

Monophonic Distance in Graphs

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Abstract

For any two vertices u and v in a connected graph G , a $u - v$ path is a *monophonic path* if it contains no chords, and the *monophonic distance* $d_m(u, v)$ is the length of a longest $u - v$ monophonic path in G . For any vertex v in G , the *monophonic eccentricity* of v is $e_m(v) = \max \{d_m(u, v) : u \in V\}$. The subgraph induced by the vertices of G having minimum monophonic eccentricity is the *monophonic center* of G , and it is proved that every graph is the monophonic center of some graph. Also it is proved that the monophonic center of every connected graph G lies in some block of G . With regard to convexity, this monophonic distance is the basis of some detour monophonic parameters such as detour monophonic number, upper detour monophonic number, forcing detour monophonic number, etc. The concept of detour monophonic sets and detour monophonic numbers by fixing a vertex of a graph would be introduced and discussed. Various interesting results based on these parameters are also discussed in this chapter.

Keywords: monophonic path, monophonic distance, detour monophonic number, upper detour monophonic number, forcing detour monophonic number, vertex detour monophonic number, upper vertex detour monophonic number, forcing vertex detour monophonic number

1. Introduction

In this chapter, we consider a finite connected graph $G = (V(G), E(G))$ having no loops and multiple edges. The *order* and *size* of G are denoted by p and q , respectively. Distance in graphs is a wide branch of graph theory having numerous scientific and real-life applications. There are many kinds of distances in graphs found in literature. For any two vertices u and v in G , the *distance* $d(u, v)$ from u to v is defined as the length of a shortest $u - v$ path in G . The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v to a vertex of G . The *radius* $rad G$ of G is the

minimum eccentricity among the vertices of G , while the *diameter* $\text{diam } G$ of G is the maximum eccentricity among the vertices of G . The distance between two vertices is a fundamental concept in pure graph theory, and this distance is a metric on the vertex set of G . More results related to this distance are found in Refs. [1–9]. This distance is used to study the central concepts like center, median, and centroid of a graph [10–22]. With regard to convexity, this distance is the basis of some geodetic parameters such as geodetic number, connected geodetic number, upper geodetic number and forcing geodetic number [23–32]. The geodesic graphs, extremal graphs, distance regular graphs and distance transitive graphs are some important classes based on the distance in graphs [33, 34]. These concepts have interesting applications in location theory and convexity theory. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v . A vertex v is an *extreme vertex* if the subgraph induced by its neighbors is complete.

The detour distance, which is defined to be the length of a longest path between two vertices of a graph, is also a metric on the vertex set of G [35, 36]. For any two vertices u and v in G , the *detour distance* $D(u, v)$ from u to v is defined as the length of a longest $u - v$ path in G . The *detour eccentricity* $e_D(v)$ of a vertex v in G is the maximum detour distance from v to a vertex of G . The *detour radius* rad_D of G is the minimum detour eccentricity among the vertices of G , while the *detour diameter* diam_D of G is the maximum detour eccentricity among the vertices of G . With regard to detour convexity, the detour number of a graph was introduced and studied in Refs. [25, 37]. The detour concepts and colorings are widely used in the channel assignment problem in FM radio technology and also in certain molecular problems in theoretical chemistry.

The parameter geodetic (detour) number of a graph is global in the sense that there is exactly one geodetic (detour) number for a graph. The concept of geodetic (detour) sets and geodetic (detour) numbers by fixing a vertex of a graph was also introduced and discussed in Refs. [38–42]. With respect to each vertex of a graph, there is a geodetic (detour) number, and so there will be at most as many geodetic (detour) numbers as there are vertices in the graph.

2. Monophonic distance

Definition 2.1. A chord of a path u_1, u_2, \dots, u_n in a connected graph G is an edge $u_i u_j$ with $j \geq i + 2$. A $u - v$ path P is called a *monophonic path* if it is a chordless path. The length of a longest $u - v$ monophonic path is called the *monophonic distance* from u to v , and it is denoted by $d_m(u, v)$. A $u - v$ monophonic path with its length equal to $d_m(u, v)$ is known as a $u - v$ monophonic.

Example 2.2. Consider the graph G given in **Figure 1**. It is easily verified that $d(v_1, v_4) = 2$, $D(v_1, v_4) = 6$, and $d_m(v_1, v_4) = 4$. Thus the monophonic distance is different from both the distance and the detour distance. The monophonic path $P : v_1, v_2, v_8, v_7, v_4$ is $v_1 - v_4$ monophonic while the monophonic path $Q : v_1, v_3, v_4$ is not $v_1 - v_4$ monophonic.

The usual distance d and the detour distance D are metrics on the vertex set V of a connected graph G , whereas the monophonic distance d_m is not a metric on V . For the graph G given in **Figure 1**, $d_m(v_4, v_6) = 5$, $d_m(v_4, v_5) = 1$ and $d_m(v_5, v_6) = 1$. Hence $d_m(v_4, v_6) > d_m(v_4, v_5) + d_m(v_5, v_6)$, and so the triangle inequality is not satisfied.

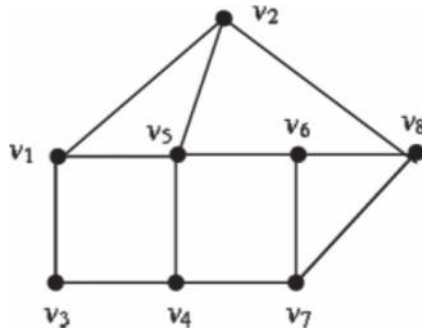


Figure 1. The graph G in Example 2.2.

The following result is an easy consequence of the respective definitions.

Proposition 2.3. Let u and v be any two vertices in a graph G of order p . Then

$$0 \leq d(u, v) \leq d_m(u, v) \leq D(u, v) \leq p - 1.$$

Result 2.4. Let u and v be any two vertices in a connected graph G . Then

- (i) $d_m(u, v) = 0$ if and only if $u = v$.
- (ii) $d_m(u, v) = 1$ if and only if uv is an edge of G .
- (iii) $d_m(u, v) = p - 1$ if and only if G is the path with endvertices u and v .
- (iv) $d(u, v) = d_m(u, v) = D(u, v)$ if and only if G is a tree.

Definition 2.5. For any vertex v in a connected graph G , the monophonic eccentricity of v is $e_m(v) = \max \{d_m(u, v) : u \in V\}$. A vertex u of G such that $d_m(u, v) = e_m(v)$ is called a monophonic eccentric vertex of v . The monophonic radius and monophonic diameter of G are defined by $rad_m G = \min \{e_m(v) : v \in V\}$ and $diam_m G = \max \{e_m(v) : v \in V\}$, respectively.

Example 2.6. Table 1 shows the monophonic distance between the vertices and also the monophonic eccentricities of vertices of the graph G given in Figure 1. It is to be noted that $rad_m G = 3$ and $diam_m G = 5$.

Remark 2.7. In any connected graph, the eccentricity of every two adjacent vertices differs by at most 1. However, this is not true in the case of monophonic distance. For the graph G given in Figure 1, $e_m(v_5) = 3$ and $e_m(v_6) = 5$.

Note 2.8. Any two vertices u and v in a tree T are connected by a unique path, and so $d(u, v) = d_m(u, v) = D(u, v)$. Hence $rad T = rad_m T = rad_D T$ and $diam T = diam_m T = diam_D T$. The monophonic radius and the monophonic diameter of some standard graphs are given in Table 2.

$d_m(v_i, v_j)$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	$e_m(v)$
v_1	0	1	1	4	1	4	3	4	4
v_2	1	0	4	3	1	5	4	1	5
v_3	1	4	0	1	2	4	4	4	4
v_4	4	3	1	0	1	5	1	4	5
v_5	1	1	2	1	0	1	3	3	3
v_6	4	5	4	5	1	0	1	1	5
v_7	3	4	4	1	3	1	0	1	4
v_8	4	1	4	4	3	1	1	0	4

Table 1. Monophonic eccentricities of the vertices of the graph G in **Figure 1**.

Graph G	K_p	C_p	$W_{1,p-1} (p \geq 4)$	$K_{1,p-1} (p \geq 2)$	$K_{m,n} (m, n \geq 2)$	P_p
$rad_m G$	1	$p - 2$	1	1	2	$\begin{bmatrix} p \\ 2 \end{bmatrix}$
$diam_m G$	1	$p - 2$	$p - 3$	2	2	$p - 1$

Table 2. Monophonic radius and monophonic diameter of some standard graphs.

The next theorem follows from Proposition 2.3.

Theorem 2.9. For a connected graph G , the following results hold:

- $e(v) \leq e_m(v) \leq e_D(v)$ for any vertex v in G .
- $rad G \leq rad_m G \leq rad_D G$.
- $diam G \leq diam_m G \leq diam_D G$.

Theorem 2.10. (a) If a, b and c are integers with $3 \leq a \leq b \leq c$, then there exists a connected graph G such that $rad G = a$, $rad_m G = b$ and $rad_D G = c$.

(b) If a, b and c are integers with $5 \leq a \leq b \leq c$, then there exists a connected graph G such that $diam G = a$, $diam_m G = b$ and $diam_D G = c$.

Proof. (a) The result is proved by considering three cases.

Case (i) $3 \leq a = b = c$. Consider $G = P_{2a+1}$, the path of order $2a + 1$. It is clear that $rad G = rad_m G = rad_D G = a$.

Case (ii) $3 \leq a \leq b < c$. Let $F_1 : u_1, u_2, \dots, u_{a-1}$ and $F_2 : v_1, v_2, \dots, v_{a-1}$ be two copies of the path P_{a-1} of order $a-1$. Let $F_3 : w_1, w_2, \dots, w_{b-a+3}$ and $F_4 : z_1, z_2, \dots, z_{b-a+3}$ be two copies of the path P_{b-a+3} of order $b-a+3$, and $F_5 = K_{c-b+1}$ the complete graph of order $c-b+1$ with $V(F_5) = \{x_1, x_2, \dots, x_{c-b+1}\}$. We construct the graph G as follows: (i) identify the vertices x_1 in F_5 and w_1 in F_3 ; also identify the vertices x_{c-b+1} in F_5 and z_1 in F_4 ; (ii) identify the vertices w_{b-a+3} in F_3 and u_2 in F_1 , and identify the vertices z_{b-a+3} in F_4 and v_2 in F_2 ; and (iii) join each vertex w_i ($1 \leq i \leq b-a+2$) in F_3 and u_1 in F_1 , and join each vertex z_i ($1 \leq i \leq b-a+2$) in F_4 and v_1 in F_2 . The resulting graph G is shown in **Figure 2**. It is easily verified that $e(v) = a$ if $v \in V(F_5)$; $e(v) > a$ if $v \in V(G) - V(F_5)$; $e_m(v) = b$ if $v \in V(F_5)$; $e_m(v) > b$ if $v \in V(G) - V(F_5)$ and $e_D(v) = c$ if $v \in V(F_5)$; and $e_D(v) > c$ if $v \in V(G) - V(F_5)$. It follows that $rad\ G = a$, $rad_m\ G = b$, and $rad_D\ G = c$.

Case (iii) $3 \leq a < b = c$. Let $E_1 : v_1, v_2, \dots, v_{2a+1}$ be a path of order $2a+1$. Let $E_2 : u_1, u_2, \dots, u_{b-a+3}$ and $E_3 : w_1, w_2, \dots, w_{b-a+3}$ be two copies of the path P_{b-a+3} of order $b-a+3$, and let E_i ($4 \leq i \leq 2(b-a)+3$) be $2(b-a)$ copies of K_1 . We construct the graph G as follows: (i) identify the vertices v_{a+1} in E_1 , u_1 in E_2 , and w_1 in E_3 ; (ii) identify the vertices v_{a-1} in E_1 and u_{b-a+3} in E_2 , and identify the vertices v_{a+3} in E_1 and w_{b-a+3} in E_3 ; and (iii) join each E_i ($4 \leq i \leq b-a+3$) with v_{a+1} in E_1 and u_{i-1} in E_2 , and join each E_i ($b-a+4 \leq i \leq 2(b-a)+3$) with v_{a+1} in E_1 and $w_{i-b+a-1}$ in E_3 . The resulting graph G is shown in **Figure 3**.

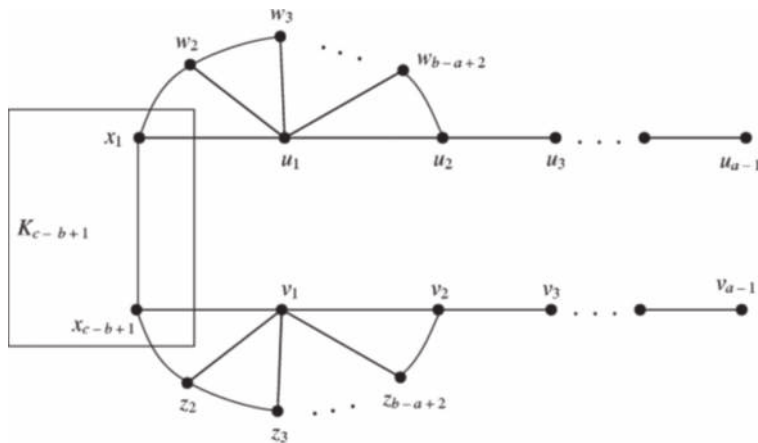


Figure 2. A graph G in Case (ii) of Theorem 2.10(a).

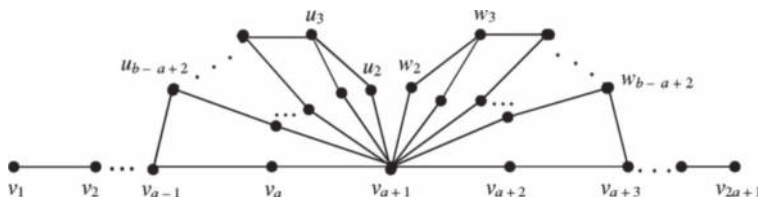


Figure 3. A graph G in Case (iii) of Theorem 2.10(a).

It is easily verified that $e(v_{a+1}) = a$; $e(v) > a$ if $v \in V(G) - \{v_{a+1}\}$; $e_m(v_{a+1}) = b$; $e_m(v) > b$ if $v \in V(G) - \{v_{a+1}\}$, and $e_D(v_{a+1}) = c$; and $e_D(v) > c$ if $v \in V(G) - \{v_{a+1}\}$. It follows that $rad\ G = a$, $rad_m\ G = b$, and $rad_D\ G = c$.

(b) This result is also proved by considering three cases.

Case (i) $5 \leq a = b = c$. Let G be a path of order $a+1$. Then $diam\ G = diam_m\ G = diam_D\ G = a$.

Case (ii) $5 \leq a \leq b < c$. Let $F_1 : u_1, u_2, \dots, u_{a-1}$ be the path P_{a-1} of order $a-1$; $F_2 : w_1, w_2, \dots, w_{b-a+3}$ be the path P_{b-a+3} of order $b-a+3$; and $F_3 = K_{c-b+1}$ be the complete graph of order $c-b+1$ with $V(F_3) = \{x_1, x_2, \dots, x_{c-b+1}\}$. We construct the graph G as follows: (i) identify the vertices x_1 in F_3 and w_1 in F_2 , and identify the vertices w_{b-a+3} in F_2 and u_2 in F_1 , and (ii) join each vertex w_i ($1 \leq i \leq b-a+2$) in F_2 and u_1 in F_1 . The resulting graph G is shown in **Figure 4**. It is easily verified that $e(v) = a$ if $v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; $e(v) < a$ if $v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\})$, and $e_m(v) = b$ if $v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; $e_m(v) < b$ if $v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\})$, and $e_D(v) = c$ if $v \in (V(F_3) - \{x_1\}) \cup \{u_{a-1}\}$; and $e_D(v) < c$ if $v \in V(F_2) \cup (V(F_1) - \{u_{a-1}\})$. It follows that $diam\ G = a$, $diam_m\ G = b$ and $diam_D\ G = c$.

Case (iii) $5 \leq a < b = c$. Let $E_1 : v_1, v_2, \dots, v_{a+1}$ be a path of order $a+1$; $E_2 : w_1, w_2, \dots, w_{b-a+3}$ be another path of order $b-a+3$; and E_i ($3 \leq i \leq b-a+2$) be $b-a$ copies of K_1 . Let G be the graph obtained from E_i for $i = 1, 2, \dots, b-a+2$ by identifying the vertices v_{a-2} and v_a of E_1 with w_1 and w_{b-a+3} of E_2 , respectively, and joining each E_i ($3 \leq i \leq b-a+2$) with v_{a-2} and w_i . The graph G is shown in **Figure 5**.

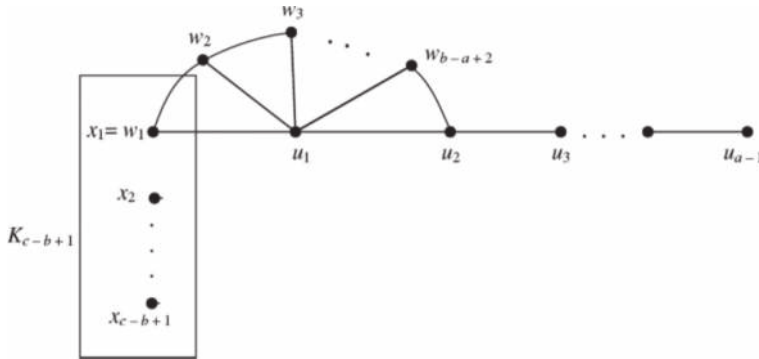


Figure 4. A graph G in Case (ii) of Theorem 2.10(b).

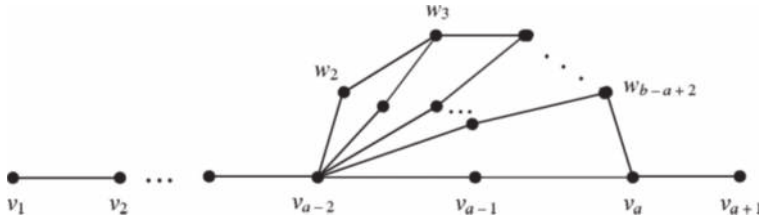


Figure 5. A graph G in Case (iii) of Theorem 2.10(b).

It is easily verified that $e(v) = a$ if $v \in \{v_1, v_{a+1}\}$; $e(v) \leq a$ if $v \in V(G) - \{v_1, v_{a+1}\}$, and $e_m(v) = b$ if $v \in \{v_1, v_{a+1}\}$; $e_m(v) \leq b$ if $v \in V(G) - \{v_1, v_{a+1}\}$, and $e_D(v) = c$ if $v \in \{v_1, v_{a+1}\}$; and $e_D(v) \leq c$ if $v \in V(G) - \{v_1, v_{a+1}\}$. It follows that $rad\ G = a$, $rad_m\ G = b$ and $rad_D\ G = c$.

For any connected graph G , the inequalities $rad\ G \leq diam\ G \leq 2\ rad\ G$ and $rad_D\ G \leq diam_D\ G \leq 2\ rad_D\ G$ hold. However, this is not true in the case of monophonic radius and monophonic diameter. For example, when the graph G is the wheel $W_{1,p-1}$ ($p \geq 6$), it is easily seen that $rad_m\ G = 1$ and $diam_m\ G = p - 3 \geq 3$ so that $diam_m\ G > 2\ rad_m\ G$.

It is proved in Ref. [6] that if a and b are any two positive integers such that $a \leq b \leq 2a$, then there is a connected graph G with $rad\ G = a$ and $diam\ G = b$. Also, it is proved in Ref. [35] that if a and b are any two positive integers such that $a \leq b \leq 2a$, then there is a connected graph G with $rad_D\ G = a$ and $diam_D\ G = b$.

Now, the following theorem gives a realization result for $rad_m\ G \leq diam_m\ G$.

Theorem 2.11. *If a and b are positive integers with $a \leq b$, then there exists a connected graph G such that $rad_m\ G = a$ and $diam_m\ G = b$.*

Proof. This result is proved by considering three cases.

Case (i) $a = b \geq 1$. Let G be the cycle C_{a+2} . Then $rad_m\ G = a$ and $diam_m\ G = b$.

Case (ii) $a < b \leq 2a$. Let $C_1 : u_1, u_2, \dots, u_{a+2}, u_1$ be a cycle of order $a + 2$ and $C_2 : v_1, v_2, \dots, v_{b-a+2}, v_1$ be a cycle of order $b - a + 2$. Let G be the graph obtained by identifying the vertex u_1 of C_1 and v_1 of C_2 . Since $b \leq 2a$, it follows that $b - a + 2 \leq a + 2$. It is clear that $d_m(u_1, x) \leq a$ for any x in G and $d_m(u_1, u_{a+1}) = a$. Therefore, $e_m(u_1) = a$. Also, it is clear that there is no vertex x with $e_m(x) < a$ and so $rad_m\ G = a$. It is clear that $d_m(u_3, v_3) = b$ and $d_m(u_3, x) \leq b$ for any vertex x in G and so $e_m(u_3) = b$. Also, it is easy to see that $e_m(x) \leq b$ for every vertex x in G so that $diam_m\ G = b$.

Case (iii) $b > 2a$. Let G be the graph obtained by identifying the central vertex of the wheel $W = K_1 + C_{b+2}$ ($b \geq 2$) and an endvertex of the path P_{2a} . Since $b > 2a$, $e_m(x) = b$ for any vertex $x \in V(C_{b+2})$. Also, $a \leq e_m(x) \leq 2a$ for any vertex $x \in V(P_{2a})$ and $e_m(v_a) = a$. Hence $rad_m\ G = a$ and $diam_m\ G = b$.

2.1. Monophonic center and monophonic periphery

Definition 2.12. A vertex v in a connected graph G is called a monophonic central vertex if $e_m(v) = rad_m\ G$, and the subgraph induced by the monophonic central vertices of G is the monophonic center $C_m(G)$ of G . A vertex v in G is called a monophonic peripheral vertex if $e_m(v) = diam_m\ G$, and the subgraph induced by the monophonic peripheral vertices of G is the monophonic periphery $P_m(G)$ of G .

Example 2.13. Consider the graph G given in **Figure 1**. It is easily verified that v_5 is the monophonic central vertex and v_2, v_4 , and v_6 are the monophonic peripheral vertices of G .

Remark 2.14. The monophonic center of a connected graph need not be connected. For the graph G given in **Figure 6**, $C_m(G) = \{v_3, v_6\}$.

Theorem 2.15. Every graph is the monophonic center of some connected graph.

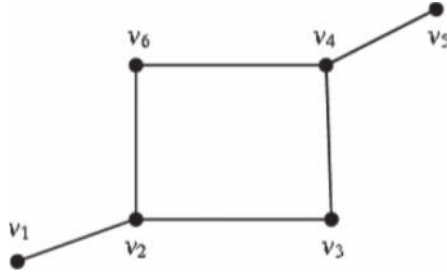


Figure 6. A graph G in Remark 2.14.

Proof. Let G be a graph. We show that G is the monophonic center of some graph. Let $l = d_m$ be the monophonic diameter of G . Let $P : u_1, u_2, \dots, u_l$ and $Q : v_1, v_2, \dots, v_l$ be two copies of the path P_l . The required graph H given in Figure 7 is got from G , P , and Q by joining each vertex of G with u_1 in P and v_1 in Q . Then $e_{mH}(x) = d_m$ for each vertex x in G and $d_m + 1 \leq e_{mH}(x) \leq 2d_m$ for each vertex x not in G . Therefore, $V(G)$ is the set of monophonic central vertices of H and so $C_m(H) = G$.

More specifically, it is proved in Ref. [43] that the center of every connected graph G lies in a single block of G . Also, it is proved in Ref. [35] that the detour center of every connected graph G lies in a single block of G . The same result is true for the monophonic center also, as proved in the following theorem.

Theorem 2.16. *The monophonic center $C_m(G)$ of every connected graph G is a subgraph of some block of G .*

Proof. Suppose that there is a connected graph G such that its monophonic center $C_m(G)$ is not a subgraph of a single block of G . Then G has a cut vertex v such that $G - v$ contains two components H_1 and H_2 , each containing vertices of $C_m(G)$. Let u be a vertex of G such that $e_m(v) = d_m(u, v)$, and let P_1 be a $u - v$ longest monophonic path in G . Then at least one of H_1 and H_2 contains no vertices of P_1 , say H_2 contains no vertex of P_1 . Now, take a vertex w in $C_m(G)$ that belongs to H_2 , and let P_2 be a $v - w$ longest monophonic path in G . Since v is a cut vertex, P_1 followed by P_2 gives a $u - w$ longest monophonic path with its length greater than that of P_1 . This gives $e_m(w) > e_m(v)$ so that w is not a monophonic central vertex of G , which is a contradiction.

Corollary 2.17. *For any tree, the monophonic center is isomorphic to K_1 or K_2 .*

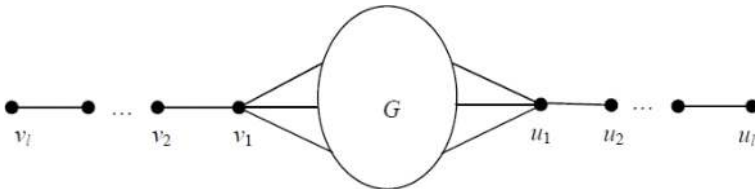


Figure 7. A graph H in Theorem 2.15.

It is proved in Ref. [44] that a nontrivial graph G is the periphery of some connected graph if and only if every vertex of G has eccentricity 1 or no vertex of G has eccentricity 1. Also, it is proved in Ref. [35] that a connected graph G of order $p \geq 3$ and radius 1 is the detour periphery of some connected graph if and only if G is Hamiltonian. A similar result is given in the next theorem, and for a proof, one may refer to Ref. [45].

Theorem 2.18. *A nontrivial graph G is the monophonic periphery of some connected graph if and only if every vertex of G has monophonic eccentricity 1 or no vertex of G has monophonic eccentricity 1.*

Definition 2.19. *A connected graph G is monophonic self-centered if $\text{rad}_m G = \text{diam}_m G$, that is, if G is its own monophonic center.*

Example 2.20. The complete graph K_n , the cycle C_n , and the complete bipartite graph $K_{m,n}$ ($m, n \geq 2$) are monophonic self-centered graphs.

The following problem is left open.

Problem 2.21. *Characterize monophonic self-centered graphs.*

Further results on monophonic distance in graphs can be found in Refs. [45, 46].

3. Detour monophonic number

Throughout this section, by a $u - v$ detour monophonic path, we mean a longest $u - v$ monophonic path.

Definition 3.1. *A set S of vertices of a connected graph G is called a detour monophonic set if every vertex of G lies on a $u - v$ detour monophonic path for some $u, v \in S$. The detour monophonic number of G is defined as the minimum cardinality of a detour monophonic set of G and is denoted by $dm(G)$.*

Example 3.2. For the graph G given in **Figure 8**, $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$, $S_3 = \{v_3, v_4, v_2\}$, $S_4 = \{v_3, v_4, v_3\}$, $S_5 = \{v_1, v_3, v_5\}$, $S_6 = \{v_1, v_3, v_6\}$, $S_7 = \{v_2, v_4, v_5\}$, and $S_8 = \{v_2, v_4, v_6\}$ are the minimum detour monophonic sets of G and so $dm(G) = 3$.

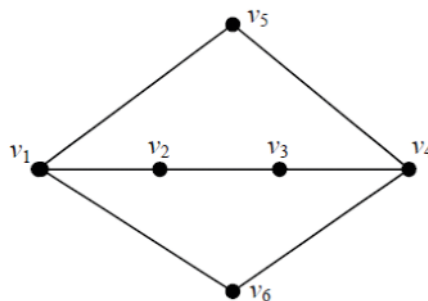


Figure 8. The graph G in Example 3.2.

If a vertex belongs to every minimum detour monophonic set of G , then it is called a *detour monophonic vertex* of G . If S is the unique minimum detour monophonic set of G , then S is the set of all detour monophonic vertices of G . In the next theorem, we show that there are certain vertices in a nontrivial connected graph G that are detour monophonic vertices of G .

Theorem 3.3. *Every detour monophonic set of a connected graph G contains all its extreme vertices. Moreover, if the set of all extreme vertices S of G is a detour monophonic set of G , then S is the unique minimum detour monophonic set of G .*

Proof. Let v be an extreme vertex and let S be a detour monophonic set of G . If v is not an element of S , then there exist two elements x and y in S such that v is an internal vertex of an $x - y$ detour monophonic path, say P . Let u and w be the vertices on P adjacent to v . Then u and w are not adjacent and so v is not an extreme vertex of G , which is a contradiction. Therefore v belongs to every detour monophonic set of G . Thus, if S is the set of all extreme vertices of G , then $dm(G) \geq |S|$. On the other hand, if S is a detour monophonic set of G , then $dm(G) \leq |S|$. Therefore $dm(G) = |S|$ and S is the unique minimum detour monophonic set of G .

The following two theorems are easy to prove.

Theorem 3.4. *Let G be a connected graph with a cut vertex v and let S be a detour monophonic set of G . Then every component of $G - v$ contains an element of S .*

Theorem 3.5. *No cut vertex of a connected graph G belongs to any minimum detour monophonic set of G .*

Since every end-block B is a branch of G at some cut vertex, it follows Theorem 3.4 and Theorem 3.5 that every minimum detour monophonic set of G contains at least one vertex from B that is not a cut vertex. Thus the following corollaries are consequences of Theorems 3.4 and 3.5.

Corollary 3.6. *If G is a connected graph with $k \geq 2$ end-blocks, then $dm(G) \geq k$.*

Corollary 3.7. *If k is the maximum number of blocks to which a vertex in a graph G belongs, then $dm(G) \geq k$.*

Theorem 3.8. *For any connected graph G , $2 \leq dm(G) \leq p$.*

Theorem 3.9. *For any connected graph G , $dm(G) = p$ if and only if G is complete.*

Proof. Let $dm(G) = p$. Suppose that G is not a complete graph. Then there exist two vertices u and v such that u and v are not adjacent in G . Since G is connected, there is a detour monophonic path from u to v , say P , with length at least 2. Clearly, $(V(G) - V(P)) \cup \{u, v\}$ is a detour monophonic set of G and hence $dm(G) \leq p - 1$, which is a contradiction. Conversely, if G is complete, then by Theorem 3.3, $dm(G) = p$.

Theorem 3.10. *If G is a nontrivial connected graph of order p and monophonic diameter d , then $dm(G) \leq p - d + 1$.*

Proof. Let $x, y \in V(G)$ such that G contains an $x - y$ detour monophonic path P of length $diam_m G = d$. Let $S = (V(G) - V(P)) \cup \{x, y\}$. Since S is a detour monophonic set of G , it follows that $dm(G) \leq |S| \leq p - d + 1$.

Theorem 3.11. *For every nontrivial tree T of order p and monophonic diameter d , $dm(T) = p - d + 1$ if and only if T is a caterpillar.*

Proof. Let T be any nontrivial tree. Let $P : u = v_0, v_1, \dots, v_d$ be a monophonic diametral path. Let k be the number of endvertices of T and let l be the number of internal vertices of T other than v_1, v_2, \dots, v_{d-1} . Then $d - 1 + l + k = p$. By Theorem 3.3 and Theorem 3.5, $dm(T) = k$ and so $dm(T) = p - d - l + 1$. Hence $dm(T) = p - d + 1$ if and only if $l = 0$, if and only if all the internal vertices of T lie on the monophonic diametral path P , and if and only if T is a caterpillar.

It is known that $rad_m G \leq diam_m G$ for a connected graph G . It is proved in Ref. [45] that if a and b are any two positive integers such that $a \leq b$, then there is a connected graph G with $rad_m G = a$ and $diam_m G = b$. The same result can also be extended so that the detour monophonic number can be prescribed when $rad_m G < diam_m G$, and for a proof, one may refer to Ref. [47].

Theorem 3.12. *For positive integers r , d and $n \geq 4$ with $r < d$, there exists a connected graph G with $rad_m G = r$, $diam_m G = d$ and $dm(G) = n$.*

Problem 3.13. For any three positive integers r , d and $n \geq 4$ with $r = d$, does there exist a connected graph G with $rad_m G = r$, $diam_m G = d$ and $dm(G) = n$?

3.1. Upper detour monophonic number

Definition 3.14. A detour monophonic set S of a connected graph G is called a minimal detour monophonic set if no proper subset of S is a detour monophonic set of G . The maximum cardinality of a minimal detour monophonic set of G is the upper detour monophonic number of G , denoted by $dm^+(G)$.

Example 3.15. Consider the graph G given in **Figure 8**. The minimal detour monophonic sets are $S_1 = \{v_1, v_2, v_3\}$, $S_2 = \{v_2, v_3, v_4\}$, $S_3 = \{v_3, v_6, v_2\}$, $S_4 = \{v_3, v_6, v_3\}$, $S_5 = \{v_1, v_3, v_5\}$, $S_6 = \{v_1, v_3, v_6\}$, $S_7 = \{v_2, v_4, v_5\}$, $S_8 = \{v_2, v_4, v_6\}$ and $S_9 = \{v_1, v_4, v_5, v_6\}$. For this graph, the upper detour monophonic number is 4, and the detour monophonic number is 3.

Note 3.16. Every minimum detour monophonic set is a minimal detour monophonic set, but the converse is not true. For the graph G given in **Figure 8**, S_9 is a minimal detour monophonic set, but it is not a minimum detour monophonic set of G .

The following three theorems are easy to prove.

Theorem 3.17. *For any connected graph G , $2 \leq dm(G) \leq dm^+(G) \leq p$.*

Theorem 3.18. *For a connected graph G , $dm(G) = p$ if and only if $dm^+(G) = p$.*

Theorem 3.19. *If G is a connected graph of order p with $dm(G) = p - 1$, then $dm^+(G) = p - 1$.*

The next theorem is an interesting realization result, and for a proof, one may refer to Ref. [48].

Theorem 3.20. *For any three positive integers a , b and n with $2 \leq a \leq n \leq b$, there is a connected graph G with $dm(G) = a$, $dm^+(G) = b$ and a minimal detour monophonic set of cardinality n .*

3.2. Forcing detour monophonic number

A connected graph G may contain more than one minimum detour monophonic sets. For example, the graph G given in **Figure 8** contains eight minimum detour monophonic sets. For each minimum detour monophonic set S in G , there is always some subset T of S that uniquely determines S as the minimum detour monophonic set containing T . Such sets are called “forcing detour monophonic subsets” and these sets are discussed in this section.

Definition 3.21. Let S be a minimum detour monophonic set of a connected graph G . A subset S' of S is a forcing detour monophonic subset for S if S is the unique minimum detour monophonic set that contains S' . A forcing detour monophonic subset for S of minimum cardinality is a minimum forcing detour monophonic subset of S . The cardinality of a minimum forcing detour monophonic subset of S is the forcing detour monophonic number $fdm(S)$ in G . The forcing detour monophonic number of G is $fdm(G) = \min \{fdm(S)\}$, where the minimum is taken over all minimum detour monophonic sets S in G .

Example 3.22. For the graph G given in **Figure 9**, $S_1 = \{z, w, v\}$, $S_2 = \{z, w, u\}$ and $S_3 = \{z, w, x\}$ are the minimum detour monophonic sets of G . It is clear that $fdm(S_1) = 1$, $fdm(S_2) = 1$ and $fdm(S_3) = 1$ so that $fdm(G) = 1$. For the graph G given in **Figure 10**, $S = \{y, v\}$ is the unique minimum detour monophonic set of G and so $fdm(G) = 0$.

The next theorem follows immediately from the definitions of the detour monophonic number and the forcing detour monophonic number of a graph G .

Theorem 3.23. For a connected graph G , $0 \leq fdm(G) \leq dm(G) \leq p$.

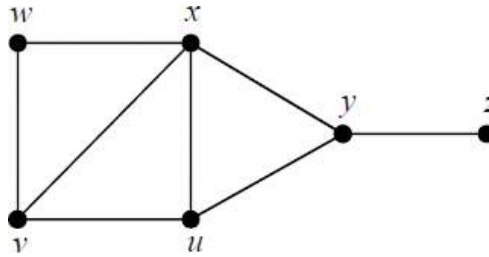


Figure 9. A graph G with $fdm(G) = 1$.

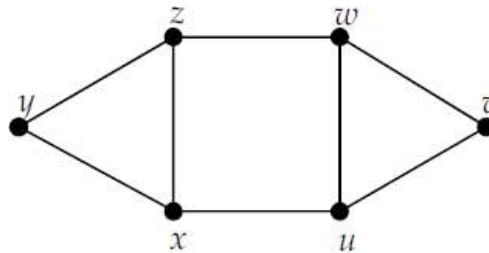


Figure 10. A graph G with $fdm(G) = 0$.

The following theorem characterizes graphs G for which $fdm(G) = 0$, $fdm(G) = 1$ and $fdm(G) = dm(G)$. The proof is an easy consequence of the definitions of the detour monophonic number and the forcing detour monophonic number.

Theorem 3.24. *Let G be a connected graph. Then*

- (i) *$fdm(G) = 0$ if and only if G contains exactly one minimum detour monophonic set.*
- (ii) *$fdm(G) = 1$ if and only if G contains two or more minimum detour monophonic sets, one of which is a unique minimum detour monophonic set that contains one of its elements.*
- (iii) *$fdm(G) = dm(G)$ if and only if no minimum detour monophonic set of G is the unique minimum detour monophonic set that contains any of its proper subsets.*

The next theorem gives a realization result for the parameters $fdm(G)$ and $dm(G)$, and for a proof, the reader may refer to Ref. [49].

Theorem 3.25. *For every pair a, b of positive integers with $0 \leq a < b$ and $b \geq 2$, there exists a connected graph G such that $fdm(G) = a$ and $dm(G) = b$.*

Further results on detour monophonic concepts in graphs can be found in Refs. [47–50].

4. Vertex detour monophonic number

The parameter detour monophonic number of a graph is global in the sense that there is exactly one detour monophonic number for a graph. The concept of detour monophonic sets and detour monophonic numbers by fixing a vertex of a graph was also introduced and discussed in this section. With respect to each vertex of a graph, there is a detour monophonic number, and so there will be at most as many detour monophonic numbers as there are vertices in the graph.

Definition 4.1. *For any vertex x in a connected graph G , a set S_x of vertices in G is called an x -detour monophonic set if every vertex of G lies on an $x - y$ detour monophonic path in G for some y in S_x . The x -detour monophonic number of G , denoted by $dm_x(G)$, is defined to be the minimum cardinality of an x -detour monophonic set of G . An x -detour monophonic set of cardinality $dm_x(G)$ is called a dm_x -set of G .*

It is easy to observe that for any vertex x in G , x does not belong to any dm_x -set of G .

Example 4.2. For the graph G given in **Figure 11**, the minimum vertex detour monophonic sets and the vertex detour monophonic numbers are given in **Table 3**.

The next two theorems are easy to prove.

Theorem 4.3. *For any vertex x in a connected graph G , the following results hold.*

- (i) *Every dm_x -set of G contains all its extreme vertices other than the vertex x (whether x is extreme vertex or not).*
- (ii) *No dm_x -set of G contains a cut vertex of G .*

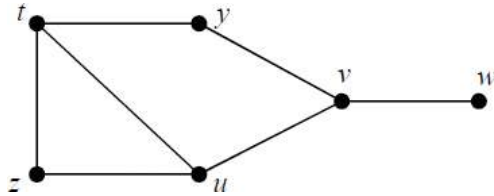


Figure 11. The graph G in Example 4.2.

Vertex	Minimum vertex detour monophonic sets	Vertex detour monophonic number
t	$\{z,w\}$	2
y	$\{w,z,t\}, \{w,z,u\}$	3
z	$\{u,w\}, \{w,y\}$	2
u	$\{w,z,y\}$	3
v	$\{w,t,z\}, \{w,u,z\}$	3
w	$\{t,z\}, \{z,u\}$	2

Table 3. Vertex detour monophonic numbers of the graph G in Figure 11.

Theorem 4.4. For any vertex x in a connected graph G of order p , $1 \leq dm_x(G) \leq p - 1$.

Theorem 4.5. For any vertex x in a connected graph G of order p , $dm_x(G) = p - 1$ if and only if $deg\ x = p - 1$.

Proof. Let x be any vertex in a connected graph G of order p . Let $dm_x(G) = p - 1$. If $deg\ x < p - 1$, then there is a vertex u in G that is not adjacent to x . Since G is connected, there is a detour monophonic path from x to u , say P , with length greater than or equal to 2. Then $(V(G) - V(P)) \cup \{u\}$ is an x -detour monophonic set of G so that $dm_x(G) \leq p - 2$, which is a contradiction. Conversely, let $deg\ x = p - 1$. Hence x is adjacent to all other vertices of G . This shows that all these vertices form the dm_x -set of G and so $dm_x(G) = p - 1$.

Corollary 4.6. A graph G is complete if and only if $dm_x(G) = p - 1$ for every vertex x in G .

4.1. Upper vertex detour monophonic number

Definition 4.7. Let x be any vertex of a connected graph G . An x -detour monophonic set S_x is called a minimal x -detour monophonic set if no proper subset of S_x is an x -detour monophonic set. The upper x -detour monophonic number is the maximum cardinality of a minimal x -detour monophonic set of G and is denoted by $dm_x^+(G)$.

Example 4.8. For the graph G given in Figure 12, the minimum vertex detour monophonic sets the vertex detour monophonic numbers, the minimal vertex detour monophonic sets and the upper vertex detour monophonic numbers are given in Table 4.

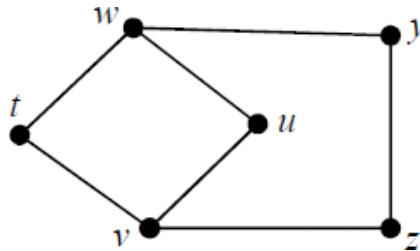


Figure 12. The graph G in Example 4.8.

Vertex x	Minimum x -detour monophonic sets	$dm_x(G)$	Minimal x -detour monophonic sets	$dm_x^+(G)$
t	$\{u,y\}, \{u,z\}$	2	$\{u,y\}, \{u,z\}$	2
u	$\{t,y\}, \{t,z\}$	2	$\{t,y\}, \{t,z\}$	2
v	$\{w,y\}, \{z,y\}$	2	$\{w,y\}, \{z,y\}, \{w,t,u\}$	3
w	$\{z,y\}, \{z,v\}$	2	$\{z,y\}, \{z,v\}, \{v,t,u\}$	3
y	$\{v,z\}, \{v,t\}, \{v,u\}$	2	$\{v,z\}, \{v,t\}, \{v,u\}, \{t,u,w\}$	3
z	$\{w,y\}, \{w,t\}, \{w,u\}$	2	$\{w,y\}, \{w,t\}, \{w,u\}, \{v,t,u\}$	3

Table 4. Upper vertex detour monophonic numbers of the graph G in Figure 12.

Since every minimum x -detour monophonic set is a minimal x -detour monophonic set, we have $1 \leq dm_x(G) \leq dm_x^+(G) \leq p - 1$. In view of this, we have the following theorems, and for proofs one may refer to Ref. [51].

Theorem 4.9. Let x be any vertex in a connected graph G of order $p \geq 3$. If $dm_x(G) = 1$, then $dm_x^+(G) \leq p - 2$.

Theorem 4.10. Let x be any vertex in a connected graph G . Then $dm_x(G) = p - 1$ if and only if $dm_x^+(G) = p - 1$.

Theorem 4.11. For any four integers j, k, l and p with $2 \leq j \leq k \leq l \leq p - 7$, there exists a connected graph G of order p with $dm_x(G) = j$, $dm_x^+(G) = l$ and a minimal x -detour monophonic set of cardinality k .

4.2. Forcing vertex detour monophonic number

Definition 4.12. Let x be any vertex of a connected graph G and let S_x be a minimum x -detour monophonic set of G . A subset S' of S_x is an x -forcing subset for S_x if S_x is the unique minimum x -detour monophonic set that contains S' . An x -forcing subset for S_x of minimum cardinality is a minimum x -forcing subset of S_x . The cardinality of a minimum x -forcing subset of S_x is the forcing x -detour monophonic number $fdm_x(S_x)$ in G . The forcing x -detour monophonic number of G is $fdm_x(G) = \min \{fdm_x(S_x)\}$, where the minimum is taken over all minimum x -detour monophonic sets S_x in G .

Definition 4.13. Let x be any vertex of a connected graph G . The upper forcing x -detour monophonic number, $fdm_x^+(G)$, of G is the maximum forcing x -detour monophonic number among all minimum x -detour monophonic sets of G .

Example 4.14. For the graph G given in **Figure 13**, the minimum vertex detour monophonic sets, the vertex detour monophonic numbers, the forcing vertex detour monophonic sets, the forcing vertex detour monophonic numbers and the upper forcing vertex detour monophonic numbers are given in **Table 5**.

Theorem 4.15. For any vertex x in a connected graph G , $0 \leq fdm_x(G) \leq fdm_x^+(G) \leq dm_x(G)$.

The following theorem gives a realization result for the parameters $fdm_x(G)$, $fdm_x^+(G)$, $dm_x(G)$, and for a proof, one may refer to Ref. [52].

Theorem 4.16. For any three integers r , s and t with $2 \leq r \leq s \leq t$ with $2r - s \geq 0$, there exists a connected graph G with $fdm_x(G) = r$, $fdm_x^+(G) = s$ and $dm_x(G) = t$ for some vertex x in G .

There are useful applications of these concepts to security-based communication network design. In the case of designing the channel for a communication network, although all the vertices are covered by the network when considering detour monophonic sets, some of the edges may be left out. This drawback is rectified in the case of edge detour monophonic sets so that considering edge detour monophonic sets is more advantageous to real-life application of communication networks. The edge detour monophonic sets are discussed in Refs. [53–55].

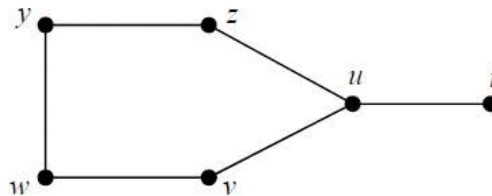


Figure 13. The graph G in Example 4.14.

Vertex x	dm_x -sets	$dm_x(G)$	x -forcing subsets	$fdm_x(G)$	$fdm_x^+(G)$
t	$\{v,w\}, \{y,z\}, \{w,y\}$	2	$\{v\}, \{z\}, \{w,y\}$	1	2
u	$\{t,v,w\}, \{t,y,z\}, \{t,w,y\}$	3	$\{v\}, \{z\}, \{w,y\}$	1	2
v	$\{t,z\}$	2	Φ	0	0
w	$\{t,v\}, \{t,z\}$	2	$\{v\}, \{z\}$	1	1
y	$\{t,v\}, \{t,z\}$	2	$\{v\}, \{z\}$	1	1
z	$\{t,v\}$	2	Φ	0	0

Table 5. Forcing and upper forcing vertex detour monophonic numbers of the graph G in Figure 13.

5. Conclusion

In this chapter, the new distance known as monophonic distance in a graph is introduced, and its properties are studied. Its relationship with the usual distance and detour distance is discussed. Various realization theorems are proved with regard to the radius (diameter), monophonic radius (monophonic diameter) and detour radius (detour diameter). Results regarding monophonic center and monophonic periphery of a graph are presented. Further, the concept of a detour monophonic set in a graph is introduced and its various properties are presented. Consequently, the parameters, viz., detour monophonic number, upper detour monophonic number and forcing detour monophonic number of a graph are introduced and studied. In a similar way, the vertex detour monophonic number, the upper vertex detour monophonic number and the forcing vertex detour monophonic number of a graph are introduced and studied. Many interesting characterization theorems and also realization theorems with regard to all these parameters are presented. The results presented in this chapter would help the researchers in graph theory to develop new results and applications to various branches of science.

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