# Monochromatic connectivity and graph products * 

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#### Abstract

The concept of monochromatic connectivity was introduced by Caro and Yuster. A path in an edge-colored graph is called a monochromatic path if all the edges on the path are colored the same. An edge-coloring of $G$ is a monochromatic connection coloring ( $M C$-coloring, for short) if there is a monochromatic path joining any two vertices in $G$. The monochromatic connection number, denoted by $m c(G)$, is defined to be the maximum number of colors used in an $M C$-coloring of a graph $G$. In this paper, we study the monochromatic connection number on the lexicographical, strong, Cartesian and direct product and present several upper and lower bounds for these products of graphs.


Keywords: Monochromatic path, $M C$-coloring, monochromatical connection num- ber, Cartesian product, lexicographical product, strong product, direct product.

AMS subject classification 2010: $05 \mathrm{C} 15 ; 05 \mathrm{C} 12 ; 05 \mathrm{C} 35$.

## 1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. For a graph $G$, we use $V(G), E(G), n(G)$, $m(G), \delta(G), \kappa(G), \kappa^{\prime}(G), \delta(G)$ and $\operatorname{diam}(G)$ to denote the vertex set, edge set, number of vertices, number of edges, connectivity, edge-connectivity, minimum degree and diameter of $G$, respectively. The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [9]. Readers can see [9, 10, 11] for details. Consider an edge-coloring (not necessarily proper) of a graph $G=(V, E)$. We say that a path of $G$ is rainbow, if no two edges on the path have the same

[^0]color. An edge-colored graph $G$ is rainbow connected if every two vertices are connected by a rainbow path. The minimum number of colors required to rainbow color a graph $G$ is called the rainbow connection number, denoted by $r c(G)$. For more results on the rainbow connection, we refer to the survey paper [21] of Li , Shi and Sun and a new book [22] of Li and Sun.

Let $G$ be a nontrivial connected graph with an edge-coloring $f: E(G) \rightarrow\{1,2, \ldots, \ell\}$, $\ell \in N$, where adjacent edges may be colored the same. A path of $G$ is a monochromatic path if all the edges on the path are colored the same. An edge-coloring of $G$ is a monochromatic connection coloring (MC-coloring, for short) if there is a monochromatic path joining any two vertices in $G$. How colorful can an $M C$-coloring be ? One can see that this question is the natural opposite of the well-studied problem on rainbow connection number of graphs. Let $m c(G)$ denote the maximum number of colors used in an $M C$-coloring of a graph $G$, which called the monochromatic connection number of $G$. Note that an $M C$-coloring does not exist if G is not connected, and in this case we simply let $m c(G)=0$.

These concepts were introduced by Caro and Yuster in [8]. For more results on monochromatic connection number, we refer to [4, 5, 8, ,15]. The following observation is immediate.

Observation 1 [8] Let $G$ be a connected graph with $n(G)$ vertices and $m(G)$ edges. Then

$$
m c(G) \geq m(G)-n(G)+2
$$

Simply color the edges of a spanning tree with one color, and each of the remaining edges may be assigned a distinct fresh color. Caro and Yuster gave some sufficient conditions for graphs attaining this lower bound.

Theorem 1 [8] Let $G$ be a connected graph with $n>3$. If $G$ satisfies any of the following properties, then $m c(G)=m-n+2$.
(a) $G$ (the complement of $G$ ) is 4-connected;
(b) $G$ is triangle-free;
(c) $\Delta(G)<n-\frac{2 m-3(n-1)}{n-3}$; In particular, this holds if $\Delta(G) \leq(n+1) / 2$, and this also holds if $\Delta(G) \leq n-2 m / n$.
(d) $\operatorname{Diam}(G) \geq 3$;
(e) $G$ has a cut vertex.

Product networks were proposed based upon the idea of using the cross product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [13. Recently, there has been an increasing interest in a class of interconnection networks called Cartesian product networks; see [2, 13]. The other standard products (Direct, strong, and lexicographic) draw a constant attention of graph research community, see some recent papers [1, 19, 24, 27].

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the monochromatic connection number. Every of these four products will be treated in one of the forthcoming sections. In Section 3, we demonstrate the usefulness of the proposed constructions by applying them to some instances of product networks.

## 2 Main results

In this section, we study the monochromatic connection number of four graph product.

Lemma 1 [8] Let $G$ be a connected graph with $n(G)$ vertices and $m(G)$ edges. Then

$$
m c(G) \leq E(G)-V(G)+\kappa(G)+1
$$

In [25], Spacapan obtained the following result.

Lemma 2 [25] Let $G$ and $H$ be two nontrivial graphs. Then

$$
\kappa(G \square H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G)+\delta(H)\} .
$$

Yang and Xu [26] investigated the classical connectivity of the lexicographic product of two graphs.

Lemma 3 [26] Let $G$ and $H$ be two graphs. If $G$ is non-trivial, non-complete and connected, then

$$
\kappa(G \circ H)=\kappa(G)|V(H)| .
$$

Let $S_{G}$ and $S_{H}$ be separating sets of connected graphs $G$ and $H$, and let $G^{\prime}$ and $H^{\prime}$ be arbitrary connected components of $G-S_{G}$ and $H-S_{H}$. Then the set of vertices

$$
\left(S_{G} \times V\left(H^{\prime}\right)\right) \cup\left(S_{G} \times S_{H}\right) \cup\left(V\left(G^{\prime}\right) \times S_{H}\right)
$$

is called $a\urcorner$-set of $G \boxtimes H$; see [16].

Lemma 4 [16] Let $G$ and $H$ be connected graphs, at least one not complete. Set $\ell(G \boxtimes H)$ be the minimum size of $a \backslash$-set of $G \boxtimes H$. Then

$$
\kappa(G \boxtimes H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} .
$$

Lemma 5 [16] Let $G$ and $H$ be nonbipartite graphs. Then

$$
\kappa^{\prime}(G \times H)=\min \left\{2 \kappa^{\prime}(G)|V(H)|, 2 \kappa^{\prime}(H)|V(G)|, \delta(G) \delta(H)\right\} .
$$

Let $d_{G}(u, v)$ denote the distance between $u$ and $v$ in $G$. Denote by $d_{G}(u)$ the degree of vertex $u$ in $G$. The following lemma is from [16].

Lemma 6 [16] Let $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ be two vertices of $G \square H$. Then

$$
d_{G \square H}\left((g, h),\left(g^{\prime}, h^{\prime}\right)\right)=d_{G}\left(g, g^{\prime}\right)+d_{H}\left(h, h^{\prime}\right) .
$$

Corollary 1 Let $G$ be a connected graph. Then

$$
\operatorname{diam}(G \square H)=\operatorname{diam}(G)+\operatorname{diam}(H)
$$

Lemma 7 [16] Let $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ be two vertices of $G \circ H$. Then

$$
d_{G \circ H}\left((g, h),\left(g^{\prime}, h^{\prime}\right)\right)= \begin{cases}d_{G}\left(g, g^{\prime}\right), & \text { if } g \neq g^{\prime} \\ d_{H}\left(h, h^{\prime}\right), & \text { if } g=g^{\prime} \text { and } d_{G}(g)=0 \\ \min \left\{d_{H}\left(h, h^{\prime}\right), 2\right\}, & \text { if } g=g^{\prime} \text { and } d_{G}(g) \neq 0\end{cases}
$$

Lemma 8 [16] Let $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ be two vertices of $G \square H$. Then

$$
d_{G \boxtimes H}\left((g, h),\left(g^{\prime}, h^{\prime}\right)\right)=\max \left\{d_{G}\left(g, g^{\prime}\right), d_{H}\left(h, h^{\prime}\right)\right\} .
$$

Corollary 2 Let $G$ be a connected graph. Then

$$
\operatorname{diam}(G \boxtimes H)=\max \{\operatorname{diam}(G), \operatorname{diam}(H)\} .
$$

### 2.1 The Cartesian product

The Cartesian product of two graphs $G$ and $H$, written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ are adjacent if and only if $g=g^{\prime}$ and $\left(h, h^{\prime}\right) \in E(H)$, or $h=h^{\prime}$ and $\left(g, g^{\prime}\right) \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \square H$ is isomorphic to $H \square G$. The Cartesian product is commutative, that is, $G \square H \cong H \square G$. Clearly, $|E(G \circ H)|=|E(H)||V(G)|+|E(G)||V(H)|$.

Theorem 2 Let $G$ and $H$ be a connected graph.
(1) If neither $G$ nor $H$ is a tree, then
$\max \{|E(G)||V(H)|,|E(H)||V(G)|\}+2 \leq m c(G \square H) \leq|E(G)||V(H)|+(|E(H)|-1)|V(G)|+1$.
(2) If $G$ is not a tree and $H$ is a tree, then

$$
|E(H)||V(G)|+2 \leq m c(G \square H) \leq|E(G)||V(H)|+1
$$

(3) If both $G$ and $H$ are trees, then

$$
|E(G)||E(H)|+1 \leq m c(G \square H) \leq|E(G)||E(H)|+2
$$

Moreover, the lower bounds are sharp.

Proof. (1) Since $H$ is not a tree, it follows that $|E(H)| \geq|V(H)|$. By Observation [1, we have

$$
\begin{aligned}
m c(G \square H) & \geq|E(G \square H)|-|V(G \square H)|+2 \\
& =|E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+2 \\
& \geq|E(G)||V(H)|+2 .
\end{aligned}
$$

From Lemmal2, $\kappa(G \square H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G)+\delta(H)\} \leq \kappa(H)|V(G)| \leq$ $(|V(H)|-1)|V(G)|$. Furthermore, by Lemma 1, we have

$$
\begin{aligned}
& m c(G \square H) \\
\leq & E(G \square H)-V(G \square H)+\kappa(G \square H)+1 \\
= & |E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+\kappa(G \square H)+1 \\
\leq & |E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+|V(G)|(|V(H)|-1)+1 \\
= & |E(G)||V(H)|+|E(H)||V(G)|-|V(G)|+1 .
\end{aligned}
$$

(2) Since $G$ is not a tree and $H$ is a tree, it follows that $|E(G)| \geq|V(G)|$ and $|E(H)|=$ $|V(H)|-1$. By Observation we have

$$
\begin{aligned}
m c(G \square H) & \geq E(G \square H)-V(G \square H)+2 \\
& =|E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+2 \\
& \geq|V(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+2 \\
& =|E(H)||V(G)|+2 .
\end{aligned}
$$

From Lemmal2, $\kappa(G \square H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G)+\delta(H)\} \leq \kappa(H)|V(G)| \leq$ $|V(G)|$. By Lemma 1, we have

$$
\begin{aligned}
m c(G \square H) & \leq E(G \square H)-V(G \square H)+\kappa(G \square H)+1 \\
& =|E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+\kappa(G \square H)+1 \\
& =|E(G)||V(H)|+(|V(H)|-1)|V(G)|-|V(G)||V(H)|+\kappa(G \square H)+1 \\
& \leq|E(G)||V(H)|-|V(G)|+|V(G)|+1 \\
& \leq|E(G)||V(H)|+1 .
\end{aligned}
$$

(3) Since both $G$ and $H$ are trees, it follows that $|E(G)|=|V(G)|-1$ and $|E(H)|=$ $|V(H)|-1$. By Observation 1, we have

$$
\begin{aligned}
m c(G \square H) & \geq E(G \square H)-V(G \square H)+2 \\
& =|E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+2 \\
& =(|V(G)|-1)|V(H)|+(|V(H)|-1)|V(G)|-|V(G)||V(H)|+2 \\
& =|E(G)||E(H)|+1 .
\end{aligned}
$$

From Lemma $2, \kappa(G \square H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G)+\delta(H)\} \leq \delta(G)+\delta(H)=$ 2. By Lemma 1 , we have

$$
\begin{aligned}
m c(G \square H) & \leq E(G \square H)-V(G \square H)+\kappa(G \square H)+1 \\
& =|E(G)||V(H)|+|E(H)||V(G)|-|V(G)||V(H)|+\kappa(G \square H)+1 \\
& =(|V(G)|-1)|V(H)|+(|V(H)|-1)|V(G)|-|V(G)||V(H)|+\kappa(G \square H)+1 \\
& =|E(G)||E(H)|+\kappa(G \square H) \\
& \leq|E(G)||E(H)|+2 .
\end{aligned}
$$

To show the sharpness of the lower bounds in Theorem 2, we consider the following example.

Example 1: (1) Let $G$ be a cycle of order at least 3, and $H$ be a cycle of order at least 4. From Corollary $\mathbb{1} \operatorname{diam}(G \square H)=\operatorname{diam}(G)+\operatorname{diam}(H) \geq 3$. By Theorem 亿 $m c(G \square H)=$ $|E(G \square H)|-|V(G \square H)|+2=|E(G)||V(H)|+2=|E(H)||V(G)|+2$.
(2) Let $G$ be a cycle of order at least 4 , and $H$ be a path of order at least 3 . From Corollary 1. $\operatorname{diam}(G \square H)=\operatorname{diam}(G)+\operatorname{diam}(H) \geq 3$. By Theorem 亿. $m c(G \square H)=|E(G \square H)|-$ $|V(G \square H)|+2=|E(H)||V(G)|+2$.
(3) Let $G=P_{2}$ and $H$ be a path of order at least 3. From Corollary $\mathbb{1} \operatorname{diam}(G \square H)=$ $\operatorname{diam}(G)+\operatorname{diam}(H) \geq 3$. Therefore, $m c(G \square H)=|E(G \square H)|-|V(G \square H)|+2=|E(G)||E(H)|+$ 1.

The following corollary is immediate from Theorem 2,

Corollary 3 Let $G$ and $H$ be a connected graph.
(1) If neither $G$ nor $H$ is a tree, then $m c(G \square H) \geq \max \{m c(G)|V(H)|+2, m c(H)|V(G)|+$ $2\}$.
(2) If $G$ is not a tree and $H$ is a tree, then $m c(G \square H) \geq m c(H)|V(G)|+2$.
(3) If both $G$ and $H$ are trees, then $m c(G \square H) \geq m c(G) m c(H)+1$.

### 2.2 The lexicographical product

The lexicographic product $G \circ H$ of graphs $G$ and $H$ has the vertex set $V(G \circ H)=$ $V(G) \times V(H)$. Two vertices $(g, h),\left(g^{\prime}, h^{\prime}\right)$ are adjacent if $g g^{\prime} \in E(G)$, or if $g=g^{\prime}$ and $h h^{\prime} \in E(H)$. The lexicographic product is not commutative and is connected whenever $G$ is connected. Note that unlike the Cartesian Product, the lexicographic product is a noncommutative product since $G \circ H$ need not be isomorphic to $H \circ G$. Clearly, $|E(G \circ H)|=$ $|E(H)||V(G)|+|E(G)||V(H)|^{2}$.

Theorem 3 Let $G$ and $H$ be a connected graph.
(1) If neither $G$ nor $H$ is a tree, then

$$
|E(G)||V(H)|^{2}+2 \leq m c(G \circ H) \leq|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(H)|+1 .
$$

(2) If $G$ not a tree and $H$ is a tree, then
$|E(H)||V(G)|(|V(H)|+1)+2 \leq m c(G \circ H) \leq|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(H)|+1$.
(3) If $H$ not a tree and $G$ is a tree, then

$$
|E(H)||V(G)|^{2}+2 \leq m c(G \circ H) \leq|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(H)|+1 .
$$

(4) If both $G$ and $H$ are trees, then

$$
|E(H)||E(G)|(|V(H)|+1)+1 \leq m c(G \circ H)) \leq|E(H)||E(G)|(|V(H)|+1)+|V(H)| .
$$

Moreover, the lower bounds are sharp.

Proof. (1) Since $H$ is not a tree, it follows that $|E(H)| \geq|V(H)|$. By Observation 1, we have

$$
\begin{aligned}
m c(G \circ H) & \geq|E(G \circ H)|-|V(G \circ H)|+2 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+2 \\
& \geq|E(G)||V(H)|^{2}+2 .
\end{aligned}
$$

From Lemma 3, $\kappa(G \circ H)=\kappa(G)|V(H)| \leq(|V(G)|-1)|V(H)|$. By Lemma [1, we have

$$
\begin{aligned}
m c(G \circ H) & \leq|E(G \circ H)|-|V(G \circ H)|+\kappa(G \circ H)+1 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
& \leq|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+(|V(G)|-1)|V(H)|+1 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(H)|+1 .
\end{aligned}
$$

(2) Since $G$ is not a tree and $H$ is a tree, it follows that $|E(G)| \geq|V(G)|$ and $|E(H)|=$ $|V(H)|-1$. By Observation [1, we have

$$
\begin{aligned}
m c(G \circ H) & \geq|E(G \circ H)|-|V(G \circ H)|+2 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+2 \\
& =(|V(H)|-1)|V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+2 \\
& \geq|E(H)||V(G)|(|V(H)|+1)+2 .
\end{aligned}
$$

From Lemma 3, $\kappa(G \circ H)=\kappa(G)|V(H)| \leq(|V(G)|-1)|V(H)|$. By Lemma [1, we have

$$
\begin{aligned}
& m c(G \circ H) \\
\leq & |E(G \circ H)|-|V(G \circ H)|+\kappa(G \circ H)+1 \\
= & |E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
= & |E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
\leq & |E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+(|V(G)|-1)|V(H)|+1 \\
= & |E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(H)|+1 .
\end{aligned}
$$

(3) Since $H$ is not a tree, it follows that $|E(H)| \geq|V(H)|$. By Observation 11 we have

$$
\begin{aligned}
m c(G \circ H) & \geq|E(G \circ H)|-|V(G \circ H)|+2 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+2 \\
& \geq|E(G)||V(H)|^{2}+2
\end{aligned}
$$

From Lemma 3, $\kappa(G \circ H)=\kappa(G)|V(H)|=|V(H)|$. By Lemma 1, we have

$$
\begin{aligned}
m c(G \circ H) & \leq|E(G \circ H)|-|V(G \circ H)|+\kappa(G \circ H)+1 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+|V(H)|+1 .
\end{aligned}
$$

(4) Since both $G$ and $H$ are trees, it follows that $|E(G)|=|V(G)|-1$ and $|E(H)|=$ $|V(H)|-1$. By Observation [1 we have

$$
\begin{aligned}
m c(G \circ H) & \geq|E(G \circ H)|-|V(G \circ H)|+2 \\
& =|E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+2 \\
& =(|V(H)|-1)|V(G)|+(|V(G)|-1)|V(H)|^{2}-|V(G)||V(H)|+2 \\
& =|E(H)||E(G)|(|V(H)|+1)+1 .
\end{aligned}
$$

From Lemma 3, $\kappa(G \circ H)=\kappa(G)|V(H)|=|V(H)|$. By Lemma 1, we have

$$
\begin{aligned}
& m c(G \circ H) \\
\leq & |E(G \circ H)|-|V(G \circ H)|+\kappa(G \circ H)+1 \\
= & |E(H)||V(G)|+|E(G)||V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
= & (|V(H)|-1)|V(G)|+(|V(G)|-1)|V(H)|^{2}-|V(G)||V(H)|+\kappa(G \circ H)+1 \\
= & |E(H)||E(G)|(|V(H)|+1)+\kappa(G \circ H) \\
= & |E(H)||E(G)|(|V(H)|+1)+V(H) .
\end{aligned}
$$

To show the sharpness of the lower bounds in Theorem ??, we consider the following example.

Example 2: (1) Let $G$ be a cycle of order at least 6 , and $H$ be a cycle of order at least 3. From Lemma 7, $\operatorname{diam}(G \circ H) \geq \operatorname{diam}(G) \geq 3$. Therefore, $m c(G \circ H)=|E(G \circ H)|-|V(G \circ H)|+2=$ $|E(G)||V(H)|^{2}+2$.
(2) Let $G$ be a cycle of order at least 6 , and $H=P_{n}, n \geq 4$. By Lemma 7, $\operatorname{diam}(G \circ H) \geq$ $\operatorname{diam}(G) \geq 3$. Therefore, $m c(G \circ H)=|E(H)||V(G)|(|V(H)|+1)+2$.
(3) Let $G$ be a path of order at least 4, and $H$ be a cycle of order at least 3. By Lemma 7. $\operatorname{diam}(G \circ H) \geq \operatorname{diam}(G) \geq 3$. Therefore, $m c(G \circ H)=|E(H) \| V(G)|^{2}+2$.
(4) Let $G$ be a path of order at least 4 , and $H=P_{2}$. By Lemma 7, $\operatorname{diam}(G \circ H) \geq$ $\operatorname{diam}(G) \geq 3$. Therefore, $m c(G \circ H)=|E(H)||E(G)|(|V(H)|+1)+1$.

The following corollary is immediate from Theorem 3

Corollary 4 Let $G$ and $H$ be a connected graph.
(1) If neither $G$ nor $H$ is a tree, then $m c(G \circ H) \geq m c(G)|V(H)|^{2}+2$.
(2) If $G$ not a tree and $H$ is a tree, then $m c(G \circ H) \geq m c(H)|V(G)|(|V(H)|+1)+2$.
(3) If $H$ not a tree and $G$ is a tree, then $m c(G \circ H) \geq m c(H)|V(G)|^{2}+2$.
(4) If both $G$ and $H$ are trees, then $m c(G \circ H)) \geq m c(G) m c(H)(|V(H)|+1)+1$.

Moreover, the lower bounds are sharp.

### 2.3 The strong product

The strong product $G \boxtimes H$ of graphs $G$ and $H$ has the vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ are adjacent whenever $g g^{\prime} \in E(G)$ and $h=h^{\prime}$, or $g=g^{\prime}$ and $h h^{\prime} \in E(H)$, or $g g^{\prime} \in E(G)$ and $h h^{\prime} \in E(H)$. Clearly, $|E(G \boxtimes H)|=|E(H)||V(G)|+$ $|E(G)||V(H)|+2|E(G)||E(H)|$.

Theorem 4 Let $G$ and $H$ be a connected graph, and at least one of $G$ and $H$ is not a complete graph.
(1) If neither $G$ nor $H$ is a tree, then
$m c(G \boxtimes H) \geq \max \{|E(G)||V(H)|+2|E(H)||E(G)|+2,|E(H)||V(G)|+2|E(H)||E(G)|+2\}$
and
$m c(G \boxtimes H) \leq|E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-\min \{|V(H)|,|V(G)|\}+1$.
(2) If $G$ not a tree and $H$ is a tree, then
$|E(H)||V(G)|+2|E(H)||E(G)|+2 \leq m c(G \boxtimes H) \leq|E(G)||V(H)|+2|E(H)||E(G)|+1$.
(3) If both $G$ and $H$ are trees, then

$$
3|E(H)||E(G)|+1 \leq m c(G \boxtimes H) \leq 3|E(H)||E(G)|+\min \{|V(G)|,|V(H)|\}
$$

Moreover, the lower bounds are sharp.

Proof. (1) Since $H$ is not a tree, it follows that $|E(H)| \geq|V(H)|$. By Observation 11 we have

$$
\begin{aligned}
& m c(G \boxtimes H) \\
\geq & |E(G \boxtimes H)|-|V(G \boxtimes H)|+2 \\
= & |E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+2 \\
\geq & |E(G)||V(H)|+2|E(H)||E(G)|+2 .
\end{aligned}
$$

From Lemma 4, $\kappa(G \boxtimes H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} \leq \min \{(|V(G)|-$ 1) $|V(H)|,(|V(H)|-1)|V(G)|\}=|V(G)||V(H)|-\min \{|V(H)|,|V(G)|\}$. By Lemma [1, we have

$$
\begin{aligned}
& m c(G \boxtimes H) \\
\leq & E(G \boxtimes H)-V(G \boxtimes H)+\kappa(G \boxtimes H)+1 \\
= & |E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+\kappa(G \boxtimes H)+1 \\
\leq & |E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-\min \{|V(H)|,|V(G)|\}+1 .
\end{aligned}
$$

(2) Since $G$ is not a tree, it follows that $|E(G)| \geq|V(G)|$. Since $H$ is a tree, we have $|E(H)|=|V(H)|-1$. By Observation 1, we have

$$
\begin{aligned}
& m c(G \boxtimes H) \\
\geq & |E(G \boxtimes H)|-|V(G \boxtimes H)|+2 \\
= & |E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+2 \\
= & |E(G)||V(H)|+(|V(H)|-1)|V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+2 \\
\geq & |E(H)||V(G)|+2|E(H)||E(G)|+2 .
\end{aligned}
$$

From Lemma 4, $\kappa(G \boxtimes H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} \leq \kappa(H)|V(G)|=$ $|V(G)|$. By Lemma 1 , we have

$$
\begin{aligned}
& m c(G \boxtimes H) \\
\leq & |E(G \boxtimes H)|-|V(G \boxtimes H)|+\kappa(G \boxtimes H)+1 \\
= & |E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+\kappa(G \boxtimes H)+1 \\
= & |E(G)||V(H)|+(|V(H)|-1)|V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+\kappa(G \boxtimes H)+1 \\
= & |E(G)||V(H)|-|V(G)|+2|E(H)||E(G)|+\kappa(G \boxtimes H)+1 \\
\leq & |E(G)||V(H)|+2|E(H)||E(G)|+1 .
\end{aligned}
$$

(3) Since both $G$ and $H$ are trees, it follows that $|E(G)|=|V(G)|-1$ and $|E(H)|=$ $|V(H)|-1$. By Observation [1 we have

$$
\begin{aligned}
& m c(G \boxtimes H) \\
\geq & |E(G \boxtimes H)|-|V(G \boxtimes H)|+2 \\
= & |E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+2 \\
= & (|V(G)|-1)|V(H)|+(|V(H)|-1)|V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+2 \\
= & 3|E(H)||E(G)|+1 .
\end{aligned}
$$

From Lemma囷 $\kappa(G \boxtimes H)=\min \{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} \leq \min \{\kappa(G)|V(H)|$, $\kappa(H)|V(G)|\} \leq \min \{|V(H)|,|V(G)|\}$. By Lemma [1, we have
$m c(G \boxtimes H)$
$\leq|E(G \boxtimes H)|-|V(G \boxtimes H)|+\kappa(G \boxtimes H)+1$
$=|E(G)||V(H)|+|E(H)||V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+\kappa(G \boxtimes H)+1$
$=|V(G)|-1)|V(H)|+(|V(H)|-1)|V(G)|+2|E(H)||E(G)|-|V(G)||V(H)|+\kappa(G \boxtimes H)+1$
$=3|E(H)||E(G)|+\kappa(G \boxtimes H)$
$\leq 3|E(H)||E(G)|+\min \{|V(H)|,|V(G)|\})$.

To show the sharpness of the lower bounds in Theorem 4, we consider the following example.

Example 3: (1) Let $G$ be a cycle of order at least 6 , and $H$ be a cycle of order at least 3 . By Corollary 2, $\operatorname{diam}(G \boxtimes H)=\max \{\operatorname{diam}(G), \operatorname{diam}(H)\} \geq 3$. Therefore, $m c(G \boxtimes H)=$ $|E(G)||V(H)|+2|E(H)||E(G)|+2=|E(H)||V(G)|+2|E(H)||E(G)|+2$.
(2) Let $G$ be a cycle of order at least 3 , and $H$ be a cycle of order at least 4. By Corollary 2 $\operatorname{diam}(G \boxtimes H)=\max \{\operatorname{diam}(G), \operatorname{diam}(H)\} \geq 3$. Therefore, $m c(G \boxtimes H)=$ $|E(H)||V(G)|+2|E(H)||E(G)|+2$.
(3) Let $G=P_{2}$ and $H$ be a cycle of order at least 4. By Corollary 2, $\operatorname{diam}(G \boxtimes H)=$ $\max \{\operatorname{diam}(G), \operatorname{diam}(H)\} \geq 3$. Therefore, $m c(G \boxtimes H)=3|E(H)||E(G)|+1$.

The following corollary is immediate from Theorem 4 .

Corollary 5 Let $G$ and $H$ be a connected graph.
(1) If neither $G$ nor $H$ is a tree, then
$m c(G \boxtimes H) \geq \max \{|m c(G)||V(H)|+2|m c(H)||m c(G)|+2,|m c(H)||V(G)|+2|m c(H)||m c(G)|+2\}$.
(2) If $G$ not a tree and $H$ is a tree, then

$$
m c(G \boxtimes H) \geq|m c(H)||V(G)|+2|m c(H)||m c(G)|+2 .
$$

(3) If both $G$ and $H$ are trees, then

$$
m c(G \boxtimes H) \geq 3|m c(H)||m c(G)|+1
$$

Moreover, the lower bounds are sharp.

### 2.4 The direct product

The direct product $G \times H$ of graphs $G$ and $H$ has the vertex set $V(G) \times V(H)$. Two vertices $(g, h)$ and $\left(g^{\prime}, h^{\prime}\right)$ are adjacent if the projections on both coordinates are adjacent, i.e., $g g^{\prime} \in E(G)$ and $h h^{\prime} \in E(H)$. Clearly, $|E(G \times H)|=2|E(G)||E(H)|$.

Theorem 5 Let $G$ and $H$ be nonbipartite graphs. Then

$$
|E(H)||E(G)|+2 \leq m c(G \times H) \leq 2|E(H)||E(G)|+1
$$

Moreover, the lower bounds are sharp.

Proof. Since $H$ is not a tree, it follows that $|E(H)| \geq|V(H)|$. By Observation 1 , we have

$$
\begin{aligned}
m c(G \times H) & \geq|E(G \times H)|-|V(G \times H)|+2 \\
& =2|E(H)||E(G)|-|V(G)||V(H)|+2 \\
& \geq|E(H)||E(G)|+2 .
\end{aligned}
$$

From Lemma 5, $\kappa(G \times H) \leq \kappa^{\prime}(G \times H)=\min \left\{2 \kappa^{\prime}(G)|V(H)|, 2 \kappa^{\prime}(H)|V(G)|, \delta(G) \delta(H)\right\} \leq$ $\delta(G) \delta(H) \leq|V(G)||V(H)|$. By Lemma [1 we have

$$
\begin{aligned}
m c(G \times H) & \leq E(G \times H)-V(G \times H)+\kappa(G \times H)+1 \\
& =2|E(H)||E(G)|-|V(G)||V(H)|+\kappa(G \times H)+1 \\
& =2|E(H)||E(G)|+1
\end{aligned}
$$

To show the sharpness of the lower bounds in Theorem 5, we consider the following example.
Example 4: Let $G$ be a cycle of order at least 3 , and $H$ be a cycle of order at least 6 . By Lemma 2 $2 \operatorname{diam}(G \times H)=\max \{\operatorname{diam}(G), \operatorname{diam}(H)\} \geq 3$. Therefore, $m c(G \times H)=$ $|E(H)||E(G)|+2$.

The following corollary is immediate from Theorem 5 5
Corollary 6 Let one of $G$ and $H$ be a non-bipartite connected graph. Then

$$
m c(G \times H) \geq|m c(H)||m c(G)|+2 .
$$

## 3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

### 3.1 Two-dimensional grid graph

A two-dimensional grid graph is an $m \times n$ graph $G_{n, m}$ that is the graph Cartesian product $P_{n} \square P_{m}$ of path graphs on $m$ and $n$ vertices. See Figure $1(a)$ for the case $m=3$. For more details on grid graph, we refer to [6, 17]. The network $P_{n} \circ P_{m}$ is the graph lexicographical product $P_{n} \circ P_{m}$ of path graphs on $m$ and $n$ vertices. For more details on $P_{n} \circ P_{m}$, we refer to [23]. See Figure $1(b)$ for the case $m=3$.

(a)

(b)

Figure 1: (a) Two-dimensional grid graph $G_{n, 3} ;(b)$ The network $P_{n} \circ P_{3}$.

Proposition 1 (i) For network $P_{n} \square P_{m}(n \geq 3, m \geq 2)$,

$$
m c\left(P_{n} \square P_{m}\right)=n m-n-m+2 .
$$

(ii) For network $P_{n} \circ P_{m}(n \geq 4, m \geq 3)$,

$$
m c\left(P_{n} \circ P_{m}\right)=m^{2} n-m^{2}-n+2 .
$$

Proof. (i) From Corollary 亿 $\operatorname{diam}(G \square H)=\operatorname{diam}(G)+\operatorname{diam}(H) \geq 3$. Therefore, From Theorem [1 we have

$$
\begin{aligned}
m c\left(P_{n} \square P_{m}\right) & =\left|E\left(P_{n} \square P_{m}\right)\right|-\left|V\left(P_{n} \square P_{m}\right)\right|+2 \\
& =\left(\left|V\left(P_{n}\right)\right|-1\right)\left|V\left(P_{m}\right)\right|+\left(\left|V\left(P_{m}\right)\right|-1\right)\left|V\left(P_{n}\right)\right|-\left|V\left(P_{n}\right)\right|\left|V\left(P_{m}\right)\right|+2 \\
& =(n-1) m+(m-1) n-n m+2 \\
& =n m-n-m+2 .
\end{aligned}
$$

(2) From Lemma $\mathbf{7} \operatorname{diam}(G \circ H) \geq \operatorname{diam}(G) \geq 3$. From Theorem $\mathbb{1}$, we have

$$
\begin{aligned}
m c\left(P_{n} \circ P_{m}\right) & =E\left(P_{n} \circ P_{m}\right)-V\left(P_{n} \circ P_{m}\right)+2 \\
& =\left|E\left(P_{m}\right)\right|\left|V\left(P_{n}\right)\right|+\left|E\left(P_{n}\right)\right|\left|V\left(P_{m}\right)\right|^{2}-\left|V\left(P_{n}\right)\right|\left|V\left(P_{m}\right)\right|+2 \\
& =(m-1) n+(n-1) m^{2}-m n+2 \\
& =m^{2} n-m^{2}-n+2 .
\end{aligned}
$$

## $3.2 n$-dimensional mesh

An $n$-dimensional mesh is the Cartesian product of $n$ linear arrays. By this definition, two-dimensional grid graph is a 2 -dimensional mesh. An $n$-dimensional hypercube is a special case of an $n$-dimensional mesh, in which the $n$ linear arrays are all of size 2 ; see [18.

Proposition 2 (i) For $n$-dimensional mesh $P_{L_{1}} \square P_{L_{2}} \square \cdots \square P_{L_{n}}(n \geq 4)$,

$$
m c\left(P_{L_{1}} \square P_{L_{2}} \square \cdots \square P_{L_{n}}\right) \geq\left(2 \ell_{1} \ell_{2}-\ell_{1}-\ell_{2}\right)\left(\ell_{3} \ell_{4} \cdots \ell_{n}\right)+2 .
$$

(ii) For network $P_{L_{1}} \circ P_{L_{2}} \circ \cdots \circ P_{L_{n}}$,

$$
m c\left(P_{L_{1}} \circ P_{L_{2}} \circ \cdots \circ P_{L_{n}}\right) \geq\left(\ell_{1} \ell_{2}^{2}+\ell_{1} \ell_{2}-\ell_{1}-\ell_{2}^{2}\right)\left(\ell_{3} \ell_{4} \cdots \ell_{n}\right)^{2}+2 .
$$

Proof. (i) By Lemma 2, we have $\operatorname{diam}\left(\left(P_{L_{1}} \square P_{L_{2}} \square \cdots \square P_{L_{n}}\right)=\sum_{i=1}^{n} \operatorname{diam}\left(P_{L_{i}}\right) \geq 3\right.$. Set $G=P_{L_{1}} \square P_{L_{2}}$ and $H=P_{L_{2}} \square \cdots \square P_{L_{n}}$. Since both $G$ and $H$ are not trees, it follows from Theorem 2 that $m c(G \square H) \geq \max \{|E(G)||V(H)|,|E(H)||V(G)|\}+2 \geq|E(G)||V(H)|+2$. From Theorem [1, we have

$$
\begin{aligned}
& m c\left(P_{L_{1}} \square P_{L_{2}} \square \cdots \square P_{L_{n}}\right) \\
\geq & \left|E\left(P_{L_{1}} \square P_{L_{2}}\right)\right|\left|V\left(P_{L_{3}} \square \cdots \square P_{L_{n}}\right)\right|+2 \\
= & \left.\left|E\left(P_{L_{1}}\right)\right|\left|V\left(P_{L_{2}}\right)\right|+\left|E\left(P_{L_{2}}\right)\right|\left|V\left(P_{L_{1}}\right)\right|\right)\left(\left|V\left(P_{L_{3}}\right)\right| \cdots\left|V\left(P_{L_{n}}\right)\right|\right)+2 \\
= & \left(2 \ell_{1} \ell_{2}-\ell_{1}-\ell_{2}\right) \ell_{3} \ell_{4} \cdots \ell_{n}+2 .
\end{aligned}
$$

(ii) By Lemma 7. we have $\operatorname{diam}\left(\left(P_{L_{1}} \circ P_{L_{2}} \circ \cdots \circ P_{L_{n}}\right)=\max \left\{\operatorname{diam}\left(R_{i}\right)\right\} \geq 3\right.$. Set $G=P_{L_{1}} \circ P_{L_{2}}$ and $H=P_{L_{2}} \circ \cdots \circ P_{L_{n}}$. Since both $G$ and $H$ are not trees, it follows from Theorem 3 that $m c(G \circ H) \geq|E(G)||V(H)|^{2}+2$. From Theorem 1, we have

$$
\begin{aligned}
& m c\left(P_{L_{1}} \circ P_{L_{2}} \circ \cdots \circ P_{L_{n}}\right) \\
\geq & \left|E\left(P_{L_{1}} \circ P_{L_{2}}\right)\right|\left|V\left(P_{L_{3}} \circ \cdots \circ P_{L_{n}}\right)\right|^{2}+2 \\
= & \left(\left|E\left(P_{L_{2}}\right)\right|\left|V\left(P_{L_{1}}\right)\right|+\left|E\left(P_{L_{1}}\right)\right|\left|V\left(P_{L_{2}}\right)\right|^{2}\right)\left(\left|V\left(P_{L_{3}}\right)\right| \cdots\left|V\left(P_{L_{n}}\right)\right|\right)^{2}+2 \\
= & \left(\ell_{1} \ell_{2}^{2}+\ell_{1} \ell_{2}-\ell_{1}-\ell_{2}^{2}\right)\left(\ell_{3} \ell_{4} \cdots \ell_{n}\right)^{2}+2 .
\end{aligned}
$$

## $3.3 n$-dimensional torus

An $n$-dimensional torus is the Cartesian product of $n$ rings $R_{1}, R_{2}, \cdots, R_{n}$ of size at least three. (A ring is a cycle in Graph Theory.) The rings $R_{i}$ are not necessary to have the same size. Here, we consider the networks constructed by $R_{1} \square R_{2} \square \cdots \square R_{n}$ and $R_{1} \circ R_{2} \circ \cdots \circ R_{n}$.

Proposition 3 (i) For network $R_{1} \square R_{2} \square \cdots \square R_{n}, n \geq 4$

$$
m c\left(R_{1} \square R_{2} \square \cdots \square R_{n}\right) \geq r_{1} r_{2} \cdots r_{n}+2 .
$$

where $r_{i}$ is the order of $R_{i}$ and $3 \leq i \leq n$.
(ii) For network $R_{1} \circ R_{2} \circ \cdots \circ R_{n}, n \geq 4$

$$
m c\left(R_{1} \circ R_{2} \circ \cdots \circ R_{n}\right) \geq r_{1}\left(r_{2} \cdots r_{n}\right)^{2}+2 .
$$

Proof. (i) By Lemma 2, we have $\operatorname{diam}\left(\left(R_{1} \square R_{2} \square \cdots \square R_{n}\right)=\sum_{i=1}^{n} \operatorname{diam}\left(R_{i}\right) \geq 3\right.$. Set $G=R_{1}$ and $H=R_{2} \square \cdots \square R_{n}$. Since both $G$ and $H$ are not trees, it follows from Theorem 2 that $m c(G \square H) \geq \max \{|E(G)||V(H)|,|E(H)||V(G)|\}+2 \geq|E(G)||V(H)|+2$. From Theorem 1. we have

$$
\begin{aligned}
m c\left(R_{1} \square R_{2} \square \cdots \square R_{n}\right) & \geq\left|E\left(R_{1}\right)\right|\left|V\left(R_{2} \square \cdots \square R_{n}\right)\right|+2 \\
& =r_{1} r_{2} \cdots r_{n}+2 .
\end{aligned}
$$

(ii) By Lemma 7, we have $\operatorname{diam}\left(\left(R_{1} \circ R_{2} \circ \cdots \circ R_{n}\right)=\max \left\{\operatorname{diam}\left(R_{i}\right)\right\} \geq 3\right.$. Set $G=R_{1}$ and $H=R_{2} \circ \cdots \circ R_{n}$. Since both $G$ and $H$ are not trees, it follows from Theorem 3 that $m c(G \circ H) \geq|E(G)||V(H)|^{2}+2$. From Theorem [1 we have

$$
\begin{aligned}
m c\left(R_{1} \circ R_{2} \circ \cdots \circ R_{n}\right) & \geq\left|E\left(R_{1}\right)\right|\left|V\left(R_{1} \circ R_{2} \circ \cdots \circ R_{n}\right)\right|^{2}+2 \\
& =r_{1}\left(r_{2} \cdots r_{n}\right)^{2}+2 .
\end{aligned}
$$

## $3.4 n$-dimensional generalized hypercube

Let $K_{m}$ be a clique of $m$ vertices, $m \geq 2$. An $n$-dimensional generalized hypercube [13, 14] is the Cartesian product of $m$ cliques. We have the following:

Proposition 4 (i) For network $K_{m_{1}} \square K_{m_{2}} \square \cdots \square K_{m_{n}}\left(m_{i} \geq 2, n \geq 3,1 \leq i \leq n\right)$

$$
m c\left(K_{m_{1}} \square K_{m_{2}} \square \cdots \square K_{m_{n}}\right) \geq\binom{ m_{1}}{2} m_{2} \cdots m_{n}+2 .
$$

(ii) For network $K_{m_{1}} \circ K_{m_{2}} \circ \cdots \circ K_{m_{n}}$,

$$
m c\left(K_{m_{1}} \circ K_{m_{2}} \circ \cdots \circ K_{m_{n}}\right)=\binom{m_{1} m_{2} \cdots m_{n}}{2} .
$$

Proof. (i) By Lemma 2, we have $\operatorname{diam}\left(K_{m_{1}} \square K_{m_{2}} \square \cdots \square K_{m_{n}}\right)=\sum_{i=1}^{n} K_{m_{i}} \geq 3$. Set $G=K_{m_{1}}$ and $H=K_{m_{2}} \square \cdots \square K_{m_{n}}$. Since both $G$ and $H$ are not trees, it follows from Theorem 2 that $m c(G \square H) \geq \max \{|E(G)||V(H)|,|E(H)||V(G)|\}+2 \geq|E(G)||V(H)|+2$. From Theorem 1, we have

$$
\begin{aligned}
m c\left(K_{m_{1}} \square K_{m_{2}} \square \cdots \square K_{m_{n}}\right) & \geq\left|E\left(K_{m_{1}}\right)\right|\left|V\left(K_{m_{2}} \square K_{m_{3}} \square \cdots \square K_{m_{n}}\right)\right|+2 \\
& =\binom{m_{1}}{2} m_{2} \cdots m_{n}+2 .
\end{aligned}
$$

(ii) Note that $K_{m_{1}} \circ K_{m_{2}} \circ \cdots \circ K_{m_{n}}$ is a complete graph of order $\prod_{i=1}^{n} m_{i}$. From Theorem 1. we have

$$
m c\left(K_{m_{1}} \circ K_{m_{2}} \circ \cdots \circ K_{m_{n}}\right)=\binom{m_{1} m_{2} \cdots m_{n}}{2} .
$$

## 3.5 n-dimensional hyper Petersen network

An $n$-dimensional hyper Petersen network $H P_{n}$ is the Cartesian product of $Q_{n-3}$ and the well-known Petersen graph [12, where $n \geq 3$ and $Q_{n-3}$ denotes an $(n-3)$-dimensional hypercube. The cases $n=3$ and 4 of hyper Petersen networks are depicted in Figure 2. Note that $H P_{3}$ is just the Petersen graph (see Figure $2(a)$ ).

The network $H L_{n}$ is the lexicographical product of $Q_{n-3}$ and the Petersen graph, where $n \geq 3$ and $Q_{n-3}$ denotes an $(n-3)$-dimensional hypercube; see [23]. Note that $H L_{3}$ is just the Petersen graph, and $H L_{4}$ is a graph obtained from two copies of the Petersen graph by add one edge between one vertex in a copy of the Petersen graph and one vertex in another copy. See Figure $2(c)$ for an example (We only show the edges $v_{1} u_{i}(1 \leq i \leq 10)$ ).


Figure 2: (a) Petersen graph; (b) The network $H P_{4} ;(c)$ The structure of $H L_{4}$.

Proposition 5 (1) For network $H P_{3}$ and $H L_{3}, m c\left(H P_{3}\right)=m c\left(H L_{3}\right)=7$;
(2) For network $H L_{4}$ and $H P_{4}, m c\left(H P_{4}\right)=22$ and $112 \leq m c\left(H L_{4}\right) \leq 121$.

Proof. (1) By Theorem 亿, we have $m c\left(H P_{3}\right)=m c\left(H L_{3}\right)=\left|E\left(H L_{3}\right)\right|-\left|V\left(H L_{3}\right)\right|+2=7$.
(2) By Lemman, we have $m c\left(H P_{4}\right)=\left|E\left(H L_{4}\right)\right|-\left|V\left(H L_{4}\right)\right|+2=22$ and $121 \geq E\left(H L_{4}\right)-$ $V\left(H L_{4}\right)+\kappa\left(H L_{4}\right)+1 \geq m c\left(H L_{4}\right) \geq E\left(H L_{4}\right)-V\left(H L_{4}\right)+2=112$.

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