Monochromatic connectivity and graph products *

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Abstract

The concept of monochromatic connectivity was introduced by Caro and Yuster. A path in an edge-colored graph is called a *monochromatic path* if all the edges on the path are colored the same. An edge-coloring of G is a *monochromatic connection coloring* (*MC*-coloring, for short) if there is a monochromatic path joining any two vertices in G. The *monochromatic connection number*, denoted by mc(G), is defined to be the maximum number of colors used in an *MC*-coloring of a graph G. In this paper, we study the monochromatic connection number on the lexicographical, strong, Cartesian and direct product and present several upper and lower bounds for these products of graphs.

Keywords: Monochromatic path, *MC*-coloring, monochromatical connection num- ber, Cartesian product, lexicographical product, strong product, direct product.

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. For a graph G, we use V(G), E(G), n(G), m(G), $\delta(G)$, $\kappa(G)$, $\kappa'(G)$, $\delta(G)$ and diam(G) to denote the vertex set, edge set, number of vertices, number of edges, connectivity, edge-connectivity, minimum degree and diameter of G, respectively. The rainbow connections of a graph which are applied to measure the safety of a network are introduced by Chartrand, Johns, McKeon and Zhang [9]. Readers can see [9, 10, 11] for details. Consider an edge-coloring (not necessarily proper) of a graph G = (V, E). We say that a path of G is rainbow, if no two edges on the path have the same

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color. An edge-colored graph G is rainbow connected if every two vertices are connected by a rainbow path. The minimum number of colors required to rainbow color a graph G is called the rainbow connection number, denoted by rc(G). For more results on the rainbow connection, we refer to the survey paper [21] of Li, Shi and Sun and a new book [22] of Li and Sun.

Let G be a nontrivial connected graph with an edge-coloring $f : E(G) \to \{1, 2, \ldots, \ell\}, \ell \in N$, where adjacent edges may be colored the same. A path of G is a monochromatic path if all the edges on the path are colored the same. An edge-coloring of G is a monochromatic connection coloring (MC-coloring, for short) if there is a monochromatic path joining any two vertices in G. How colorful can an MC-coloring be? One can see that this question is the natural opposite of the well-studied problem on rainbow connection number of graphs. Let mc(G) denote the maximum number of colors used in an MC-coloring of a graph G, which called the monochromatic connection number of G. Note that an MC-coloring does not exist if G is not connected, and in this case we simply let mc(G) = 0.

These concepts were introduced by Caro and Yuster in [8]. For more results on monochromatic connection number, we refer to [4, 5, 8, 15]. The following observation is immediate.

Observation 1 [8] Let G be a connected graph with n(G) vertices and m(G) edges. Then

$$mc(G) \ge m(G) - n(G) + 2.$$

Simply color the edges of a spanning tree with one color, and each of the remaining edges may be assigned a distinct fresh color. Caro and Yuster gave some sufficient conditions for graphs attaining this lower bound.

Theorem 1 [8] Let G be a connected graph with n > 3. If G satisfies any of the following properties, then mc(G) = m - n + 2.

- (a) G (the complement of G) is 4-connected;
- (b) G is triangle-free;

(c) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$; In particular, this holds if $\Delta(G) \leq (n+1)/2$, and this also holds if $\Delta(G) \leq n - 2m/n$.

- (d) $Diam(G) \ge 3;$
- (e) G has a cut vertex.

Product networks were proposed based upon the idea of using the cross product as a tool for "combining" two known graphs with established properties to obtain a new one that inherits properties from both [13]. Recently, there has been an increasing interest in a class of interconnection networks called Cartesian product networks; see [2, 13]. The other standard products (Direct, strong, and lexicographic) draw a constant attention of graph research community, see some recent papers [1, 19, 24, 27].

In this paper, we consider four standard products: the lexicographic, the strong, the Cartesian and the direct with respect to the monochromatic connection number. Every of these four products will be treated in one of the forthcoming sections. In Section 3, we demonstrate the usefulness of the proposed constructions by applying them to some instances of product networks.

2 Main results

In this section, we study the monochromatic connection number of four graph product.

Lemma 1 [8] Let G be a connected graph with n(G) vertices and m(G) edges. Then

$$mc(G) \le E(G) - V(G) + \kappa(G) + 1.$$

In [25], Špacapan obtained the following result.

Lemma 2 [25] Let G and H be two nontrivial graphs. Then

$$\kappa(G\Box H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\}.$$

Yang and Xu [26] investigated the classical connectivity of the lexicographic product of two graphs.

Lemma 3 [26] Let G and H be two graphs. If G is non-trivial, non-complete and connected, then

$$\kappa(G \circ H) = \kappa(G)|V(H)|.$$

Let S_G and S_H be separating sets of connected graphs G and H, and let G' and H' be arbitrary connected components of $G - S_G$ and $H - S_H$. Then the set of vertices

$$(S_G \times V(H')) \cup (S_G \times S_H) \cup (V(G') \times S_H)$$

is called a \neg -set of $G \boxtimes H$; see [16].

Lemma 4 [16] Let G and H be connected graphs, at least one not complete. Set $\ell(G \boxtimes H)$ be the minimum size of a \neg -set of $G \boxtimes H$. Then

$$\kappa(G \boxtimes H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\}.$$

Lemma 5 [16] Let G and H be nonbipartite graphs. Then

$$\kappa'(G \times H) = \min\{2\kappa'(G)|V(H)|, 2\kappa'(H)|V(G)|, \delta(G)\delta(H)\}.$$

Let $d_G(u, v)$ denote the distance between u and v in G. Denote by $d_G(u)$ the degree of vertex u in G. The following lemma is from [16].

Lemma 6 [16] Let (g, h) and (g', h') be two vertices of $G \Box H$. Then $d_{G \Box H}((q, h), (q', h')) = d_G(q, q') + d_H(h, h').$

Corollary 1 Let G be a connected graph. Then

 $diam(G\Box H) = diam(G) + diam(H).$

Lemma 7 [16] Let (g,h) and (g',h') be two vertices of $G \circ H$. Then

$$d_{G\circ H}((g,h),(g',h')) = \begin{cases} d_G(g,g'), & \text{if } g \neq g'; \\ d_H(h,h'), & \text{if } g = g' \text{ and } d_G(g) = 0; \\ \min\{d_H(h,h'),2\}, & \text{if } g = g' \text{ and } d_G(g) \neq 0. \end{cases}$$

Lemma 8 [16] Let (g,h) and (g',h') be two vertices of $G\Box H$. Then

 $d_{G \boxtimes H}((g,h), (g',h')) = \max\{d_G(g,g'), d_H(h,h')\}.$

Corollary 2 Let G be a connected graph. Then

 $diam(G \boxtimes H) = \max\{diam(G), diam(H)\}.$

2.1 The Cartesian product

The Cartesian product of two graphs G and H, written as $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (g, h) and (g', h') are adjacent if and only if g = g' and $(h, h') \in E(H)$, or h = h' and $(g, g') \in E(G)$. Clearly, the Cartesian product is commutative, that is, $G \Box H$ is isomorphic to $H \Box G$. The Cartesian product is commutative, that is, $G \Box H \cong H \Box G$. Clearly, $|E(G \circ H)| = |E(H)||V(G)| + |E(G)||V(H)|$.

Theorem 2 Let G and H be a connected graph.

(1) If neither G nor H is a tree, then

 $\max\{|E(G)||V(H)|, |E(H)||V(G)|\} + 2 \le mc(G \Box H) \le |E(G)||V(H)| + (|E(H)| - 1)|V(G)| + 1.$

(2) If G is not a tree and H is a tree, then

 $|E(H)||V(G)| + 2 \le mc(G\Box H) \le |E(G)||V(H)| + 1.$

(3) If both G and H are trees, then

 $|E(G)||E(H)| + 1 \le mc(G\Box H) \le |E(G)||E(H)| + 2.$

Moreover, the lower bounds are sharp.

Proof. (1) Since H is not a tree, it follows that $|E(H)| \ge |V(H)|$. By Observation 1, we have

$$\begin{split} mc(G \Box H) &\geq |E(G \Box H)| - |V(G \Box H)| + 2 \\ &= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + 2 \\ &\geq |E(G)||V(H)| + 2. \end{split}$$

From Lemma 2, $\kappa(G \Box H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\} \le \kappa(H)|V(G)| \le (|V(H)| - 1)|V(G)|$. Furthermore, by Lemma 1, we have

$$mc(G\Box H)$$

$$\leq E(G\Box H) - V(G\Box H) + \kappa(G\Box H) + 1$$

$$= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + \kappa(G\Box H) + 1$$

$$\leq |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + |V(G)|(|V(H)| - 1) + 1$$

$$= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)| + 1.$$

(2) Since G is not a tree and H is a tree, it follows that $|E(G)| \ge |V(G)|$ and |E(H)| = |V(H)| - 1. By Observation 1, we have

$$\begin{aligned} mc(G \Box H) &\geq E(G \Box H) - V(G \Box H) + 2 \\ &= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + 2 \\ &\geq |V(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + 2 \\ &= |E(H)||V(G)| + 2. \end{aligned}$$

From Lemma 2, $\kappa(G \Box H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\} \le \kappa(H)|V(G)| \le |V(G)|$. By Lemma 1, we have

$$\begin{split} mc(G \Box H) &\leq E(G \Box H) - V(G \Box H) + \kappa(G \Box H) + 1 \\ &= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + \kappa(G \Box H) + 1 \\ &= |E(G)||V(H)| + (|V(H)| - 1)|V(G)| - |V(G)||V(H)| + \kappa(G \Box H) + 1 \\ &\leq |E(G)||V(H)| - |V(G)| + |V(G)| + 1 \\ &\leq |E(G)||V(H)| + 1. \end{split}$$

(3) Since both G and H are trees, it follows that |E(G)| = |V(G)| - 1 and |E(H)| = |V(H)| - 1. By Observation 1, we have

$$\begin{split} mc(G \Box H) &\geq E(G \Box H) - V(G \Box H) + 2 \\ &= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + 2 \\ &= (|V(G)| - 1)|V(H)| + (|V(H)| - 1)|V(G)| - |V(G)||V(H)| + 2 \\ &= |E(G)||E(H)| + 1. \end{split}$$

From Lemma 2, $\kappa(G \Box H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \delta(G) + \delta(H)\} \le \delta(G) + \delta(H) = 2$. By Lemma 1, we have

$$\begin{split} mc(G\Box H) &\leq E(G\Box H) - V(G\Box H) + \kappa(G\Box H) + 1 \\ &= |E(G)||V(H)| + |E(H)||V(G)| - |V(G)||V(H)| + \kappa(G\Box H) + 1 \\ &= (|V(G)| - 1)|V(H)| + (|V(H)| - 1)|V(G)| - |V(G)||V(H)| + \kappa(G\Box H) + 1 \\ &= |E(G)||E(H)| + \kappa(G\Box H) \\ &\leq |E(G)||E(H)| + 2. \end{split}$$

To show the sharpness of the lower bounds in Theorem 2, we consider the following example.

Example 1: (1) Let G be a cycle of order at least 3, and H be a cycle of order at least 4. From Corollary 1, $diam(G\Box H) = diam(G) + diam(H) \ge 3$. By Theorem 1, $mc(G\Box H) = |E(G\Box H)| - |V(G\Box H)| + 2 = |E(G)||V(H)| + 2 = |E(H)||V(G)| + 2$.

(2) Let G be a cycle of order at least 4, and H be a path of order at least 3. From Corollary 1, $diam(G\Box H) = diam(G) + diam(H) \ge 3$. By Theorem 1, $mc(G\Box H) = |E(G\Box H)| - |V(G\Box H)| + 2 = |E(H)||V(G)| + 2$.

(3) Let $G = P_2$ and H be a path of order at least 3. From Corollary 1, $diam(G\Box H) = diam(G) + diam(H) \ge 3$. Therefore, $mc(G\Box H) = |E(G\Box H)| - |V(G\Box H)| + 2 = |E(G)||E(H)| + 1$.

The following corollary is immediate from Theorem 2.

Corollary 3 Let G and H be a connected graph.

(1) If neither G nor H is a tree, then $mc(G\Box H) \ge \max\{mc(G)|V(H)|+2, mc(H)|V(G)|+2\}$.

(2) If G is not a tree and H is a tree, then $mc(G\Box H) \ge mc(H)|V(G)| + 2$.

(3) If both G and H are trees, then $mc(G\Box H) \ge mc(G)mc(H) + 1$.

2.2 The lexicographical product

The lexicographic product $G \circ H$ of graphs G and H has the vertex set $V(G \circ H) = V(G) \times V(H)$. Two vertices (g,h), (g',h') are adjacent if $gg' \in E(G)$, or if g = g' and $hh' \in E(H)$. The lexicographic product is not commutative and is connected whenever G is connected. Note that unlike the Cartesian Product, the lexicographic product is a non-commutative product since $G \circ H$ need not be isomorphic to $H \circ G$. Clearly, $|E(G \circ H)| = |E(H)||V(G)| + |E(G)||V(H)|^2$.

Theorem 3 Let G and H be a connected graph.

(1) If neither G nor H is a tree, then

$$|E(G)||V(H)|^{2} + 2 \le mc(G \circ H) \le |E(H)||V(G)| + |E(G)||V(H)|^{2} - |V(H)| + 1.$$

(2) If G not a tree and H is a tree, then

$$|E(H)||V(G)|(|V(H)|+1)+2 \le mc(G \circ H) \le |E(H)||V(G)|+|E(G)||V(H)|^2-|V(H)|+1.$$

(3) If H not a tree and G is a tree, then

$$|E(H)||V(G)|^{2} + 2 \le mc(G \circ H) \le |E(H)||V(G)| + |E(G)||V(H)|^{2} - |V(H)| + 1.$$

(4) If both G and H are trees, then

$$|E(H)||E(G)|(|V(H)|+1) + 1 \le mc(G \circ H)) \le |E(H)||E(G)|(|V(H)|+1) + |V(H)|.$$

Moreover, the lower bounds are sharp.

Proof. (1) Since H is not a tree, it follows that $|E(H)| \ge |V(H)|$. By Observation 1, we have

$$mc(G \circ H) \geq |E(G \circ H)| - |V(G \circ H)| + 2$$

= $|E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + 2$
 $\geq |E(G)||V(H)|^2 + 2.$

From Lemma 3, $\kappa(G \circ H) = \kappa(G)|V(H)| \leq (|V(G)| - 1)|V(H)|$. By Lemma 1, we have

$$\begin{split} mc(G \circ H) &\leq |E(G \circ H)| - |V(G \circ H)| + \kappa(G \circ H) + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ &\leq |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + (|V(G)| - 1)|V(H)| + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(H)| + 1. \end{split}$$

(2) Since G is not a tree and H is a tree, it follows that $|E(G)| \ge |V(G)|$ and |E(H)| = |V(H)| - 1. By Observation 1, we have

$$mc(G \circ H) \geq |E(G \circ H)| - |V(G \circ H)| + 2$$

= $|E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + 2$
= $(|V(H)| - 1)|V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + 2$
 $\geq |E(H)||V(G)|(|V(H)| + 1) + 2.$

From Lemma 3, $\kappa(G \circ H) = \kappa(G)|V(H)| \le (|V(G)| - 1)|V(H)|$. By Lemma 1, we have

$$\begin{split} mc(G \circ H) \\ &\leq |E(G \circ H)| - |V(G \circ H)| + \kappa(G \circ H) + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ &\leq |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + (|V(G)| - 1)|V(H)| + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(H)| + 1. \end{split}$$

(3) Since H is not a tree, it follows that $|E(H)| \ge |V(H)|$. By Observation 1, we have

$$mc(G \circ H) \geq |E(G \circ H)| - |V(G \circ H)| + 2$$

= $|E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + 2$
 $\geq |E(G)||V(H)|^2 + 2.$

From Lemma 3, $\kappa(G \circ H) = \kappa(G)|V(H)| = |V(H)|$. By Lemma 1, we have

$$\begin{split} mc(G \circ H) &\leq |E(G \circ H)| - |V(G \circ H)| + \kappa(G \circ H) + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + |V(H)| + 1. \end{split}$$

(4) Since both G and H are trees, it follows that |E(G)| = |V(G)| - 1 and |E(H)| = |V(H)| - 1. By Observation 1, we have

$$\begin{split} mc(G \circ H) &\geq |E(G \circ H)| - |V(G \circ H)| + 2 \\ &= |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + 2 \\ &= (|V(H)| - 1)|V(G)| + (|V(G)| - 1)|V(H)|^2 - |V(G)||V(H)| + 2 \\ &= |E(H)||E(G)|(|V(H)| + 1) + 1. \end{split}$$

From Lemma 3, $\kappa(G \circ H) = \kappa(G)|V(H)| = |V(H)|$. By Lemma 1, we have

$$\begin{split} & mc(G \circ H) \\ \leq & |E(G \circ H)| - |V(G \circ H)| + \kappa(G \circ H) + 1 \\ = & |E(H)||V(G)| + |E(G)||V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ = & (|V(H)| - 1)|V(G)| + (|V(G)| - 1)|V(H)|^2 - |V(G)||V(H)| + \kappa(G \circ H) + 1 \\ = & |E(H)||E(G)|(|V(H)| + 1) + \kappa(G \circ H) \\ = & |E(H)||E(G)|(|V(H)| + 1) + V(H). \end{split}$$

To show the sharpness of the lower bounds in Theorem ??, we consider the following example.

Example 2: (1) Let G be a cycle of order at least 6, and H be a cycle of order at least 3. From Lemma 7, $diam(G \circ H) \ge diam(G) \ge 3$. Therefore, $mc(G \circ H) = |E(G \circ H)| - |V(G \circ H)| + 2 = |E(G)||V(H)|^2 + 2$.

(2) Let G be a cycle of order at least 6, and $H = P_n, n \ge 4$. By Lemma 7, $diam(G \circ H) \ge diam(G) \ge 3$. Therefore, $mc(G \circ H) = |E(H)||V(G)|(|V(H)| + 1) + 2$.

(3) Let G be a path of order at least 4, and H be a cycle of order at least 3. By Lemma 7, $diam(G \circ H) \ge diam(G) \ge 3$. Therefore, $mc(G \circ H) = |E(H)||V(G)|^2 + 2$.

(4) Let G be a path of order at least 4, and $H = P_2$. By Lemma 7, $diam(G \circ H) \ge diam(G) \ge 3$. Therefore, $mc(G \circ H) = |E(H)||E(G)|(|V(H)| + 1) + 1$.

The following corollary is immediate from Theorem 3.

Corollary 4 Let G and H be a connected graph.

- (1) If neither G nor H is a tree, then $mc(G \circ H) \ge mc(G)|V(H)|^2 + 2$.
- (2) If G not a tree and H is a tree, then $mc(G \circ H) \ge mc(H)|V(G)|(|V(H)|+1)+2$.
- (3) If H not a tree and G is a tree, then $mc(G \circ H) \ge mc(H)|V(G)|^2 + 2$.
- (4) If both G and H are trees, then $mc(G \circ H) \ge mc(G)mc(H)(|V(H)|+1)+1$.

Moreover, the lower bounds are sharp.

2.3 The strong product

The strong product $G \boxtimes H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent whenever $gg' \in E(G)$ and h = h', or g = g' and $hh' \in E(H)$, or $gg' \in E(G)$ and $hh' \in E(H)$. Clearly, $|E(G \boxtimes H)| = |E(H)||V(G)| + |E(G)||V(H)| + 2|E(G)||E(H)|$.

Theorem 4 Let G and H be a connected graph, and at least one of G and H is not a complete graph.

(1) If neither G nor H is a tree, then

 $mc(G \boxtimes H) \ge \max\{|E(G)||V(H)| + 2|E(H)||E(G)| + 2, |E(H)||V(G)| + 2|E(H)||E(G)| + 2\}$

and

$$mc(G \boxtimes H) \le |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| - \min\{|V(H)|, |V(G)|\} + 1.$$

(2) If G not a tree and H is a tree, then

$$|E(H)||V(G)| + 2|E(H)||E(G)| + 2 \le mc(G \boxtimes H) \le |E(G)||V(H)| + 2|E(H)||E(G)| + 1.$$

(3) If both G and H are trees, then

 $3|E(H)||E(G)| + 1 \le mc(G \boxtimes H) \le 3|E(H)||E(G)| + \min\{|V(G)|, |V(H)|\}.$

Moreover, the lower bounds are sharp.

Proof. (1) Since H is not a tree, it follows that $|E(H)| \ge |V(H)|$. By Observation 1, we have

$$\begin{split} &mc(G \boxtimes H) \\ \geq & |E(G \boxtimes H)| - |V(G \boxtimes H)| + 2 \\ = & |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| - |V(G)||V(H)| + 2 \\ \geq & |E(G)||V(H)| + 2|E(H)||E(G)| + 2. \end{split}$$

From Lemma 4, $\kappa(G \boxtimes H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} \le \min\{(|V(G)| - 1)|V(H)|, (|V(H)| - 1)|V(G)|\} = |V(G)||V(H)| - \min\{|V(H)|, |V(G)|\}$. By Lemma 1, we have

$$mc(G \boxtimes H)$$

$$\leq E(G \boxtimes H) - V(G \boxtimes H) + \kappa(G \boxtimes H) + 1$$

$$= |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| - |V(G)||V(H)| + \kappa(G \boxtimes H) + 1$$

 $\leq |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| - \min\{|V(H)|, |V(G)|\} + 1.$

(2) Since G is not a tree, it follows that $|E(G)| \ge |V(G)|$. Since H is a tree, we have |E(H)| = |V(H)| - 1. By Observation 1, we have

$mc(G \boxtimes H)$

- $\geq |E(G \boxtimes H)| |V(G \boxtimes H)| + 2$
- = |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + 2
- = |E(G)||V(H)| + (|V(H)| 1)|V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + 2
- $\geq |E(H)||V(G)| + 2|E(H)||E(G)| + 2.$

From Lemma 4, $\kappa(G \boxtimes H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} \le \kappa(H)|V(G)| = |V(G)|$. By Lemma 1, we have

 $mc(G \boxtimes H)$

- $\leq \quad |E(G\boxtimes H)|-|V(G\boxtimes H)|+\kappa(G\boxtimes H)+1$
- $= |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + \kappa(G \boxtimes H) + 1$
- $= |E(G)||V(H)| + (|V(H)| 1)|V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + \kappa(G \boxtimes H) + 1$
- $= |E(G)||V(H)| |V(G)| + 2|E(H)||E(G)| + \kappa(G \boxtimes H) + 1$
- $\leq ||E(G)||V(H)| + 2|E(H)||E(G)| + 1.$

(3) Since both G and H are trees, it follows that |E(G)| = |V(G)| - 1 and |E(H)| = |V(H)| - 1. By Observation 1, we have

 $mc(G \boxtimes H)$

 $\geq |E(G\boxtimes H)| - |V(G\boxtimes H)| + 2$

- = |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + 2
- = (|V(G)| 1)|V(H)| + (|V(H)| 1)|V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + 2|E(H)||E(H)||E(H)| |V(G)||V(H)| + 2|E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E(H)||E
- = 3|E(H)||E(G)| + 1.

From Lemma 4, $\kappa(G \boxtimes H) = \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|, \ell(G \boxtimes H)\} \le \min\{\kappa(G)|V(H)|, \kappa(H)|V(G)|\} \le \min\{|V(H)|, |V(G)|\}$. By Lemma 1, we have

 $mc(G \boxtimes H)$

- $\leq |E(G\boxtimes H)| |V(G\boxtimes H)| + \kappa(G\boxtimes H) + 1$
- $= |E(G)||V(H)| + |E(H)||V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + \kappa(G \boxtimes H) + 1$
- $= |V(G)| 1)|V(H)| + (|V(H)| 1)|V(G)| + 2|E(H)||E(G)| |V(G)||V(H)| + \kappa(G \boxtimes H) + 1$
- $= 3|E(H)||E(G)| + \kappa(G \boxtimes H)$
- $\leq 3|E(H)||E(G)| + \min\{|V(H)|, |V(G)|\}).$

To show the sharpness of the lower bounds in Theorem 4, we consider the following example.

Example 3: (1) Let G be a cycle of order at least 6, and H be a cycle of order at least 3. By Corollary 2, $diam(G \boxtimes H) = \max\{diam(G), diam(H)\} \ge 3$. Therefore, $mc(G \boxtimes H) = |E(G)||V(H)| + 2|E(H)||E(G)| + 2 = |E(H)||V(G)| + 2|E(H)||E(G)| + 2$.

(2) Let G be a cycle of order at least 3, and H be a cycle of order at least 4. By Corollary 2, $diam(G \boxtimes H) = \max\{diam(G), diam(H)\} \ge 3$. Therefore, $mc(G \boxtimes H) = |E(H)||V(G)| + 2|E(H)||E(G)| + 2$.

(3) Let $G = P_2$ and H be a cycle of order at least 4. By Corollary 2, $diam(G \boxtimes H) = \max\{diam(G), diam(H)\} \ge 3$. Therefore, $mc(G \boxtimes H) = 3|E(H)||E(G)| + 1$.

The following corollary is immediate from Theorem 4.

Corollary 5 Let G and H be a connected graph.

(1) If neither G nor H is a tree, then

 $mc(G\boxtimes H) \geq \max\{|mc(G)||V(H)|+2|mc(H)||mc(G)|+2,|mc(H)||V(G)|+2|mc(H)||mc(G)|+2\}.$

(2) If G not a tree and H is a tree, then

$$mc(G \boxtimes H) \ge |mc(H)||V(G)| + 2|mc(H)||mc(G)| + 2.$$

(3) If both G and H are trees, then

$$mc(G \boxtimes H) \ge 3|mc(H)||mc(G)| + 1.$$

Moreover, the lower bounds are sharp.

2.4 The direct product

The direct product $G \times H$ of graphs G and H has the vertex set $V(G) \times V(H)$. Two vertices (g,h) and (g',h') are adjacent if the projections on both coordinates are adjacent, i.e., $gg' \in E(G)$ and $hh' \in E(H)$. Clearly, $|E(G \times H)| = 2|E(G)||E(H)|$.

Theorem 5 Let G and H be nonbipartite graphs. Then

 $|E(H)||E(G)| + 2 \le mc(G \times H) \le 2|E(H)||E(G)| + 1.$

Moreover, the lower bounds are sharp.

Proof. Since H is not a tree, it follows that $|E(H)| \ge |V(H)|$. By Observation 1, we have

$$mc(G \times H) \geq |E(G \times H)| - |V(G \times H)| + 2$$

= 2|E(H)||E(G)| - |V(G)||V(H)| + 2
\geq |E(H)||E(G)| + 2.

From Lemma 5, $\kappa(G \times H) \leq \kappa'(G \times H) = \min\{2\kappa'(G)|V(H)|, 2\kappa'(H)|V(G)|, \delta(G)\delta(H)\} \leq \delta(G)\delta(H) \leq |V(G)||V(H)|$. By Lemma 1, we have

$$mc(G \times H) \leq E(G \times H) - V(G \times H) + \kappa(G \times H) + 1$$

= 2|E(H)||E(G)| - |V(G)||V(H)| + \kappa(G \times H) + 1
= 2|E(H)||E(G)| + 1.

To show the sharpness of the lower bounds in Theorem 5, we consider the following example.

Example 4: Let G be a cycle of order at least 3, and H be a cycle of order at least 6. By Lemma 2, $diam(G \times H) = max\{diam(G), diam(H)\} \ge 3$. Therefore, $mc(G \times H) = |E(H)||E(G)| + 2$.

The following corollary is immediate from Theorem 5.

Corollary 6 Let one of G and H be a non-bipartite connected graph. Then

$$mc(G \times H) \ge |mc(H)||mc(G)| + 2.$$

3 Applications

In this section, we demonstrate the usefulness of the proposed constructions by applying them to some instances of Cartesian and lexicographical product networks.

3.1 Two-dimensional grid graph

A two-dimensional grid graph is an $m \times n$ graph $G_{n,m}$ that is the graph Cartesian product $P_n \Box P_m$ of path graphs on m and n vertices. See Figure 1 (a) for the case m = 3. For more details on grid graph, we refer to [6, 17]. The network $P_n \circ P_m$ is the graph lexicographical product $P_n \circ P_m$ of path graphs on m and n vertices. For more details on $P_n \circ P_m$, we refer to [23]. See Figure 1 (b) for the case m = 3.



Figure 1: (a) Two-dimensional grid graph $G_{n,3}$; (b) The network $P_n \circ P_3$.

Proposition 1 (i) For network $P_n \Box P_m$ $(n \ge 3, m \ge 2)$,

$$mc(P_n \Box P_m) = nm - n - m + 2.$$

(ii) For network $P_n \circ P_m$ $(n \ge 4, m \ge 3)$,

$$mc(P_n \circ P_m) = m^2 n - m^2 - n + 2.$$

Proof. (i) From Corollary 1, $diam(G \Box H) = diam(G) + diam(H) \ge 3$. Therefore, From Theorem 1, we have

$$mc(P_n \Box P_m) = |E(P_n \Box P_m)| - |V(P_n \Box P_m)| + 2$$

= $(|V(P_n)| - 1)|V(P_m)| + (|V(P_m)| - 1)|V(P_n)| - |V(P_n)||V(P_m)| + 2$
= $(n - 1)m + (m - 1)n - nm + 2$
= $nm - n - m + 2.$

(2) From Lemma 7, $diam(G \circ H) \ge diam(G) \ge 3$. From Theorem 1, we have

$$mc(P_n \circ P_m) = E(P_n \circ P_m) - V(P_n \circ P_m) + 2$$

= $|E(P_m)||V(P_n)| + |E(P_n)||V(P_m)|^2 - |V(P_n)||V(P_m)| + 2$
= $(m-1)n + (n-1)m^2 - mn + 2$
= $m^2n - m^2 - n + 2.$

3.2 *n*-dimensional mesh

An *n*-dimensional mesh is the Cartesian product of *n* linear arrays. By this definition, two-dimensional grid graph is a 2-dimensional mesh. An *n*-dimensional hypercube is a special case of an *n*-dimensional mesh, in which the *n* linear arrays are all of size 2; see [18].

Proposition 2 (i) For n-dimensional mesh $P_{L_1} \Box P_{L_2} \Box \cdots \Box P_{L_n}$ $(n \ge 4)$,

$$mc(P_{L_1} \Box P_{L_2} \Box \cdots \Box P_{L_n}) \ge (2\ell_1\ell_2 - \ell_1 - \ell_2)(\ell_3\ell_4 \cdots \ell_n) + 2.$$

(ii) For network $P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}$,

$$mc(P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) \ge (\ell_1 \ell_2^2 + \ell_1 \ell_2 - \ell_1 - \ell_2^2)(\ell_3 \ell_4 \cdots \ell_n)^2 + 2.$$

Proof. (i) By Lemma 2, we have $diam((P_{L_1} \Box P_{L_2} \Box \cdots \Box P_{L_n}) = \sum_{i=1}^n diam(P_{L_i}) \ge 3$. Set $G = P_{L_1} \Box P_{L_2}$ and $H = P_{L_2} \Box \cdots \Box P_{L_n}$. Since both G and H are not trees, it follows from Theorem 2 that $mc(G \Box H) \ge \max\{|E(G)||V(H)|, |E(H)||V(G)|\} + 2 \ge |E(G)||V(H)| + 2$. From Theorem 1, we have

$$mc(P_{L_{1}} \Box P_{L_{2}} \Box \cdots \Box P_{L_{n}})$$

$$\geq |E(P_{L_{1}} \Box P_{L_{2}})||V(P_{L_{3}} \Box \cdots \Box P_{L_{n}})| + 2$$

$$= |E(P_{L_{1}})||V(P_{L_{2}})| + |E(P_{L_{2}})||V(P_{L_{1}})|)(|V(P_{L_{3}})| \cdots |V(P_{L_{n}})|) + 2$$

$$= (2\ell_{1}\ell_{2} - \ell_{1} - \ell_{2})\ell_{3}\ell_{4} \cdots \ell_{n} + 2.$$

(*ii*) By Lemma 7, we have $diam((P_{L_1} \circ P_{L_2} \circ \cdots \circ P_{L_n}) = \max\{diam(R_i)\} \geq 3$. Set $G = P_{L_1} \circ P_{L_2}$ and $H = P_{L_2} \circ \cdots \circ P_{L_n}$. Since both G and H are not trees, it follows from Theorem 3 that $mc(G \circ H) \geq |E(G)||V(H)|^2 + 2$. From Theorem 1, we have

$$mc(P_{L_{1}} \circ P_{L_{2}} \circ \cdots \circ P_{L_{n}})$$

$$\geq |E(P_{L_{1}} \circ P_{L_{2}})||V(P_{L_{3}} \circ \cdots \circ P_{L_{n}})|^{2} + 2$$

$$= (|E(P_{L_{2}})||V(P_{L_{1}})| + |E(P_{L_{1}})||V(P_{L_{2}})|^{2})(|V(P_{L_{3}})|\cdots |V(P_{L_{n}})|)^{2} + 2$$

$$= (\ell_{1}\ell_{2}^{2} + \ell_{1}\ell_{2} - \ell_{1} - \ell_{2}^{2})(\ell_{3}\ell_{4}\cdots \ell_{n})^{2} + 2.$$

3.3 *n*-dimensional torus

An *n*-dimensional torus is the Cartesian product of *n* rings R_1, R_2, \dots, R_n of size at least three. (A ring is a cycle in Graph Theory.) The rings R_i are not necessary to have the same size. Here, we consider the networks constructed by $R_1 \square R_2 \square \dots \square R_n$ and $R_1 \circ R_2 \circ \dots \circ R_n$.

Proposition 3 (i) For network $R_1 \Box R_2 \Box \cdots \Box R_n$, $n \ge 4$

 $mc(R_1 \Box R_2 \Box \cdots \Box R_n) \ge r_1 r_2 \cdots r_n + 2.$

where r_i is the order of R_i and $3 \le i \le n$.

(ii) For network $R_1 \circ R_2 \circ \cdots \circ R_n$, $n \ge 4$

$$mc(R_1 \circ R_2 \circ \cdots \circ R_n) \ge r_1(r_2 \cdots r_n)^2 + 2.$$

Proof. (i) By Lemma 2, we have $diam((R_1 \Box R_2 \Box \cdots \Box R_n) = \sum_{i=1}^n diam(R_i) \ge 3$. Set $G = R_1$ and $H = R_2 \Box \cdots \Box R_n$. Since both G and H are not trees, it follows from Theorem 2 that $mc(G \Box H) \ge \max\{|E(G)||V(H)|, |E(H)||V(G)|\} + 2 \ge |E(G)||V(H)| + 2$. From Theorem 1, we have

$$mc(R_1 \Box R_2 \Box \cdots \Box R_n) \geq |E(R_1)| |V(R_2 \Box \cdots \Box R_n)| + 2$$

= $r_1 r_2 \cdots r_n + 2.$

(*ii*) By Lemma 7, we have $diam((R_1 \circ R_2 \circ \cdots \circ R_n) = \max\{diam(R_i)\} \ge 3$. Set $G = R_1$ and $H = R_2 \circ \cdots \circ R_n$. Since both G and H are not trees, it follows from Theorem 3 that $mc(G \circ H) \ge |E(G)||V(H)|^2 + 2$. From Theorem 1, we have

$$mc(R_1 \circ R_2 \circ \cdots \circ R_n) \geq |E(R_1)||V(R_1 \circ R_2 \circ \cdots \circ R_n)|^2 + 2$$

= $r_1(r_2 \cdots r_n)^2 + 2.$

3.4 *n*-dimensional generalized hypercube

Let K_m be a clique of m vertices, $m \ge 2$. An *n*-dimensional generalized hypercube [13, 14] is the Cartesian product of m cliques. We have the following:

Proposition 4 (i) For network $K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_n}$ $(m_i \ge 2, n \ge 3, 1 \le i \le n)$ $mc(K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_n}) \ge \binom{m_1}{2} m_2 \cdots m_n + 2.$

(ii) For network $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}$,

$$mc(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}) = \binom{m_1 m_2 \cdots m_n}{2}$$

Proof. (i) By Lemma 2, we have $diam(K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_n}) = \sum_{i=1}^n K_{m_i} \ge 3$. Set $G = K_{m_1}$ and $H = K_{m_2} \Box \cdots \Box K_{m_n}$. Since both G and H are not trees, it follows from Theorem 2 that $mc(G \Box H) \ge \max\{|E(G)||V(H)|, |E(H)||V(G)|\} + 2 \ge |E(G)||V(H)| + 2$. From Theorem 1, we have

$$mc(K_{m_1} \Box K_{m_2} \Box \cdots \Box K_{m_n}) \geq |E(K_{m_1})||V(K_{m_2} \Box K_{m_3} \Box \cdots \Box K_{m_n})| + 2$$
$$= \binom{m_1}{2} m_2 \cdots m_n + 2.$$

(*ii*) Note that $K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}$ is a complete graph of order $\prod_{i=1}^n m_i$. From Theorem 1, we have

$$mc(K_{m_1} \circ K_{m_2} \circ \cdots \circ K_{m_n}) = \binom{m_1 m_2 \cdots m_n}{2}.$$

3.5 *n*-dimensional hyper Petersen network

An *n*-dimensional hyper Petersen network HP_n is the Cartesian product of Q_{n-3} and the well-known Petersen graph [12], where $n \ge 3$ and Q_{n-3} denotes an (n-3)-dimensional hypercube. The cases n = 3 and 4 of hyper Petersen networks are depicted in Figure 2. Note that HP_3 is just the Petersen graph (see Figure 2 (a)).

The network HL_n is the lexicographical product of Q_{n-3} and the Petersen graph, where $n \geq 3$ and Q_{n-3} denotes an (n-3)-dimensional hypercube; see [23]. Note that HL_3 is just the Petersen graph, and HL_4 is a graph obtained from two copies of the Petersen graph by add one edge between one vertex in a copy of the Petersen graph and one vertex in another copy. See Figure 2 (c) for an example (We only show the edges v_1u_i $(1 \leq i \leq 10)$).



Figure 2: (a) Petersen graph; (b) The network HP_4 ; (c) The structure of HL_4 .

Proposition 5 (1) For network HP_3 and HL_3 , $mc(HP_3) = mc(HL_3) = 7$;

(2) For network HL_4 and HP_4 , $mc(HP_4) = 22$ and $112 \le mc(HL_4) \le 121$.

Proof. (1) By Theorem 1, we have $mc(HP_3) = mc(HL_3) = |E(HL_3)| - |V(HL_3)| + 2 = 7$.

(2) By Lemma 1, we have $mc(HP_4) = |E(HL_4)| - |V(HL_4)| + 2 = 22$ and $121 \ge E(HL_4) - V(HL_4) + \kappa(HL_4) + 1 \ge mc(HL_4) \ge E(HL_4) - V(HL_4) + 2 = 112.$

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