The p-Domination Number of Complete Multipartite Graphs^{*}

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Abstract

Let G = (V, E) be a graph and p be a positive integer. A subset $S \subseteq V$ is called a p-dominating set of G if every vertex not in S has at least p neighbors in S. The p-domination number $\gamma_p(G)$ is the minimum cardinality of a p-dominating set in G. In this paper, we establish an exact formula of the p-domination number of all complete multipartite graphs for arbitrary positive integer p.

Keywords: p-domination set, p-domination number, complete multipartite graph

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1 Induction

For notation and graph-theoretical terminology not defined here, we refer the reader to [3]. Let G = (V, E) be a finite simple graph with vertex set V = V(G) and edge set E = E(G). The *neighborhood* and *degree* of a vertex $v \in V$ are $N_G(v) = \{u \in V : uv \in E\}$ and $d_G(v) = |N_G(v)|$, respectively. A *dominating set* of G is a subset $S \subseteq V$ such

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that every vertex of V - S has at least one neighbor in S. The domination number $\gamma(G)$ is the minimum cardinality of all dominating sets in G. The domination is a classical concept in graph theory. The early literature on the domination with related topics is, in detail, surveyed in the two outstanding books by Haynes, Hedetniemi and Slater [10, 11].

Fink and Jacobson [8, 9] generalized the concept of dominating set. Let p be a positive integer. A subset $D \subseteq V$ is a p-dominating set of G if $|N_G(v) \cap D| \ge p$ for each $v \in V - D$. The p-domination number $\gamma_p(G)$ is the minimum cardinality of all p-dominating sets in G. A p-dominating set D with $|D| = \gamma_p(G)$ is called a γ_p -set of G (for short, $\gamma_p(G)$ -set). For $S, T \subseteq V, S$ p-dominate T in G if $|N_G(v) \cap S| \ge p$ for each $v \in T - S$. Clearly, the 1-dominating set is the well-known dominating set in a graph G, and so $\gamma_1(G) = \gamma(G)$. By the definition of p-dominating set, the following observation is obvious.

Observation 1 Every p-dominating set contains all the vertices with degree at most p-1.

The determination of the *p*-domination number for graphs seems to be a difficult problem. In 1989, Jacobson and Peters [12] showed that the problem is NP-complete in general graphs. In 1994, Bean, Henning and Swart [1] proved the problem remains NP-complete in bipartite or chordal graphs. These results show that the following study is of important significance.

- Find the lower and upper bounds of γ_p with difference as small as possible.
- Determine exact values of γ_p for some graphs, specially well-known networks.

Many works focused on the bounds of γ_p for general graphs or some special classes of graphs (see, for example, [2, 5, 6, 7, 13]). Very recently, Chellali et al. [4] have given an excellent survey on this topics. Until now, however, no research has been done on calculating the exact values of γ_p even for some particular graphs except [14]. In [14], the author obtained the exact 2-domination number of the toroidal grid graphs $C_m \Box C_n$ in some cases.

In this paper, we give an exact formula of γ_p for arbitrary positive integer p and the complete *t*-partite graph K_{n_1,n_2,\cdots,n_t} .

Through this paper, the graph G always denotes a complete t-partite graph K_{n_1,n_2,\cdots,n_t} with t-partition $\{V_1, V_2, \cdots, V_t\}, N_t = \{1, 2, \cdots, t\}$ and

$$f(I) = \sum_{i \in I} n_i \text{ for } I \subseteq N_t.$$

Note that if t = 1 or t = 2 and $f(N_t) \leq p$ then $\gamma_p(G) = |V(G)|$ by Observation 1. Thus, we always assume $t \geq 2$ and $f(N_t) > p$.

2 Optimal γ_p -sets of G

For any $D \subseteq V(G)$, define

 $D_i = V_i \cap D$ for each $i \in N_t$, and $I_D = \{i \in N_t : |D_i| = |V_i|\}.$

Lemma 2 If $t \ge 2$ and $f(N_t) > p$, then

$$\gamma_p(G) \le \min\{f(I) : I \subseteq N_t \text{ with } f(I) \ge p\}.$$

with equality if G has a γ_p -set D with $f(I_D) \ge p$.

Proof. Let $I \subseteq N_t$ with $f(I) \ge p$ and $S = \bigcup_{i \in I} V_i$. Then

$$|S| = \sum_{i \in I} |V_i| = \sum_{i \in I} n_i = f(I) \ge p.$$

Since G is a complete t-partite graph, for any $v \in V(G) - S$, we have $S \subseteq N_G(v)$ and so $|N_G(v) \cap S| = |S| \ge p$. This implies that S is a p-dominating set of G, and so

 $\gamma_p(G) \le \min\{f(I) : I \subseteq N_t \text{ with } f(I) \ge p\}.$

On the other hand, let D be a $\gamma_p(G)$ -set with $f(I_D) \ge p$. Then

$$\gamma_p(G) = |D| \geq \sum_{i \in I_D} |V_i| = \sum_{i \in I_D} n_i = f(I_D)$$

$$\geq \min\{f(I) : I \subseteq N_t \text{ with } f(I) \ge p\}.$$

The lemma follows.

Lemma 3 If $t \ge 2$ and $f(N_t) > p$, then $|I_D| \le t - 2$ for any $\gamma_p(G)$ -set D with $f(I_D) < p$.

Proof. Clearly $|I_D| \leq t - 1$ by $f(N_t) > p > f(I_D)$. If $|I_D| = t - 1$, then there is a unique index $i_0 \in N_t$ such that $N_t - I_D = \{i_0\}$. By the definition of I_D , $V(G) - V_{i_0} \subseteq D$ and there exists a vertex x in V_{i_0} but not in D. Since D is a $\gamma_p(G)$ -set and $f(I_D) < p$, we can deduce a contradiction as follows:

$$p \le |N_G(x) \cap D| = |V(G) - V_{i_0}| = \sum_{i \in I_D} n_i = f(I_D) < p_i$$

Hence $|I_D| \leq t - 2$.

For a $\gamma_p(G)$ -set D with $|I_D| < t$, $|D| = f(I_D) + \sum_{i \in N_t - I_D} |D_i|$. By Lemma 3, the value of $\left| |D_i| - \frac{|D| - f(I_D)}{t - |I_D|} \right|$ is well-defined for any $i \in N_t - I_D$ if $t \ge 2$ and $f(N_t) > p$. Define

$$\mu(D) = \sum_{i \in N - I_D} \left| |D_i| - \frac{|D| - f(I_D)}{t - |I_D|} \right|.$$

Definition 2.1 A $\gamma_p(G)$ -set D is called to be optimal if the following conditions hold: 1) $f(I_D) < p$; 2) $|I_D| \ge |I_S|$ for any $\gamma_p(G)$ -set S; 3) $\mu(D) \le \mu(S)$ for any $\gamma_p(G)$ -set S with $I_D = I_S$.

By the definition, if each $\gamma_p(G)$ -set D has $f(I_D) < p$, then there must be at least one optimal γ_p -set in G. To obtain the upper bound of $\gamma_p(G)$, by Lemma 2, we only need to consider the case that every $\gamma_p(G)$ -set D satisfies $f(I_D) < p$. We investigate properties of optimal γ_p -sets starting with the following critical lemma.

Lemma 4 $||D_i| - |D_j|| \le 1$ for any optimal $\gamma_p(G)$ -set D and $i, j \in N_t - I_D$.

Proof. By Lemma 3, $t - |I_D| \ge 2$ and so $N_t - I_D \ne \emptyset$. Let

$$|D_s| = \max\{|D_i|: i \in N_t - I_D\}$$
 and $|D_w| = \min\{|D_i|: i \in N_t - I_D\}.$

Suppose, to be contrary, that $|D_s| - |D_w| \ge 2$. Clearly, $|D_s| \ge 2$. Since $w \in N_t - I_D$, $D_w \subsetneq V_w$. Hence there are $x \in D_s$ and $y \in V_w - D_w$. Let

$$D^* = (D - \{x\}) \cup \{y\}.$$

Then

$$I_{D^*} = \begin{cases} I_D & \text{if } |D_w| < |V_w| - 1; \\ I_D \cup \{w\} & \text{if } |D_w| = |V_w| - 1. \end{cases}$$
(2.1)

Thus $I_D \subseteq I_{D^*}$. We first claim that D^* is a $\gamma_p(G)$ -set. In fact, it is easy to see that D^* can p-dominate $V(G) - V_w$. By the choice of $s \in N_t - I_D$, $V_s - D_s \neq \emptyset$. Since $D - D_s$ can p-dominate $V_s - D_s$, we have $|D| - |D_s| \ge p$. It follows that, for any vertex $z \in V_w - D^*$,

$$|N_G(z) \cap D^*| = |D^*| - |D^*_w| = |D| - (|D_w| + 1) \ge |D| - |D_s| + 1 \ge p + 1,$$

which means that D^* can p-dominate z and, hence, D^* is a $\gamma_p(G)$ -set.

By the second condition of the optimality of D, we have $|I_D| \ge |I_{D^*}|$. Thus $I_D = I_D^*$ by $I_D \subseteq I_{D^*}$. Combined with $|D| = |D^*| = \gamma_p(G)$, we can obtain that

$$\frac{|D| - f(I_D)}{t - |I_D|} = \frac{|D^*| - f(I_{D^*})}{t - |I_{D^*}|}.$$

For convenience, we use the notation λ to represent them.

We now show $\mu(D^*) - \mu(D) < 0$. Since $|D| = \gamma_p(G) = f(I_D) + \sum_{i \in N_t - I_D} |D_i|$,

$$\lambda = \frac{1}{t - |I_D|} \sum_{i \in N_t - I_D} |D_i|.$$

By $|D_s| - |D_w| \ge 2$ and the choices of s and w, we have that

$$|D_w| + 1 \le |D_s| - 1$$
 and $|D_w| < \lambda < |D_s|.$

It follows that

$$\begin{split} \mu(D^*) - \mu(D) &= \sum_{i \in N_t - I_{D^*}} |(|D_i^*| - \lambda)| - \sum_{i \in N_t - I_D} |(|D_i| - \lambda)| \\ &= |(|D_w^*| - \lambda)| + |(|D_s^*| - \lambda)| - (\lambda - |D_w|) - (|D_s| - \lambda) \\ &= |(|D_w| + 1 - \lambda)| + |(|D_s| - 1 - \lambda)| - (|D_s| - |D_w|) \\ &= \begin{cases} 2(|D_w| - \lambda) & \text{if } \lambda < |D_w| + 1 \\ -2 & \text{if } |D_w| + 1 \le \lambda \le |D_s| - 1 \\ 2(\lambda - |D_s|) & \text{if } \lambda > |D_s| - 1 \end{cases} \\ &< 0. \end{split}$$

This contradicts with the third condition of the optimality of D, and so $|D_s| - |D_w| \le 1$.

The lemma follows.

For an optimal $\gamma_p(G)$ -set $D, t - |I_D| \ge 2$ by Lemma 3, and so $N_t - I_D \neq \emptyset$. Thus we denote

$$k = \max\{|D_i|: i \in N_t - I_D\} \text{ and } \ell = \min\{|D_i|: i \in N_t - I_D\}.$$
 (2.2)

If $k \neq \ell$, then $k = \ell + 1$ by Lemma 4. Define

$$A = \begin{cases} \{i \in N_t - I_D : |D_i| = \ell + 1\} & \text{if } k = \ell + 1; \\ \emptyset & \text{if } k = \ell, \end{cases}$$

$$B = \{i \in N_t - I_D : |D_i| = \ell\}.$$

(2.3)

Then $\{A, B\}$ is a partition of $N_t - I_D$ and $B \neq \emptyset$.

Lemma 5 |A| = 0 or $2 \le |A| \le t - |I_D| - 1$ for any optimal $\gamma_p(G)$ -set D.

Proof. Since $\{A, B\}$ is a partition of $N_t - I_D$ and $B \neq \emptyset$, it is obvious that $|A| \leq t - |I_D| - 1$. We now show $|A| \neq 1$. Assume to the contrary that |A| = 1.

Let $A = \{i_1\}$. Then $|D_{i_1}| = \ell + 1 \ge 1$ and $V_{i_1} - D_{i_1} \ne \emptyset$ since $i_1 \in N_t - I_D$. Since $D - D_{i_1}$ p-dominates $V_{i_1} - D_{i_1}$, we have $|D| - |D_{i_1}| \ge p$. Take any vertex $x \in D_{i_1}$ and let

$$D' = D - \{x\}.$$

Consider any vertex y in V(G) - D'. If $y \in V_{i_1}$, then

$$|N_G(y) \cap D'| = |D'| - |D'_{i_1}| = (|D| - 1) - (|D_{i_1}| - 1)| = |D| - |D_{i_1}| \ge p.$$

If $y \notin V_{i_1}$, then there exists some $j \in B$ such that $y \in V_j$. Noting $|D_j| = |D_{i_1}| - 1$, we have that

$$|N_G(y) \cap D'| = |D'| - |D'_j| = (|D| - 1) - |D_j| = |D| - |D_{i_1}| \ge p.$$

Hence D' is a *p*-dominating set of G with $|D'| = |D| - 1 = \gamma_p(G) - 1$, a contradiction. The lemma follows. **Lemma 6** $\gamma_p(G) \ge p + \ell + \delta_A$ for any optimal $\gamma_p(G)$ -set D, where ℓ and A are defined in (2.2) and (2.3), respectively, δ_A is the characteristic function on A, i.e., $\delta_A = 0$ if |A| = 0 and $\delta_A = 1$ otherwise.

Proof. Note that $N_t - I_D \neq \emptyset$ and $V_i - D_i \neq \emptyset$ for $i \in N_t - I_D$. To *p*-dominate $V_i - D_i$, $|D - D_i| = |D| - |D_i| \ge p$ for $i \in N_t - I_D$.

If |A| = 0, then $\delta_A = 0$ and $N_t - I_D = B$. For any $i \in B$, $|D_i| = \ell$ by (2.3), and so $\gamma_p(G) = |D| \ge p + |D_i| = p + \ell = p + \ell + \delta_A$.

If $|A| \neq 0$, then $\delta_A = 1$. For $i \in A$, $|D_i| = \ell + 1$ by (2.3). Thus $\gamma_p(G) = |D| \ge p + |D_i| = p + \ell + 1 = p + \ell + \delta_A$.

The lemma follows.

Lemma 7 $\left\lceil \frac{p-f(I_D)}{t-|I_D|-1} \right\rceil \leq n_i \text{ for any optimal } \gamma_p(G) \text{-set } D \text{ and } i \in N_t - I_D.$

Proof. Let $N - I_D = A \cup B$ as defined in (2.3). Then $|D_i| = \ell + 1$ for $i \in A$ and $|D_j| = \ell$ for $j \in B$. Note that $n_i = |V_i| \ge |D_i| + 1 \ge \ell + 1$ for any $i \in N_t - I_D$. It follows that

$$|D| = f(I_D) + f(A) + f(B)$$

= $f(I_D) + |A|(\ell + 1) + (t - |I_D| - |A|)\ell$
= $f(I_D) + (t - |I_D| - 1)\ell + \ell + |A|,$

from which we have

$$\begin{aligned} \frac{p-f(I_D)}{t-|I_D|-1} \rceil &= \ell + \left\lceil \frac{|A|-\delta_A}{t-|I_D|-1} - \frac{|D|-(p+\ell+\delta_A)}{t-|I_D|-1} \right\rceil \\ &\leq \ell + \left\lceil \frac{|A|-\delta_A}{t-|I_D|-1} \right\rceil \quad \text{(by Lemma 6)} \\ &\leq \ell + \delta_A \qquad \text{(by Lemma 5)} \\ &\leq \ell + 1 \\ &\leq n_i \quad \text{for any } i \in N_t - I_D \end{aligned}$$

as desired, and so the lemma follows.

3 Main results

In this section, we will give an exact formula of γ_p for a complete *t*-partite graph $G = K_{n_1, n_2, \dots, n_t}$. By Lemma 2, if G contains a γ_p -set D with $f(I_D) \ge p$, then

$$\gamma_p(G) = \min\{f(I) : I \subseteq N_t \text{ with } f(I) \ge p\}.$$

Thus, we only need to consider the case of $f(I_D) < p$ for any $\gamma_p(G)$ -set D. In this case, G must have optimal $\gamma_p(G)$ -sets. Moreover, for any optimal $\gamma_p(G)$ -set D, $|I_D| \leq t - 2$ by Lemma 3, and $\lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil \leq n_i$ for any $i \in N_t - I_D$ by Lemma 7. Thus, the following family \mathscr{I}_p of the subsets of N_t is well-defined.

$$\mathscr{I}_p = \{ I \subset N_t : |I| \le t - 2, f(I)$$

p	s_1	\mathscr{I}_p	s_2	$\gamma_p(G)$
1	2	$\{\emptyset\}$	1	$s_1 = p + s_2 = 2$
2	2	$\{\emptyset\}$	1	$s_1 = 2$
3	4	$\{\emptyset, \{1\}, \{2\}\}$	1	$s_1 = p + s_2 = 4$
4	4	$\{\emptyset, \{1\}, \{2\}\}$	1	$s_1 = 4$
5	10	$\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$	1	$p + s_2 = 6$
6	10	$\{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$	2	$p + s_2 = 8$
7	10	$\{\{1,2\}\}$	3	$s_1 = p + s_2 = 10$
9	10	$\{\{1,2\}\}$	5	$s_1 = 10$
11	12	$\{\{1\}, \{2\}, \{3\}, \{1, 2\}\}$	1	$s_1 = p + s_2 = 12$
13	14	$\{\{3\},\{1,2\},\{1,3\},\{2,3\}\}$	1	$s_1 = p + s_2 = 14$
14	17	$\{\{3\},\{1,2\},\{1,3\},\{2,3\}\}$	2	$p + s_2 = 16$
15	17	Ø	∞	$s_1 = 17$

Table 1: Examples of s_1 , \mathscr{I}_p , s_2 and $\gamma_p(G)$ for $G = K_{2,2,10,17}$, where $N_4 = \{1, 2, 3, 4\}$.

Some examples of \mathscr{I}_p for $G = K_{2,2,10,17}$ can be found in Table 1.

Let

$$s_{1} = \min\{f(I) : I \subseteq N_{t} \text{ with } f(I) \ge p\} \text{ and}$$
$$s_{2} = \begin{cases} \min\{\lceil \frac{p-f(I)}{t-|I|-1}\rceil : I \in \mathscr{I}_{p}\} & \text{if } \mathscr{I}_{p} \neq \emptyset;\\ \infty & \text{if } \mathscr{I}_{p} = \emptyset. \end{cases}$$

Lemma 8 Let $G = K_{n_1, n_2, \dots, n_t}$ with $t \ge 2$ and $f(N_t) > p$. Then $\gamma_p(G) \le p + s_2$.

Proof. If $\mathscr{I}_p = \emptyset$, then $s_2 = \infty$ and so $\gamma_p(G) . Assume that <math>\mathscr{I}_p \neq \emptyset$ below. Let $I \in \mathscr{I}_p$ (without loss of generality, say $I = \{1, \dots, k\}$) with

$$k \leq t-2$$
, $f(I) < p$ and $s_2 = \lceil \frac{p-f(I)}{t-k-1} \rceil \leq n_i$ for each $i \in \{k+1, \cdots, t\}$.

Since $t-k-1 \ge 1$ and p-f(I) > 0, there are two integers q and r with $q \ge 0$ and $0 \le r \le t-k-2$ such that

$$p - f(I) = q(t - k - 1) + r.$$

Then for each $i \in \{k+1, \cdots, t\}$,

$$n_i \ge s_2 = \begin{cases} q+1 & \text{if } r \ne 0; \\ q & \text{if } r = 0. \end{cases}$$
(3.1)

Thus, we can choose $D \subseteq V(G)$ such that

$$D = (V_1 \cup \dots \cup V_k) \cup (V'_{k+1} \cup \dots \cup V'_{k+r}) \cup (V'_{k+r+1} \cup \dots \cup V'_{t-1}) \cup V'_t$$

where, for each $i \in \{k + 1, \dots, t\}$, V'_i is a subset of V_i satisfying

$$|V_i'| = \begin{cases} q+1 & \text{if } k+1 \le i \le k+r \\ q & \text{if } k+r+1 \le i \le t-1 \\ s_2 & \text{if } i=t. \end{cases}$$
(3.2)

Thus,

$$|D| = \sum_{i=1}^{k} |V_i| + \sum_{i=k+1}^{k+r} |V'_i| + \sum_{j=k+r+1}^{t-1} |V'_j| + |V'_t|$$

= $(n_1 + \dots + n_k) + r(q+1) + (t-k-r-1)q + s_2$
= $(f(I) + q(t-k-1) + r) + s_2$
= $p + s_2$.

To complete the proof, we only need to show that D is a p-dominating set of G. To this aim, let v be any vertex in V(G) - D. By the choice of D, there is some $i_0 \in \{k + 1, \dots, t\}$ such that $v \in V_{i_0} - V'_{i_0}$. Since G is a complete t-partite graph,

$$|N_G(v) \cap D| = |D| - |V'_{i_0}| = p + s_2 - |V'_{i_0}|$$

By (3.1) and (3.2), we have

$$s_2 - |V'_{i_0}| = \begin{cases} 1 & \text{if } r \neq 0 \text{ and } k + r + 1 \leq i_0 \leq t - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\geq 0.$$

It follows that $|N_G(v) \cap D| = p + s_2 - |V'_{i_0}| \ge p$, which implies that D can p-dominate x. Hence D is a p-dominating set of G. The lemma follows.

We now state our main result as follows.

Theorem 9 For any integer $p \ge 1$ and a complete t-partite graph $G = K_{n_1, n_2, \dots, n_t}$ with $t \ge 2$ and $f(N_t) > p$,

$$\gamma_p(G) = \min\{s_1, p + s_2\}.$$

Proof. From Lemmas 2 and 8, we can obtain that $\gamma_p(G) \leq \min\{s_1, p + s_2\}$, and if G has a γ_p -set D with $f(I_D) \geq p$ then $\gamma_p(G) = s_1 \geq \min\{s_1, p + s_2\}$.

In the following, assume that every $\gamma_p(G)$ -set D satisfies $f(I_D) < p$. Let D be an optimal $\gamma_p(G)$ -set. To the end, we only need to show $\gamma_p(G) \ge p + s_2$.

Since $|I_D| \leq t-2$ by Lemma 3 and $\lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil \leq n_i$ for any $i \in N_t - I_D$ by Lemma 7, we have $I_D \in \mathscr{I}_p$, and so $\lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil \geq s_2$. From the proof of Lemma 7, we know that $\ell + \delta_A \geq \lceil \frac{p-f(I_D)}{t-|I_D|-1} \rceil$. Hence, by Lemma 6,

$$\gamma_p(G) \ge p + \ell + \delta_A \ge p + \left\lceil \frac{p - f(I_D)}{t - |I_D| - 1} \right\rceil \ge p + s_2.$$

The theorem follows.

Some illustrations of s_1 , s_2 and $\gamma_p(G) = \min\{s_1, p + s_2\}$ for the complete 4-partite graph $G = K_{2,2,10,17}$ are shown in Table 1.

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