# The $p$-Domination Number of Complete Multipartite Graphs* 

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#### Abstract

Let $G=(V, E)$ be a graph and $p$ be a positive integer. A subset $S \subseteq V$ is called a $p$-dominating set of $G$ if every vertex not in $S$ has at least $p$ neighbors in $S$. The $p$-domination number $\gamma_{p}(G)$ is the minimum cardinality of a $p$-dominating set in $G$. In this paper, we establish an exact formula of the $p$-domination number of all complete multipartite graphs for arbitrary positive integer $p$.


Keywords: $p$-domination set, $p$-domination number, complete multipartite graph
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## 1 Induction

For notation and graph-theoretical terminology not defined here, we refer the reader to [3]. Let $G=(V, E)$ be a finite simple graph with vertex set $V=V(G)$ and edge set $E=E(G)$. The neighborhood and degree of a vertex $v \in V$ are $N_{G}(v)=\{u \in V: u v \in$ $E\}$ and $d_{G}(v)=\left|N_{G}(v)\right|$, respectively. A dominating set of $G$ is a subset $S \subseteq V$ such

[^0]that every vertex of $V-S$ has at least one neighbor in $S$. The domination number $\gamma(G)$ is the minimum cardinality of all dominating sets in $G$. The domination is a classical concept in graph theory. The early literature on the domination with related topics is, in detail, surveyed in the two outstanding books by Haynes, Hedetniemi and Slater [10, 11].

Fink and Jacobson [8, 9] generalized the concept of dominating set. Let $p$ be a positive integer. A subset $D \subseteq V$ is a $p$-dominating set of $G$ if $\left|N_{G}(v) \cap D\right| \geq p$ for each $v \in V-D$. The $p$-domination number $\gamma_{p}(G)$ is the minimum cardinality of all $p$-dominating sets in $G$. A $p$-dominating set $D$ with $|D|=\gamma_{p}(G)$ is called a $\gamma_{p}$-set of $G$ (for short, $\gamma_{p}(G)$-set). For $S, T \subseteq V, S p$-dominate $T$ in $G$ if $\left|N_{G}(v) \cap S\right| \geq p$ for each $v \in T-S$. Clearly, the 1-dominating set is the well-known dominating set in a graph $G$, and so $\gamma_{1}(G)=\gamma(G)$. By the definition of $p$-dominating set, the following observation is obvious.

Observation 1 Every p-dominating set contains all the vertices with degree at most $p-1$.

The determination of the $p$-domination number for graphs seems to be a difficult problem. In 1989, Jacobson and Peters [12] showed that the problem is NP-complete in general graphs. In 1994, Bean, Henning and Swart [1] proved the problem remains NP-complete in bipartite or chordal graphs. These results show that the following study is of important significance.

- Find the lower and upper bounds of $\gamma_{p}$ with difference as small as possible.
- Determine exact values of $\gamma_{p}$ for some graphs, specially well-known networks.

Many works focused on the bounds of $\gamma_{p}$ for general graphs or some special classes of graphs (see, for example, [2, 5, 6, 7, 13]). Very recently, Chellali et al. [4] have given an excellent survey on this topics. Until now, however, no research has been done on calculating the exact values of $\gamma_{p}$ even for some particular graphs except [14]. In [14], the author obtained the exact 2-domination number of the toroidal grid graphs $C_{m} \square C_{n}$ in some cases.

In this paper, we give an exact formula of $\gamma_{p}$ for arbitrary positive integer $p$ and the complete $t$-partite graph $K_{n_{1}, n_{2}, \cdots, n_{t}}$.

Through this paper, the graph $G$ always denotes a complete $t$-partite graph $K_{n_{1}, n_{2}, \cdots, n_{t}}$ with $t$-partition $\left\{V_{1}, V_{2}, \cdots, V_{t}\right\}, N_{t}=\{1,2, \cdots, t\}$ and

$$
f(I)=\sum_{i \in I} n_{i} \text { for } I \subseteq N_{t} .
$$

Note that if $t=1$ or $t=2$ and $f\left(N_{t}\right) \leq p$ then $\gamma_{p}(G)=|V(G)|$ by Observation 1. Thus, we always assume $t \geq 2$ and $f\left(N_{t}\right)>p$.

## 2 Optimal $\gamma_{p}$-sets of $G$

For any $D \subseteq V(G)$, define
$D_{i}=V_{i} \cap D$ for each $i \in N_{t}$, and $I_{D}=\left\{i \in N_{t}:\left|D_{i}\right|=\left|V_{i}\right|\right\}$.
Lemma 2 If $t \geq 2$ and $f\left(N_{t}\right)>p$, then

$$
\gamma_{p}(G) \leq \min \left\{f(I): I \subseteq N_{t} \text { with } f(I) \geq p\right\}
$$

with equality if $G$ has a $\gamma_{p}$-set $D$ with $f\left(I_{D}\right) \geq p$.
Proof. Let $I \subseteq N_{t}$ with $f(I) \geq p$ and $S=\bigcup_{i \in I} V_{i}$. Then

$$
|S|=\sum_{i \in I}\left|V_{i}\right|=\sum_{i \in I} n_{i}=f(I) \geq p
$$

Since $G$ is a complete $t$-partite graph, for any $v \in V(G)-S$, we have $S \subseteq N_{G}(v)$ and so $\left|N_{G}(v) \cap S\right|=|S| \geq p$. This implies that $S$ is a $p$-dominating set of $G$, and so

$$
\gamma_{p}(G) \leq \min \left\{f(I): I \subseteq N_{t} \text { with } f(I) \geq p\right\}
$$

On the other hand, let $D$ be a $\gamma_{p}(G)$-set with $f\left(I_{D}\right) \geq p$. Then

$$
\begin{aligned}
\gamma_{p}(G)=|D| & \geq \sum_{i \in I_{D}}\left|V_{i}\right|=\sum_{i \in I_{D}} n_{i}=f\left(I_{D}\right) \\
& \geq \min \left\{f(I): I \subseteq N_{t} \text { with } f(I) \geq p\right\}
\end{aligned}
$$

The lemma follows.

Lemma 3 If $t \geq 2$ and $f\left(N_{t}\right)>p$, then $\left|I_{D}\right| \leq t-2$ for any $\gamma_{p}(G)$-set $D$ with $f\left(I_{D}\right)<p$.

Proof. Clearly $\left|I_{D}\right| \leq t-1$ by $f\left(N_{t}\right)>p>f\left(I_{D}\right)$. If $\left|I_{D}\right|=t-1$, then there is a unique index $i_{0} \in N_{t}$ such that $N_{t}-I_{D}=\left\{i_{0}\right\}$. By the definition of $I_{D}, V(G)-V_{i_{0}} \subseteq D$ and there exists a vertex $x$ in $V_{i_{0}}$ but not in $D$. Since $D$ is a $\gamma_{p}(G)$-set and $f\left(I_{D}\right)<p$, we can deduce a contradiction as follows:

$$
p \leq\left|N_{G}(x) \cap D\right|=\left|V(G)-V_{i_{0}}\right|=\sum_{i \in I_{D}} n_{i}=f\left(I_{D}\right)<p
$$

Hence $\left|I_{D}\right| \leq t-2$.
For a $\gamma_{p}(G)$-set $D$ with $\left|I_{D}\right|<t,|D|=f\left(I_{D}\right)+\sum_{i \in N_{t}-I_{D}}\left|D_{i}\right|$. By Lemma 3, the value of $\left|\left|D_{i}\right|-\frac{|D|-f\left(I_{D}\right)}{t-\left|I_{D}\right|}\right|$ is well-defined for any $i \in N_{t}-I_{D}$ if $t \geq 2$ and $f\left(N_{t}\right)>p$. Define

$$
\mu(D)=\sum_{i \in N-I_{D}}| | D_{i}\left|-\frac{|D|-f\left(I_{D}\right)}{t-\left|I_{D}\right|}\right| .
$$

Definition 2.1 $A \gamma_{p}(G)$-set $D$ is called to be optimal if the following conditions hold: 1) $f\left(I_{D}\right)<p$; 2) $\left|I_{D}\right| \geq\left|I_{S}\right|$ for any $\gamma_{p}(G)$-set $S$; 3) $\mu(D) \leq \mu(S)$ for any $\gamma_{p}(G)$-set $S$ with $I_{D}=I_{S}$.

By the definition, if each $\gamma_{p}(G)$-set $D$ has $f\left(I_{D}\right)<p$, then there must be at least one optimal $\gamma_{p}$-set in $G$. To obtain the upper bound of $\gamma_{p}(G)$, by Lemma 2, we only need to consider the case that every $\gamma_{p}(G)$-set $D$ satisfies $f\left(I_{D}\right)<p$. We investigate properties of optimal $\gamma_{p}$-sets starting with the following critical lemma.

Lemma $4 \| D_{i}\left|-\left|D_{j}\right|\right| \leq 1$ for any optimal $\gamma_{p}(G)$-set $D$ and $i, j \in N_{t}-I_{D}$.

Proof. By Lemma 3, $t-\left|I_{D}\right| \geq 2$ and so $N_{t}-I_{D} \neq \emptyset$. Let

$$
\left|D_{s}\right|=\max \left\{\left|D_{i}\right|: i \in N_{t}-I_{D}\right\} \quad \text { and } \quad\left|D_{w}\right|=\min \left\{\left|D_{i}\right|: i \in N_{t}-I_{D}\right\} .
$$

Suppose, to be contrary, that $\left|D_{s}\right|-\left|D_{w}\right| \geq 2$. Clearly, $\left|D_{s}\right| \geq 2$. Since $w \in N_{t}-I_{D}$, $D_{w} \nsubseteq V_{w}$. Hence there are $x \in D_{s}$ and $y \in V_{w}-D_{w}$. Let

$$
D^{*}=(D-\{x\}) \cup\{y\} .
$$

Then

$$
I_{D^{*}}= \begin{cases}I_{D} & \text { if }\left|D_{w}\right|<\left|V_{w}\right|-1 ;  \tag{2.1}\\ I_{D} \cup\{w\} & \text { if }\left|D_{w}\right|=\left|V_{w}\right|-1\end{cases}
$$

Thus $I_{D} \subseteq I_{D^{*}}$. We first claim that $D^{*}$ is a $\gamma_{p}(G)$-set. In fact, it is easy to see that $D^{*}$ can $p$-dominate $V(G)-V_{w}$. By the choice of $s \in N_{t}-I_{D}, V_{s}-D_{s} \neq \emptyset$. Since $D-D_{s}$ can $p$-dominate $V_{s}-D_{s}$, we have $|D|-\left|D_{s}\right| \geq p$. It follows that, for any vertex $z \in V_{w}-D^{*}$,

$$
\left|N_{G}(z) \cap D^{*}\right|=\left|D^{*}\right|-\left|D_{w}^{*}\right|=|D|-\left(\left|D_{w}\right|+1\right) \geq|D|-\left|D_{s}\right|+1 \geq p+1,
$$

which means that $D^{*}$ can $p$-dominate $z$ and, hence, $D^{*}$ is a $\gamma_{p}(G)$-set.
By the second condition of the optimality of $D$, we have $\left|I_{D}\right| \geq\left|I_{D^{*}}\right|$. Thus $I_{D}=I_{D}^{*}$ by $I_{D} \subseteq I_{D^{*}}$. Combined with $|D|=\left|D^{*}\right|=\gamma_{p}(G)$, we can obtain that

$$
\frac{|D|-f\left(I_{D}\right)}{t-\left|I_{D}\right|}=\frac{\left|D^{*}\right|-f\left(I_{D^{*}}\right)}{t-\left|I_{D^{*}}\right|} .
$$

For convenience, we use the notation $\lambda$ to represent them.
We now show $\mu\left(D^{*}\right)-\mu(D)<0$. Since $|D|=\gamma_{p}(G)=f\left(I_{D}\right)+\sum_{i \in N_{t}-I_{D}}\left|D_{i}\right|$,

$$
\lambda=\frac{1}{t-\left|I_{D}\right|} \sum_{i \in N_{t}-I_{D}}\left|D_{i}\right| .
$$

By $\left|D_{s}\right|-\left|D_{w}\right| \geq 2$ and the choices of $s$ and $w$, we have that

$$
\left|D_{w}\right|+1 \leq\left|D_{s}\right|-1 \text { and }\left|D_{w}\right|<\lambda<\left|D_{s}\right| .
$$

It follows that

$$
\begin{aligned}
\mu\left(D^{*}\right)-\mu(D) & =\sum_{i \in N_{t}-I_{D^{*}}}\left|\left(\left|D_{i}^{*}\right|-\lambda\right)\right|-\sum_{i \in N_{t}-I_{D}}\left|\left(\left|D_{i}\right|-\lambda\right)\right| \\
& =\left|\left(\left|D_{w}^{*}\right|-\lambda\right)\right|+\left|\left(\left|D_{s}^{*}\right|-\lambda\right)\right|-\left(\lambda-\left|D_{w}\right|\right)-\left(\left|D_{s}\right|-\lambda\right) \\
& =\left|\left(\left|D_{w}\right|+1-\lambda\right)\right|+\left|\left(\left|D_{s}\right|-1-\lambda\right)\right|-\left(\left|D_{s}\right|-\left|D_{w}\right|\right) \\
& = \begin{cases}2\left(\left|D_{w}\right|-\lambda\right) & \text { if } \lambda<\left|D_{w}\right|+1 \\
-2 & \text { if }\left|D_{w}\right|+1 \leq \lambda \leq\left|D_{s}\right|-1 \\
2\left(\lambda-\left|D_{s}\right|\right) & \text { if } \lambda>\left|D_{s}\right|-1\end{cases} \\
& <0 .
\end{aligned}
$$

This contradicts with the third condition of the optimality of $D$, and so $\left|D_{s}\right|-\left|D_{w}\right| \leq 1$.
The lemma follows.
For an optimal $\gamma_{p}(G)$-set $D, t-\left|I_{D}\right| \geq 2$ by Lemma 3, and so $N_{t}-I_{D} \neq \emptyset$. Thus we denote

$$
\begin{equation*}
k=\max \left\{\left|D_{i}\right|: i \in N_{t}-I_{D}\right\} \text { and } \ell=\min \left\{\left|D_{i}\right|: i \in N_{t}-I_{D}\right\} . \tag{2.2}
\end{equation*}
$$

If $k \neq \ell$, then $k=\ell+1$ by Lemma 4. Define

$$
\begin{align*}
& A= \begin{cases}\left\{i \in N_{t}-I_{D}:\left|D_{i}\right|=\ell+1\right\} & \text { if } k=\ell+1 ; \\
\emptyset & \text { if } k=\ell\end{cases}  \tag{2.3}\\
& B=\left\{i \in N_{t}-I_{D}:\left|D_{i}\right|=\ell\right\} .
\end{align*}
$$

Then $\{A, B\}$ is a partition of $N_{t}-I_{D}$ and $B \neq \emptyset$.

Lemma $5|A|=0$ or $2 \leq|A| \leq t-\left|I_{D}\right|-1$ for any optimal $\gamma_{p}(G)$-set $D$.

Proof. Since $\{A, B\}$ is a partition of $N_{t}-I_{D}$ and $B \neq \emptyset$, it is obvious that $|A| \leq$ $t-\left|I_{D}\right|-1$. We now show $|A| \neq 1$. Assume to the contrary that $|A|=1$.

Let $A=\left\{i_{1}\right\}$. Then $\left|D_{i_{1}}\right|=\ell+1 \geq 1$ and $V_{i_{1}}-D_{i_{1}} \neq \emptyset$ since $i_{1} \in N_{t}-I_{D}$. Since $D-D_{i_{1}} p$-dominates $V_{i_{1}}-D_{i_{1}}$, we have $|D|-\left|D_{i_{1}}\right| \geq p$. Take any vertex $x \in D_{i_{1}}$ and let

$$
D^{\prime}=D-\{x\} .
$$

Consider any vertex $y$ in $V(G)-D^{\prime}$. If $y \in V_{i_{1}}$, then

$$
\left|N_{G}(y) \cap D^{\prime}\right|=\left|D^{\prime}\right|-\left|D_{i_{1}}^{\prime}\right|=(|D|-1)-\left(\left|D_{i_{1}}\right|-1\right)\left|=|D|-\left|D_{i_{1}}\right| \geq p\right.
$$

If $y \notin V_{i_{1}}$, then there exists some $j \in B$ such that $y \in V_{j}$. Noting $\left|D_{j}\right|=\left|D_{i_{1}}\right|-1$, we have that

$$
\left|N_{G}(y) \cap D^{\prime}\right|=\left|D^{\prime}\right|-\left|D_{j}^{\prime}\right|=(|D|-1)-\left|D_{j}\right|=|D|-\left|D_{i_{1}}\right| \geq p
$$

Hence $D^{\prime}$ is a $p$-dominating set of $G$ with $\left|D^{\prime}\right|=|D|-1=\gamma_{p}(G)-1$, a contradiction. The lemma follows.

Lemma $6 \gamma_{p}(G) \geq p+\ell+\delta_{A}$ for any optimal $\gamma_{p}(G)$-set $D$, where $\ell$ and $A$ are defined in (2.2) and (2.3), respectively, $\delta_{A}$ is the characteristic function on $A$, i.e., $\delta_{A}=0$ if $|A|=0$ and $\delta_{A}=1$ otherwise.

Proof. Note that $N_{t}-I_{D} \neq \emptyset$ and $V_{i}-D_{i} \neq \emptyset$ for $i \in N_{t}-I_{D}$. To $p$-dominate $V_{i}-D_{i}$, $\left|D-D_{i}\right|=|D|-\left|D_{i}\right| \geq p$ for $i \in N_{t}-I_{D}$.

If $|A|=0$, then $\delta_{A}=0$ and $N_{t}-I_{D}=B$. For any $i \in B,\left|D_{i}\right|=\ell$ by (2.3), and so $\gamma_{p}(G)=|D| \geq p+\left|D_{i}\right|=p+\ell=p+\ell+\delta_{A}$.

If $|A| \neq 0$, then $\delta_{A}=1$. For $i \in A,\left|D_{i}\right|=\ell+1$ by (2.3). Thus $\gamma_{p}(G)=|D| \geq$ $p+\left|D_{i}\right|=p+\ell+1=p+\ell+\delta_{A}$.

The lemma follows.

Lemma $7\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil \leq n_{i}$ for any optimal $\gamma_{p}(G)$-set $D$ and $i \in N_{t}-I_{D}$.

Proof. Let $N-I_{D}=A \cup B$ as defined in (2.3). Then $\left|D_{i}\right|=\ell+1$ for $i \in A$ and $\left|D_{j}\right|=\ell$ for $j \in B$. Note that $n_{i}=\left|V_{i}\right| \geq\left|D_{i}\right|+1 \geq \ell+1$ for any $i \in N_{t}-I_{D}$. It follows that

$$
\begin{aligned}
|D| & =f\left(I_{D}\right)+f(A)+f(B) \\
& =f\left(I_{D}\right)+|A|(\ell+1)+\left(t-\left|I_{D}\right|-|A|\right) \ell \\
& =f\left(I_{D}\right)+\left(t-\left|I_{D}\right|-1\right) \ell+\ell+|A|,
\end{aligned}
$$

from which we have

$$
\begin{array}{rlr}
\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil & =\ell+\left\lceil\frac{|A|-\delta_{A}}{t-\left|I_{D}\right|-1}-\frac{|D|-\left(p+\ell+\delta_{A}\right)}{t-\left|I_{D}\right|-1}\right\rceil \\
& \leq \ell+\left\lceil\frac{1 A \mid \delta_{A}}{t-\left|I_{D}\right|-1}\right\rceil \quad \text { (by Lemma 6) } \\
& \leq \ell+\delta_{A} \quad \text { (by Lemma (5) } \\
& \leq \ell+1 & \\
& \leq n_{i} \quad \text { for any } i \in N_{t}-I_{D}
\end{array}
$$

as desired, and so the lemma follows.

## 3 Main results

In this section, we will give an exact formula of $\gamma_{p}$ for a complete $t$-partite graph $G=K_{n_{1}, n_{2}, \cdots, n_{t}}$. By Lemma 2, if $G$ contains a $\gamma_{p}$-set $D$ with $f\left(I_{D}\right) \geq p$, then

$$
\gamma_{p}(G)=\min \left\{f(I): I \subseteq N_{t} \text { with } f(I) \geq p\right\} .
$$

Thus, we only need to consider the case of $f\left(I_{D}\right)<p$ for any $\gamma_{p}(G)$-set $D$. In this case, $G$ must have optimal $\gamma_{p}(G)$-sets. Moreover, for any optimal $\gamma_{p}(G)$-set $D,\left|I_{D}\right| \leq t-2$ by Lemma 3, and $\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil \leq n_{i}$ for any $i \in N_{t}-I_{D}$ by Lemma 7. Thus, the following family $\mathscr{I}_{p}$ of the subsets of $N_{t}$ is well-defined.

$$
\mathscr{I}_{p}=\left\{I \subset N_{t}:|I| \leq t-2, f(I)<p \text { and }\left\lceil\frac{p-f(I)}{t-|I|-1}\right\rceil \leq n_{i} \text { for each } i \in N_{t}-I\right\} .
$$

| $p$ | $s_{1}$ | $\mathscr{I}_{p}$ | $s_{2}$ | $\gamma_{p}(G)$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | $\{\emptyset\}$ | 1 | $s_{1}=p+s_{2}=2$ |
| 2 | 2 | $\{\emptyset\}$ | 1 | $s_{1}=2$ |
| 3 | 4 | $\{\emptyset,\{1\},\{2\}\}$ | 1 | $s_{1}=p+s_{2}=4$ |
| 4 | 4 | $\{\emptyset,\{1\},\{2\}\}$ | 1 | $s_{1}=4$ |
| 5 | 10 | $\{\emptyset,\{1\},\{2\},\{1,2\}\}$ | 1 | $p+s_{2}=6$ |
| 6 | 10 | $\{\emptyset,\{1\},\{2\},\{1,2\}\}$ | 2 | $p+s_{2}=8$ |
| 7 | 10 | $\{\{1,2\}\}$ | 3 | $s_{1}=p+s_{2}=10$ |
| 9 | 10 | $\{\{1,2\}\}$ | 5 | $s_{1}=10$ |
| 11 | 12 | $\{\{1\},\{2\},\{3\},\{1,2\}\}$ | 1 | $s_{1}=p+s_{2}=12$ |
| 13 | 14 | $\{\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ | 1 | $s_{1}=p+s_{2}=14$ |
| 14 | 17 | $\{\{3\},\{1,2\},\{1,3\},\{2,3\}\}$ | 2 | $p+s_{2}=16$ |
| 15 | 17 | $\emptyset$ | $\infty$ | $s_{1}=17$ |

Table 1: Examples of $s_{1}, \mathscr{I}_{p}, s_{2}$ and $\gamma_{p}(G)$ for $G=K_{2,2,10,17}$, where $N_{4}=\{1,2,3,4\}$.

Some examples of $\mathscr{I}_{p}$ for $G=K_{2,2,10,17}$ can be found in Table 1.
Let

$$
\begin{aligned}
& s_{1}=\min \left\{f(I): I \subseteq N_{t} \text { with } f(I) \geq p\right\} \text { and } \\
& s_{2}= \begin{cases}\min \left\{\left\lceil\frac{p-f(I)}{t-|I|-1}\right\rceil: I \in \mathscr{I}_{p}\right\} & \text { if } \mathscr{I}_{p} \neq \emptyset \\
\infty & \text { if } \mathscr{I}_{p}=\emptyset\end{cases}
\end{aligned}
$$

Lemma 8 Let $G=K_{n_{1}, n_{2}, \cdots, n_{t}}$ with $t \geq 2$ and $f\left(N_{t}\right)>p$. Then $\gamma_{p}(G) \leq p+s_{2}$.

Proof. If $\mathscr{I}_{p}=\emptyset$, then $s_{2}=\infty$ and so $\gamma_{p}(G)<p+s_{2}$. Assume that $\mathscr{I}_{p} \neq \emptyset$ below. Let $I \in \mathscr{I}_{p}$ (without loss of generality, say $I=\{1, \cdots, k\}$ ) with

$$
k \leq t-2, f(I)<p \text { and } s_{2}=\left\lceil\frac{p-f(I)}{t-k-1}\right\rceil \leq n_{i} \text { for each } i \in\{k+1, \cdots, t\}
$$

Since $t-k-1 \geq 1$ and $p-f(I)>0$, there are two integers $q$ and $r$ with $q \geq 0$ and $0 \leq r \leq t-k-2$ such that

$$
p-f(I)=q(t-k-1)+r .
$$

Then for each $i \in\{k+1, \cdots, t\}$,

$$
n_{i} \geq s_{2}= \begin{cases}q+1 & \text { if } r \neq 0  \tag{3.1}\\ q & \text { if } r=0\end{cases}
$$

Thus, we can choose $D \subseteq V(G)$ such that

$$
D=\left(V_{1} \cup \cdots \cup V_{k}\right) \cup\left(V_{k+1}^{\prime} \cup \cdots \cup V_{k+r}^{\prime}\right) \cup\left(V_{k+r+1}^{\prime} \cup \cdots \cup V_{t-1}^{\prime}\right) \cup V_{t}^{\prime}
$$

where, for each $i \in\{k+1, \cdots, t\}, V_{i}^{\prime}$ is a subset of $V_{i}$ satisfying

$$
\left|V_{i}^{\prime}\right|= \begin{cases}q+1 & \text { if } k+1 \leq i \leq k+r  \tag{3.2}\\ q & \text { if } k+r+1 \leq i \leq t-1 \\ s_{2} & \text { if } i=t\end{cases}
$$

Thus,

$$
\begin{aligned}
|D| & =\sum_{i=1}^{k}\left|V_{i}\right|+\sum_{i=k+1}^{k+r}\left|V_{i}^{\prime}\right|+\sum_{j=k+r+1}^{t-1}\left|V_{j}^{\prime}\right|+\left|V_{t}^{\prime}\right| \\
& =\left(n_{1}+\cdots+n_{k}\right)+r(q+1)+(t-k-r-1) q+s_{2} \\
& =(f(I)+q(t-k-1)+r)+s_{2} \\
& =p+s_{2} .
\end{aligned}
$$

To complete the proof, we only need to show that $D$ is a $p$-dominating set of $G$. To this aim, let $v$ be any vertex in $V(G)-D$. By the choice of $D$, there is some $i_{0} \in\{k+1, \cdots, t\}$ such that $v \in V_{i_{0}}-V_{i_{0}}^{\prime}$. Since $G$ is a complete $t$-partite graph,

$$
\left|N_{G}(v) \cap D\right|=|D|-\left|V_{i_{0}}^{\prime}\right|=p+s_{2}-\left|V_{i_{0}}^{\prime}\right| .
$$

By (3.1) and (3.2), we have

$$
\begin{aligned}
s_{2}-\left|V_{i_{0}}^{\prime}\right| & = \begin{cases}1 & \text { if } r \neq 0 \text { and } k+r+1 \leq i_{0} \leq t-1 \\
0 & \text { otherwise }\end{cases} \\
& \geq 0
\end{aligned}
$$

It follows that $\left|N_{G}(v) \cap D\right|=p+s_{2}-\left|V_{i_{0}}^{\prime}\right| \geq p$, which implies that $D$ can $p$-dominate $x$. Hence $D$ is a $p$-dominating set of $G$. The lemma follows.

We now state our main result as follows.
Theorem 9 For any integer $p \geq 1$ and a complete $t$-partite graph $G=K_{n_{1}, n_{2}, \cdots, n_{t}}$ with $t \geq 2$ and $f\left(N_{t}\right)>p$,

$$
\gamma_{p}(G)=\min \left\{s_{1}, p+s_{2}\right\} .
$$

Proof. From Lemmas 2 and 8, we can obtain that $\gamma_{p}(G) \leq \min \left\{s_{1}, p+s_{2}\right\}$, and if $G$ has a $\gamma_{p}$-set $D$ with $f\left(I_{D}\right) \geq p$ then $\gamma_{p}(G)=s_{1} \geq \min \left\{s_{1}, p+s_{2}\right\}$.

In the following, assume that every $\gamma_{p}(G)$-set $D$ satisfies $f\left(I_{D}\right)<p$. Let $D$ be an optimal $\gamma_{p}(G)$-set. To the end, we only need to show $\gamma_{p}(G) \geq p+s_{2}$.

Since $\left|I_{D}\right| \leq t-2$ by Lemma 3 and $\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil \leq n_{i}$ for any $i \in N_{t}-I_{D}$ by Lemma 7 , we have $I_{D} \in \mathscr{I}_{p}$, and so $\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil \geq s_{2}$. From the proof of Lemma 7, we know that $\ell+\delta_{A} \geq\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil$. Hence, by Lemma 6,

$$
\gamma_{p}(G) \geq p+\ell+\delta_{A} \geq p+\left\lceil\frac{p-f\left(I_{D}\right)}{t-\left|I_{D}\right|-1}\right\rceil \geq p+s_{2} .
$$

The theorem follows.
Some illustrations of $s_{1}, s_{2}$ and $\gamma_{p}(G)=\min \left\{s_{1}, p+s_{2}\right\}$ for the complete 4-partite graph $G=K_{2,2,10,17}$ are shown in Table 1 .

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