

Contractible Hamiltonian Cycles in Polyhedral Maps

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Abstract

We present a necessary and sufficient condition for existence of a contractible Hamiltonian Cycle in the edge graph of equivelar maps on surfaces. We also present an algorithm to construct such cycles. This is further generalized and shown to hold for more general maps.

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1 Introduction and Definitions

Recall that a graph $G := (V, E)$ is a simple graph with vertex set V and edge set E . A *surface* S is a connected, compact, 2-dimensional manifold without boundary. A *map* on a surface S is an embedding of a finite graph G such that the closure of components of $S \setminus G$ are p -gonal 2-disks where ($p \geq 3$). The components are also called *facets*. The map M is called a *polyhedral map* if non - empty intersection of any two facets of the map is either a vertex or an edge see [4]. We call G the edge graph of the map and denote it by $EG(M)$. The vertices and edges of G are also called vertices and edges of the map, respectively. In what follows we will use the terms map and polyhedral map interchangeably to denote a polyhedral map. A polyhedral map M is called $\{p, q\}$ *equivelar*, $p, q \geq 3$ if each vertex in M is incident with exactly q numbers of p -gonal facets. If $p = 3$ then the map is called a q -equivelar triangulation or a degree - regular triangulation of type q . A path P in a graph G is a subgraph $P : [v_1 v_2 \dots v_n]$ of G , such that the vertex set of P is $V(P) = \{v_1, v_2, \dots, v_n\} \subseteq V(G)$ and $v_i v_{i+1}$ are edges in P for $1 \leq i \leq n - 1$. A path $P : [v_1, v_2, \dots, v_n]$ in G is said to be a cycle if $v_n v_1$ is also an edge in P . A graph without any cycles is called a tree. Length $l(P)$ of path P is the number of edges in P . See [10] for details about graphs on surfaces and [3] for graph theory related terminology.

In this article we are interested in finding out whether a Hamiltonian cycle exists in the edge graph of a polyhedral map?. Such cycles in planar graphs have been extensively studied. For example, in [12] Tutte showed that every 4-connected planar graph has a Hamiltonian cycle. In 1970, Grünbaum, [6] conjectured that every 4-connected graph which admits an embedding in the torus has a Hamiltonian cycle. In [7] it is shown that a 3-connected bipartite graph embeddable in torus has a Hamiltonian cycle if it is balanced

and each vertex of one of its partite sets has degree four. We are thus led to think about vertex degree considerations in graphs while looking for such cycles. The work in this direction has been going on for quite some time. A. Altshuler [1], [2] studied Hamiltonian cycles in the edge graph of equivelar maps on the torus. He showed that in the graph consisting of vertices and edges of equivelar maps of above type there exists a Hamiltonian cycle. Continuous analogues of such cycles in maps may or may not be homotopic to the generators of fundamental group of the surface on which they lie. The cycles which are homotopic to a generator are essential cycles and those which are homotopic to a point are inessential cycles. We are trying to figure out such inessential cycles combinatorially.

We will call a cycle in the edge graph of a map to be *contractible* if it bounds a 2-disk (2-cell). For example boundary cycle of a facet is contractible. Study of such cycles are being actively pursued and they are being applied to obtain important results. For example, in [11] the authors use contractible Hamiltonian cycle in triangulations of projective plane to determine a linear upper bound on the number of diagonal flips to mutually transform any two triangulations into one another. In [5] the authors produce a contractible Hamiltonian cycle in every 5-connected triangulation of the Klein bottle. In [13] the second author presented a necessary and sufficient condition for existence of contractible Hamiltonian cycles in equivelar triangulation of surfaces. In this article we extend the results of [13] to present a necessary and sufficient condition for existence of a contractible Hamiltonian cycle in edge graph of an equivelar map. In section 4 we give computer implementable steps of the ideas used in proof to determine (possibly!!) all contractible Hamiltonian cycles in a given map. This is extended in section 5 to maps other than the equivelar maps.

We begin with some definitions which will be needed in the course of proof of main Theorem 1. For more details on these topics one may refer to [9].

If v is a vertex of a map K then the number of edges incident with v is called the *degree* of v and is denoted by $\deg(v)$. If the number of vertices, edges and facets of K are denoted by $f_0(K)$, $f_1(K)$ and $f_2(K)$ respectively, then the integer $\chi(K) = f_0(K) - f_1(K) + f_2(K)$ is called the *Euler characteristic* of K . The *dual map* M , of K is defined to be the map on same surface as K which has for its vertices the set of facets of K and two vertices u_1 and u_2 of M are ends of an edge of M if the corresponding facets in K have an edge in common. The well known maps of type $\{3, 6\}$ and $\{6, 3\}$ on the surface of torus are examples of mutually dual maps. Let K' be a subset of set of faces of K and $D = \cup_{\sigma \in K'} \sigma$. If D is connected then we will call it a *disc* in K . An edge e of a facet σ in D is said to be a *free* edge of σ , if e is not contained in any other facet in D . The process of deleting a facet which has a free edge in a D is called an *elementary collapse* on D . Applying a sequence of elementary collapses to D results into another disc \mathcal{K} of K . We say that D collapses to \mathcal{K} . If D collapses to a point then we say that D is a collapsible map. It is a fact that collapsible maps are contractible [compare [8], pp. 32].

Consider a $\{p, q\}$ equivelar map K on a surface S that has n vertices. :

Definition 1 Let M denote the dual map of K . Let $T := (V, E)$ denote a tree in the edge graph $EG(M)$ of M . We say that T is a *proper tree* if the following conditions hold:

1. $\sum_{i=1}^k \deg(v_i) = n + 2(k - 1)$, where $V = \{v_1, v_2, \dots, v_k\}$, $\deg(v)$ denotes degree of v in $EG(M)$
2. whenever two vertices u_1 and u_2 of T lie on a face F in M , a path $P[u_1, u_2]$ joining u_1 and u_2 in the boundary ∂F of F is a part of T , and

3. any path P in T which lies in a face F of M is of length at most $q - 2$, where $q = \text{length of } (\partial F)$.

Remark 1 If the map K is $\{p, q\}$ equivelar then $k = \frac{n-2}{p-2}$. Thus, for an equivelar triangulation on n vertices the proper tree has exactly $n - 2$ vertices.

Remark 2 Note that the disc D in K which is corresponding dual of the proper tree T in M is collapsible and therefore it is a topological 2-disc.

Definition 2 A proper tree T is called an admissible proper tree if the boundary of corresponding dual 2-disc D in K is a Hamiltonian Cycle in $EG(K)$.

Main result of this article is :

THEOREM 1 The edge graph $EG(K)$ of an equivelar map K has a contractible Hamiltonian cycle if and only if the edge graph of corresponding dual map of K has a proper tree.

More generally, we prove :

THEOREM 2 The edge graph $EG(K)$ of a map K on a surface has a contractible Hamiltonian cycle if and only if the edge graph of corresponding dual map of K has a proper tree.

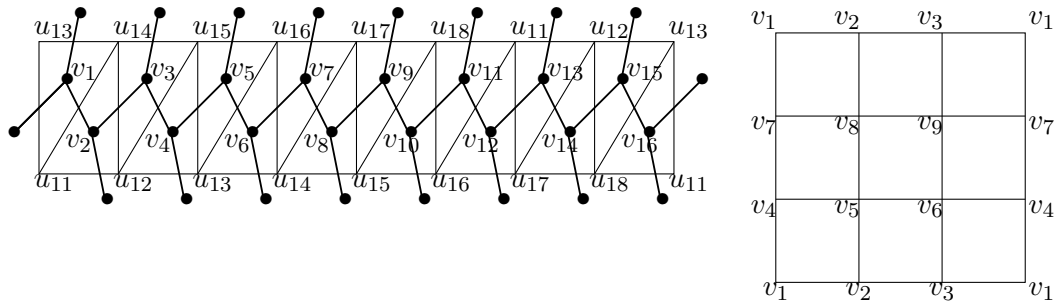
These results rely on :

LEMMA 1.1 Let M denote the dual map of an n vertex $\{p, q\}$ -equivelar map K on a surface S . If $\frac{n-2}{p-2} = m$ is an integer then M has an admissible proper tree on m vertices.

In the next section we give an example of a dual map of a triangulation. These are $\{3, 6\}$ and $\{6, 3\}$ maps on torus. The second example is of self dual map of type $\{4, 4\}$ on the torus. There does not exist any Hamiltonian cycle in the second example. In the section following it we present some facts and properties of a proper tree and proceed to prove the main result of this article. In section 4 we also give computer implementable steps to find out a Hamiltonian cycle.

2 Example and Results

EXAMPLE 1 $\{3, 6\}$ and $\{6, 3\}$ -equivelar maps (left) and $\{4, 4\}$ -equivelar map (right) on the torus



3 Proper tree in equivelar maps :

LEMMA 3.1 *Let T be a proper tree in a $\{q, p\}$ equivelar map M . Then $T \cap F \neq \emptyset$ for any face F of M .*

PROOF OF LEMMA 3.1 Let $V(T)$ and $E(T)$ respectively be the set of vertices and edges of the tree T . We construct two sets E and \tilde{F} as follows. Let E be a singleton set which contains a vertex $v_1 \in V(T)$ at 1st(initial) step and $F_{1,1}, F_{1,2}, \dots, F_{1,p}$ are the faces of M such that all the faces $F_{1,1}, F_{1,2}, \dots, F_{1,p}$ are incident at vertex v_1 . Put $\tilde{F} = \{F_{1,1}, F_{1,2}, \dots, F_{1,p}\}$. At i^{th} step, we choose a vertex $v_i \in V(T) \setminus E$ such that $\{w, v_i\} \in E(T)$, where $w \in E$ and put in set E . Since v_i be the new vertex of E , so there are some adjacent faces of v_i which are not in the set \tilde{F} . Now claim is, there are exactly $r = (p - 2)$ faces $F_{i,1}, F_{i,2}, \dots, F_{i,p-2}$ where each are different from all the faces of \tilde{F} and all are incident at vertex v_i . Suppose $r \neq (p - 2)$. Then there are two possibilities:

One: Suppose $r < p - 2$, then there are at least three faces F_1^i, F_2^i and F_3^i which are in \tilde{F} and all are incident at vertex v_i . Suppose $\{w, v_i\}$ be a common edge of F_1^i and F_2^i and there exist a vertex $u \in E$ such that $u \in V(F_3^i)$. Since v_1 and u are the vertices of tree T , so there exist a path $P_1(v_1 \rightarrow u)$. Similarly there exist a path from $P_2(v_1 \rightarrow v_i)$. Also, since v_i and u lie on the face F_3^i and $u, v_i \in V(T)$. Therefore the path $P_3(v_i \rightarrow u)$ in F_3^i also party of tree. Hence $P_1 \cup P_2 \cup P_3$ contains a *cycle* or collection of cycles. This can not happen because T is a tree. Hence $r \geq (p - 2)$.

Two: Suppose $r > (p - 2)$. We know at each vertex there are exactly p faces adjacent. Suppose we are choosing an edge $\{u_1, u_2\}$ in $E(T)$ such that $u_1 \in E$ and $u_2 \in V(T) \setminus E$ and this edge is the common edge between exactly two face and both are incident at the vertices u_1, u_2 . So both the faces are repetition at the vertex u_2 . Hence $r \leq (p - 2)$.

Hence $r = p - 2$ and at each step there are $(p - 2)$ faces which are different from all the faces of \tilde{F} and collect them in \tilde{F} . Thus the number of faces in $\tilde{F} = p + \underbrace{(p - 2) + (p - 2) + \dots + (p - 2)}_{(p-2) \text{ times}}$

(repetitions of $(p - 2) = (\#|V| - 1)$). That is $\#\tilde{F} = p + (p - 2)(\frac{n-2}{p-2} - 1) = p + n - p = n$.

So after $\#|V|^{th}$ step \tilde{F} will contain all the faces of polyhedral map. Hence tree touches all the faces of M . This proves the Lemma

LEMMA 3.2 *Let K be a n vertex $\{p, q\}$ equivelar map of a surface S . Let M denote the dual polyhedron corresponding to K and T be a $\frac{n-2}{p-2}$ vertex proper tree in M . Let D denote the subcomplex of K which is dual of T . Then D is a 2 - disk and $bd(D)$ is a Hamiltonian cycle in K .*

PROOF OF LEMMA 3.2: By definition of a dual, D consists of $\frac{n-2}{p-2}$ p - gons corresponding to $\frac{n-2}{p-2}$ vertices of T . Two p - gons in D have an edge in common if the corresponding vertices are adjacent in T . Here the set D is a collapsible *simplicial* complex and hence it is a 2 - disk. Since T has vertices of degree one, $bd(D) \neq \emptyset$, and being boundary complex of a 2 - disk, it is a connected cycle. Observe that the number of edges in $\frac{n-2}{p-2}$ p - gons is $p(\frac{n-2}{p-2})$ and for each edge of T exactly 2 edges are identified. Hence the number of edges which remains unidentified in D is $p(\frac{n-2}{p-2}) - 2(\frac{n-2}{p-2} - 1) = \frac{pn-2p-2n+2p}{p-2} = \frac{n(p-2)}{p-2} = n$. Hence the number of vertices in $bd(D) := \partial D = n$. If the vertices $v_1, v_2 \in \partial D$ such that v_1, v_2 lie on a path of length $< n$ and $v_1 = v_2$. This means the faces F_1 and F_2 in D with $v_1 \in F_1, v_2 \in F_2, F_1 \neq F_2$ and F_1 not adjacent to F_2 . Thus there exist a face F' in D such that the vertex $u_{F'}$ in T corresponding to F' does not belong to the face $F(v_1)$

corresponding to vertex v_1 . But this contradicts that T is a proper tree. Thus a the cycle ∂D contains exactly n distinct vertices. Since $\#V(K) = n$, ∂D is a Hamiltonian cycle in K . This proves the Lemma.

LEMMA 3.3 *Let M denote the dual map of an n vertex $\{p, q\}$ equivelar map K on a surface S . If $\frac{n-2}{p-2}$ is not an integer then M does not have any admissible proper tree.*

PROOF OF LEMMA 3.3: Let T denote a tree on m vertices and $V(T) = \{v_1, v_2, \dots, v_m\}$ be the set of vertices of T . Let $\frac{n-2}{p-2}$ be as in statement of Lemma 3.3. Then, there are two possibilities, namely, $m \leq \lfloor \frac{n-2}{p-2} \rfloor$ or $m \geq \lceil \frac{n-2}{p-2} \rceil$. As shown in the proof of Lemma 3.1, at a vertex v_1 exactly p faces of M are incident and if v_i is adjacent to v_1 then exactly $p-2$ faces distinct from the p faces containing v_1 are incident at v_i ($i \neq 1$). We take union of all the faces and denote this union by \tilde{F} .

When $m \leq \lfloor \frac{n-2}{p-2} \rfloor$, the number of faces in $\tilde{F} = p + \underbrace{(p-2) + (p-2) + \dots + (p-2)}_{(m-1) \text{ repetitions of } (p-2)}$

$$\begin{aligned}
&= p + (m-1)(p-2) \\
&\leq p + (\lfloor \frac{n-2}{p-2} \rfloor - 1)(p-2) \\
&= p + (\frac{n-2}{p-2} - \{\frac{n-2}{p-2}\} - 1)(p-2) \\
&= p + (\frac{n-p}{p-2} - \{\frac{n-2}{p-2}\})(p-2) \\
&= n - (p-2)\{\frac{n-2}{p-2}\} \\
&< n.
\end{aligned}$$

Therefore the tree T does not touch all the faces of M . Hence T is not an admissible proper tree. Similarly, when $m \geq \lceil \frac{n-2}{p-2} \rceil$, the number of faces in $\tilde{F} > n$. But total number of faces in M are n . This can not happen. Hence T is not an admissible proper tree. This proves the Lemma.

PROOF OF LEMMA 1.1: First we prove, if $\frac{n-2}{p-2} = m$ is an integer then M has a proper tree on m vertices. Let K and M be as in the statement of Lemma. We construct a vertex set $V(T)$ and an edge set $E(T)$ of a tree T in M . For this, choose a vertex v_1 in M and form $V(T) = \{v_1\}$. There is a facet F_{v_1} in K corresponding to v_1 . Define a set $W = \{u \in V(K) : u \text{ is incident with } F_{v_1}\}$. At the i^{th} step of construction, $1 < i \leq m$, form $V(T) = V(T) \cup v_i$ by choosing a vertex v_i in $V(M) \setminus V(T)$ such that $v_i v_j$ is an edge in M iff $j = i-1$. Define $E(T) = \{v_{i-1} v_i : 1 < i \leq m\}$. We get facets F_{v_i} in K corresponding to v_i and put all the $p-2$ vertices of F_{v_i} into W . In this construction there is no subset U of $V(T)$ for which U is equal to $V(F_j^*)$ for any facet F_j^* in M . Thus, at the m -th step the graph $T := (V(T), E(T))$, which is a tree by construction, satisfies the conditions two and three in the Definition 1. Hence it is a proper tree. The number of elements in W will be $p + \underbrace{(p-2) + (p-2) + \dots + (p-2)}_{(m-1) \text{ repetitions of } (p-2)} = p + (p-2)(m-1) = p + n - p = n$, since $m = \frac{n-2}{p-2}$. Hence by Lemma 3.2 the dual corresponding to T bounds a 2-disk D and $bd(D)$ is a Hamiltonian cycle in K . So, T is an admissible proper tree. This proves the Lemma.

4 The Steps for Searching a Separating Hamiltonian cycle in equivelar maps :

The following steps may be implemented as a computer program to located separating Hamiltonian cycles :

ALGORITHM 1 *Let $EG(K)$ be the edge graph of a $\{p, q\}$ equivelar map K on a surface S , $\#V(EG(K)) = n$ and $w = \frac{n-2}{p-2}$ be an integer (by lemma 3.3 and lemma ??). Let M be the set of all p -gonal facet and i denote the no of steps. We construct two set D and V as follows. Choose an element $P_0 \in M$. Define $D := \{P_0\}$, $V := \{V(P_0)\}$ and $i = 1$.*

1. *At next step, if $w = 1$ then, $n = p$ i.e. surface is a 2-disk bounded by a p -gon. In this case, the p -gon itself a Hamiltonian cycle. we stop here.*
2. *Suppose $w > 1$. This follows $n > p$ as $n = p + (w-1)(p-2)$ and $p \geq 3$. This follows $\#V < \#V(EG(K))$ i.e. there is a vertex which is not in V . Then at next step, we go to the only one of the following steps.*

(a) *Suppose there is a vertex $v \in V(EG(K)) \setminus V$ and $v \in V(P)$ with $V(P) \cap V = \{v_1, v_2\}$ such that $E(P) \cap E(P_{-1}) = \{\{v_1, v_2\}\}$ where $P_{-1} \in D$. Then we take $D = D \cup \{P\}$, $V = V \cup V(P)$ and we increase i by 1 and go to the next step.*

(b) *Suppose there is a vertex $v \in V(EG(K)) \setminus V$ and $v \in V(P)$ with $V(P) \cap V = \emptyset$. Then we get a sequence of facets in order P_1, P_2, \dots, P_r with the following properties-*

- i. $P_i \cap P_{i+1}$ is an edge for $1 \leq i \leq r-1$
- ii. $P_1 = P$
- iii. *There exist only one facet P_{-1} ($P_{-1} \in D$) with $V(P_{-1}) \cap V(P_r) = \{u_1, w_1\}$, where $E(P_{-1}) \cap E(P_r) = \{\{u_1, w_1\}\}$*
- iv. *For all P'_i s, $V(P_i) \cap V = \emptyset, i = 1, 2, \dots, r-1$.*

Let W be the dual of K . Also G denote the edge-graph of the dual map of $(V, E(D), D)$ and u denote the dual vertex of P_1 in W .

Then the above order sequence P_1, P_2, \dots, P_r of facets exist because-

- i. *There always exist a path $Q(v \rightarrow u)$ in W with the following properties-*

- A. $v \in V(G)$
- B. $u \in K$
- C. $V(G) \cap V(Q) = \{v\}$

as G and W are connected.

- ii. *Dual of Q in K is $\{P_1, P_2, \dots, P_r\}$*

Here we choose P_r and we take $D = D \cup \{P_r\}$ and $V = V \cup V(P_r)$. Now we increase i by 1 and go to the next step.

Hence after a step we will go to one of the above two steps until we get a condition $V = V(EG(K))$ and $i = w$.

At last step, $\#D = w$ as $V = V(EG(K))$. Let P_1, P_2, \dots, P_w be the facets in D then $P_1 \cup P_2 \cup \dots \cup P_w$, $P_j \in D$ for $1 \leq j \leq w$ is a 2-disk and $\partial(\cup P_i)$ is a Hamiltonian cycle. We stop here.

5 Proper tree in polyhedral maps :

LEMMA 5.1 *Let T be a proper tree in a general polyhedral map M on a surface S . Then $T \cap F \neq \emptyset$ for any facet F of M .*

PROOF OF LEMMA 5.1 Let $V(T)$ and $E(T)$ respectively be the set of vertices and edges of the tree T . We construct two sets E and \tilde{F} as follows-

1. Choose a vertex $v_1 \in V(T)$ of degree m and let $\{F_{1,1}, F_{1,2}, \dots, F_{1,m}\}$ be the set of facets of M such that the facets $F_{1,1}, F_{1,2}, \dots, F_{1,m}$ are adjacent to the vertex v_1 . Put $\tilde{F}_1 = \{F_{1,1}, F_{1,2}, \dots, F_{1,m}\}$ and $E_1 = \{v_1\}$.
2. At 2nd step, choose a vertex v_2 other than v_1 where $\{v_1, v_2\}$ is an edge of T . Suppose the degree of v_2 is l . Then there are l facets $F_{2,1}, F_{2,2}, \dots, F_{2,l}$ adjacent to the vertex v_2 . And here exactly two facets adjacent to v_1 and v_2 as $\{v_1, v_2\}$ is an edge in polyhedral map. Hence, there are exactly $l - 2$ no of facets of M adjacent to v_2 which do not belong to the set \tilde{F} . Hence, put all the new facets at v_2 in \tilde{F}_2 and $E_2 := E_1 \cup \{v_2\}$.
3. At a general step, say at i^{th} step, choose a vertex $v_i \in V(T) \setminus E_{i-1}$ such that for some $w \in E$, $\{w, v_i\}$ is an edge of T and assume $\deg(v_i) = t$. We define $E_i := E_{i-1} \cup \{v_i\}$. We claim, since $v_i \notin E_{i-1}$, there are exactly $r = (t - 2)$ distinct facets $F_{i,1}, F_{i,2}, \dots, F_{i,r}$ incident at v_i and each are different from all the facets of \tilde{F} . Suppose $r \neq (t - 2)$. Then there are following possibilities:

Suppose $r < t - 2$ i.e. $t - r \geq 3$ then there are at least three facets F_1^i, F_2^i and F_3^i incident with v_i and contained in \tilde{F} . Suppose $\{w, v_i\}$ is a common edge of F_1^i and F_2^i and there exist a vertex $u \in E(T)$ such that $u \in V(F_3^i)$. This implies two distinct sub paths in T have u and v_i as their end vertices. These two paths would hence constitute a cycle in T contradicting that T is a tree. Hence $r \not< (t - 2)$. Arguing in similar way, we see that $r > (t - 2)$ is also not possible. Hence $r = t - 2$ i.e. at i^{th} step exactly $\deg(v_i) - 2$ new facets get

added to the set \tilde{F} . After k^{th} step, the number of facets in \tilde{F} is $\deg(v_1) + \sum_{i=2}^k (\deg(v_i) - 2)$,

where number of elements in $V(T) = k$. In other words number of elements in \tilde{F} is $\sum_{i=1}^k \deg(v_i) - \sum_{i=2}^k 2 = n + 2(k - 1) - 2(k - 1) = n$. So, after $V(T) = k^{th}$ step \tilde{F} will contain all the facets of the polyhedral map. Hence the tree T touches all the facets of M . This proves the Lemma.

LEMMA 5.2 *Let K be a n vertex polyhedral map and M denote the dual polyhedron corresponding to K . Let T be a k vertex proper tree in M . If D denotes the subcomplex of K which is dual of T then D is a 2-disk and the boundary ∂D of D is a Hamiltonian cycle in $EG(K)$.*

PROOF OF LEMMA 5.2: Since T has k vertices, D consists of k facets F_1, F_2, \dots, F_k . Two facets in D have an edge in common if the corresponding vertices are adjacent in T . Here the set D is a collapsible simplicial complex and hence it is a 2-disk. Since T has vertex of degree one, $bd(D) \neq \emptyset$, and being boundary complex of a 2-disk, it is a connected cycle. Observe that the number of edges in k polygons are $\sum_{i=1}^k \text{length}(F_i)$ and for each edge of T exactly 2 edges are identified. Hence the total number of edges which remains unidentified

in D is $\sum_{i=1}^k \text{degree}(v_i) - \sum_{i=2}^k 2 = n + 2(k-1) - 2(k-1) = n$. Hence the number of vertices in $bd(D) := \partial D = n$. If the vertices $v_1, v_2 \in \partial D$ such that v_1, v_2 lie on a path of length $< n$ in ∂D and $v_1 = v_2$. This means the facets F_1 and F_2 in D with $v_1 \in F_1, v_2 \in F_2, F_1 \neq F_2$ and F_1 not adjacent to F_2 . Thus there exist a facet F' in D such that the vertex $u_{F'}$ in T corresponding to F' does not belong to the facet $F(v_1)$ corresponding to vertex v_1 . But this contradicts that T is a proper tree. Thus a the cycle ∂D contains exactly n distinct vertices. Since $\#V(K) = n$, ∂D is a Hamiltonian cycle in K . This proves the Lemma.

PROOF OF THEOREM 2: The above lemma shows the if part. Conversely, let K denote a polyhedral map and $H := (v_1, v_2, \dots, v_n)$ denote a contractible Hamiltonian cycle in $EG(K)$. Let F_1, F_2, \dots, F_m denote the facets of length $l_i = \text{length}(F_i)$ such that $H = \partial(\bigcup_{j=1}^m F_j = D)$. We claim that all the facets F_i have their vertices on H . For, otherwise, there will be identifications on the surface due to the hypothesis that H is Hamiltonian. Thus, if x denotes the number of facets in the subpolyhedra D which is topologically a 2-disc then the Euler characteristic relation gives us $1 = n - (\frac{\sum_{i=1}^m l_i - n}{2} + n) + m$ i.e. $\sum_{i=1}^m l_i = n + 2(m-1)$. In the edge graph of dual map M of K , consider the graph corresponding to D with m vertices u_1, u_2, \dots, u_m . This graph is a tree which is also a proper tree. This is so because $\sum_{i=1}^m \text{length}(F_i) = n + 2(m-1)$, where $l_i = \text{length}(F_i)$ and there does not exist any subset S_1 of $\{F_1, F_2, \dots, F_m\}$ such that union of elements of S_1 is a 2-disc subpolyhedra of K , whose boundary is a link of a vertex in K . \square

COROLLARY 1 *The edge graph $EG(K)$ of a $\{p, q\}$ equivelar map K on a surface has a contractible Hamiltonian cycle if and only if the edge graph of corresponding dual map of K has a proper tree.*

PROOF OF COROLLARY 1: In the proof of above theorem 2 we choose length $l_i = \text{length}(F_i) = p$ and $H := (v_1, v_2, \dots, v_n)$, where H denotes a contractible Hamiltonian cycle in $EG(K)$. If x denotes the number of p -gons in the disk resulting as a union of facets corresponding to v_i s then the Euler characteristic relation gives us $1 = n - (\frac{p \times x - n}{2} + n) + x$. Thus $x = \frac{n-2}{p-2}$. So that $m = \frac{n-2}{p-2}$. Hence, the edge graph $EG(K)$ of a $\{p, q\}$ equivelar map K on a surface has a contractible Hamiltonian cycle. This proves the corollary.

PROOF OF THEOREM 1: The proof follows by corollary 1. \square

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