

Very Well-Covered Graphs of Girth at least Four and Local Maximum Stable Set Greedoids*

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Abstract

A *maximum stable set* in a graph G is a stable set of maximum cardinality. S is a *local maximum stable set* of G , and we write $S \in \Psi(G)$, if S is a maximum stable set of the subgraph induced by $S \cup N(S)$, where $N(S)$ is the neighborhood of S .

Nemhauser and Trotter Jr. [20], proved that any $S \in \Psi(G)$ is a subset of a maximum stable set of G . In [12] we have shown that the family $\Psi(T)$ of a forest T forms a greedoid on its vertex set. The cases where G is bipartite, triangle-free, well-covered, while $\Psi(G)$ is a greedoid, were analyzed in [14], [15], [17], respectively.

In this paper we demonstrate that if G is a very well-covered graph of girth ≥ 4 , then the family $\Psi(G)$ is a greedoid if and only if G has a unique perfect matching.

Keywords: very well-covered graph, local maximum stable set, greedoid, triangle-free graph, König-Egerváry graph.

1 Introduction

Throughout this paper $G = (V, E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V = V(G)$ and edge set $E = E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X . K_n, C_n denote respectively, the complete graph on $n \geq 1$ vertices and the chordless cycle on $n \geq 3$ vertices. If $A, B \subset V$ and $A \cap B = \emptyset$, then (A, B) stands for the set $\{e = ab : a \in A, b \in B, e \in E\}$.

The *neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u : u \in V \text{ and } vu \in E\}$. For $A \subset V$, we denote $N(A) = \{v \in V - A : N(v) \cap A \neq \emptyset\}$ and $N[A] = A \cup N(A)$.

A *stable set* in G is a set of pairwise non-adjacent vertices. A stable set of maximum size will be referred to as a *maximum stable set* of G , and the *stability number* of G , denoted by $\alpha(G)$, is the cardinality of a maximum stable set in G . Let $\Omega(G)$ stand for the set of all maximum stable sets of G .

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A *matching* in a graph $G = (V, E)$ is a set of edges $M \subseteq E$ such that no two edges of M share a common vertex. A *maximum matching* is a matching of maximum cardinality. By $\mu(G)$ is denoted the cardinality of a maximum matching. A matching is *perfect* if it saturates all the vertices of the graph.

Let us recall that G is a *König-Egerváry graph* provided $\alpha(G) + \mu(G) = |V(G)|$ [4], [23]. As a well-known example, any bipartite graph is a König-Egerváry graph [5], [10].

Theorem 1.1 [13] *If G is a König-Egerváry graph, then every maximum matching is contained in $(S, V(G) - S)$, for each $S \in \Omega(G)$.*

A matching $M = \{a_i b_i : a_i, b_i \in V(G), 1 \leq i \leq k\}$ of graph G is called a *uniquely restricted matching* if M is the unique perfect matching of $G[\{a_i, b_i : 1 \leq i \leq k\}]$ [8]. For instance, all the maximum matchings of the graph G in Figure 1 are uniquely restricted, while the graph H from the same figure has both uniquely restricted maximum matchings (e.g., $\{uv, xw\}$) and non-uniquely restricted maximum matchings (e.g., $\{xy, tv\}$).

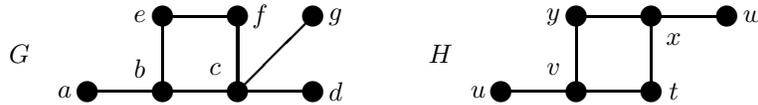


Figure 1: The unique cycle of H is alternating with respect to the matching $\{yv, tx\}$.

Recall that G is *well-covered* if all its maximal stable sets have the same cardinality [21], and G is *very well-covered* if, in addition, it has no isolated vertices and $|V(G)| = 2\alpha(G)$ [6].

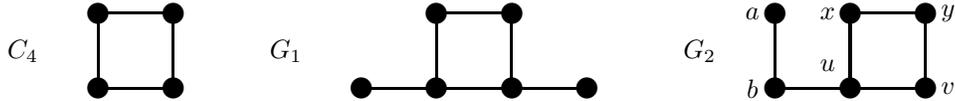


Figure 2: Only C_4 and G_1 are very well-covered graphs.

It is easy to prove that every graph having a perfect matching consisting of pendant edges is very well-covered. The converse is not generally true; e.g., the graphs C_4 and G_1 depicted in Figure 2. Moreover, there are well-covered graphs without perfect matchings; e.g., K_3 . Nevertheless, having a perfect matching is a necessary condition for very well-coveredness.

Theorem 1.2 [6] *For a graph G without isolated vertices the following are equivalent:*

- (i) G is very well-covered;
- (ii) there exists a perfect matching in G that satisfies property P;
- (iii) there exists at least one perfect matching in G and every perfect matching in G satisfies property P.

A matching M in a graph G satisfies *Property P* if

$$"N(x) \cap N(y) = \emptyset, \text{ and each } v \in N(x) - \{y\} \text{ is adjacent to all vertices of } N(y) - \{x}"$$

hold for every edge $xy \in M$.

For example, the perfect matching $M = \{ab, xy, uv\}$ of the graph G_2 from Figure 2 does not satisfy Property P , since $uv \in M, b \in N(u), y \in N(v)$, but $by \notin E(G_2)$. Hence, G_2 is not a very well-covered graph. Moreover, G_2 is not well-covered, because no maximum stable set of G_2 includes the stable set $\{b, v\}$. However, G_2 is a König-Egerváry graph. Notice that K_4 is well-covered, has perfect matchings, but is neither a König-Egerváry graph, nor a very well-covered graph.

Theorem 1.3 [11] *A graph is very well-covered if and only if it is a well-covered König-Egerváry graph.*

A set $A \subseteq V(G)$ is a *local maximum stable set* of G if $A \in \Omega(G[N[A]])$ [12]; by $\Psi(G)$ we denote the family of all local maximum stable sets of the graph G . For instance, $\{a\}, \{a, e\} \in \Psi(G)$, while $\{c\}, \{b, f\} \notin \Psi(G)$, where G is from Figure 1. Notice also that in the same graph, the stable sets $\{a, e\}, \{b, f\}$ are contained in some maximum stable sets of G , while for $\{a, c\}, \{c, e\}$ this is not true.

Theorem 1.4 [20] *Every local maximum stable set of a graph is a subset of a maximum stable set.*

Definition 1.5 [1], [9] *A greedoid is a pair (V, \mathcal{F}) , where $\mathcal{F} \subseteq 2^V$ is a non-empty set system satisfying the following conditions:*

- Accessibility: *for every non-empty $X \in \mathcal{F}$ there is an $x \in X$ such that $X - \{x\} \in \mathcal{F}$;*
- Exchange: *for $X, Y \in \mathcal{F}, |X| = |Y| + 1$, there is an $x \in X - Y$ such that $Y \cup \{x\} \in \mathcal{F}$.*

In the sequel we use \mathcal{F} instead of (V, \mathcal{F}) , as the ground set V will be, usually, the vertex set of some graph.



Figure 3: $\Psi(G)$ is not a greedoid, $\Psi(H)$ is a greedoid.

The graphs from Figure 3 are non-bipartite König-Egerváry graphs, and all their maximum matchings are uniquely restricted. Let us remark that both graphs are also triangle-free, but only $\Psi(H)$ is a greedoid. It is clear that $\{b, c\} \in \Psi(G)$, while $\{b\}, \{c\} \notin \Psi(G)$. Notice also that $G[N[\{b, c\}]]$ is not a König-Egerváry graph, and, as one can see from the following theorem, this is a good reason for $\Psi(G)$ not to be a greedoid.

Theorem 1.6 [15] *If G is a triangle-free graph, then the following assertions are equivalent:*

- (i) $\Psi(G)$ is a greedoid;
- (ii) *all maximum matchings of G are uniquely restricted and the closed neighborhood of every local maximum stable set of G induces a König-Egerváry graph.*

The cases of trees, bipartite graphs, unicycle graphs, whose family of local maximum stable sets forms a greedoid, were analyzed in [12], [14], [18], respectively.

In this paper we characterize very well-covered graphs of girth at least four, whose families of local maximum stable sets are greedoids.

2 Results

Let us remark that the very well-covered graph G_1 in Figure 2 has a C_4 and one of the edges of this C_4 belongs to the unique perfect matching of G_1 .

Lemma 2.1 *No edge of some C_q , for $q = 3$ or $q \geq 5$, belongs to a perfect matching in a very well-covered graph.*

Proof. If the graph G is very well-covered, then by Theorem 1.2, G has a perfect matching, say M , and each perfect matching satisfies Property P .

Let $xy \in M$. Then, Property P implies that $N(x) \cap N(y) = \emptyset$, i.e., xy belongs to no C_3 in G . Further, if $v \in N(x) - \{y\}$ and $u \in N(y) - \{x\}$, Property P assures that $vu \in E(G)$, i.e., xy belongs to no C_q , for $q \geq 5$. ■

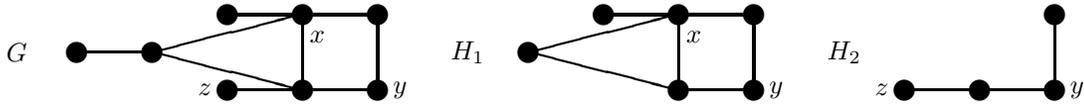


Figure 4: Both $H_1 = G[N[\{x, y\}]]$ and $H_2 = G[N[\{y, z\}]]$ are König-Egerváry graphs.

Let us mention that if G is very well-covered, S is a stable set such that $G[N[S]]$ is a König-Egerváry graph, then S does not necessarily belong to $\Psi(G)$; e.g., the set $S_1 = \{x, y\}$ is stable in the graph G depicted in Figure 4, and $S_1 \notin \Psi(G)$, while $H_1 = G[N[S_1]]$ is a König-Egerváry graph. Notice that $S_2 = \{y, z\} \in \Psi(G)$ and $H_2 = G[N[S_2]]$ is a König-Egerváry graph. The following finding, firstly presented in [15], shows that this phenomenon is true for very well-covered graphs in general. We repeat the proof for the sake of self-containment.

Theorem 2.2 *If G is a very well-covered graph, then $G[N[S]]$ is a König-Egerváry graph, for every $S \in \Psi(G)$.*

Proof. By Theorem 1.3, G is a König-Egerváry graph. According to Theorem 1.2, G has a perfect matching, say M , and each perfect matching satisfies Property P .

Suppose by way of contradiction that there is $S = \{v_i : 1 \leq i \leq k\} \in \Psi(G)$, such that $G[N[S]]$ is not a König-Egerváry graph.

Since G is well-covered, there exists some $W \in \Omega(G)$, with $S \subseteq W$. By Theorem 1.1, $M \subseteq (W, V(G) - W)$, and because M is a perfect matching and $S \subseteq W$, we infer that S is matched by M into $N(S)$, and this implies $|S| \leq |N(S)|$. The assumption that $G[N[S]]$ is not a König-Egerváry graph leads to $|N(S)| > |S|$. It means that there exists a vertex $x \in N(S) - M(S)$, where $M(S) = \{w_i : v_i w_i \in M, 1 \leq i \leq k\}$.

In the following, we will prove that the set $\{x\} \cup M(S)$ is stable.

Firstly, x must be adjacent to some vertex, say v_1 , from S , otherwise $S \cup \{x\}$ is a stable set larger than S in $G[N[S]]$, in contradiction with $S \in \Psi(G)$. By Lemma 2.1, x is not adjacent to w_1 , since $v_1 w_1 \in M$. Thus, $\{x, w_1\}$ is a stable set.

One of x, w_1 must be adjacent to one vertex, say v_2 , from S , because, otherwise, the set $\{x, w_1\} \cup \{v_i : 2 \leq i \leq k\}$ would be stable in $G[N[S]]$, larger than S . If $w_1 v_2 \in E(G)$, then Property P , applied to the edge $v_1 w_1 \in M$, ensures that $x v_2 \in E(G)$.

In other words, x must be adjacent to v_1 . Moreover, the set $\{x, w_1, w_2\}$ is stable, because $x w_2 \notin E(G)$ according to Lemma 2.1, while for $w_1 w_2 \in E(G)$ we get, by Property P , that $x w_1 \in E(G)$, in contradiction with the fact that $\{x, w_1\}$ is a stable set.

Assume that for some $j < k$, the set

$$A_j = \{x\} \cup \{w_i : 1 \leq i \leq j\}$$

is stable, and x is adjacent to each $v_i, 1 \leq i \leq j$. Then, there is an edge joining a vertex, say a , belonging to A , and a vertex, say v_{j+1} , from the set $\{v_i : j+1 \leq i \leq k\}$. Otherwise,

$$A_j \cup \{v_i : j+1 \leq i \leq k\}$$

is a stable set in $G[N[S]]$, larger than S . If $a = w_t$, then by Property P , when the edge $v_t w_t$ is concerned, the vertex x must be adjacent to v_{j+1} . Thus, no matter where a is located, the vertex x is adjacent to the vertex v_{j+1} (see Figure 5(a)).

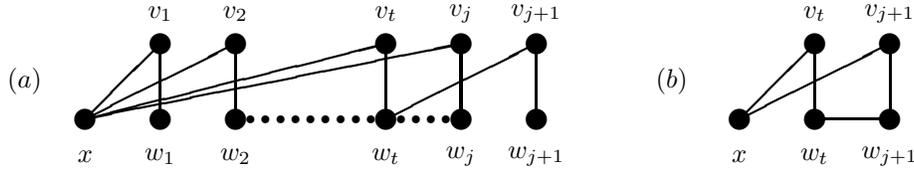


Figure 5: (a) The vertex x is adjacent to all vertices from $\{v_i : 1 \leq i \leq j\}$. (b) The vertices $x, v_{j+1}, w_{j+1}, w_t, v_t$ span a five vertex cycle.

Since $xv_{j+1} \in E(G)$ and $v_{j+1}w_{j+1} \in M$, Lemma 2.1 implies that the vertices x and w_{j+1} are not adjacent. Moreover, no vertex from the set $\{w_i : 1 \leq i \leq j\}$ is adjacent to w_{j+1} . Otherwise, if some w_t is adjacent to w_{j+1} , then $\{x, v_{j+1}, w_{j+1}, w_t, v_t\}$ spans a five vertex cycle in $G[N[S]]$ (see Figure 5(b)). In accordance with Property P , when the edge $v_t w_t$ is concerned, the vertex x must be adjacent to w_{j+1} . Hence, $\{x, v_{j+1}, w_{j+1}\}$ spans a triangle, which is impossible, by Lemma 2.1.

Therefore, the set A_{j+1} is stable. In this way one can eventually reach the set $\{x\} \cup M(S)$, which must be stable in $G[N[S]]$ like all its predecessors. Now the inequality

$$|\{x\} \cup M(S)| > |S|$$

stays in contradiction with the following facts:

$$\{x\} \cup M(S) \subseteq N[S] \quad \text{and} \quad S \in \Psi(G).$$

Consequently, $G[N[S]]$ is a König-Egerváry graph. ■

Theorem 2.3 *Let G be a very well-covered graph of girth at least 4. Then the following assertions are equivalent:*

- (i) $\Psi(G)$ is a greedoid;
- (ii) G has a unique maximum matching.

Proof. Firstly, Theorem 1.2 implies that each maximum matching of G is perfect.

(i) \implies (ii) Since the girth of G is greater or equal to 4, the graph G is triangle-free. Hence, according to Theorem 1.6, a perfect matching of G is unique.

(ii) \implies (i) In fact, G has a unique perfect matching. Consequently, every maximum matching of G is uniquely restricted. Combining the fact that G is triangle-free with Theorems 2.2 and 1.6, we conclude that $\Psi(G)$ is a greedoid. ■

The structure of very well-covered graphs of girth at least 5 is more specific.

Theorem 2.4 [3], [16] *Let G be a graph of girth at least 5. Then G is very well-covered if and only if $G = H \circ K_1$, for some graph H of girth ≥ 5 .*

Consequently, a very well-covered graph of girth ≥ 5 has a unique perfect matching. Therefore, by Theorem 2.3, we get the following.

Corollary 2.5 [17] *Each very well-covered graph of girth at least 5 generates a local maximum stable set greedoid.*

It is known that the recognition of well-covered graphs is a co-**NP**-complete problem [2], [22]. Nevertheless, very well-covered graphs can be recognized in polynomial time. Actually, it goes directly from Favaron's characterization. Namely, to recognize a graph as being very well-covered, we just need to show that it has a perfect matching which satisfies property P . To find a maximum matching one needs $O(|V|^{\frac{1}{2}} \bullet |E|)$ time [19]. To check property P one has to handle $O(|V|^3)$ pairs of vertices in the worst case. All in all, it gives us an $O(|V|^3)$ algorithm.

If our goal is to recognize very well-covered graphs with unique perfect matchings, then we may do better. The reason for this is that one can test whether the graph has a unique perfect matching, and find it if it exists, in $O(|E| \bullet \log^4 |V|)$ time [7]. Finally, Theorem 2.3 and Corollary 2.5 justify that one can decide in $O(|E| \bullet \log^4 |V|)$ time whether $\Psi(G)$ is a greedoid, for a given very well-covered graph G of girth ≥ 4 .

3 Conclusions

In this paper we have proved that $\Psi(G)$ is a greedoid for those very well-covered graphs G of girth ≥ 4 that have a unique perfect matching.

Problem 3.1 *Characterize very well-covered graphs of girth three producing local maximum stable set greedoids.*

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