

(30) 
$$\psi_{jj}(b) = \frac{1}{\operatorname{Tr}(\rho_j)} \sum_{i \in \Lambda} \operatorname{Tr}\left(B_j^i \rho_j B_j^{i^*} \otimes |i\rangle\langle i|b\right).$$

3.1. Quantum Markov chain associated with the disordered phase. In this section, we are going to discuss about QMC associated with the disordered phase of the system. Here, by the disordered phase, it is meant a QMC corresponding to the trivial solution of (22). Indeed, we have the following result.

**Theorem 3.6.** Assume that Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$  are finite dimensional, then  $h_0 = \mathbb{1}_{\mathcal{B}(\mathcal{H})} \otimes \mathbb{1}_{\mathcal{B}(\mathcal{K})}$  defines a homogeneous boundary condition  $\mathbf{h}_0 := (h_u = \alpha_u(h_0))_{u \in V}$  satisfying (22). Moreover for any normalized density matrix  $\omega_o \in \mathcal{A}_{o;+}$  the quantum Markov chain  $\varphi$  associated with the triplet  $(\omega_o, \mathcal{E}, h_0)$  is given by

(31) 
$$\varphi(a) = \sum_{j} \operatorname{Tr} \left( \mathcal{M}_{jj}(\omega_o) a_o \right) \prod_{u \in \Lambda_{[1,n]}} \psi_{j,j}(a_u)$$

for every  $a = \bigotimes_{u \in \Lambda_n} a_u \in \mathcal{A}_{\Lambda_n}$ .

*Proof.* For each  $u \in V$  and  $j, j' \in \Lambda$ , one has

$$\varphi_{jj'}(\mathbf{I}) := \frac{1}{\operatorname{Tr}(\rho_j)^{1/2} \operatorname{Tr}(\rho_{j'})^{1/2}} \operatorname{Tr}\left(\rho_j^{1/2} \rho_{j'}^{1/2} \otimes |j'\rangle\langle j|\right) = \delta_{jj'}$$

From (17), it follows that

$$\mathcal{E}_{u}(\mathbf{1}^{(u)} \otimes \mathbf{1}^{(u,1)} \otimes \cdots \otimes \mathbf{1}^{(u,k)}) = \sum_{\substack{(i,j,j') \in \Lambda^{3}}} M_{j}^{i*} M_{j'}^{i} \delta_{jj'}$$

$$= \sum_{\substack{(i,j) \in \Lambda^{2} \\ (i,j) \in \Lambda^{2}}} M_{j}^{i*} M_{j}^{i}$$

$$\stackrel{(2)}{=} \mathbf{1}^{(u)}.$$

This proves that  $\mathbf{h}_0$  defines a boundary condition. Let  $\varphi$  be a QMC associated with this boundary condition. From (23) one gets (31).

**Remark 3.7.** We notice that QMC given in (23) generalizes the Markov chains associated with open quantum random walks studied in [25] to trees. In the one dimensional setting, they propose a more general class of QMC associated with OQRW than the ones considered in [25] due to the existence of boundary conditions.

## 4. Phase transition for QMCs on trees associated with two-state OQRWs

In this section, we focus on several applications of quantum Markov chains associated open quantum random walks on trees to a phenomena of phase transition, the following definition of phase transition within QMC scheme was first introduced in [41].

**Definition 4.1.** We say that there exists a phase transition for the constructed QMC associated with OQRW if the following conditions are satisfied:

- (a) EXISTENCE: The equations (21), (22) have at least two  $(u_0, \{h^x\}_{x \in L})$  and  $(v_0, \{s^x\}_{x \in L})$  solutions;
- (c) NOT OVERLAPPING SUPPORTS: there is a projector  $P \in B_L$  such that  $\varphi_{u_0,\mathbf{h}}(P) < \varepsilon$  and  $\varphi_{v_0,\mathbf{s}}(P) > 1 \varepsilon$ , for some  $\varepsilon > 0$ .
- (b) NOT QUASI-EQUIVALENCE: the corresponding quantum Markov chains  $\varphi_{u_0,\mathbf{h}}$  and  $\varphi_{v_0,\mathbf{s}}$  are not quasi equivalent

Otherwise, we say there is no phase transition.

In this section, for the sake of simplicity, we reduce the study to the case  $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$ . For each  $u \in V$ , we take  $A_u = \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \equiv M_4(\mathbb{C})$  and let  $\Lambda = \{1, 2\}$ . The interactions are given by

(32) 
$$B_1^1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad B_2^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_1^2 = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix}, \quad B_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

where  $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$ ,  $ac \neq 0$ . Put

$$(33) p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

and

$$|1\rangle = \left[\begin{array}{c} 1 \\ 0 \end{array}\right], |2\rangle = \left[\begin{array}{c} 0 \\ 1 \end{array}\right]$$

Notice that  $(|1\rangle, |2\rangle)$  is an ortho-normal basis of  $\mathcal{K} \equiv \mathbb{C}^2$ . In the sequel elements of  $\mathcal{B}(\mathcal{H})$  will be denoted by means of  $2 \times 2$  complex matrices, while elments of  $\mathcal{B}(\mathcal{K})$  will be written using Dirac notation  $|i\rangle < j|$ .

4.1. Existence of boundary conditions and their associated quantum Markov chains. In this section, we are goong to determine all the translation invariant boundary conditions associated with the considered two-state OQRW (32). This means that we describe positive solutions  $h \in \mathcal{A}_{u,;+}$  of the compatibility equation (22).

**Lemma 4.2.** The translation invariant boundary conditions associated with the two-states OQRW (32) are given by:

(34) 
$$h^{(u)} = \sum_{j,j' \in \Lambda} h_{j,j'} \otimes |j\rangle\langle j'| \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}),$$

where

$$h_{j,j'} = \left(\frac{\operatorname{Tr}(\rho_{j'}^{1/2}\rho_j^{1/2}h_{j,j'})}{\sqrt{\operatorname{Tr}(\rho_j)\operatorname{Tr}(\rho_j')}}\right)^k \sum_{i \in \Lambda} B_j^{i*} B_j^{i'}.$$

*Proof.* From (22), one has

$$h^{(u)} = \sum_{i,j,i',j'=1,2} M_j^{i*(u)} M_{j'}^{i'(u)} \prod_{\ell=1}^k \text{Tr}(A_j^i h^{(u,\ell)} A_{j'}^{i'*}).$$

Since the boundary condition is translation invariant (i.e.  $h^{(u)} = h$  for all  $u \in V$ ), one gets

(35) 
$$h = \sum_{i,j,i',j'=1,2} M_j^{i*} M_{j'}^{i'} \operatorname{Tr}(A_j^i h A_{j'}^{i'*})^k.$$

Now, using

$$M_i^{i*}M_{i'}^{i'} = B_i^{i*}B_{i'}^{i'} \otimes |j\rangle\langle j'|\delta_{i,i'}$$

and

$$\operatorname{Tr}(A_{j}^{i}hA_{j'}^{i'*}) = \operatorname{Tr}(A_{j'}^{i'*}A_{j}^{i}h) = \frac{1}{\sqrt{\operatorname{Tr}(\rho_{j})\operatorname{Tr}(\rho'_{j})}}\operatorname{Tr}(\rho_{j'}^{1/2}\rho_{j}^{1/2}\otimes|j'\rangle\langle j|h)\delta_{i,i'},$$

The (35) becomes

$$h = \sum_{i,j,j' \in \Lambda} \left( \frac{\operatorname{Tr}(\rho_{j'}^{1/2} \rho_j^{1/2} \otimes |j'\rangle\langle j|h)}{\sqrt{\operatorname{Tr}(\rho_j)\operatorname{Tr}(\rho_j')}} \right)^k B_j^{i*} B_{j'}^i \otimes |j\rangle\langle j'|.$$

Finally, by identification with (34), we are led to

$$h_{j,j'} = \sum_{i \in \Lambda} \left( \frac{\operatorname{Tr}(\rho_{j'}^{1/2} \rho_j^{1/2} \otimes |j'\rangle\langle j|h)}{\sqrt{\operatorname{Tr}(\rho_j)\operatorname{Tr}(\rho_j')}} \right)^k B_j^{i*} B_{j'}^i = \left( \frac{\operatorname{Tr}(\rho_{j'}^{1/2} \rho_j^{1/2} h_{j,j'})}{\sqrt{\operatorname{Tr}(\rho_j)\operatorname{Tr}(\rho_j')}} \right)^k \sum_{i \in \Lambda} B_j^{i*} B_{j'}^i.$$

**Theorem 4.3.** Let  $\{E_{jj'}, 1 \geq j, j' \leq 4\}$  denotes the canonical basis of  $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K}) \equiv M_4(\mathbb{C})$ . Write the translation invariant boundary conditions associated with the two-states OQRW (32) in the form

$$h = \sum_{j,j'=1}^{4} e_{jj'} E_{jj'}.$$

Then

(36) 
$$\begin{cases} e_{11} = e_{33} = \frac{\operatorname{Tr}(\rho_{1} \otimes |1\rangle\langle 1|h_{1,1})^{k}}{\operatorname{Tr}(\rho_{1})^{k}} \\ e_{22} = e_{44} = \frac{\operatorname{Tr}(\rho_{2} \otimes |2\rangle\langle 2|h_{2,2})^{k}}{\operatorname{Tr}(\rho_{2})^{k}} \\ e_{12} = \frac{\overline{c}}{\overline{a}}e_{14} = \overline{c}\frac{\operatorname{Tr}(\rho_{2}^{1/2}\rho_{1}^{1/2} \otimes |2\rangle\langle 1|h_{1,2})^{k}}{(\operatorname{Tr}(\rho_{1})\operatorname{Tr}(\rho_{2}))^{k/2}} \\ e_{21} = \frac{c}{a}e_{41} = c\frac{\operatorname{Tr}(\rho_{1}^{1/2}\rho_{1}^{1/2} \otimes |1\rangle\langle 2|h_{2,1})^{k}}{(\operatorname{Tr}(\rho_{1})\operatorname{Tr}(\rho_{2}))^{k/2}} \\ e_{j,j'} = 0, \quad otherwise \end{cases}$$

In particular, for the initial states  $\rho_1 = \rho_2 = |1\rangle\langle 1|$  and k = 2, there are exactly 4 non-trivial solutions given by

(37) 
$$h_0 = \mathbf{1}_{M_4}, \quad h_1 = \mathbf{1}_{M_2} \otimes |1\rangle\langle 1|, \quad h_2 = \mathbf{1}_{M_2} \otimes |2\rangle\langle 2|, \quad h_3 = h_0 + h_c,$$

where

$$h_c = \frac{1}{c}B_2^2 \otimes B_2^1 + \frac{1}{c}B_2^2 \otimes B_2^{1*}$$
 defined only if  $|c| = 1$ ,

Moreover, if for each  $i \in \{0, 1, 2, 3\}$  an initial density matrix  $\omega_i$  associated with the boundary condition  $h_i$  satisfying (21) then there exists four quantum Markov chains  $\varphi_{\omega_i, h_i}$ ,  $0 \le i \le 3$  and

(38) 
$$\varphi_{\omega_j,h_j}(a) = \operatorname{Tr} \left( \mathcal{M}_{jj}(\omega_o) a_o \right) \prod_{u \in \Lambda_{[1,n]}} \psi_{j,j}(a_u), \quad \forall j \in \{1,2\}$$

*Proof.* From Lemma 4.2 and the fact that  $|a|^2 + |c|^2 = |b|^2 + |d|^2 = 1$ , a straightforward computation leads to (36). Next, for the initial states  $\rho_1 = \rho_2 = |1\rangle\langle 1|$  and k = 2, the system (36) reduces to,

$$\begin{cases}
e_{11} = e_{33} = e_{11}^2 \\
e_{22} = e_{44} = e_{22}^2 \\
e_{12} = \frac{\overline{c}}{\overline{a}}e_{14} = \overline{c}e_{12}^2 \\
e_{21} = \frac{c}{a}e_{41} = ce_{21}^2 \\
e_{ij'} = 0, \quad otherwise
\end{cases}$$

for  $a \neq 0$  and

$$\begin{cases}
e_{11} = e_{33} = e_{11}^{2} \\
e_{22} = e_{44} = e_{22}^{2} \\
e_{12} = \overline{c} e_{12}^{2} \\
e_{21} = c e_{21}^{2} \\
e_{jj'} = 0, \quad otherwise
\end{cases}$$

when a = 0 (and so |c| = 1). Therefore, the non-negative Hermitian solutions satisfy

$$\begin{cases}
e_{11} = e_{33} & \in \{0, 1\} \\
e_{22} = e_{44} & \in \{0, 1\} \\
e_{12} = \overline{e_{21}} & \in \{0, \frac{1}{\overline{c}}\} \\
e_{jj'} = 0, & otherwise
\end{cases}$$

This leads to the solutions (37).

Now since for i each the solutions  $h_i$  is positive there exists an initial  $\omega_i \in \mathcal{A}_o$  such that  $\text{Tr}(\omega_i h_i) = 1$ . Let  $\mathcal{E}_u$  given by (16) then from Theorem 3.4 the triplet  $(\omega_i, (\mathcal{E}_u)_u, h_i)$  defines a quantum Markov chain  $\varphi_{\omega_i, h_i}$  on  $\mathcal{A}_V$ . One can easily check that

$$\varphi_{j,j'}(h_1) = \delta_{j,1}\delta_{1,j'}, \quad \varphi_{j,j'}(h_2) = \delta_{j,2}\delta_{2,j'},$$

Therefore, according to (23) one gets (38).

4.2. Not quasi-equivalence property. Recall that two states  $\varphi$  and  $\psi$  on a  $C^*$ -algebra  $\mathcal{A}$  are said be quasi-equivalent if the GNS representations  $\pi_{\varphi}$  and  $\pi_{\psi}$  are quasi-equivalent. The reader is referred to [16] for the notion of quasi-equivalence of representations. The following result proposes a criteria for the non-quasi equivalence, we are going to use the following result (see [16, Corollary 2.6.11]).

**Lemma 4.4.** Let  $\varphi_1$ ,  $\varphi_2$  be two factor states on a quasi-local algebra  $\mathfrak{A} = \bigcup_{\Lambda} \mathfrak{A}_{\Lambda}$ . The states  $\varphi_1$ ,  $\varphi_2$  are quasi-equivalent if and only if for any given  $\varepsilon > 0$  there exists a finite volume  $\Lambda \subset V$  such that  $|\varphi_1(a) - \varphi_2(a)| < \varepsilon ||a||$  for all  $a \in B_{\Lambda'}$  with  $\Lambda' \cap \Lambda = \emptyset$ .

**Theorem 4.5.** Assume that |c| = 1. If  $\mathcal{M}_{11}(\omega_1) = \mathcal{M}_{22}(\omega_2)$  then the quantum Markov chains  $\varphi_{h_1,\omega_1}$  and  $\varphi_{h_2,\omega_2}$  are quasi-equivalent.

*Proof.* First from (38) the two states  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$  are product states then they are factor states. The assumption |c| = 1 implies |a| = 0, then according to (29), (30) and (32) the two states  $\psi_{11}$  and  $\psi_{22}$  coincide on  $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{K})$ . It follows that,

$$|\varphi_{\omega_1,h_1}(a) - \varphi_{\omega_2,h_2}(a)| = |\operatorname{Tr}((\mathcal{M}_{11}(\omega_1) - \mathcal{M}_{22}(\omega_2)) a_o)| \prod_{u \in \Lambda_{[1,n]}} |\psi_{1,1}(a_u)|$$

Hence, clearly if we have  $\mathcal{M}_{11}(\omega_1) = \mathcal{M}_{22}(\omega_2)$ , then the quantum Markov chains  $\varphi_{h_1,\omega_1}$  and  $\varphi_{h_2,\omega_2}$  are quasi-equivalent.

**Theorem 4.6.** Assume that |c| < 1. The quantum Markov chains  $\varphi_{h_1,\omega_1}$  and  $\varphi_{h_2,\omega_2}$  are not quasi-equivalent.

*Proof.* Let us define an element of  $\mathcal{A}_{\Lambda_n}$  as follows

$$E_{\Lambda_n} = \sigma^{x_{W_n}(1)} \otimes \mathbf{1}_{\Lambda_n \setminus \{x_{W_n}(1)\}}$$

where

$$\sigma^{x_{W_n}(1)} = \mathbf{1}_{M_2} \otimes p$$

here  $x_{W_n}(1)$  is defined by (8). Then, we have

$$\psi_{11}(\sigma^{x_{W_n}(1)}) = \text{Tr}(B_1^1 p B_1^{1*}) = |a|^2 \text{ and } \psi_{22}(\sigma^{x_{W_n}(1)}) = \text{Tr}(B_2^1 p B_2^{1*}) = 0.$$

On the other hand,

$$\psi_{j,j}(\mathbf{1}_{M_2} \otimes \mathbf{1}_{M_2}) = \sum_{i=1}^2 \text{Tr}\left(pB_j^{i*}B_{j'}^i\right) = \text{Tr}\left(p\sum_{i=1}^2 B_j^{i*}B_j^i\right) = \text{Tr}(p) = 1.$$

Hence,

$$\varphi_{\omega_1,h_1}(E_{\Lambda_n}) = \operatorname{Tr}\left(\mathcal{M}_{11}(\omega_1)\right)|a|^2 = |a|^2 \text{ and } \varphi_{\omega_2,h_2}(E_{\Lambda_n}) = 0.$$

One gets

$$\varphi_{\omega_2,h_2}(E_{\Lambda_n}) = 0$$
  
$$\varphi_{\omega_1,h_1}(E_{\Lambda_n}) = |a|^2.$$

Now, since |c|<1, and  $\|E_{\Lambda_n}\|=1$  then  $\varepsilon_0:=|a|^2>0$ 

$$|\varphi_{\omega_2,h_2}(E_{\Lambda_n}) - \varphi_{\omega_1,h_1}(E_{\Lambda_n})| = |a|^2 \ge \varepsilon_0 ||E_{\Lambda_n}||$$

for every  $n \geq 0$ . Therefore from Lemma 4.4 the two quantum Markov chains  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$  are not quasi-equivalent.

4.3. Not overlapping support. Recall that two states  $\varphi$  and  $\psi$  on  $A_V$  have not overlapping supports if there is a projector  $P \in B_V$  such that  $\varphi(P) < \varepsilon$  and  $\psi(P) > 1 - \varepsilon$ , for some  $\varepsilon > 0$ .

**Lemma 4.7.** Any rank-1 projection in  $M_2(\mathbb{C})$  has the form

(39) 
$$p(\varepsilon, z) = \begin{pmatrix} \varepsilon & z\sqrt{\varepsilon(1-\varepsilon)} \\ \overline{z}\sqrt{\varepsilon(1-\varepsilon)} & 1-\varepsilon \end{pmatrix}$$

where  $\varepsilon \in [0,1], z \in \mathbb{C}$  with |z|=1.

Remark 4.8. Notice that,

$$\operatorname{Tr}(\mathcal{M}_{jj}(w)) = \sum_{i \in \Lambda} \operatorname{Tr}(M_j^i w M_j^{i*}) = \sum_{i \in \Lambda} \operatorname{Tr}(M_j^{i*} M_j^i w)$$
$$= \operatorname{Tr}(\sum_{i \in \Lambda} B_j^{i*} B_j^i \otimes |j\rangle \langle j|w)$$
$$= \operatorname{Tr}(\mathbb{I}_{\mathcal{H}} \otimes |j\rangle \langle j|w).$$

Writing w as a block matrix:

$$w = \sum_{j,j' \in \lambda} \omega_{j,j'} \otimes |j\rangle\langle j'|,$$

one gets

$$\operatorname{Tr}(\mathcal{M}_{jj}(w)) = \operatorname{Tr}(\omega_{j,j}).$$

More generally, one has

$$M_{jj'}(w) = \sum_{i \in \Lambda} M_{j'}^i w M_j^{i*} = \sum_{i \in \Lambda} B_{j'}^i \omega_{j',j} B_j^{i*} \otimes |i\rangle\langle i|.$$

For each  $n \in \mathbb{N}$ , we denote

$$p_n(\varepsilon, z) = \bigotimes_{u \in \Lambda} (p(\varepsilon, z) \otimes I)^{(u)} \in \mathcal{A}_{\Lambda_n}$$

**Theorem 4.9.** The quantum Markov chains  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$  have not overlapping supports if and only if  $\operatorname{Tr}(\omega_1 p \otimes \mathbb{1}_{M_2}) \neq 1$ .

Proof. One has

$$\varphi_{\omega_1,h_1}(p_n(\varepsilon,z)) = \varepsilon^{2^{n+1}-2} \operatorname{Tr} \left( \mathcal{M}_{11}(\omega_1) p(\varepsilon,z) \otimes \mathbf{1}_{M_2} \right),$$

and

$$\varphi_{\omega_2,h_2}(p_n(\varepsilon,z)) = \varepsilon^{2^{n+1}-2} \operatorname{Tr} \left( \mathcal{M}_{22}(\omega_2) p(\varepsilon,z) \otimes \mathbf{1}_{M_2} \right),$$

Let  $P_n = p_n(\varepsilon, z)$  be a rank-1 projector.

• If  $\varepsilon < 1$ , then one has

$$\lim_{n \to \infty} \varphi_{\omega_1, h_1}(P_n) = \lim_{n \to \infty} \varphi_{\omega_2, h_2}(P_n) = 0.$$

• If  $\varepsilon = 1$ , then using Remark 4.8 one has

$$\varphi_{\omega_2,h_2}(P_n) = \operatorname{Tr}(\omega_2 h_2) = 1.$$

On the other hand one has

$$\varphi_{\omega_1,h_1}(P_n) = \operatorname{Tr}(\omega_1 p \otimes \mathbf{1}_{M_2}) \neq 1 = \varphi_{\omega_2,h_2}(P_n).$$

The above results leads to the following concluding result.

**Theorem 4.10.** if |c| < 1, then there exists a phase transitions for the quantum Markov chains associated with the two-state OQRW (32).

*Proof.* Take

$$\omega_1 = q \otimes |1\rangle\langle 1|$$
 and  $\omega_2 = p \otimes |2\rangle\langle 2|$ .

Using (24), (32) and (13) we find

$$\mathcal{M}_{11}(\omega_1) = q \otimes \left( |b|^2 |1\rangle\langle 1| + |d|^2 |2\rangle\langle 2| \right)$$

and

$$\mathcal{M}_{22}(\omega_2) = p \otimes |2\rangle\langle 2|.$$

On the one hand, by Lemma 4.4, the two quantum Markov chains  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$  are not quasi-equivalent since

$$|\varphi_{\omega_2,h_2}(E_{\Lambda_n}) - \varphi_{\omega_1,h_1}(E_{\Lambda_n})| = |a|^2$$

for every  $n \ge 0$ . On the other hand, clearly (by Theorem 4.9) the two quantum Markov chains  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$  are not overlapping supports since

$$\operatorname{Tr}(\omega_1(p\otimes \mathbf{1}_{M_2}))=0\neq 1.$$

## 5. Mean entropy for QNCs on trees associated with OQRWs

This section is devoted to the computation of mean entropies for the two quantum Markov chains  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$  given by (38). In the notations of section 4, the algebra  $\mathcal{A}_x \equiv \mathbf{M}_4(\mathbb{C})$ . Let  $\varphi$  be a state on  $\mathcal{A}_V$ . For each bounded region I of the vertex set V, the density matrix of the restriction  $\varphi[_{\mathcal{A}_I}$  will be denoted by  $D_I^{\varphi}$ . The von Neumann entropy of  $\varphi[_{\mathcal{A}_I}$  is defined to be

(40) 
$$S(\varphi) = -\text{Tr}(D_{\varphi} \log D_{\varphi})$$

In [50] the mean entropy for quantum Markov states on trees was defined as follows

(41) 
$$s(\varphi) := \lim_{n \to \infty} \frac{1}{|\Lambda_n|} S(\varphi \lceil_{\mathcal{A}_{\Lambda_n}})$$

We use the same formulae for the quantum Markov chains  $\varphi_{\omega_1,h_1}$  and  $\varphi_{\omega_2,h_2}$ 

**Theorem 5.1.** For each  $j \in \{1,2\}$ , let  $\varphi_{\omega_j,h_j}$  be given by (38). The mean entropy of the quantum Markov chin  $\varphi_{\omega_j,h_j}$  coincides with the von Neumann entropy of the state  $\psi_{jj}$ :

$$(42) s(\varphi_{\omega_j,h_j}) = S(\psi_{jj})$$

where  $\psi_{jj}$  is as in (25). Therefore, one has

$$s(\varphi_{\omega_1,h_1}) = -2\left(|a|^2 \log |a| + |c|^2 \log |c|\right)$$

and

$$s(\varphi_{\omega_2,h_2}) = 0$$

*Proof.* From (38) the density matrices of  $\varphi_{\omega_j,h_j}$  is given as follows

$$D_{\varphi_{\omega_j,h_j}\lceil_{\mathcal{A}_{\Lambda_n}}} = \mathcal{M}_{jj}(\omega_1)^{(o)} \otimes \bigotimes_{x \in \Lambda_{[1,n]}} D_{\psi_{jj}}^{(x)}$$

One finds

(44) 
$$\log D_{\varphi_{\omega_j,h_j}\lceil_{\mathcal{A}_{\Lambda_n}}} = \log \mathcal{M}_{jj}(\omega_1)^{(o)} + \sum_{x \in \Lambda_{[1,n]}} \log D_{\psi_{jj}}^{(x)}$$

One has

$$S\left(\varphi_{\omega_{j},h_{j}}\lceil_{\mathcal{A}_{\Lambda_{n}}}\right) = -\operatorname{Tr}\left(D_{\varphi_{\omega_{j},h_{j}}\lceil_{\mathcal{A}_{\Lambda_{n}}}}\log D_{\varphi_{\omega_{j},h_{j}}\lceil_{\mathcal{A}_{\Lambda_{n}}}}\right)$$

$$= -\varphi_{\omega_{j},h_{j}}\left(\log D_{\varphi_{\omega_{j},h_{j}}\lceil_{\mathcal{A}_{\Lambda_{n}}}}\right)$$

$$\stackrel{(44)}{=} -\varphi_{\omega_{j},h_{j}}\left(\log \mathcal{M}_{jj}(\omega_{1})^{(o)}\right) - \sum_{x\in\Lambda_{[1,n]}} \varphi_{\omega_{j},h_{j}}\left(\log D_{\psi_{jj}}^{(x)}\right)$$

$$\stackrel{(25)}{=} -\operatorname{Tr}\left(\mathcal{M}_{jj}(\omega_{1})^{(o)}\log \mathcal{M}_{jj}(\omega_{1})^{(o)}\right) - \sum_{x\in\Lambda_{[1,n]}} \psi_{jj}\left(\log D_{\psi_{jj}}^{(x)}\right)$$

$$= -\operatorname{Tr}\left(\mathcal{M}_{jj}(\omega_{1})^{(o)}\log \mathcal{M}_{jj}(\omega_{1})^{(o)}\right) - |\Lambda_{[1,n]}|\psi_{jj}\left(\log D_{\psi_{jj}}\right)$$

$$= S(\varphi_{o}) + (2^{n+1} - 2)S(\psi_{jj})$$

It follows that the mean entropy of the quantum Markov chains  $\varphi_{\omega_i,h_i}$  is given by

$$s(\varphi_{\omega_{j},h_{j}}) = \lim_{n \to \infty} \frac{S\left(\varphi_{\omega_{j},h_{j}} \lceil_{\mathcal{A}_{\Lambda_{n}}}\right)}{|\Lambda_{n}|}$$

$$= \lim_{n \to \infty} \left(-\frac{S(\varphi_{o})}{2^{n+1}-1} - \frac{2^{n+1}-2}{2^{n+1}-1}S(\psi_{jj})\right)$$

$$= S(\psi_{jj})$$

Now, from (25) one can see that the density matrix of  $\psi_{ij}$  is defined by

$$D_{\psi_{jj}} = \frac{1}{\operatorname{Tr}(\rho_j)} \sum_{i \in \Lambda} B_j^i \rho_j B_j^{i*} \otimes |i\rangle\langle i| = \sum_{i \in \Lambda} B_j^i p B_j^{i*} \otimes |i\rangle\langle i|$$

since  $\rho_j = |1\rangle\langle 1| = p$ . It follows that

$$D_{\psi_{11}} = p \otimes \left( |a|^2 |1\rangle\langle 1| + |c|^2 |2\rangle\langle 2| \right)$$

and

$$D_{\psi_{22}} = q \otimes |2\rangle\langle 2|$$

Hence we get

$$s(\varphi_{\omega_1,h_1}) = -\text{Tr}(D_{\psi_{11}}\log D_{\psi_{11}}) = -2\left(|a|^2\log|a| + |c|^2\log|c|\right)$$

and

$$s(\varphi_{\omega_2,h_2}) = -\text{Tr}(D_{\psi_{22}} \log D_{\psi_{22}}) = 0$$

That completes the proof.

## ACKNOWLEDGMENTS

The authors gratefully acknowledge Qassim University, represented by the Deanship of Scientific Research, on the financial support for this research under the number (10173-cba-2020-1-3-I) during the academic year 1442 AH / 2020 AD.

# References

- [1] L. Accardi, Noncommutative Markov chains, Proc. of Int. School of Math. Phys. Camerino (1974), 268–295.
- [2] L. Accardi, On noncommutative Markov property, Funct. Anal. Appl. 8 (1975), 1–8.
- [3] L. Accardi, A. Frigerio, Markovian cocycles, *Proc. Royal Irish Acad.* **83A** (1983) 251-263.
- [4] L. Accardi, C. Cecchini, Conditional expectations in von Neumann algebras and a Theorem of Takesaki, J. Funct. Anal. 45, 245–273 (1982).
- [5] L. Accardi, F. Fidaleo, F. Mukhamedov, Markov states and chains on the CAR algebra, *Inf. Dim. Analysis, Quantum Probab. Related Topics* **10** (2007), 165–183.

- [6] L. Accardi, A. Souissi, E. Soueidy, Quantum Markov chains: A unification approach, Inf. Dim. Analysis, Quantum Probab. Related Topics 23(2020), 2050016.
- [7] L. Accardi, F. Mukhamedov, M. Saburov, On Quantum Markov Chains on Cayley tree I: uniqueness of the associated chain with XY -model on the Cayley tree of order two, *Inf. Dim. Analysis, Quantum Probab. Related Topics* 14(2011), 443–463.
- [8] L. Accardi, F. Mukhamedov, M. Saburov, On Quantum Markov Chains on Cayley tree II: Phase transitions for the associated chain with XY -model on the Cayley tree of order three, *Ann. Henri Poincare* 12(2011), 1109-1144.
- [9] L. Accardi, F. Mukhamedov, M. Saburov, On Quantum Markov Chains on Cayley tree III: Ising model, J. Stat. Phys. 157 (2014), 303-329.
- [10] L. Accardi, H. Ohno, F. Mukhamedov, Quantum Markov fields on graphs, Inf. Dim. Analysis, Quantum Probab. Related Topics 13(2010), 165–189.
- [11] L. Accardi, F. Mukhamedov., A. Souissi, Construction of a new class of quantum Markov fields, *Adv. Oper. Theory* 1 (2016), no. 2, 206-218.
- [12] L. Accardi, G.S. Watson, Quantum random walks, in book: L. Accardi, W. von Waldenfels (eds) Quantum Probability and Applications IV, Proc. of the year of Quantum Probability, Univ. of Rome Tor Vergata, Italy, 1987, LNM, 1396(1987), 73–88.
- [13] S. Attal, F. Petruccione, C. Sabot, I. Sinayskiy. Open Quantum Random Walks. J. Stat. Phys. 147(2012), 832-852.
- [14] I. Bardet, D. Bernard, Y. Pautrat, Passage times, exit times and Dirichlet problems for open quantum walks, J. Stat. Phys. 167(2017), 173-204.
- [15] T. Beboist, V. Jaksic, Y. Pautrat, C.A. Pillet, On entropy production of repeated quantum measurament I. General Theory, Commun. Math. Phys., 57 (2018), no. 1, 77?-123.
- [16] O. Bratteli, D.W. Robinson, Operator algebras and quantum statistical mechanics I, Springer-Verlag, New York, 1987.
- [17] D. Burgarth, V. Giovannetti, The generalized Lyapunov theorem and its application to quantum channels. New J. Phys. 9 (2007) 150.
- [18] R. Carbone, Y. Pautrat. Homogeneous open quantum random walks on a lattice. J. Stat. Phys. 160(2015), 1125-1152.
- [19] R. Carbone, Y. Pautrat. Open quantum random walks: reducibility, period, ergodic properties. Ann. Henri Poincaré 17(2016), 99-135.
- [20] C. M. Chandrashekar, R. Laflamme, Quantum phase transition using quantum walks in an optical lattice, Phys. Rev. A 78(2008), 022314.
- [21] J.I. Cirac, F. Verstraete, Renormalization and tensor product states in spin chains and lattices, J. Phys. A. Math. Theor. 42 (2009), 504004.
- [22] R.L. Dobrushin, The description of a random field by means of conditional probabilities and conditions of its regularity, *Theor. Probab. Appl.* 13 (1968), 197-224.
- [23] A. Dhahri. C.K. Ko, H.J. Yoo, Quantum Markov chains associated with open quantum random walks, J. Stat. Phys. 176(2019), 1272–1295
- [24] A. Dhahri, F. Mukhamedov, Open Quantum Random Walks and Quantum Markov Chains. Funct. Anal. Appl. 53(2019), 137–142.
- [25] A. Dhahri, F. Mukhamedov, Open quantum random walks, quantum Markov chains and recurrence. Rev. Math. Phys. 31(2019), 1950020.
- [26] M. Fannes, B. Nachtergaele, R.F. Werner, Finitely correlated states on quantum spin chains, Commun. Math. Phys. 144 (1992), 443–490.
- [27] Y. Feng, N. Yu and M. Ying, Model checking quantum Markov chains, J. Computer Sys. Sci. 79, 1181—1198 (2013).
- [28] F. Fidaleo, F. Mukhamedov, Diagonalizability of non homogeneous quantum Markov states and associated von Neumann algebras, Probab. Math. Stat. 24 (2004), 401–418.
- [29] Y. H. Goolam Hossen, I. Sinayskiy, F. Petruccione, Non-reversal Open Quantum Walks, Open Syst. & Inf. Dyn. 25(2018), 1850017.
- [30] S. Gudder, Quantum Markov chains, J. Math. Phys. 49, 072105 (2008).
- [31] J. Kempe, "Quantum random walks—an introductory overview," Contemporary Physics, 44, 307–327 (2003).
- [32] A. Khrennikov, M. Ohya. N. Watanabe, Quantum probability from classical signal theory, *Inter. J. Quantum Infor.* **9**(2011), 281-292.
- [33] T. Kitagawa, M.S. Rudner, E. Berg, E. Demler, Exploring topological phases with quantum walks, Phys. Rev. A 82(2010), 033429.
- [34] T., Kitagawa, Topological phenomena in quantum walks: elementary introduction to the physics of topological phases, Quantum Information Processing 11(2012), 1107–1148.
- [35] N. Konno, H. J. Yoo. Limit theorems for open quantum random walks. J. Stat. Phys. 150 (2013), 299-319.
- [36] B. Kümmerer, Quantum Markov processes and applications in physics. In book: Quantum independent increment processes. II, 259–330, Lecture Notes in Math., 1866, Springer, Berlin, 2006.

- [37] C. F. Lardizabal, R. R. Souza. On a class of quantum channels, open random walks and recurrence. J. Stat. Phys. 159(2015), 772-796.
- [38] C. Liu, N. Petulante. On Limiting distributions of quantum Markov chains. Int. J. Math. and Math. Sciences. 2011(2011), ID 740816.
- [39] T. Machida, Phase transition of an open quantum walk, Inter. J. Quantum Inform. 19(2021), 2150028.
- [40] D. A. Meyer, From quantum cellular automata to quantum lattice gases, J. Stat. Phys. 85(1996), 551-574.
- [41] F. Mukhamedov, A. Barhoumi, A. Souissi, Phase transitions for quantum Markov chains associated with Ising type models on a Cayley tree, J. Stat. Phys. 163, 544–567 (2016).
- [42] F. Mukhamedov, A. Barhoumi, A. Souissi, On an algebraic property of the disordered phase of the Ising model with competing interactions on a Cayley tree, *Math. Phys. Anal. Geom.* **19**, 21 (2016).
- [43] F. Mukhamedov, A. Barhoumi, A. Souissi, S. El Gheteb, A quantum Markov chain approach to phase transitions for quantum Ising model with competing XY-interactions on a Cayley tree, *J. Math. Phys.* **61**, 093505 (2020).
- [44] F. Mukhamedov, S. El Gheteb, Uniqueness of quantum Markov chain associated with XY-Ising model on the Cayley tree of order two, *Open Sys. & Infor. Dyn.* **24**(2017), 175010.
- [45] F. Mukhamedov, S. El Gheteb, Clustering property of Quantum Markov Chain associated to XY-model with competing Ising interactions on the Cayley tree of order two, *Math. Phys. Anal. Geom.* **22**, 10 (2019).
- [46] F. Mukhamedov and S. El Gheteb, Factors generated by XY-model with competing Ising interactions on the Cayley tree, Ann. Henri Poincare 21(2020), 241–253.
- [47] F. Mukhamedov, U. Rozikov, On Gibbs measures of models with competing ternary and binary interactions on a Cayley tree and corresponding von Neumann algebras, J. Stat. Phys. 114, 825–848 (2004).
- [48] F. Mukhamedov, A. Souissi, Quantum Markov States on Cayley trees, J. Math. Anal. Appl. 473(2019), 313–333.
- [49] F. Mukhamedov, A. Souissi, Diagonalizability of quantum Markov States on trees , J. Stat. Phys. 182(2021), Article 9.
- [50] F. Mukhamedov, A. Souissi, Entropy for quantum Markov states on trees, preprint (2021).
- [51] B. Nachtergaele. Working with quantum Markov states and their classical analogoues, in *Quantum Probability and Applications V*, Lect. Notes Math. **1442**, Springer-Verlag, 1990, pp. 267–285.
- [52] M. A. Nielsen, I. L. Chuang. Quantum computation and quantum information. Cambridge Univ. Press, 2000.
- [53] J. R. Norris. Markov chains. Cambridge Univ. Press, 1997.
- [54] J. Novotný, G. Alber, I. Jex. Asymptotic evolution of random unitary operations. Cent. Eur. J. Phys. 8(2010), 1001-1014.
- [55] K. Ohmura, N. Watanabe, Quantum dynamical mutual entropy based on AOW entropy, *Open Syst. & Inf. Dyn.* **26**(2019), 1950009
- [56] R. Orus, A practical introduction of tensor networks: matrix product states and projected entangled pair states, Ann of Physics 349 (2014) 117-158.
- [57] Y.M. Park, H.H. Shin, Dynamical entropy of generalized quantum Markov chains over infinite dimensional algebras, J. Math. Phys. 38 (1997), 6287–6303.
- [58] R. Portugal. Quantum walks and search algorithms. Springer, 2013.
- [59] S. Rommer, S. Ostlund, A class of ansatz wave functions for 1D spin systems and their relation to DMRG, Phys. Rev. B 55 (1997) 2164.
- [60] N. Watanabe, Note on entropies of quantum dynamical systems, Foundations of Phys. 41(2011), 549 -563.