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ROUND WEIGHTING PROBLEM AND GATHERING IN RADIO NETWORKS WITH SYMMETRICAL INTERFERENCE*

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Abstract

In this article we consider the problem of gathering information in a gateway in a radio mesh access network. Due to interferences, calls (transmissions) cannot be performed simultaneously. This leads us to define a *round* as a set of non-interfering calls. Following the work of Klasing, Morales and Pérennes, we model the problem as a Round Weighting Problem (RWP) in which the objective is to minimize the overall period of non-interfering calls activations (total number of rounds) providing enough capacity to satisfy the throughput demand of the nodes.

We develop tools to obtain lower and upper bounds for general graphs. Then, more precise results are obtained considering a symmetric interference model based on distance of graphs, called the distance-d interference model (the particular case $d = 1$ corresponds to the *primary node model*).

We apply the presented tools to get lower bounds for grids with the gateway either in the middle or in the corner. We obtain upper bounds which in most of the cases match the lower bounds, using strategies that either route the demand of a single node or route simultaneously flow from several source nodes. Therefore, we obtain exact and constructive results for grids, in particular for the case of uniform demands answering a problem asked by Klasing, Morales and Pérennes.

Keywords Radio networks, wireless networks, interference, grids, gathering, bounds, approximation algorithms.

1 Introduction

Routing steady traffic demands has been extensively studied in the literature for wired networks, but also for multi-hop radio networks where interferences have to be taken into account.

We consider the case where the information of the nodes must be gathered in a special node called gateway (or base station) in order, for example, to access Internet. This problem was asked by FRANCE TELECOM R&D (now ORANGE LABS) under the name of "How to bring

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Internet in the villages" [12] where there is no high speed access everywhere. In this context, the gateway providing high speed access to the village (for example via an antenna) receives the demand of the houses (equipped with radio devices). This creates a many-to-one communication. The converse problem (personalized broadcasting) where the gateway acts as a source and sends personalized information to each device is also of interest in this context and results obtained here can be used for solving it by reversing the protocols.

The goal consists in minimizing the completion time for gathering. This problem is known as the *Round Scheduling Problem (RSP)* (or Minimum Time Gathering problem) (see the survey in [13]). The RSP is also important in sensor networks (see example in [19]) where the idea is to collect data (alerts) in a Base Station. However, a major goal in sensor network protocols is to minimize energy consumption and most research assumes that data can be combined (or *aggregated*) to reduce transmission costs.

In [24], it was shown that if traffic demands are sufficiently steady, the problem can be relaxed to the *Round Weighting Problem (RWP)* (to be defined precisely in the next subsection), where we want to minimize the number of rounds; a round consists of non interfering calls. The RWP and the RSP have similar behavior when the network links are completely "filled" from the source to the destination (steady state). They differ by the additional time to "fill" and "drain" the network (transient states) that is only taken into account by RSP.

Results and protocols depend on the way interferences are modeled. We consider a binary model of interference based on distances in the communication graph, but differently from most of the RSP works cited here, we consider a symmetrical interference model called *distance-d*. In particular this model assumes that all message transmissions have a confirmation of reception. That is motivated by reliable protocols in which the nodes answer with an acknowledgment message. Although the model remains theoretical, it gives lower bounds on the gathering time for real networks or equivalently upper bounds on the number of users (or on the traffic demand) the network can accommodate, which is useful for an operator when planning their networks. Indeed we can consider instead of rounds consisting of calls at distance-d, the subset of rounds with acceptable signal-to-interference-plus-noise ratios (SINR) as the time to check if a round has an acceptable SINR is very small.

1.1 Problem Statement

Fist of all, let us state precisely the problem and the interference model. Let $G = (V, E)$ be the *communication graph* where the vertices in V represent the nodes of the network. We suppose that the communications are symmetric; so an edge $\{u, v\} \in E$ means that vertex u is into the communication area of vertex v and vice-versa. G is a symmetric digraph, but for simplicity's sake we use an undirected graph to represent it. We consider that a transmission between two nodes u and v is done via a *call*, and it is represented by the activation of the edge $\{u, v\}$ of G.

In radio networks, signals are subject to interference constraints and so, two calls which are "too near" cannot be performed simultaneously. We suppose here a *binary symmetric model of interference*. We define a *round* R as a set of pairwise non-interfering calls (calls which can be performed simultaneously). A round is therefore a set of edges of G. The interference model induces a set of possible rounds $\mathcal{R} \subseteq 2^E$ (exponential in the size of E).

We consider a synchronous communication network (the nodes are time-synchronized) so all the edges of a round can be activated at the same time. The weight of a round represents a capacity assigned to the edges of this round. The rounds are activated one after the other. We consider that the value assigned to the edge capacity is enough to support the same amount of flow, e.g. a capacity of 1 supports a flow of 1 (see later about the round weight function w).

The Round Weighting Problem (RWP) has been formalized in [24] for general demands from any source node u to any destination v. We restrict ourselves to the gathering instances where each node $v \in V$ has to send a demand $b(v)$ to the gateway node g. The demand of each node v may be split over multiple paths to g.

The part sent through the path P is in fact a flow denoted $\phi_v(P)$. When the demand is satisfied, the following condition applies

$$(\forall v \in V) \sum_{P \in \mathcal{P}_{v,g}} \phi_v(P) \geq b(v), \tag{1}$$

where $\mathcal{P}_{v,g}$ denotes the set of all paths between v and g. Therefore, the RWP consists in assigning a weight w (real number representing the duration of the time slot) to the rounds in such a way that the total weight of the rounds containing an edge is greater than the flow going through this edge. More precisely, the function w induces a capacity over each edge e given by the sum of the weights of the rounds containing the edge e. We denote $c_w(e)$ this induced capacity of the edge e. In this way $c_w(e) = \sum_{R \in \mathcal{R}: e \in R} w(R)$. We say that *a solution w is admissible* if there exist a set of paths and flow ϕ satisfying the demand such that:

$$(\forall e \in E) \sum_{v \in V} \sum_{P \in \mathcal{P}_{v,g}: e \in P} \phi_v(P) \leq c_w(e) = \sum_{R \in \mathcal{R}: e \in R} w(R). \tag{2}$$

The goal is to find an admissible weight function w in such a way that the overall weight $W = \sum_{R \in \mathcal{R}} w(R)$ is minimized. Therefore, the Round Weighting Problem may be summarized as follows.

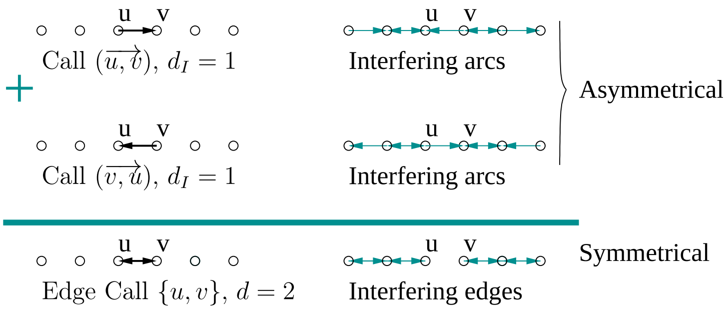
Problem:	Round Weighting for gathering instances.
Input:	A graph $G = (V, E)$, a gateway $g \in V$, a set of all rounds $\mathcal{R} \subseteq 2^E$ (exponential size), the sets of paths $\cup_{v \in V} \mathcal{P}_{v,g}$ (exponential size) and a demand function $b : V \to \mathbb{R}^+$.
Solution:	An admissible round weight function w defined over $\mathcal{R}$.
Goal:	Minimize the overall weight of w, i.e. $W = \sum_{R \in \mathcal{R}} w(R)$.

In some applications, the time slots are of fixed integer size and, consequently, the weight of the rounds has to be integer (defined in number of time slots). The RWP with this integrality assumption is called the *Integer Round Weighting Problem (IRWP)*. Notice that if the demands are integer, a solution of IRWP has integer flows. A more restricted problem is the *mono-routing problem* (or *unsplittable flow*), that routes the demands of a node v using an unique path. It avoids dealing with the packet-reordering problem, as the packets arrive at the destination in the same order they were sent. Notice that, if $b(v)$ is integer, the solution for the mono-routing problem has necessarily integer rounds (and also integer flows).

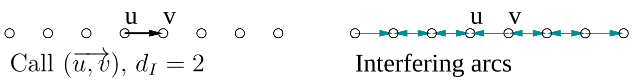
1.2 Distance-d model of interference

Although the tools developed in this article apply for any binary interference model, in order to give precise results, we use a model of interference based on distances (in number of edges) in graphs. The model can be viewed as a (symmetric) variant of the interference model used for example in [4, 24] where a node causes interference in all the nodes at distance at most d_I from it (nodes in its interference zone). In their model, two directed calls (s,r) and (s',r') interfere if $d(s, r') \leqslant d_I$ or $d(s', r) \leqslant d_I$ where $d(u, v)$ denotes the distance from u to v (asymmetrical interference model).

Note that, if device u calls device v, it is desirable that v has a way to let u know that the transmission has been successful sending an acknowledgment (confirmation of reception). Such feedback is performed by a transmission from v to u. For this reason, most interference models



(a) Call (u, v) in an asymmetrical and symmetrical model with, respectively, $d_I = 1$ and $d = 2$.



(b) Call (u, v) in an asymmetrical model with $d_I = 2$.

Figure 1: Relation between the symmetrical model (using distance-d model) and the asymmetrical model (with $d_I = d - 1$).

assume that interferences are symmetrical. As this model is used in the protocol 802.11, some authors named it *the* 802.11 *interference model* [32].

In this work, we consider a symmetrical interference model called distance-d model. In such model, two calls interfere if there is an end vertex of one call at distance at most $d-1$ from an end vertex of the other call. More precisely, let the distance between two calls (edges) $e = \{u, v\}$, $e' = \{u', v'\}$ be the minimum distance $d(e, e') = \min_{x \in \{u,v\}, y \in \{u',v'\}} d(x, y)$ between their end vertices. Using this distance we get the following:

Definition 1 *In the distance-d model, two calls (edges) e and e' interfere if their distance $d(e, e') < d$.*

Consequently, a *round consists of edges which are pairwise at distance $\geqslant d$*. The particular case $d = 1$ is called the primary node interference model [18, 23] or node-exclusive interference model [29]. In that case, a round is a matching. In the case $d = 2$, we obtain the so called distance-2 interference model [27, 13, 32, 33]. In this case, a round is an induced matching.

One of the reasons to use $d = d_I - 1$ (and not $= d_I$) is to be coherent with these two particular models. Furthermore, let the conflict graph be the graph whose vertices represent the edges (possible calls) of G, two vertices being joined if the corresponding calls interfere. Then, in the case $d = 1$, the conflict graph is nothing else than the line graph $L(G)$ of G. (The vertices of $L(G)$ represent the edges of G and two vertices are joined in $L(G)$ if their corresponding edges in G intersect). More generally, for any d, the conflict graph is the d-th power of $L(G)$. (The k-th power of a graph being the graph with two vertices joined if their distance is less than or equal to k).

A comparison between symmetrical (using distance-d model) and asymmetrical interference model in a path-like network is depicted in Figure 1(a). The arcs which interfere with the communication between the nodes u and v are indicated. For the asymmetrical model, the figure shows first the case of a directed call $\overrightarrow{u,v}$ and then of the directed call $\overrightarrow{v,u}$. We can see that, for the symmetrical model, the edge call $\{u, v\}$ corresponds to the activation of the arcs $\overrightarrow{u,v}$ and $\overrightarrow{v,u}$ (in both directions). Consequently, the interfering edges correspond to the sum of all interfering arcs for the calls $\overrightarrow{u,v}$ and $\overrightarrow{v,u}$ (in the asymmetrical model) as illustrated in Figure 1(a).

Observe that, the set of interfering arcs in the asymmetrical model is smaller than the set for the symmetrical model with $d_I = d-1$. Then, any solution for the RWP with the asymmetrical model is a lower bound for the RWP with the symmetrical model.

Now, considering $d_I = d$, the set of interfering arcs in the symmetrical model (see Figure 1(a) for $d = 2$) is smaller than the corresponding set for the asymmetrical model. (Compare with the set of arcs illustrated in Figure 1(b) for the asymmetrical model with $d_I = d = 2$). Consequently, any solution for the RWP using the symmetrical model corresponds to a lower bound for the RWP using the asymmetrical model (with $d_I = d$). So, as the RWP with symmetrical model considers less constraints (interfering arcs) with $d_I = d$, it can be considered as a "relaxation" of the RWP with asymmetrical model.

As our problem deals with a gathering, the additional interference of the symmetrical model with $d_I = d-1$ makes some difference only if we need to use some paths backwards the gateway (as there are more interfering backwards arcs in the symmetrical model). However, in general the formulae that we obtain using the distance-d (symmetrical) model are similar (upper bound with small gap) to that of [10] that uses the asymmetrical model considering $d_I = d-1$.

Note that our model is a simplification of reality in which a node can be subject to interference from all of the other nodes, and models based on signal-to-interference-plus-noise ratios (SINR) are more accurate. However, our model is more accurate than the classical half duplex model of wired networks or the primary node model of interference. Furthermore, it is still tractable and we can give precise results. Finally the general tools we have given can apply by considering instead of rounds consisting of calls at distance-d, the subset of rounds with acceptable SINR; indeed the time to check if a round has an acceptable SINR is very small.

1.3 Related Work

Given the importance of wireless networks, fundamental problems arises such as routing, scheduling and gathering data under interference [1]. Thus, many models have been introduced and studied as well as different variants of interference [13].

In [24], the authors introduced the Round Weighting Problem (RWP) for asymmetrical interference in the general case, with demands between any pair of nodes. In this case of an arbitrary traffic pattern (analogous to a multicommodity flow), they showed that the problem is very difficult to approximate; indeed, to approximate the RWP within $n^{1-\varepsilon}$ is NP-Hard [24]. Even for the case of gathering, the problem was shown to be NP-hard. Furthermore, a 4-approximation algorithm for general topologies was presented. For paths, the RWP was shown to be polynomial. One of the open questions of the paper was to find simple efficient algorithms for grids. This question will be answered in the following sections of this article. The approach of [24] consisted in the study of the dual of the corresponding optimization problem. This method was also used in [10] and also in [17], where the authors propose a Lagrangian relaxation and then, they prove the convergence of their method towards the optimal solution.

As said at the beginning, the RWP can be seen as a relaxation of the *Round Scheduling Problem (RSP)* (or Minimum Time Gathering problem). Many results have been obtained on this problem but with hypothesis different from ours. We refer to the survey in [13]. The most closed model uses a binary distance model of interference but asymmetric. In the case of asymmetrical interference a protocol for general graphs with an arbitrary amount of information to be transmitted from each vertex is presented in [4]. The protocol is an approximation algorithm with performance ratio at most 4. It is also shown in [4] that there is no fully polynomial time approximation scheme for gathering if $d_I > d_T$, unless $\mathcal{P} = \mathcal{NP}$, and the problem is $\mathcal{NP}$-hard if

$d_I = d_T$, where d_T denotes the maximum distance of transmission (in our case $d_T = 1$). If each vertex has exactly one piece of information to transmit, the problem is $\mathcal{NP}$-hard if $d_I > d_T$ [4] and if $d_I = d_T = 1$ [26]. A modified version of the problem in which messages can be released over time is considered in [14] and a 4-approximation algorithm is presented. For specific topologies, polynomial or 1-approximation algorithms are also given for paths [3, 7] and trees [11] . For grids, the problem has been solved in the unitary-traffic case (a variant of uniform demand where each node has one unit of traffic to send to the gateway) and also in hexagonal grids (see [10]).

Many other articles consider also a binary distance model of interference, but do not allow buffering; in that case the tools and results are different. When no buffering is allowed, the problem has been solved for trees for $d_I = 1$ [6] and for general d_I [5, 20] (where a closed-form expression is given when all vertices have exactly one piece of information to transmit). For square grids with the gateway in the center, a multiplicative 1.5-approximation algorithm is given in [30] and an additive +1 approximation algorithm is given in [8].

In this article, we are considering a symmetrical interference model that has also been studied in [16, 21]. The authors in [16] proved that RWP remains NP-hard even on a bipartite graph with one source, for any $d \geqslant 3$ fixed. For $d = 2$, they also proved NP-hardness on a bipartite graph with multiples sources. For $d = 1$, they show that the problem is polynomial in 3-connected graphs and in bipartite graphs. Finally, they show that a list version of the problem is inapproximable in polynomial time by a factor of $O(\log n)$ even on n-vertex paths, for any $d \geqslant 1$. In [21], the authors perform simulations for the RWP in general graphs.

Within the same model another communication problem the distributed "link scheduling problem" is considered in [9, 15, 22], but there the main objective is to insure the stability of the system with random arrivals.

A related problem consists in finding the longest round satisfying the distance-2 (symmetric) interference model. This problem is called *maximum induced matching* [31] and *maximum distance 2 matching*(D2EMIS) [2]. D2EMIS is known to be APX-complete for regular graphs, but admits a PTAS for disk graphs) [2]. The problem is generalized for arbitrary interference distance in [29] by considering different weights to the edges.

1.4 Our Results

In this article, we study the *Round Weighting Problem (RWP)* with symmetrical interferences. We first develop tools to obtain lower bounds in Section 3 for general graphs, in particular for the distance-d interference model. Then, in Section 4, we propose routing strategies finding upper bounds with the gateway placed anywhere.

Then we consider grids as they model well both access networks and also random networks (see [25]). We answer the question asked in [24] of finding simple efficient algorithms and the complexity of the problem for grids. For this purpose, we apply the tools presented in Section 3 to obtain in Section 5 precise lower bounds for grids with the gateway either in the middle or in the corner. Then, in Section 6 and in Section 7 of this article, we use the general tools of Section 4 and specific tools to obtain upper bounds both in the case of routing the demand of a single node or of a combination of nodes. In most of the cases they match the lower bounds; in particular we get exact and constructive results for grids for the case of uniform demands.

Our results are of theoretical nature, but they can be of interest for applications in particular in designing such radio networks. Indeed they give an upper bound on the demand or on the number of users the network can accommodate. Therefore, if the number of planned users is near from or greater than this bound, the operator should for example implement more gateways (or increase the bandwidth).

2 Definitions

In this section, we present some definitions that are useful later.

Definitions related to the edges of G

- $G(V, E)$: communication graph with V as set of nodes (vertices) and E as set of edges (possible calls).
- g: a specific node $g \in V$ called gateway.
- $L(G)$: line graph of G, i.e. the graph whose vertices represent the edges of G and where two vertices are joined in $L(G)$ if their corresponding edges in G intersect.
- $d(u, v)$ with $u, v \in V$: distance between u and v, that is the length of a shortest path between them (e.g. the neighbors of g are at distance 1 of g).
- $d(e, e')$ with $e, e' \in E$: distance between edges $e = (u, v)$ and $e' = (u', v')$ which corresponds to $\min_{x \in \{u,v\}, y \in \{u',v'\}} d(x, y)$.
- E_l: set of edges at level l, i.e. edges joining a node at distance l from the gateway to a node at distance $l - 1$. More precisely, $E_l = \{e = (u, v) \in E \mid d(g, u) = l \text{ and } d(g, v) = l - 1\}$. For example, E_1 are all the edges incident to the gateway g.
- K_0: set of edges from all levels $l \leqslant \lceil \frac{d}{2} \rceil$, with d as the interference parameter. That is, $K_0 = \bigcup_{1 \leqslant l \leqslant \lceil \frac{d}{2} \rceil} E_l$.
- V_K: set of nodes in G incident to the set of edges K. Usually K is a call-clique (see definition below). For the case of K_0, the set V_{K_0} is $V_{K_0} = \{v \in V \setminus g | d(v, g) \leqslant \lceil \frac{d}{2} \rceil\}$. Notice that, according to the definition, the gateway g is not included in V_{K_0}.

Definitions related to interferences and cliques

- d: interference distance. Sometimes, we use the auxiliary parameter $k = \lceil \frac{d}{2} \rceil$. So, $d = 2k$ for d even and, $d = 2k - 1$ for d odd.
- Distance-d model: binary model of interference where two edge calls e and e' *interfere* if $d(e, e') < d$ (see definition in Section 1.2).
- $C(G)$: conflict graph of G, that is the graph whose vertices represent the edges of G, two vertices are joined if the corresponding edges (which represent calls) interfere. Consequently in the distance-d model, the conflict graph C(G) is the d-th power of the line graph $L(G)$.
- call-clique: set of pairwise interfering edges in G. In $C(G)$, the corresponding vertices form a clique. For example in the distance-d model, K_0 is a call-clique, because the distance between any pair of edges is less than d and therefore they interfere. Given that we are dealing with grids, sometimes it is easy to describe the call-clique by means of its adjacent vertices.

Definitions related to flows, rounds and the weight function

- $b(v)$: demand of the node $v \in V$.
- $\mathcal{P}_{v,g}$: the set of all paths between v and g in G (exponential size).
- $\phi_v(P)$ with $P \in \mathcal{P}_{v,g}$: flow from node v sent to g using path P. Notice that according to Constraint 1 the demand must be completely satisfied.
- $\phi_v(e)$: flow sourced at node v traversing the edge e. More precisely $\phi_v(e) = \sum_{P \in \mathcal{P}_{v,g};\, e \in P} \phi_v(P)$.
- $\phi(e)$: flow traversing the edge e. $\phi(e) = \sum_{v \in V} \phi_v(e)$.
- $\phi_v(E')= \sum_{e \in E'} \phi_v(e)$: sum of the flow from v on a set of edges E'.
- $\phi(E')= \sum_{v \in V} \phi_v(E')$: sum of the flow on a set of edges E'.
- R (Round): set of non-interfering edges, i.e. an independent set in $C(G)$.
- $\mathcal{R}$: set of all rounds of G (it has an exponential size).
- $\mathcal{R}_e \subset \mathcal{R}$: set of all the rounds containing the edge e.
- $w(R)$: round weight function $w : R \to \mathbb{R}^+$ giving the weight of round R.
- $c_w(e)$: the capacity of the edge e induced by the weights of the rounds in $\mathcal{R}_e$. More precisely, $c_w(e) = \sum_{R \in \mathcal{R}_e} w(R) = \sum_{R \in \mathcal{R}} w(R)|R \cap \{e\}|$.
- $c_w(E')= \sum_{e \in E'} c_w(e) = \sum_{e \in E'} \sum_{R \in \mathcal{R}_e} w(R) = \sum_{R \in \mathcal{R}} w(R)|R \cap E'|$, the capacity of the edges $E' \subseteq E$ is a measure derived from the weights of the rounds covering these edges.
- Admissible solutions: let us recall first that, as seen in Section 1.1, a solution corresponds to a function assigning weights to rounds. Therefore, we say that the weights $w(R)$ assigned to the rounds $R \in \mathcal{R}$ are *admissible* if there exist a set of paths and a flow ϕ satisfying Equation 2.
- W: the overall weight $W = \sum_{R \in \mathcal{R}} w(R)$. Moreover, as seen in Section 1.1, W is the value of the objective function in the RWP.
- $W_{\min}$: the minimum value of W over all the admissible weight functions w. Therefore, it corresponds to the optimal solution of RWP (see Section 1.1).

Definitions related to grids

- Rectangular Grid: a rectangular $p \times q$ grid is the graph with $N = pq$ vertices, denoted (x, y) where $-p_1 \leqslant x \leqslant p_2$ with $p_1 + p_2 + 1 = p$ and $-q_1 \leqslant y \leqslant q_2$ with $q_1 + q_2 + 1 = q$ (p_1, p_2, q_1, q_2 being integers). Any vertex (x, y) is joined (if they exist) to its four neighboring vertices $(x + 1, y)$, $(x - 1, y)$, $(x, y + 1)$ and $(x, y - 1)$. We assume that the gateway has the coordinates $(0, 0)$.
- Gateway Position: it is represented by a position (x, y) in the grid. We see later that the results strongly depend on the position of the gateway. In the distance-d model, we consider mainly two extremal places:
  - gateway in the *corner*: g is the vertex $(0, 0)$ and $p_1 = q_1 = 0$.
  - gateway in the *middle*: g is far enough of the borders and $\min(p_1, p_2, q_1, q_2) \geqslant \lceil \frac{d+1}{2} \rceil$.

- Rotation function: we define also the *Rotation function* ρ as the one to one vertex mapping $\rho((x,y)) = (-y,x)$ which corresponds to a rotation in the plane of $\frac{\pi}{2}$ around the central node $(0,0)$. Let the rotation of an edge $e = (v_1, v_2)$ be $\rho(e) = (\rho(v_1), \rho(v_2))$ and let the rotation of a path $P = \{e_1, e_2...\}$ be $\rho(P) = \{\rho(e_1), \rho(e_2)...\}$. This definition works well when $p_1 = p_2 = q_1 = q_2$. We can extend it to any grid by doing the rotation in a super grid with size $(2p'+1, 2p'+1)$ with $p' = \max(p_1, p_2, q_1, q_2)$ and ignoring the vertices not in the original grid.

3 Lower bounds: general results

In this section, we show how to use call-cliques (see definition in Section 2) to obtain lower bounds for the RWP in particular for the distance-d model. We give some illustrative examples using grids (see more ingenious bounds for grids in Section 5).

3.1 Lower bounds using one call-clique

Recall that a call-clique is a set of edges pairwise interfering. So, two transmissions in a call-clique cannot be performed simultaneously. Thus, the sum of the capacities of the edges in a call-clique sets up a lower bound for the RWP as follows.

Lemma 1 *Let $K \subseteq E$ be a call-clique, then $c_w(K) \leqslant W$.*

Proof: We know that $c_w(K) = \sum_{R\in\mathcal{R}} w(R)|R\cap K|$. As each round R is a set of non-interfering edges, R contains at most one edge of K. Therefore, $|R\cap K| \leqslant 1$ and consequently $c_w(K) \leqslant \sum_{R\in\mathcal{R}} w(R) = W$. ■

For $F \subseteq E$ and a path $P \in \mathcal{P}_{v,g}$ (between v and g), let $\mathrm{LB}(P,F)$ denote the number of edges that P and F have in common. Therefore, $\mathrm{LB}(P,F) = |P\cap F|$. We define $\mathrm{LB}(v,F)$ as the minimum $\mathrm{LB}(P,F)$ over all the paths $P \in \mathcal{P}_{v,g}$.

Lemma 2 *Let $F \subseteq E$, then $c_w(F) \geqslant \sum_{v\in V} b(v)\,\mathrm{LB}(v,F)$.*

Proof: For any flow $\phi(F)$, $c_w(F) \geqslant \phi(F) = \sum_{v\in V} \phi_v(F) \geqslant \sum_{v\in V} b(v)\,\mathrm{LB}(v,F)$. ■

The first idea consists in choosing particular sets F. A natural candidate is the set E_l (of edges at level l). The paths from the nodes *outside* E_l, i.e. the nodes at distance at least l of the gateway must cross at least one edge in E_l to reach the gateway. So, if $d(v,g) \geqslant l$, then $\mathrm{LB}(v, E_l) \geqslant 1$ and we have the following.

Corollary 1 $c_w(E_l) \geqslant \sum_{v;d(v,g)\geqslant l} b(v)$.

We use Corollary 1 to give a lower bound for $c_w(K_0)$ in the distance-d model. The bound uses the value S_0 defined below. Recall that the call-clique K_0 is the set of edges around the gateway at level at most $\lceil \frac{d}{2} \rceil$.

Definition 2 $S_0 = \sum_{v\in V_{K_0}} d(v,g)b(v) + \lceil \frac{d}{2} \rceil \sum_{v\notin V_{K_0}} b(v)$.

Lemma 3 *In the distance-d model, $c_w(K_0) \geqslant S_0$.*

Proof: As $K_0 = \bigcup_{l\leqslant\lceil \frac{d}{2}\rceil} E_l$ and the levels E_l for $1 \leqslant l \leqslant \lceil \frac{d}{2} \rceil$ are pairwise disjoints, then $c_w(K_0) = \sum_{l\leqslant\lceil \frac{d}{2}\rceil} c_w(E_l) \geqslant \sum_{l\leqslant\lceil \frac{d}{2}\rceil} \sum_{v;d(v,g)\geqslant l} b(v) = S_0$. ■

Note that the value S_0 is independent of the function w. Therefore,

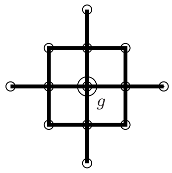


Figure 2: Call-clique for $d = 3$ with g in the middle. It corresponds to K_0 for the distance-3 model.

Proposition 1 *In the distance-d model,* $W_{\min} \geqslant S_0$.

We see later that, in some cases, the lower bound S_0 is attained. It happens for the grid with the gateway in the middle and d odd (see Theorem 6). Recall that S_0 uses the call-clique K_0. There may, however, be other call-cliques larger than K_0 giving better lower bounds.

For example, Figure 2 shows an example of a call-clique K_0 for the distance-3 model (d odd). Notice that in this case, K_0 is a *maximal call-clique*: if a new edge is added to K_0, the resulting set is not a call-clique. However, for some considered parameters of the problem, K_0 may not be a maximal call-clique (see Figure 9). In these cases, Lemma 2 is used with a maximum call-clique K (containing K_0) as the set F. For example, for the grid with d odd and the gateway in the corner, the maximum call-clique is larger than K_0 (see Figure 9 and 10) and gives a better bound than S_0 (see Theorem 4). We show later that the lower bound is optimal for uniform demand. However, using only one call-clique does not necessarily give a tight bound.

3.2 Lower bounds using many call-cliques

We present a result similar to Lemma 2, but improved by using multiple sets of edges. Recall that $\mathcal{P}_{v,g}$ is the set of all the paths between v and g.

Lemma 4 *Given the sets of edges* $F_1, \ldots, F_q$, *then*

$$\sum_{i=1}^{q} c_w(F_i) \geqslant \sum_{v} b(v) \min_{P \in \mathcal{P}_{v,g}} \left(\sum_{i=1}^{q} \mathrm{LB}(P, F_i) \right)$$

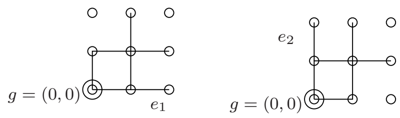
Proof: For any flow ϕ and any node v,

$$\sum_{i=1}^{q} \phi_v(F_i) \geqslant b(v) \min_{P \in \mathcal{P}_{v,g}} \sum_{i=1}^{q} \mathrm{LB}(P, F_i).$$

■

Consider first the example of a grid with the gateway at the corner and the distance-2 model depicted in Figure 3. Notice that in this model $K_0 = E_1$ and is not a maximal clique. In fact, we have two maximum call-cliques containing K_0: K_1 and K_2 which also contain the four edges adjacent to vertex $(1,1)$. Furthermore, K_1 contains the edge $e_1 = ((1,0),(2,0))$ and K_2 contains the edge $e_2 = ((0,1),(0,2))$.

Let us now calculate $\min_{P \in \mathcal{P}_{v,g}}(\mathrm{LB}(P, K_1) + \mathrm{LB}(P, K_2))$ for each vertex v in the grid. First, for any vertex v at distance 1 of g (vertices $(0,1)$ and $(1,0)$), $\mathrm{LB}(v, K_1) = \mathrm{LB}(v, K_2) = 1$. Similarly, for vertex $v^* = (1,1)$ both $\mathrm{LB}(v^*, K_1) = \mathrm{LB}(v^*, K_2) = 2$. For any other vertex $v \notin \{(0,1),(1,0),(1,1)\}$, a path between v and g use at least **3** edges in $K_1 \cup K_2$: one edge at level 2 in at least one clique and 1 edge at level 1 in both cliques, so $\mathrm{LB}(P, K_1) + \mathrm{LB}(P, K_2) \geqslant 3$.



(a) Call-clique K_1. (b) Call-clique K_2.

Figure 3: Two maximum call-cliques K_1 and K_2 in the distance-d interference model (with $d = 2$).

Using Lemma 4, we get that

$$\begin{aligned} c_w(K_1) + c_w(K_2) &\geqslant \sum_v b(v) \min_{P \in \mathcal{P}_{v,g}} (\mathrm{LB}(P, K_1) + \mathrm{LB}(P, K_2)) \\ &\geqslant 2b((0,1)) + 2b((1,0)) + 4b((1,1)) + 3 \sum_{v \notin \{(0,1),(1,0),(1,1)\}} b(v) \end{aligned}$$

and so, one of these two call-cliques has capacity c_w greater than or equal to one half of the right side value. Therefore, we have the following bound.

Proposition 2 *For the grid with g in the corner in the distance-2 model*

$$W_{min} \geqslant b(0,1) + b(1,0) + 2b(1,1) + \frac{3}{2} \sum_{v \notin \{(0,1),(1,0),(1,1)\}} b(v)$$

As a remark, we can see that in this case the use of multiple call-cliques effectively improve the lower bounds obtained in Proposition 1 which only gives that $W_{min} \geqslant \sum_v b(v)$. We see later (in Proposition 17) that the lower bound given by Proposition 2 is not only better, but also optimal. In general, we have the following.

Lemma 5 *Let $K_1, \ldots, K_q$ be a family of call-cliques. Then one of the call-cliques, K^*, satisfies $c_w(K^*) \geqslant \frac{1}{q} \sum_{v \in V} b(v) \min_{P \in \mathcal{P}_{v,g}} \sum_{i=1}^{q} \mathrm{LB}(P, K_i)$*

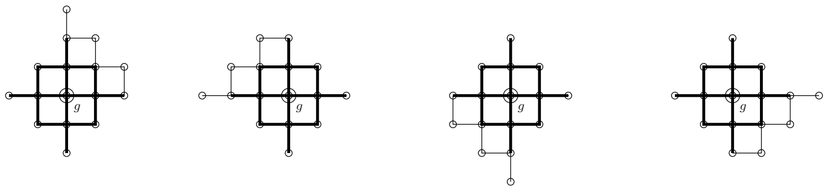
Proof: By Lemma 4, $\sum_{i=1}^{q} c_w(K_i) \geqslant \sum_{v \in V} b(v) \min_{P \in \mathcal{P}_{v,g}} \sum_{i=1}^{q} \mathrm{LB}(P, K_i)$ and so one of the call-cliques, denoted K^*, has capacity $c_w(K^*)$ greater than or equal to the average (the right side value over q). ■

Corollary 2 *Let $K_1, \ldots, K_q$ be a family of call-cliques such that each edge of E_l appears at least λ_l times in the call-cliques, then $W_{min} \geqslant \sum_l \sum_{v; d(v,g) \geqslant l} \frac{\lambda_l}{q} b(v)$.*

Proposition 3 *Let G be the grid with g in the middle and d be even ($d = 2k$).*

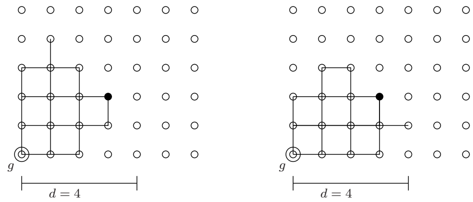
$$W_{\min} \geqslant S_0 + \frac{1}{4} \sum_{v; d(v,g) \geqslant k+1} b(v)$$

Proof: Consider the four following call-cliques (see Figure 4 for $d = 4$): they all contain the edges of K_0. Furthermore, K_1 contains the edges at level $k+1$ with positive coordinates: $((0, k+1), (0, k))$, $((k+1-i, i), (k-i, i))$ and $((k+1-i, i), (k+1-i, i-1))$ for $1 \leqslant i \leqslant k$. The call-cliques K_2, K_3 and K_4 are obtained by successive rotation of $\frac{\pi}{2}$ of the previously described call-clique K_1. In this way, the edges in E_l with $1 \leqslant l \leqslant k$ are covered 4 times and the edges in E_{k+1} are covered once (we see later that this lower bound is optimal for several cases). ■



(a) call-clique K_1. (b) call-clique K_2. (c) call-clique K_3. (d) call-clique K_4.

Figure 4: Case d even ($d = 4$) and g in the middle. The four call-cliques combined covers E_i, $1 \leqslant i \leqslant k+1$ for $d = 2k$.



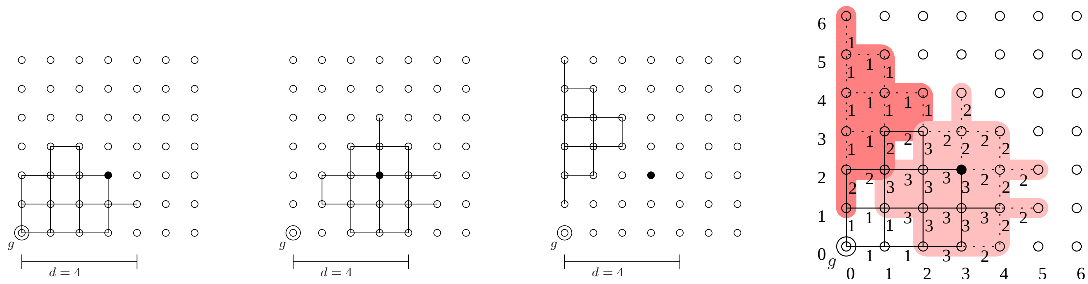
(a) call-clique K_a (b) call-clique K_b

Figure 5: Example of grid with demand concentrated at node $(3, 2)$ and $d = 4$. A lower bound of $\frac{5}{2}b((3, 2))$ is obtained using the two call-cliques K_a and K_b.

These symmetric call-cliques around the gateway (as in Figure 4) do not always give the best bounds. In some cases, there are more complex call-cliques which can be used with the Lemma 5. They may not be easy to find and furthermore, they do not necessarily contain the gateway. An example can be seen in Figures 5 and 6 where the demand is concentrated in one node (the node is $(3, 2)$ and $d = 4$). The symmetrical call-cliques K_a and K_b in Figure 5 only give a lower bound of $\frac{5}{2}$.

A better lower bound is obtained with more complex call-cliques. Notice that the solution depicted in Figure 6 improves the solution of Figure 5. The call-clique K_2 (see Figure 6(b)) is used twice and K_1 (see Figure 6(a)) and K_3 (see Figure 6(c)) once. Figure 6(d) shows all the 4 call-cliques which overlap each other and the value in each edge e represents $\lambda(e)$ (the number of call-cliques using e).

A vertex v_i represents a node at distance i of g with $i \in \{2 \dots 5\}$. Table 1 indicates, for each vertex v_i, the minimum number of edges of K_1, K_2 (repeated twice) and K_3 used by a path from $(3, 2)$ to the gateway $(0, 0)$. It follows that there is no path using less than 11 edges. Therefore,



(a) call-clique K_1 (b) call-clique K_2 (repeated 2x) (c) call-clique K_3. (d) Covering of the edges by K_1, 2xK_2 and K_3.

Figure 6: Four call-cliques are needed to obtain a better (tight) lower bound of $\frac{11}{4}b((3, 2))$ for the example in Figure 5.

Table 1: Possible paths from $(3,2)$ to the gateway $(0,0)$ use at least 11 call-cliques edges.

v_5	v_4	v_3	v_2	K_1	$K_2(\times 2)$	K_3	Total
-	$(4,0)$	$(3,0)$	-	3	4	-	11
-	$(3,1)$	$(3,0)$	-	5	3	-	11
-	-	$(2,1)$	-	5	3	-	11
-	$(2,2)$	$(1,2)$	$(1,1)$	5	3	-	11
-	$(2,2)$	$(1,2)$	$(0,2)$	5	2	2	11
-	$(1,3)$	$(1,2)$	$(1,1)$	4	3	2	12
-	$(1,3)$	$(1,2)$	$(0,2)$	4	2	4	12
-	$(0,4)$	$(0,3)$	-	2	2	5	11
$(1,4)$	$(1,3)$	$(0,3)$	-	2	2	5	11
$(2,3)$	$(1,3)$	$(0,3)$	-	3	2	4	11

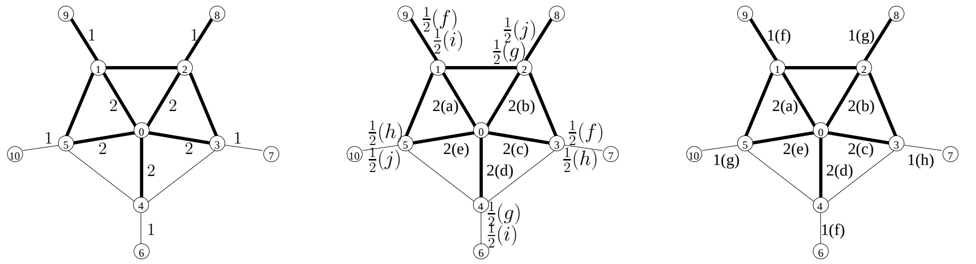


Figure 7: Example of lower bound calculation ($d = 2$).

$\min_{P \in \mathcal{P}_{(3,2),g}}(\text{LB}(P, K_1) + 2\,\text{LB}(P, K_2) + \text{LB}(P, K_3)) \geqslant 11$. Furthermore, one of the call-cliques, K^*, satisfies $c_w(K^*) \geqslant \frac{11}{4} b((3,2))$. The mathematical model in [21] showed that this bound is optimal.

3.3 Lower bounds using Critical Edges

Lemma 5 (or Corollary 2) does not attain the best lower bounds in all cases. Consider the example of Figure 7, where $d = 2$. We have 5 maximal call-cliques all containing the edges at level 1 plus two edges at level 2. Then, noting that each edge at level 2 appears exactly in two call-cliques, we apply Corollary 2 to obtain $W_{\text{min}} \geqslant \sum_{v; d(v,g) \geqslant 1} b(v) + \frac{2}{5} \sum_{v; d(v,g) \geqslant 2} b(v)$. In the particular case where $b(v) = 1$ for the 10 vertices, we obtain $W_{\text{min}} \geqslant 10 + \frac{2}{5} \cdot 5 = 12$ (see Figure 7(a)). In the next paragraphs, we show that the lower bound can be increased for the cases where it is not already optimal.

At level 1, each round R can contain at most 1 edge (they are a,b,c,d and e in Figure 7(b)). So, a flow of 10 (from the 10 nodes) contributes with a weight of 10 to cross this level as in Corollary 2. At level 2, each round R can contain at most 2 edges.

Then, the contribution to W from the 5 vertices at level 2 is, in fact, at least $\frac{5}{2}$. We need a weight of 2.5 to transfer a flow of 5, so $W_{\text{min}} \geqslant 10 + \frac{5}{2} = 12.5$. Figure 7(b) shows a fractional solution with $W_{\text{min}} = 12.5$, that uses 5 rounds (f,g,h,i and j) with a weight of $\frac{1}{2}$ each.

In the IRWP, the weight of the rounds can not be fractional. So, we need at least $\lceil \frac{5}{2} \rceil = 3$ rounds (f, g and h) with a weight of 1 each, as shown in Figure 7(c). An integer solution for IRWP with $W_{\text{min}} = 13$ is depicted in this figure.

This result is not surprising if we consider the conflict graph. Indeed, the conflict graph induced by the edges at level 2 is a cycle of length 5. Even though it contains a maximal independent set of size 2, we need 3 *labels* (with integer weight equals to 1 each) to color the whole cycle. It gives a weight of 3 to cross this level, therefore $W_{\min} \geqslant 13$. In the fractional case, it is known that we can use a fractional coloring with $\frac{5}{2}$ *labels*, so $W_{\min} \geqslant 12.5$. For information, the $W_{\min}$ for the IRWP can be different of the round up of the $W_{\min}$ for the RWP (e.g. the instance in Figure 6 has $W_{\min} = 2.75b((3,2))$ for RWP and $W_{\min} = 5b((3,2))$ for IRWP).

For a set of edges F, let us denote by $\alpha(F)$ the maximum number of non-interfering edges (it corresponds to the size of an independent set of the conflict graph generated by F).

Definition 3 *Let K be a call-clique. An edge $e \notin K$ is said to be* critical *for K if $K \cup \{e\}$ is a call-clique.*

Lemma 6 *Let K be a call-clique and F a set of critical edges for K, then $W \geqslant c_w(K) + \frac{c_w(F)}{\alpha(F)}$.*

Proof: As $K \cup \{e\}$ is a call-clique, for any e in F, a round can contain at most one edge of $K \cup \{e\}$. Then,

$$W = \sum_{R \in \mathcal{R}} w(R) \geqslant \sum_{R;|R \cap K| \neq 0} w(R) + \sum_{R;|R \cap F| \neq 0} w(R) \tag{3}$$

First, as K is a call-clique, $|R \cap K|$ is either 0 or 1 for any round R, then $\sum_{R;|R\cap K|\neq 0} w(R) = c_w(K)$. By definition, R contains non-interfering edges, so $|R \cap F| \leqslant \alpha(F)$ and $c_w(F) = \sum_R w(R)|R \cap F| = \sum_{R;|R\cap F|\neq 0} w(R)|R \cap F| \leqslant \alpha(F) \sum_{R;|R\cap F|\neq 0} w(R)$. Finally, by (3), we have that $W \geqslant c_w(K) + \frac{c_w(F)}{\alpha(F)}$.

■

Consider the specific case where $K = K_0$ and F=$E_{\lceil \frac{d}{2} \rceil+1}$ (the set of edges at level $\lceil \frac{d}{2} \rceil + 1$). Notice that, any path towards g from a vertex at distance at least $\lceil \frac{d}{2} \rceil + 1$ must use an edge of $E_{\lceil \frac{d}{2} \rceil+1}$. Consequently, we obtain the following result.

Corollary 3 *If all the edges of $E_{\lceil \frac{d}{2} \rceil+1}$ are critical for K_0, then*

$$W \geqslant S_0 + \frac{1}{\alpha\left(E_{\lceil \frac{d}{2} \rceil+1}\right)} \sum_{v;d(v,g) \geqslant \lceil \frac{d}{2} \rceil+1} b(v)$$

For example, if we apply Corollary 3 for the grid with the gateway in the middle and $d = 2k$, we have a new proof for Proposition 3. It is due to the fact that all the edges of E_{k+1}=$\{(k+1,0),(k,0))$, $((0,k+1),(0,k))$, $((0,-k-1),(0,-k))$, $((-k-1,0),(-k,0)\}$ are critical for K_0 and furthermore, they are non-interfering.

3.4 Relationship with duality

In the following, we show how our method to compute lower bounds is related to the dual of the RWP. The dual formulation of RWP has been studied in [24]. A dual version of the RWP for gathering instances can be described as follows.

Definition 4 ([24]) *The dual problem of the* RWP *consists of finding a metric $l : E \to \mathbb{R}^+$ satisfying the constraint that the maximum length of a round is 1 (i.e. $\sum_{e \in R} l(e) \leq 1, \forall R \in \mathcal{R}$). The goal consists of maximizing the total distance that the traffic needs to travel $\Lambda = \sum_{v \in V} d_l(g,v)b(v)$, where $d_l(v,g) \leqslant \sum_{e \in P} l(e), \forall P \in \mathcal{P}_{v,g}$.*

Shortly, the problem defines dual values (the $l(e)$ variables) maximizing the path of minimum length (note that the $l(e)$ variables are limited by the constraints $\sum_{e \in R} l(e) \leq 1, \forall R \in \mathcal{R}$ that prevent d_l goes to infinity).

Now, we show that it is possible to construct a feasible dual solution for RWP starting from the call-cliques. In fact, the result given by Corollary 2 can be obtained by defining the appropriate dual solution. Recall that, according to Corollary 2, $\mathcal{K} = \{K_1, \dots, K_q\}$ is a family of call-cliques and, λ_e is the number of call-cliques using the edge e (see Figure 6). Defining the metric $l : E \to \mathbb{R}^+$ as $l(e) = \frac{\lambda_e}{q}$, we can verify that l is a feasible dual solution. For that, we should verify that $\sum_{e \in R} \frac{\lambda_e}{q} \leqslant 1, \forall R \in \mathcal{R}$, that is $\sum_{e \in R} \lambda_e \leqslant q, \forall R \in \mathcal{R}$. It is true, as a round can not use a call-clique more than once (so touching at most q call-cliques), or it would have interfering edges which contradicts the round definition.

Moreover, the lower bound given by lemma 6 can also be obtained by the dual approach defining the metric l as: 1 for the edges in K; $1/\alpha(F)$ for the edges in F and 0 for the remaining edges.

4 Upper bounds: general results

In this section, we deal with finding upper bounds for the RWP in a *general graph* with the gateway g placed *anywhere*. The interference model is the distance-d model with any d (odd or even), unless specified differently.

To find upper bounds, we propose routing strategies giving a small total weight W. For that, to each vertex v, we associate some paths from v to g carrying the demand $b(v)$. Furthermore, we assign labels (or colors) c_j to these paths. Each label c_j corresponds to a round R_j and so we have to ensure that the edges with the same label do not interfere. Therefore, we introduce the notion of interference free γ-labeled *paths*.

4.1 Interference free γ-labeled *paths*

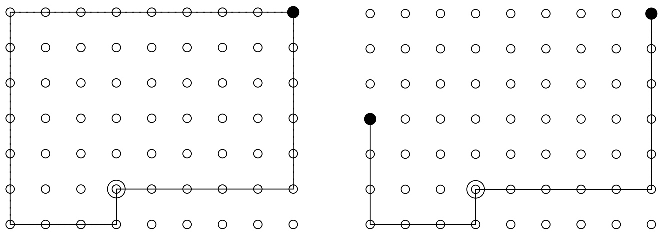
Definition 5 (Interference free γ-labeled *paths*) *A set of paths (or cycles) are said to be interference free γ-labeled if we can assign to the edges γ labels such that two edges with the same label do not interfere.*

In order to obey the inequalities in (2), $c_w(e) \geqslant \phi(e)$, we give to each round R_j a weight $w(R_j)$ equal to the maximum $\phi(e)$ among all edges e labeled c_j. With this strategy, we obtain the following proposition.

Proposition 4 *Let G be a general graph with the gateway g placed anywhere and let $V' = \{v_1, ..., v_\pi\}$ be a family of nodes (not necessarily different, but a node v can be repeated at most $b(v)$ times in V') with $g \notin V'$. For any binary interference model, if there exist π pairwise interference free γ-labeled paths, each one connecting one element of V' to g, then we can satisfy a demand of 1 from each $v_i \in V'$ with a total weight $W = \gamma$.*

Proof: We send a flow of 1 in each path. After that, each edge labeled with one of the γ labels c_j is associated with a round R_j of weight 1. The set of edges used by R_j are non-interfering, as the paths are interference free γ-labeled. Furthermore, the inequalities in (2) are respected as $c_w(e) = \phi(e) = 1$. ■

We use Proposition 4 mainly in two cases: all the v in V' are different or all the v in V' correspond to the same vertex. In the latter case, Proposition 4 gives the following.



(a) One node using 2 paths. (b) Two nodes using 2 paths.

Figure 8: Main routing strategies.

Corollary 4 *If there exist π pairwise interference free γ-labeled paths from v to g, then we can route the demand $b(v)$ with γ rounds with a total weight $W = \frac{\gamma}{\pi}b(v)$.*

Proof: By Proposition 4, with all $v_i = v$, we can route a flow of π in γ rounds of weight 1 and so a flow of $b(v)$ in γ rounds of weight $\frac{b(v)}{\pi}$ each. ■

We use two main *routing strategies*. Either we route the total demand $b(v)$ of a vertex v by finding interference free paths from v to g and applying Corollary 4 (see Figure 8(a)); or we combine paths issued from v with paths issued from other nodes (see Figure 8(b)). We have to do different combinations to be able to route all the demands.

4.2 Distance-d model of interference and the *Width*

Here we present upper bound results concerning the distance-d model of interference. In some applications, we need to route the demand from v via a single path. If we use a shortest path and we give to each edge a different label, we obtain:

Proposition 5 *For any binary model of interference, we can route the demand $b(v)$ of a node v using a single path with a weight $W \leqslant b(v)d(v, g)$.*

The particular case of a node $v \in V_{K_0}$ has $W \geqslant b(v)d(v, g)$ by Corollary 1, then we obtain:

Corollary 5 *In the distance-d model of interference, the total demand $B = b(v)$ with $v \in V_{K_0}$ can be satisfied with $W_{min} = b(v)d(v, g)$ with a single shortest path.*

If $v \notin V_{K_0}$, by proposition 1 the lower bound is $W \geqslant \lceil \frac{d}{2} \rceil b(v)$ and so it cannot be attained using Proposition 5, (as $d(v, g) > \lceil \frac{d}{2} \rceil$ for all $v \notin V_{K_0}$). Indeed, in the case of a single path of length $\geqslant d + 1$, we need at least $d + 1$ labels due to $d + 1$ consecutive edges always interfere. If we want to have an interference free path with $d+1$ labels, the only way is to repeat a sequence of $d + 1$ different labels in order such that $d + 1$ consecutive edges have different labels. We use this notion of repeated sequences many times, so we introduce the following definition:

Definition 6 (C-labeling) *Let $C = (c_1, c_2, \ldots, c_k)$ be an ordered sequence of k labels a C-labeling of a path (or a cycle) consists of repeated sequences of C. More precisely if an edge is labeled c_j then the next edge is labeled c_{j+1} (where $c_{k+1} = c_1$). Note that if we give the label of an edge, then all the labels of the path are uniquely determined. We use the sentence "we C-label a path" for short to mean that we use a C-labeling for this path.*

This construction does not always work (see later an example in Figure 12 in which the path in a grid turns back at distance shorter than d making a "short U"). In that case, there are two edges far away (that are at distance $\geqslant d$ on the path), but at distance $< d$ in the graph. Thus, the path can not be interference-free $d+1$-labeled if these two edges have the same label. Therefore, we introduce the following definition:

Definition 7 (Width d) *A path (or a cycle) has* width d*, if two edges at distance $\geqslant d$ in the path (or cycle) are also at distance $\geqslant d$ in the graph.*

Proposition 6 *In the distance-d model of interference, a path of width d can be interference free $(d+1)$-labeled.*

Proof: We C-label the path with an ordered sequence C of $d+1$ labels. If two edges have the same label, then they are necessarily at distance $\geqslant d$ in the path and, by definition of the width, they are also at distance $\geqslant d$ in the graph and so they do not interfere. ■

Proposition 7 *In the distance-d model of interference, a shortest path between any pair of points of the graph can be interference free $(d+1)$-labeled.*

Proof: By Proposition 6, it suffices to prove that a shortest path has width d. Two edges at distance $\geqslant d$ in a shortest path are also at distance $\geqslant d$ in G; otherwise, we have a shortcut creating a shorter path, that is a contradiction. ■

Corollary 6 *Considering the distance-d model of interference and a general graph, we can route a demand of $b(v)$ using a shortest path with weight $W \leqslant (d+1)b(v)$.*

Consequently, if we route the demand of each node with a shortest path, we obtain the following approximation.

Theorem 1 *In the distance-d model with $d \geqslant 1$, there exists a $\frac{d+1}{\lceil \frac{d}{2} \rceil}$-approximation for the* RWP *problem.*

Proof: We have, by Proposition 1 (for the nodes $v \notin V_{K_0}$), a lower bound of $\lceil \frac{d}{2} \rceil b(v)$ and by Corollary 6, an upper bound of $d+1$. ■

Note that it gives a 2-approximation for d odd and, for d even, an $\frac{2}{d}+2$-approximation and so in the worst case ($d=2$) a 3-approximation.

We can also use Proposition 7 to design 2 interference free $d+1$-labeled paths in the following case.

Corollary 7 *If $d(v_1, v_2) = d(v_1, g) + d(g, v_2)$ then we can send a flow of 1 from v_1 and a flow of 1 from v_2 with $d+1$ rounds.*

Proof: The path formed by the union of a shortest path from v_1 to g and the shortest path from v_2 to g is a shortest path between v_1 and v_2, then it can be $d+1$-labeled by Proposition 7. ■

Theorem 2 *Let G be a general graph with the gateway g placed anywhere and let d be* odd *$(d = 2k-1)$. In the distance-d model of interference, if we can associate to v a family of nodes V_v such that*

- *$d(v, v_j) = d(v, g) + d(g, v_j)$, for all v_j in V_v (there exists a path with width d between v and v_j containing g), and*
- *$\sum_j b(v_j) \geqslant b(v)$,*

then the demand $b(v)$ from v and a demand $b(v)$ from nodes in V_v can be satisfied with a weight $W = (d+1)b(v)$. Summarizing, a demand of $2b(v)$ can be satisfied with a weight $(d+1)b(v)$ obtaining a ratio of $\frac{d+1}{2}$ per unit of demand.

Proof: We start choosing a node v_j in V_v. Corollary 7 guarantees that we can send to g a flow of $\min\{b(v), b(v_j)\}$ from v_j and the same quantity of flow $\min\{b(v), b(v_j)\}$ from v using $d+1$ rounds. Then, we have used a weight of $(d+1)\min\{b(v), b(v_j)\}$. We repeat the process choosing a node from V_v with positive remaining demand. The process is repeated until all the demand $b(v)$ from v has been routed. ■

Cycles play an important role and are used by the *routing strategies* as illustrated in Figure 8. Indeed, a cycle containing g induces for any vertex v (of the cycle) two paths from v to g. For any pair of vertices v_1 and v_2, it induces two paths, one from v_1 to g and another from v_2 to g.

Proposition 8 *In the distance-d model of interference, a cycle of width d can be interference free $(d+1)$-labeled if and only if its length is a multiple of $d+1$.*

Proof: Let C-label the cycle with $C = c_1...c_{d+1}$ an ordered sequence of $d+1$ labels. Notice that this labeling pattern is the only interference-free candidate using only $d+1$ labels.

If the length of the cycle is a multiple of $d+1$, then the edges labeled c_i are at a distance multiple of d on the cycle, and so by definition of width at distance $\geqslant d$ in the graph. If the length is not a multiple of $d+1$ then the last edge of the path labeled c_1 is at distance $< d$ of the first edge e_1 also labeled c_1, therefore these edges interfere. ■

Corollary 8 *In the distance-d model of interference, if there exists a cycle containing v and g of width d and of length multiple of $d+1$ then the demand $b(v)$ of a node v can be satisfied with a weight $W \leqslant \frac{d+1}{2}b(v)$.*

Proof: By Proposition 8, we have two interference free $(d+1)$-labeled paths from v to g. Then, we can route half of the demand on each path obtaining, by Corollary 4, $W \leqslant \frac{d+1}{2}b(v)$. ■

If d is odd, we have a lower bound of $\frac{d+1}{2}b(v)$ (see Proposition 1) so, by Corollary 8, we obtain:

Theorem 3 *In the distance-d model of interference with d odd $(d = 2k-1)$, if there exists a cycle containing $v \notin K_0$ and g (two paths from v to g) of width d and length multiple of $d+1$, then the demand $b(v)$ of v can be satisfied with a weight $W_{min} = \frac{d+1}{2}b(v) = kb(v)$.*

Special results for the IRWP in the Primary Node Model

We can also use two paths interference free $(d+1)$-labeled issued from two different vertices. In the following, we show applications of Corollary 4.

Corollary 9 *Let $d = 1$ (primary node model) and let G be a 3-connected graph. If $\sum_v b(v)$ is even then $W_{min} = \sum_v b(v)$ is solution for IRWP.*

Proof: As the graph is 3-connected, there exists an even cycle from any node v to g. In this way, for all v, we first route a demand of $\lfloor \frac{b(v)}{2} \rfloor$ by each one of the two paths of its even cycle (see Corollary 8). After this step, there are an even number of nodes with demand 1 and the remaining nodes with demand 0. More precisely, it remains $b(v) = 0$ for all nodes with $b(v)$ even; otherwise, $b(v) = 1$. For this remaining demand, which keeps satisfying the condition ($\sum_v b(v)$ even), we route it by using a pair of disjoint paths (so interference free as $d = 1$, that is 2-labeled) from u to g and from v to g where u and v are two nodes with demand 1. ■

The hypothesis of Corollary 9 may be weakened. Instead of requiring a 3-connected graph, it is only needed that there exists an even cycle from g to any node with demand. As is true for 2-connected bipartite graphs (e.g. grids), we obtain the following.

Corollary 10 *Let $d = 1$ (primary node model) and let G be a 2-connected bipartite graph. In the distance-d model of interference, if $\sum_v b(v)$ is even then $W_{min} = \sum_v b(v)$ is solution for IRWP.*

Corollary 11 *Let $d = 1$ (primary node model) and let G be a 2-connected graph. In the distance-d model of interference, if $\sum_v b(v)$ is even and for each v, $b(v)$ is integer such that $b(v) \leqslant \frac{1}{2}\sum_u b(u)$, then $W_{min} = \sum_v b(v)$ is solution for IRWP.*

Proof: We can always route together a flow of 1 from each of the two greatest demands, as $b(v) \leqslant \frac{1}{2}\sum_u b(u)$. That is done by using a pair of disjoint paths (so interference free as $d = 1$) from u to g and from v to g, which exists as G is 2-connected [28]. Then, we repeat this process on the remaining demands, which keep satisfying the conditions. ■

Note that the condition in Corollary 11 can be weakened. For example, if there exists only one node with positive integer demand, which is a neighbor of g, its demand can be directly routed to g and so the condition $\sum_v b(v)$ being even is not necessary. Similarly, if G is bipartite, we can route in two rounds a demand of two for any vertex using an even cycle (by Corollary 8). So condition $b(v) \leqslant \frac{1}{2}\sum_u b(u)$ is not needed, as we can first route for each vertex v the maximum even integer $\leqslant b(v)$.

In some cases, we also need more complicated routing protocols (like 4 paths or 2 cycles). In the Sections 6 and 7 of this article, we give solutions for the case of grids as an application of the presented methodology.

5 Lower bounds for grids

In the next sections, we answer the question asked in [24] of finding simple efficient algorithms for grids. For this purpose, we apply the tools presented in Section 3 to obtain lower bounds for grids. We consider the gateway placed either in the middle or in the corner. In both cases, the results depend on the parity of d.

In Sections 6 and 7, we give upper bounds which in most of the cases match the lower bounds given in the following section, therefore obtaining tight results.

5.1 Gateway in the middle: a lower bound

As shown before, in Proposition 1, a lower bound for the RWP with the distance-d model is $S_0 = \sum_{v \in V_{K_0}} d(v, g)b(v) + \lceil \frac{d}{2} \rceil \sum_{v \notin V_{K_0}} b(v)$. When d is even ($d = 2k$), this lower bound can be improved to $S_0 + \frac{1}{4}\sum_{v; d(v,g)>k} b(v)$ as shown in Proposition 3. We present now specific results for grids when the demand is uniform and the gateway is in the middle. Propositions 9 and 10 present closed formulas for d odd and d even respectively. In Theorems 7 and 16, we prove that these formulas give the optimal solution.

In the following we consider a grid of size $p \times q$ with N vertices. Any node in the grid is represented by coordinates (x, y) with $-p_1 \leqslant x \leqslant p_2$ and $-q_1 \leqslant y \leqslant q_2$. The gateway corresponds to coordinate $(0, 0)$. (Refer to Section 2 for more details about the coordinate system). Recall that by definition, gateway in the *middle* means a gateway far from the borders with $\min(p_1, p_2, q_1, q_2) \geqslant \lceil \frac{d+1}{2} \rceil$ (see Section 2). We denote by N_i the number of vertices at distance i of the gateway. For $i \leqslant \min(p_1, p_2, q_1, q_2)$ (in particular for $i \leqslant k + 1$), we have $N_i = 4i$.

Proposition 9 *Let $d = 2k - 1$ be odd and let G be a grid $p \times q$ with $\min(p_1, p_2, q_1, q_2) \geqslant \lceil \frac{d+1}{2} \rceil$, N vertices and the gateway in the middle. Considering uniform demand ($b(v) = b, \forall v$), then $W_{\min} \geqslant b(k(N - 1) - \frac{4}{6}k(k + 1)(k - 1))$.*

Proof: By Proposition 1, $W_{\min} \geqslant S_0 = \sum_{v \in V_{K_0}} d(v,g)b(v) + \lceil \frac{d}{2} \rceil \sum_{v \notin V_{K_0}} b(v)$. As $b(v) = b$, we obtain $\sum_{v \in V_{K_0}} d(v,g)b(v) = b \sum_{i \leqslant k} iN_i$ and $\lceil \frac{d}{2} \rceil \sum_{v \notin V_{K_0}} b(v) = kb((N-1) - \sum_{i \leqslant k} N_i)$. Then we have:

$$\begin{aligned} W_{\min} &\geqslant b \sum_{i \leqslant k} iN_i + kb((N-1) - \sum_{i \leqslant k} N_i) \\ &= b \sum_{i \leqslant k} 4i^2 + kb(N-1) - kb \sum_{i \leqslant k} 4i \\ &= kb(N-1) - 4b[k\frac{k(k+1)}{2} - \frac{k(k+1)(2k+1)}{6}] \\ &= kb(N-1) - b\frac{4k(k+1)(k-1)}{6} \end{aligned}$$

■

Proposition 10 *Let $d = 2k$ be even and G be a grid $p \times q$ with $\min(p_1, p_2, q_1, q_2) \geqslant \lceil \frac{d+1}{2} \rceil$, N vertices, gateway in the middle and uniform demand ($b(v) = b, \forall v$). Then $W_{\min} \geqslant b((k+\frac{1}{4})(N-1) - \frac{k(k+1)(4k-1)}{6})$.*

Proof: By Proposition 3, $W_{\min} \geqslant S_0 + \frac{1}{4} \sum_{v; d(v,g) > k} b(v) = S_0 + \frac{1}{4} b((N-1) - \sum_{i \leqslant k} N_i)$. From Proposition 9, $S_0 = kb(N-1) - \frac{4}{6} kb(k+1)(k-1)$ and

$$W_{\min} \geqslant b(k + \frac{1}{4})(N-1) - b\frac{k(k+1)(4k-1)}{6}.$$

■

5.2 Gateway in the corner: a lower bound

Using the same technique as above, we now present lower bounds when the gateway is placed in the corner of the grid. Similar to the case with the gateway in the middle, the results depend on the parity of d.

Notice that, when the gateway is placed at the corner, we can construct call-cliques bigger than K_0. We define $K_{\max}$ as the call-clique composed by the edges delimited by the vertices $V_{K_0} \cup S$ where $S = \{v \mid d(v,g) \leqslant d+1 \text{ and } d(v^*, v) \leqslant \lceil \frac{d}{2} \rceil\}$ and v^* denotes the node $(\lceil \frac{d}{2} \rceil, \lceil \frac{d}{2} \rceil) = (k,k)$. An example of $K_{\max}$ for $d = 9$ ($k = 5$) is depicted in Figure 9. Another example for $d = 15$ ($k = 8$) is depicted in Figure 10 where the values of the lower bound, given in the next lemma, are also indicated.

In the following, we introduce the *individual lower bound* called $\text{lb}(v)$ which is a lower bound for routing the demand of v and is reached in many cases (see Subsection 6.2). The value depends on the zone of the grid to which v belongs (see Figure 21).

$$\text{lb}(v) = \begin{cases} d(v,g) & \text{for } v \in V_{K_0} \\ \min\left\{3\frac{(d+1)}{2} - d(v,g); d+1-d(v,v^*)\right\} & \text{for } v \in S \\ \frac{d+1}{2} & \text{otherwise.} \end{cases} \tag{4}$$

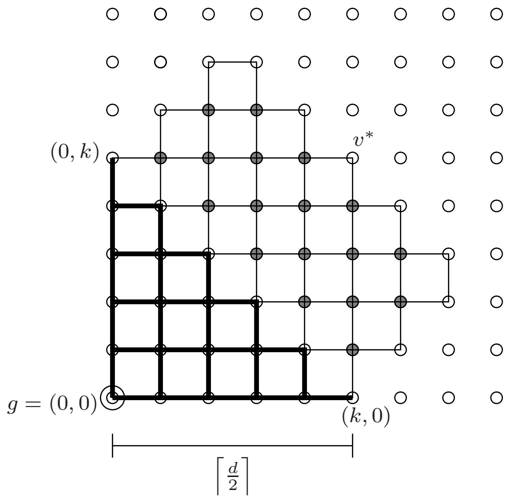


Figure 9: Call-clique $K_{\max}$ for d odd with g at the corner ($d = 9$, then $k = 5$). The call-clique K_0 consists in all the bold edges.

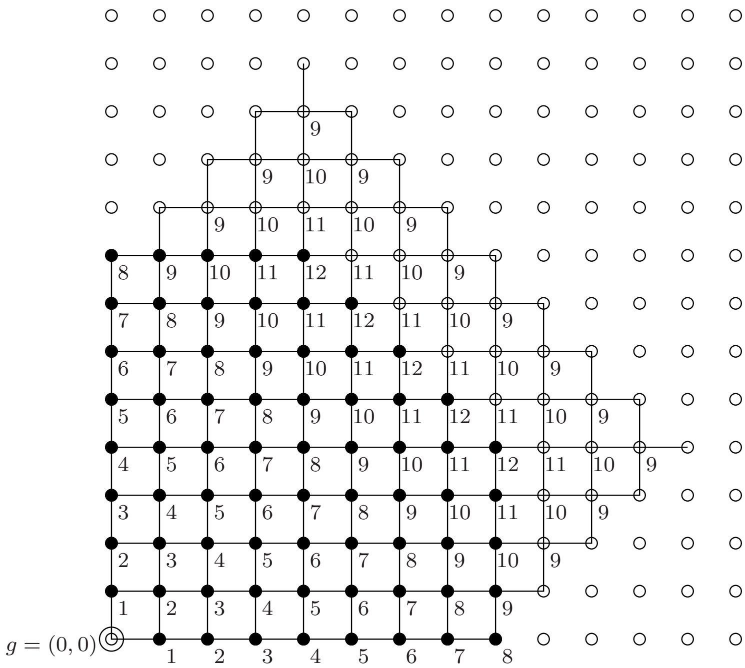


Figure 10: Lower bound per node in uniform demand case. The black nodes indicate the nodes whose lower bound corresponds to their distance to the gateway ($d = 15$, then $k = 8$).

5.2.1 Case d odd

In this section, we study the case when d is odd ($d = 2k - 1$). We use $K_{\max}$, the maximum call-clique containing K_0, to obtain better lower bounds. (We see later that this bound is already optimal). In this section, we suppose that the grid is large enough to contain the vertices of $K_{\max}$, that is $\min(p, q) \geqslant \lfloor \frac{3k}{2} \rfloor$.

Lemma 7 *Let $d = 2k-1$ be odd and let G be a grid $p \times q$ with $\min(p, q) \geqslant \lfloor \frac{3k}{2} \rfloor$ and the gateway at the corner, then* $\mathrm{LB}(v, K_{\max}) = \mathrm{lb}(v)$

Proof: If $v \in V_{K_0}$, any shortest path from v to g uses $d(v, g)$ edges in K_0 (and so in $K_{\max}$). Note that, in that case, $d + 1 - d(v, v^*) = d(v, g)$ as $d(v^*, g) = d + 1$. Otherwise, any path has to use k edges in K_0 giving the lower bound for $v \notin S$. In the remaining part of the proof, we deal with the case when $v \in S$. If $v \in S$ any path from v to g use k edges in K_0 plus certain edges connecting nodes in S. This number of used edges is either $d(v, g) - k$ needed to attain a vertex of K_0; or $2k - d(v, g)$ to attain the diagonal bordering S composed by the vertices at distance $2k$ from g (the vertices $v = (x, y)$ such that $x + y = 2k$) ; or $k - d(v, v^*)$ to attain the diagonals bordering S below ($y = x + k$) or above ($x = y + k$). Depending on the position of v in S, the minimum is attained by one of the values as summarized in Statement 4. ■

Theorem 4 *Let $d = 2k - 1$ be odd and let G be a grid $p \times q$ with $\min(p, q) \geqslant \lfloor \frac{3k}{2} \rfloor$ and the gateway at the corner, then*

$$W_{\min} \geqslant \sum_v b(v)\,\mathrm{lb}(v).$$

Proof: $W \geqslant c_w(K_{\max}) \geqslant \phi(K_{\max}) \geqslant \sum_v \phi_v(K_{\max}) \geqslant \sum_v b(v)\,\mathrm{lb}(v)$. ■

Using Theorem 4, we can derive an explicit formula for the lower bound when the demand is uniform.

Proposition 11 *Let $d = 2k - 1$ be odd and G be a grid $p \times q$ with $\min(p, q) \geqslant \lfloor \frac{3k}{2} \rfloor$, N vertices and the gateway at the corner. Assuming uniform demand (i.e. $b(v) = b, \forall v$), then*

$$W \geqslant b\left(\frac{d+1}{2}(N-1) + f(d)\right)$$

where $f(d) = \frac{d+1}{192}(d-3)(d-19)$ if $d = 4\lambda - 1$ (k even); and $f(d) = \frac{d+1}{192}(d-1)(d-21)$ if $d = 4\lambda + 1$ (k odd).

Proof: We have to count $\sum_v \mathrm{lb}(v)$. For all the vertices not in $V_{K_0} \cup S$, $\mathrm{lb}(v) = k$. (Recall that S is defined as $\{v \mid d(v, g) \leqslant 2k$ and $d(v, v^*) \leqslant k\}$). For the other vertices, we compute the value $|\mathrm{lb}(v) - k|$. We obtain $W \geqslant b\left(\frac{d+1}{2}(N-1) + B_k - A_k\right)$, where $A_k = \sum_{v \in V_{K_0}}(k - \mathrm{lb}(v))$ and $B_k = \sum_{v \in S}(\mathrm{lb}(v) - k)$.

For the vertices in V_{K_0}, $\mathrm{lb}(v) = d(v, g) \leqslant k$. We have $i + 1$ vertices at distance i. So, $A_k = \sum_{i=1}^{k-1}(i+1)(k-i) = \frac{(k-1)k(k+4)}{6}$.

For $v \in S, \mathrm{lb}(v) \geqslant k$. Consider the 4 diagonals delimiting S namely $x + y = k$; $x + y = 2k$; $x = y + k$; $y = x + k$. For the vertices (x, y) in S at distance $i > 0$ of one of the 4 diagonals, we have $\mathrm{lb}(v) = k + i$. In order to compute B_k, we distinguish two cases depending on the parity of k. We use an auxiliary parameter λ to make the calculation clear. For the case even, $k = 2\lambda$, the number of vertices in S with value of $\mathrm{lb}(v) = k + i$ is $3k - 4i$ for $1 \leqslant i \leqslant \lambda - 1$, and $\lambda + 1$ for $i = \lambda$. For the case odd, $k = 2\lambda + 1$, they are in number $3k - 4i$ for $1 \leqslant i \leqslant \lambda$. Altogether, for the case $k = 2\lambda$, $B_k = \sum_{i=1}^{\lambda-1} i(3k-4i) + \lambda(\lambda+1) = \frac{k}{6}(5\lambda^2+1)$. For the case $k = 2\lambda+1$, $B_k = \sum_{i=1}^{\lambda} i(3k-4i) = \frac{k}{6}(5\lambda(\lambda+1))$.

Finally, the value of $B_k - A_k$ to be added to $\frac{d+1}{2}(N-1)$ is: for the case $k = 2\lambda$, $B_k - A_k = \frac{k}{6}(\lambda-1)(\lambda-5)$ and for the case $k = 2\lambda+1$, $B_k - A_k = \frac{k}{6}\lambda(\lambda-5)$. ■

5.2.2 Case d even.

Let us now consider the case when d is even, $d = 2k$. As we have seen in the example of Figure 3, we have to consider in that case two cliques K_1 and K_2. These two cliques contain a clique $K_{\max}$. The clique $K_{\max}$ is the set of edges connected by the vertices of $V_{K_0} \cup S$. In addition to $K_{\max}$, K_1 contains the $\lfloor \frac{k}{2} \rfloor + 1$ vertices $v = (x, y)$ such that $x + y \leqslant 2k+1$ and $x = y + k + 1$. In the same way, K_2 contains the $\lfloor \frac{k}{2} \rfloor + 1$ vertices v such that $x + y \leqslant 2k+1$ and $y = x + k + 1$ (see example in Figure 11).

Theorem 5 *Let $d = 2k$ be even and let G be a grid $p \times q$ with $\min(p,q) \geqslant \frac{3k}{2}$ and the gateway at the corner. Then*

$$W_{\min} \geqslant \sum_v b(v)\,\mathrm{lb}(v).$$

Proof: We use Lemma 5 with the two cliques K_1 and K_2. If $v \in V_{K_0}$, any path from v to g uses $d(v,g)$ edges in $K_{\max}$. If $v \notin V_{K_0} \cup S$, any path from v to g use k edges in $K_{\max}$ and at least one in K_1 or one in K_2 giving a lower bound of $k + 1/2 = \frac{d+1}{2}$. If $v \in S$, we distinguish 3 cases depending on the number of edges needed to attain the border of S:

- $d(v,g) - k$ to attain a vertex $v = (x,y)$ such that $x + y = k$ of K_0, and so $d(v,g)$ edges in $K_{\max}$;
- $2k+1-d(v,g)$ to attain the diagonal composed by the vertices at distance $2k+1$ (i.e, $x+y = 2k+1$), but then it is needed $\frac{d+1}{2}$ (as seen before) to attain the gateway;
- $k - d(v,v^*)$ to attain the diagonal below (i.e, $x = y + k$) or above ($y = x + k$). Then it is needed k edges in $K_{\max}$ and either 2 in K_1 (vertices below) or 2 in K_2 (vertices above) or 1 in both K_1 and K_2, so altogether $2k - d(v,v^*) + 1/2 \cdot 2 = d + 1 - d(v,v^*)$.

According to the position of v, the minimum is attained by one of the values summarized in Statement 4. We conclude using the same proof as in Theorem 4.

■

Note that the formula is identical to that of the case d odd. When the demand is uniform and the grid is large enough to contain the vertices in K_1 and K_2 (that is $\min(p,q) \geqslant \frac{3k}{2} + 1$), calculations similar to those in Proposition 11 give the following result.

Proposition 12 *Let $d = 2k$ be even and let G be a grid $p \times q$ with $\min(p,q) \geqslant \frac{3k}{2}$ and g at the corner. Considering uniform demand (i.e. $b(v) = b, \forall v$), then*

$$W \geqslant b\left(\frac{d+1}{2}(N-1) + f(d)\right)$$

where $f(d) = \frac{\lambda}{12}(4\lambda^2 - 21\lambda - 1)$ if $d = 4\lambda$; and $f(d) = -\frac{1}{2} + \frac{\lambda}{12}(\lambda+1)(4\lambda-19)$ if $d = 4\lambda+2$.

Proof: The proof is the same than Proposition 11 except for the values of B_k and A_k. Now $B_k = \sum_{i=1}^{\lambda}(i-\frac{1}{2})(3(k+1)-4i) + (\lambda+\frac{1}{2})(\lambda+1) + \frac{1}{2}\sum_{i=1}^{\lambda}(k-i+1) = \frac{1}{12}(6+31\lambda+45\lambda^2+20\lambda^3)$ if $d = 4\lambda+2$ and $B_k = \sum_{i=1}^{\lambda}(i-\frac{1}{2})(3(k+1)-4i) + \frac{1}{2}\sum_{i=1}^{\lambda}(k-i) = \frac{\lambda}{12}(1+15\lambda+20\lambda^2)$ if $d = 4\lambda$.
So, as $A_k = \sum_{i=1}^{k}(i-\frac{1}{2})(k-i+2)$ then $f(d) = B_k - A_k$ and the result follows. ■

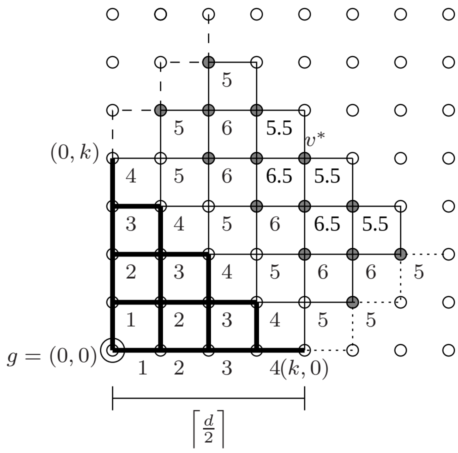


Figure 11: Two overlapped cliques for d even with g at the corner ($d = 8$). Dotted edges belong exclusively to K_1 and dashed edges belong exclusively to K_2.

6 Upper bounds for grids with single source routing

In a grid the paths or cycles have a specific structure. Indeed, they are formed by a succession of horizontal and vertical subpaths. To describe such a path or cycle, we only give the vertices where there is a change of direction. So between two vertices (x, y_0) —(x', y_0), we have an *horizontal* path consisting of all vertices (u, y_0) with $x \leqslant u \leqslant x'$ if $x < x'$, or $x' \leqslant u \leqslant x$ if $x > x'$. Similarly between two vertices $(x_0, y)(x_0, y')$, we have a *vertical* path (x_0, y) —(x_0, y') (see examples of paths in Figure 13(a)). We introduce the following definitions that are necessary to describe our methods of routing.

Definition 8 (Monotonic path) *We say that a path is monotonic (has a "stair" shape), if the first and second coordinates of the vertices where there is a change of direction are ordered in a monotonic way.*

We have 2 types of monotonic paths according to x_i and y_i vary in the same way or not. For example, a monotonic path $P = (x_0, y_0)$ —(x_1, y_0) —(x_1, y_1) —(x_1, y_2) —(x_2, y_2) —(x_2, y_3)... − −(x_m, y_n) —(x_m, y_{n+1}) is a monotonic of *negative type* +- (or -+), if the x_i are increasing $x_0 \leqslant x_1 < x_2 ... < x_m$ and the y_i are decreasing $y_1 > y_2 ... > y_n \geqslant y_{n+1}$ (as the path is undirected by considering the vertices in the opposite order we have decreasing x and increasing y). See Figure 13(b) for an example of this case. When the vertices have both increasing (resp. decreasing) x and y, the path is said to be monotonic of *positive type* ++ (resp. --).

Proposition 13 *Let G be a monotonic path in a 2-dimensional grid. It can be interference free $(d + 1)$-labeled.*

Proof: The x and y in this path are monotonic, then it has *width* d as the distance in the path is exactly that in the graph. Thus, Proposition 6 says it can be interference free $(d+1)$-labeled. See Figure 12 for an example of non monotonic path that interferes itself ("short U"). ■

Definition 9 (Path distance $d(P, Q)$) *The distance $d(P, Q)$ between two paths P and Q is the minimum of the distance between any edge of P and any edge of Q, $d(P, Q) = \min_{e_1 \in P, e_2 \in Q} d(e_1, e_2)$.*

Proposition 14 *In the distance-d model of interference, two monotonic paths, P and Q, at distance $\geqslant d$ do not interfere. So they are $d + 1$-interference free labeled.*

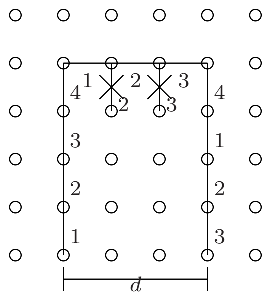
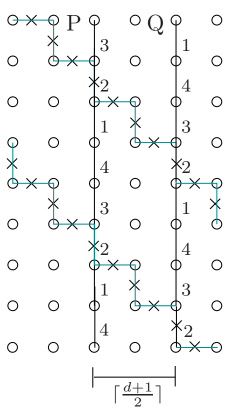
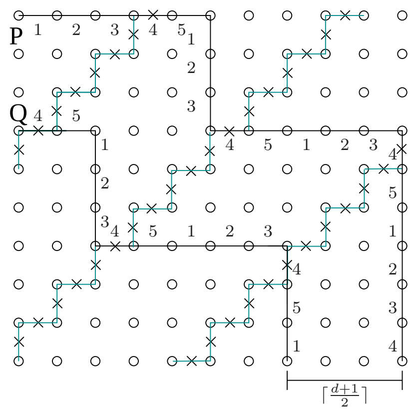


Figure 12: To turn back, a path keeps a width $\geqslant d$ to be interference free $(d+1)$-labeled $(d=3)$.



(a) Vertical paths and the negative diagonals associated with the edges labeled 2 $(d=3)$.



(b) Parallel negative paths and the positive diagonals associated with the edges labeled 4 $(d=4)$.

Figure 13: Interference free $(d+1)$-labeled paths.

Proof: As $d(P,Q) \geqslant d$, we can apply directly the Proposition 13 for each path. ■

Definition 10 (Diagonal of an edge) *The* positive *(resp.* negative*) diagonal of an edge e, denoted S_e^+ (resp. S_e^-) consists of the edges of a monotonic positive (resp. negative) path where all the subpaths are of length* 1 *(stairs of step* 1*).*

Figure 13(a) shows the negative diagonal associated with the edges labeled 2 and Figure 13(b), positive diagonal associated with the edges labeled 4. Now, we define relations between monotonic paths.

Definition 11 (d-Parallel paths) *Two monotonic paths P and Q are said d-parallel, if they are of the same* type *negative (respectively positive) and, if $e' \in Q$ and $e' \in S_e^+$ (respectively S_e^-) with $e \in P$, then $d(e,e') \geqslant d$. See Figure 13(b) for an example with two parallel negative paths.*

Proposition 15 *Given two d-parallel paths P and Q in a 2-dimensional grid G, they can be interference free $(d+1)$-labeled.*

Proof: We start labeling the path P with $d+1$ labels. Each edge of Q, that is in a *diagonal set* S_e of an edge e in P, receives the same label of e. If there exist edges in Q that are not in a *diagonal set* of P, they receive the continuation of the sequence of labels derived from the edges in *diagonal sets* of P. There is no interference between the edges with the same label as, by definition of d-parallel paths, two edges in the same diagonal are at distance $\geqslant d$. ■

In Figure 13, we illustrate the Proposition 15 with two pair of parallel paths. In particular two horizontal (or vertical) paths P and Q at distance $d(P,Q) \geqslant \lceil \frac{d+1}{2} \rceil$ are d-parallel. In that case, the distance between two edges of P and Q in the same diagonal is $2d(P,Q) - 1 \geqslant d$ (see Figure 13(a) for $d = 3$). Similarly, if two general monotonic paths have their horizontal and vertical sub-paths at distance $\geqslant \lceil \frac{d+1}{2} \rceil$, the distance between two edges at the same diagonal is $\geqslant d$. It is the case when one path is obtained from the other by translation of vector $(\lceil \frac{d+1}{2} \rceil, \lceil \frac{d+1}{2} \rceil)$, see Figure 13(b). Note that, paths uniquely horizontal or vertical can be considered of any type, either ++ or +- (see example in Figure 13(a)).

6.1 Gateway in the middle: routing each node separately

We consider here a strategy where the demands of each single node are routed separately, one at a time, that we call *single* routing.

Definition 12 (Regions of the grid) *We split the grid in* 4 *regions: R_A,R_B,R_C and R_D, as shown in Figure 14.*

Notice that we could chose different splittings. The results are valid as soon as the regions are obtained by the rotation ρ of $\frac{\pi}{2}$ of the first one. In this article, let the first region be R_A, and let it be composed by the vertices (x,y) with $x \geqslant 0$, $y \geqslant 1$ and $x + y \geqslant \lceil \frac{d+1}{2} \rceil$ (to exclude the vertices of K_0). Indeed, for a vertex of $v \in K_0$, we can route its demand $b(v)$ in $b(v)d(v,g)$ rounds by using a shortest path with $d(v,g)$ different labels (see Corollary 5). For a vertex $v \notin V_{K_0}$, we have:

Proposition 16 *Let d be odd or even, and let G be a 2-dimensional grid with* $\min(p_1,p_2,q_1,q_2) \geqslant d$*, and with gateway g in the middle. If $v \notin V_{K_0}$, then there exists a cycle C containing v and g that can be interference free $(d+1)$-labeled.*

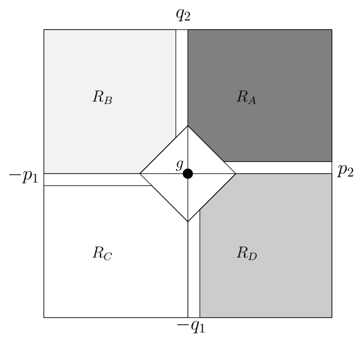


Figure 14: The grid regions and K_0.

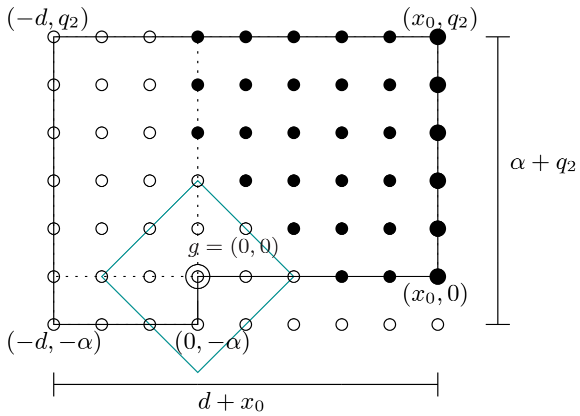


Figure 15: Routing method for a node v with a cycle and d odd (d=3).

Proof: We consider only the nodes $v = (x, y)$ of region R_A. For the nodes of other regions, we consider the cycle obtained by applying the rotation ρ to cycle C.

We construct, for any $x_0 \geqslant 0$, a generic cycle containing all the vertices of column x_0 with $y \geqslant 0$ and satisfying the hypothesis of Proposition 8 (length multiple of $d+1$ and width $\geqslant d$). Therefore, it can be interference free $(d+1)$-labeled. The reader can see Figure 15 for an example with $d = 3$ $(k = 2)$, $x_0 = 5$ and $q_2 = 5$. The cycle C consists of the following subdipaths (we indicate only the vertices where there is a change of direction).

$$(0,0) — (x_0, 0) — (x_0, q_2) — (-d, q_2) — (-d, -\alpha) — (0, -\alpha) — (0, 0).$$

The length of the cycle is $2(x_0 + d + q_2 + \alpha)$. Le us choose α as the smallest possible integer such that $2(x_0+d)+2(q_2+\alpha) \equiv 0$ $(mod\ d+1)$. Therefore the cycle has a length multiple of $d+1$. In the example the length of the cycle is $26 + 2\alpha$; so we choose $\alpha = 1$ (length $28 \equiv 0$ $(mod\ 4)$). Note that $0 \leqslant \alpha \leqslant d$ and therefore the cycle fits in the grid as $q_1 \geqslant d$. As $q_2 \geqslant d$, the horizontal paths are at distance $\geqslant d$. As we chose the vertical line at $-d$ (choice possible, as we have $p_1 \geqslant d$), the vertical paths are also at distance $\geqslant d$. So, the cycle has width $\geqslant d$. ■

6.1.1 Case d odd

In the case d odd with non-uniform demand, we obtain a complete solution for our problem in Theorem 6.

Theorem 6 *Let d be odd and let G be a 2-dimensional grid with the gateway in the middle and with* $\min(p_1, p_2, q_1, q_2) \geqslant d$. *Then* $W_{min} = S_0 = \sum_{v \in V_{K_0}} d(v, g)b(v) + \frac{d+1}{2} \sum_{v \notin V_{K_0}} b(v)$.

Proof: The lower bound follows from Proposition 1. The upper bound follows by applying Corollary 5 for the vertices $v \in V_{K_0}$ and Proposition 16 and Corollary 8 for the other vertices. ■

In the particular case of an uniform demand, we obtain a closed formula in Theorem 7 by using Proposition 9.

Theorem 7 *Let $d = 2k-1$ be odd and let G be a grid $p \times q$ with $\min(p_1, p_2, q_1, q_2) \geqslant d$, with N vertices and with the gateway in the middle. Considering uniform demand ($b(v) = b, \forall v$) then $W_{\min} = b(k(N-1) - \frac{4}{6}k(k+1)(k-1))$.*

6.1.2 Case d even

We already know that, for vertices in K_0, a lower bound of $d(v,g)$ is attained by using a shortest path. But for $v \notin V_{K_0}$, we have only a lower bound of $k + \frac{1}{4}$ by Proposition 3. Proposition 16 and Corollary 8 give for these vertices an upper bound of $k + \frac{1}{2}$ and therefore we get the following approximation.

Theorem 8 *Let d be even ($d = 2k$) and let G be a 2-dimensional grid with the gateway in the middle and with $\min(p_1, p_2, q_1, q_2) \geqslant d$. Then there exists a $1 + \frac{1}{4k+1}$-approximation for the RWP.*

Determining precisely the value of W_{min} for the single routing of any vertex is a difficult task. However, we will see later that for most of the vertices we can reach the lower bound of $k + \frac{1}{4}$. Based on Corollary 4, the only way to reach the lower bound is to find 4 pairwise interference free $(4k+1)$-labeled paths from v to g. These paths have to cross the $k+1$-th level E_{k+1} using the 4 non-interfering edges $(0, k+1)(0, k)$, $(k+1, 0)(k, 0)$, $(0, -(k+1)(0, -k)$ and $(-(k+1), 0)(-k, 0)$. As the end part of these 4 paths, we consider the shortest path from $(0, k)$ to g and its rotations (from $(-k, 0), (0, -k)$ and $(k, 0)$ to g). Let us call the set containing the edges of these 4 shortest paths a *cross* centered in g. More generally, a cross centered in $v = (x, y)$ consists of the $4k+4$ edges $(x, y+i)(x, y+i+1)$, $(x-i, y)(x-i-1, y)$, $(x, y-i)(x, y-i-1)$,$(x+i, y)(x+i+1, y)$ for $0 \leqslant i \leqslant k$. The cross in v represent the beginning of the 4 paths going to v.

A cross centered in (x, y) requires $4k+1$ labels, with the same label being given for the 4 edges $(x+k, y)(x+(k+1), y)$, $(x-k, y)(x-(k+1), y)$, $(x, y+k)(x, y+(k+1)$ and $(x, y-k)(x, y-(k+1))$. If v is "too close" from g, it may not be possible to label the two crosses with only $4k+1$ labels (see Figure 16). That happens if $d(v, g) \leqslant d+1$ and $v = (x, y)$ with $x \neq 0$, $y \neq 0$. But when $x = 0$ (or $y = 0$), we can label the 4 paths (the two crosses and the connection edges between them) with $4k+1$ labels, then reaching the lower bound (see example of Figure 20(b)). Similarly, if v is too near from the boarder of the grid we cannot have a cross centered in v and so cannot reach the lower bound.

In order to obtain better results, the grid is divided into several zones. We distinguish three disjoint zones of region $R_A = X_A \cup Y_A \cup Z_A$ (see Figure 17):

- X_A: zone composed by all nodes at distance $\leqslant d+1$ of the gateway, with $x \neq 0$ that are not in K_0.
- Y_A: zone composed by all nodes that are at distance at most k of the vertical or horizontal borders.
- Z_A: zone composed by all nodes that are not in X_A, Y_A or K_0. There exist special sub-zones in Z_A, they are:

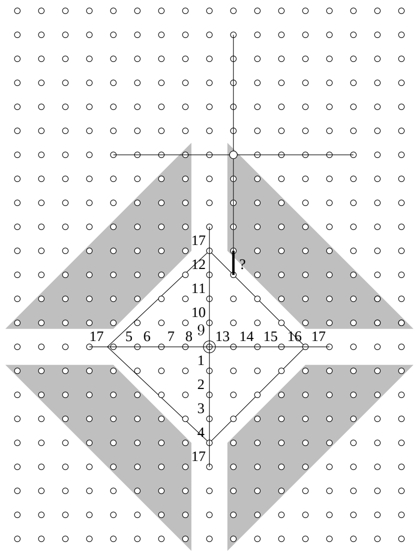


Figure 16: Routing the demand of node $(1, 8)$ in S_A, it is impossible to assign a label to the bold edge using only $4k+1$ labels ($d = 8$).

- Z'_A: the first part of Z'_A corresponds to the nodes of Z_A with $1 \leqslant x \leqslant k+1$; the second part of Z'_A corresponds to the nodes with $1 \leqslant y \leqslant k+1$.
- Z''_A: the nodes $v \in Z_A$ that are on the axes ($x = 0$).

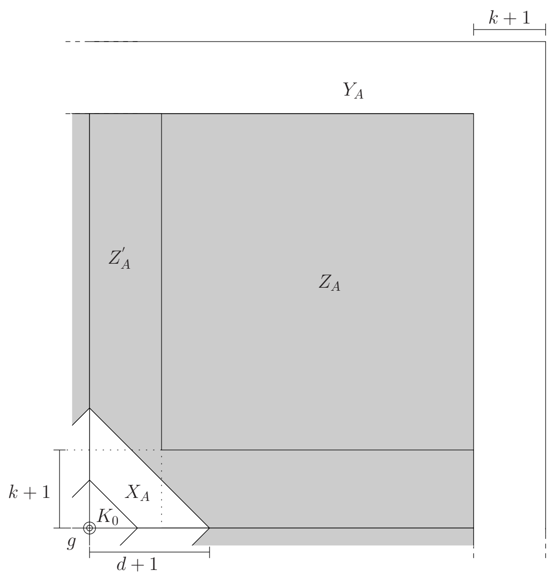


Figure 17: Subzones of R_A.

For nodes in X_A, and Y_A, as indicated before the lower bound ($> k + \frac{1}{4}$) can not be reached as it is not possible to construct a cross $(4k+1)$-labeled, but it is not easy to give a precise value. However, the following theorem shows that the lower bound can be reached for nodes in Z_A by considering the single routing of a node in that region. Note that the zone Z_A contains the majority of the nodes in grids with large p_1,p_2,q_1 and q_2.

Theorem 9 *Let d be even ($d = 2k$) and G be a 2-dimensional grid with the gateway g in the middle and with $\min(p_1, p_2, q_1, q_2) \geqslant 2d$ and let $v \in Z_A$. There exist 4 paths from v to g that can be interference free $(4k+1)$-labeled. Therefore, $W_{min} = \frac{4k+1}{4}b(v)$ for the single routing of v.*

Proof: We distinguish three cases according to the node position:

- Case 1 (see scheme in Figure 18(a) and example in Figure 18(b)): node $v \in Z_A \backslash \{Z'_A \cup Z''_A\}$, that is a node with coordinates satisfying $k+1 < x < p_1 - k$ and $k+1 < y < q_1 - k$. The 4 paths from g to v are:

  - $(0,0)$ —$(x,0)$ —(x,y);
  - $(0,0)$ —$(0,y)$ —(x,y);
  - $(0,0)$ —$(0,-(k+1))$ —$(x+(k+1),-(k+1))$ —$(x+(k+1),y)$ —(x,y)
  - $(0,0)$ —$(-(k+1),0)$ —$(-(k+1),y+(k+1))$ —$(x,y+(k+1))$ —(x,y).

  We use an ordered set of $4k+1$ labels partitioned into two disjoint ordered sequences C and C' of $2k+1$ and $2k$ labels, respectively. For example $C = 1, 2, \ldots, 2k+1$ and $C' = 2k+2, 2k+3, \ldots, 4k+1$. In the example $d = 8$, $k = 4$ and $C = 1, 2, \ldots, 9$ and $C' = 10, 11, \ldots, 17$. We have to be careful that the paths are interference free labeled and that the end edges of the crosses in g and v get the same label.

  We first C-label (see Definition 6) the following 4 subpaths (drawn in blue in Figures 18(a) and 18(b)):

  - P_1: $(-(k+1), y+(k+1))$ —$(x, y+(k+1))$ —$(x, y+k)$;
  - P_2: $(x+k, y)$ —$(x+(k+1), y)$ —$(x+(k+1), -(k+1))$;
  - P_3: $(0,y)$ —(x,y) —$(x,0)$;
  - P_4: $(-k,0)$ —$(0,0)$ —$(0,-k)$.

  We first C-label P_1 starting with label 1 for the first edge $(-(k+1), y+(k+1)), (-k, y+(k+1))$. Let ℓ be the label of edge $(x, y+k+1)(x, y+k)$. In the example, $\ell = 7$ In order to get the same label on the end edges of the cross in v, we also label ℓ the edges $(x+k, y)(x+k+1, y)$, $(x-(k+1), y)(x-k, y)$ and $(x, y-k)(x, y-(k+1))$.

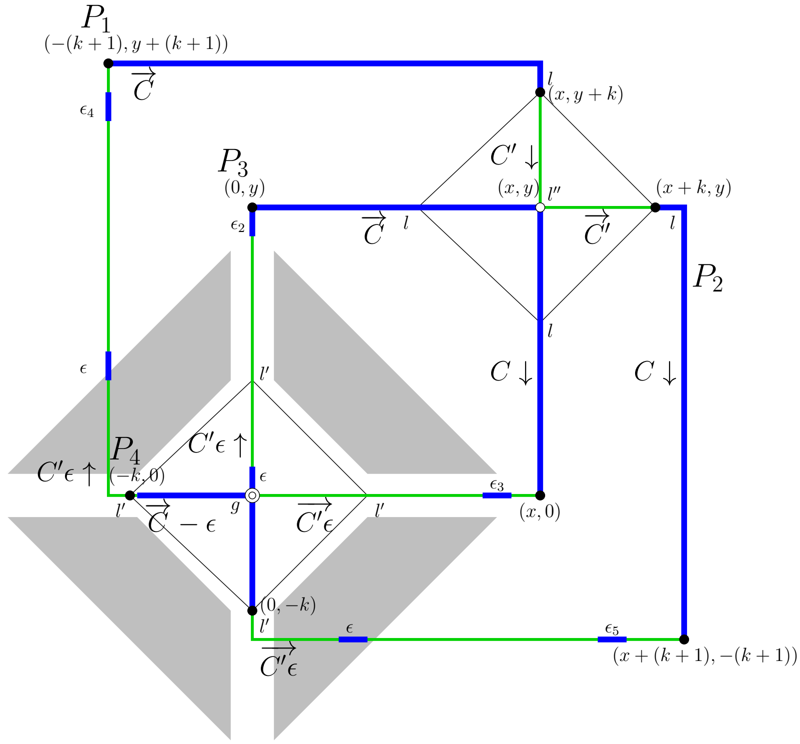
  Now we C-label P_2 using the label ℓ for the first edge $(x+k, y)(x+k+1, y)$ (the other labels are then determined). Then we C-label P_3 using the label ℓ for the edge $(x-(k+1), y)(x-k, y)$ (or the edge $(x, y-k)(x, y-(k+1))$, as these two labels are compatible the distance between the edges being d).

  It remains to label P_4. Let ϵ be the label assigned to the edge in P_3 on the positive diagonal of $(0,1)(0,0)$. In the example $\epsilon = 2$. We use a $C - \{\epsilon\}$-labeling for the subpath P_4 of length $2k$ in such a way the edge $(-1,0)(0,0)$ gets label $\epsilon - 1$ (so edge $(0,0)(0,-1)$ gets label $\epsilon + 1$).

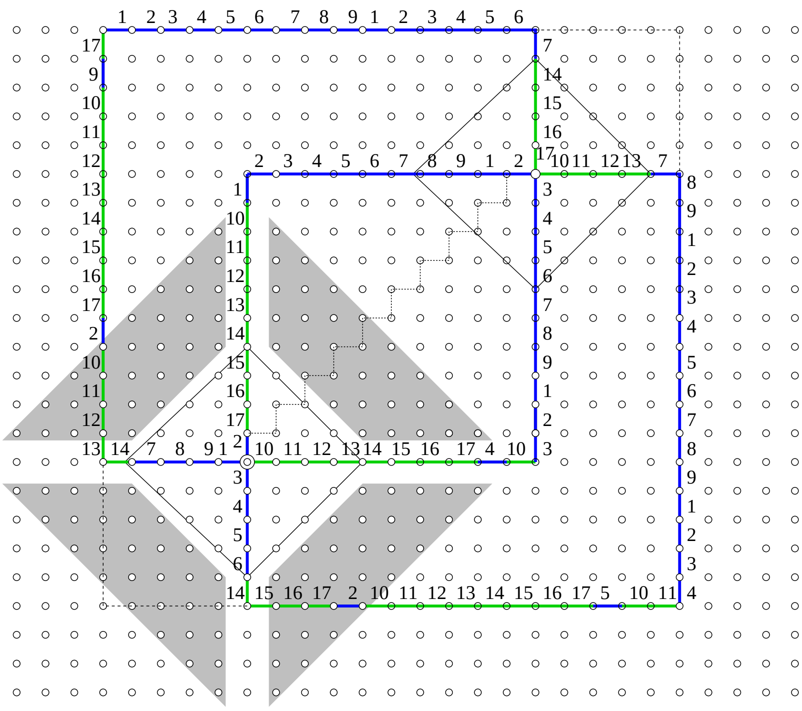
  There is no interference as the paths $P_1 \cup P_2$, P_3 and P_4 are by construction d-parallel and the labels are those given in the proof of Proposition 15.

  We label now the other subpaths (in green in the Figures) which are in fact *reflected paths* of the preceding ones:

  - P'_1: $(0,-k)$ —$(0,-(k+1))$ —$(x+(k+1),-(k+1))$;



(a) Labeling paths for the nodes in $Z_A \setminus \{Z'_A \cup Z''_A\}$.



(b) Routing the demand of node $(10, 10)$ in Z_A. In this example, $d = 8$.

Figure 18: Case 1: node $v \in Z_A \setminus \{Z'_A \cup Z''_A\}$.

- P_2': $(-(k+1), y+(k+1))$ — $(-(k+1), 0)$ — $(-k, 0)$;
- P_3': $(0, y)$ — $(0,0)$ — $(x, 0)$;
- P_4': $(x, y+k)$ — (x, y) — $(x+k, y)$.

We first give a $(C' \cup \epsilon)$-labeling to P_3' in such a way the edge $(0,1)(0,0)$ gets the label ϵ (defined above for the labeling of P_4). So we avoid interference with P_4 as the label ϵ is not used for P_4. Doing so, it might be that the first edge on the path P_3' labeled ϵ interferes with some edge of P_3 in that case we change its label to $\epsilon_2 = j-1$ where j is the label of the first edge of P_3 $(0,y),(1,y)$. Similarly, if the last edge on the path P_3' labeled ϵ interferes with some edge of P_3, we relabel it $\epsilon_3 = j'+1$ where j' is the label of the last edge of P_3 $(x,1),(x,0)$. In the example $\epsilon = 2$ and so we have to change the label ϵ of edge $(0,10),(0,9)$ to the label $j-1=1$, as the label of the first edge of P_3 $(0,10),(1,10)$ is $j=2$. We have also to change the label ϵ of edge $(8,0),(9,0)$ to 4 as the label of the last edge of P_3 $(10,1),(10,0)$ is $j'=3$.

Let ℓ' be the label of edge $(0,k)(0,k+1)$; it is also by construction the label of edge $(k,0),(k+1,0)$. To respect the labeling of the cross in $g=(0,0)$, we give also the label ℓ' to the edges $(-(k+1),0)(-k,0)$ and $(0,-k)(0,-(k+1))$. Now we give a $(C' \cup \epsilon)$-labeling to P_2' in such a way the edge $(-(k+1),0)(-k,0)$ gets the label ℓ'. It might be that the first edge of P_1' labeled ϵ interferes with some edge of P_1 in that case we change its label to $\epsilon_4 = 2k+1$ as the label of the first edge of P_1 is 1. In the example $\ell' = 14$ and we have to change the label of edge $(-5,14),(-5,13)$ to 9.

We also give a $(C' \cup \epsilon)$-labeling to P_1' in such a way the edge $(0,-k)(0,-(k+1))$ gets the label ℓ'. If the last edge on the path P_2' labeled ϵ interferes with some edge of P_2 we relabel it $\epsilon_5 = j''+1$ where j'' is the label of the last edge of P_2 $(x+(k+1),-k),(x+(k+1),-(k+1))$. In the example we have to change the label of edge $(12,-5),(13,-5)$ to $j''+1=5$ as the label of the last edge of P_3 $(15,-4),(15,-5)$ is $j''=4$.

Finally we use a C'-labeling for P_4' which is possible as P_4' is of length $2k$. Doing so all the subpaths are interference free.

- Case 2 (see scheme in Figure 19(a) and example in Figure 19(b)): node v in the first part of Z_A' (the case where v is in the second part of can be solved similarly by exchanging the role of the coordinates). The proof is similar to the case 1. We consider the following 4 paths between g and v:

  - $(0,0)$ — $(x+(k+1),0)$ — $(x+(k+1), y-(k+1))$ — $(x, y-(k+1))$ — (x,y);
  - $(0,0)$ — $(0,k+1)$ — $(-(k+1),k+1)$ — $(-(k+1),y)$ — (x,y);
  - $(0,0)$ — $(0,-k)$ — $(x+2(k+1),-(k+1))$ — $(x+2(k+1),y)$ — (x,y)
  - $(0,0)$ — $(-2(k+1),0)$ — $(-2(k+1), y+(k+1))$ — $(x, y+(k+1))$ — (x,y).

  We similarly use an ordered set of $4k+1$ labels partitioned into two disjoint ordered sequences C and C' of $2k+1$ and $2k$ labels, respectively.

  Then we use a $(C' \cup \epsilon)$-labeling for the reflected paths indicated in green in the figures (doing some changes of the ϵ when there is an interference); in the example $\ell = 3$, $\epsilon = 8$ and $\ell' = 10$.

- Case 3 (see scheme in Figure 20(a) and example in Figure 20(b)): node $v \in Z''_A$ ($x = 0$). We first C-label the following 4 subpaths (in blue in Figures 19(a) and 19(b)).

  - P_1: $(-2(k+1), y+(k+1))$ —$(x, y+(k+1))$ —$(x, y+k)$;
  - P_2: $(x+k, y)$ —$(x+2(k+1), y)$ —$(x+2(k+1), -(k+1))$;
  - P_3: $(-(k+1), y)$—(x, y)—$(x, y-(k+1))$—$(x+(k+1), y-(k+1))$—$(x+(k+1), 0)$;
  - P_4: $(-k, 0)$ —$(0, 0)$ —$(0, -k)$.

  The construction presented below is simpler than that for the case 1 but we need a little more space in the grid). We consider the following 4 paths between g and $v = 0, y$):

  - $(0, 0)$ —$(0, y)$;
  - $(0, 0)$ —$(d, 0)$ —(d, y) —$(0, y)$;
  - $(0, 0)$ —$(-d, 0)$ —$(-d, y)$ —$(0, y)$;
  - $(0, 0)$ —$(0, -(k+1))$ —$(2d, -(k+1))$ —$(2d, y+(k+1))$ —$(0, y+(k+1))$ —$(0, y)$.

  We use again two disjoint ordered sequences C and C' of $2k+1$ and $2k$ labels, respectively. We first C-label the following 2 subpaths (drawn in blue in Figures 20(a) and 20(b)).

  - P_1: $(0, y+k)$ —$(0, y+(k+1)$ —$(2d, y+(k+1))$ —$(2d, -(k+1))$;
  - P_2: $(-d, y)$ —(x, y) —$(0, 0)$ —$(d, 0)$.

  We first C-label P_1, starting with label 1 to edge $(0, y+k)(0, y+(k+1)$. Then, we C-label P_2 in such a way edge $-((k+1), y)(-k, y)$ receives label 1 to respect the labeling of the cross in $v = (0, y)$ (edge $(0, y-k)(0, y-(k+1)$ gets also label 1). Then we $(C' \cup \epsilon)$-label the reflected path $P'_1 = (-d, y)$ —$(-d, 0)$ —$(0, 0)$ —$(0, -(k+1))$ —$(2d, -(k+1))$ in such a way the edge $(-(k+1), 0)(-k, 0)$ receives the same label ϵ as the edge $(0, k)(0, k+1)$ (to respect the labeling of the cross in $(0, 0)$. We also $(C' \cup 1)$-label the reflected path P'_2 in such a way the edge $(k, y)(k+1, y)$ receives label 1.

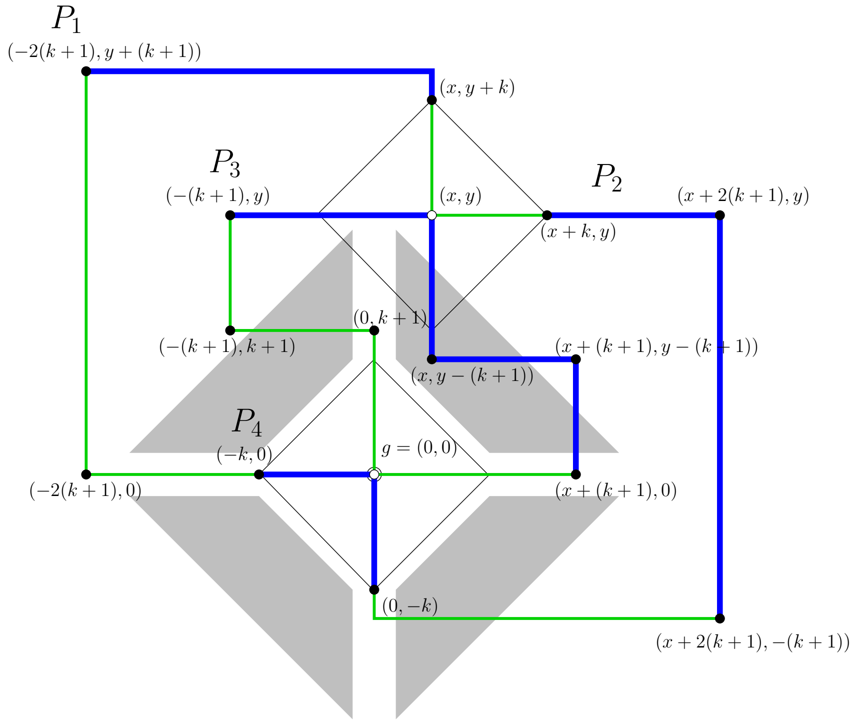
■

6.2 Gateway in the corner: routing the demand of a single node

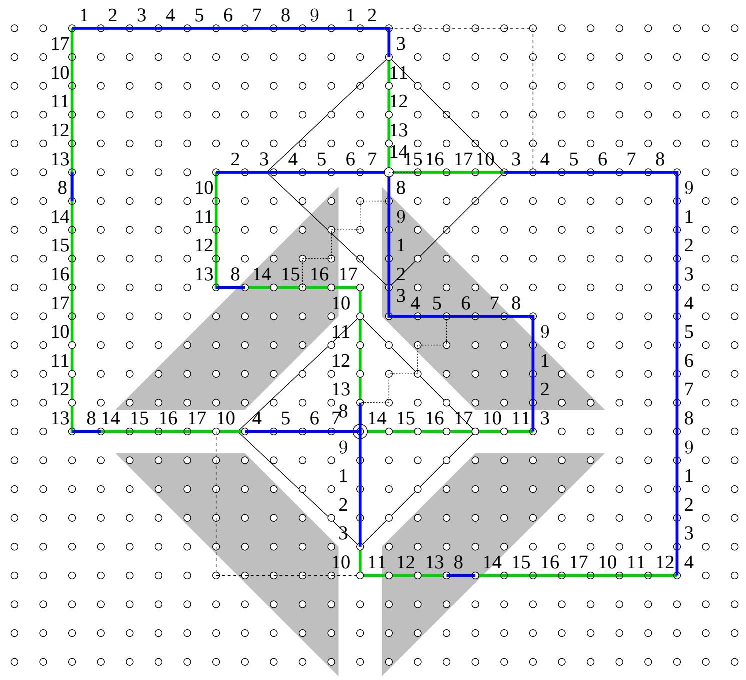
Now, we consider the case where the gateway is in the corner. Recall that we suppose that the gateway g is placed at vertex $(0, 0)$ and we consider a $p \times q$ grid with vertices (x, y) where $0 \leqslant x \leqslant p$ and $0 \leqslant y \leqslant q$. In view of the example in Figure 5 and Figure 6 (vertex $(3, 2)$ for $d = 4$), determining W_{min} when the demand is concentrated in a node can be very difficult for specific vertices. However, we show later that for most of the vertices we can determine it.

In Section 5.2, we define the *individual lower bound* $\text{lb}(v)$ of a node v, and proves that $W_{min} \geqslant \text{lb}(v)b(v)$. Now we give a routing with $W_{min} = \text{lb}(v)b(v)$ for the following subregions of the grid where we recall that $v^* = (k, k)$ (see Figure 21):

- $Z_{\boldsymbol{SP}}$ the vertices for which the lower bound is the distance of a shortest path ($\text{lb}(v) = d(v, g)$). This zone includes the vertices of K_0. They are according the proofs in Section 5.2 the vertices (x, y) such that $x \leqslant k$, $y \leqslant k$ and $d(v, g) \leqslant \frac{3k}{2}$. Indeed, in that case $d + 1 - d(v, v^*) = d(v, g)$ as $d(g, v^*) = d + 1$.

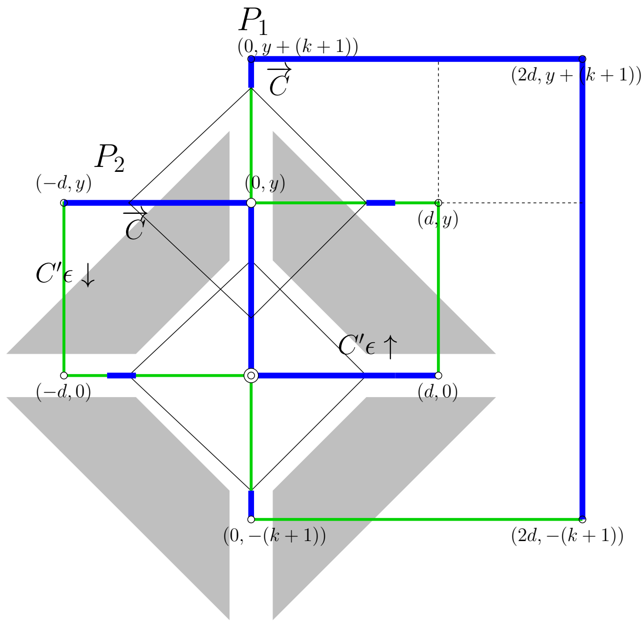


(a) Labeling paths for the nodes in Z'_A.

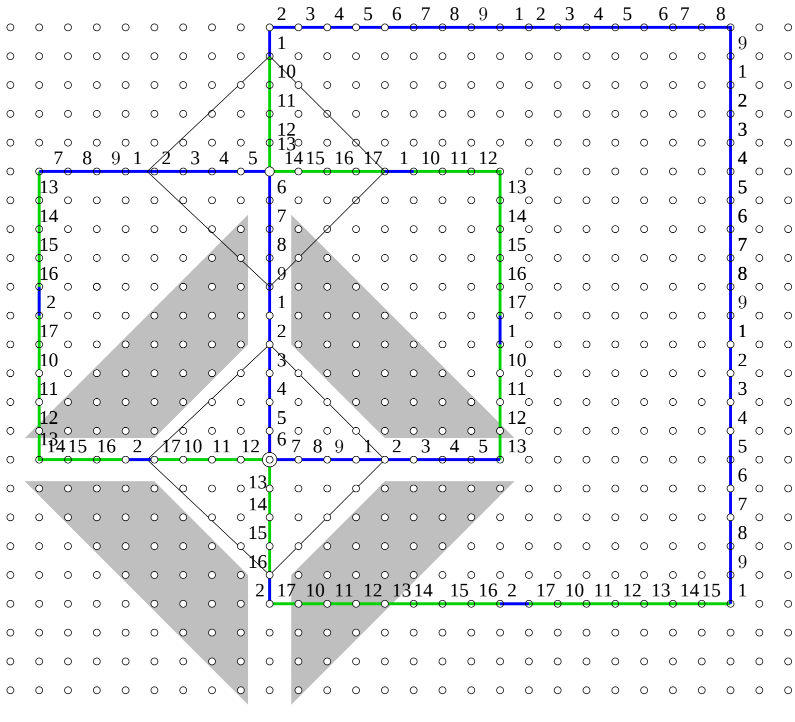


(b) Routing the demand of node $(1, 9)$ in Z'_A. In this example, $d = 8$.

Figure 19: Case 2: node v in the first part of Z'_A.



(a) Labeling paths for the nodes in Z''_A.



(b) Routing the demand of node $(0, 6)$ in Z''_A. In this example, $d = 8$.

Figure 20: Case 3: node $v \in Z''_A$ $(x = 0)$.

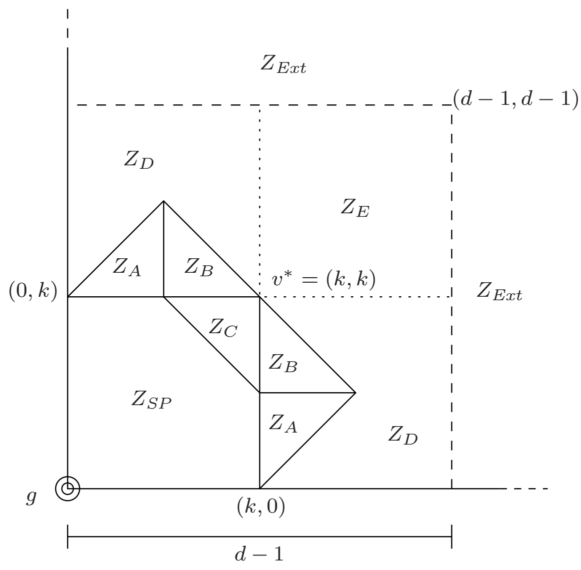


Figure 21: Zones of the grid.

- $Z_{\boldsymbol{Ext}}$ the nodes not in the square $\{0, d-1\} \times \{0, d-1\}$, plus those on the axis at distance $\geqslant \frac{d+1}{2}$. For them the lower bound is $\mathrm{lb}(v) = \frac{d+1}{2}$.
- Z_E the nodes contained in the square delimited by the nodes $\lceil\frac{d+1}{2}\rceil, \lceil\frac{d+1}{2}\rceil$ and $(d-1, d-1)$ (in the case d odd the first node is v^*). For them the lower bound is $\mathrm{lb}(v) = \frac{d+1}{2}$.
- Z_C (in the case d odd) the vertices (x, y) such that $x \leqslant k$, $y \leqslant k$ and $d(v, g) > \frac{3k}{2}$. For them the lower bound is $\mathrm{lb}(v) = d(v, v^*) + \mathrm{lb}(v^*)$

By Proposition 5 the lower bound is attained for the vertices of Z_{SP} for the single routing of v. In the next Theorem, we show that for the vertices of Z_{Ext} the lower bound $\frac{d+1}{2}$ is attained.

Theorem 10 *Let G be a 2-dimensional grid $p \times q$ with $p \geqslant 3d$, $q \geqslant 2d$ and gateway g in the corner. If $v \in Z_{\mathrm{Ext}}$, there exists a cycle containing both v and g that can be interference free $(d+1)$-labeled, therefore $W_{min} = \frac{d+1}{2}b(v)$ for the single routing of the demand $b(v)$.*

Proof: If $v \notin \{0, d-1\} \times \{0, d-1\}$, the lower bound is $\frac{d+1}{2}b(v)$ by Theorems 4 and 5. We construct a generic cycle for all the vertices not in $\{0, d-1\} \times \{0, d-1\}$, satisfying the hypothesis of Corollary 8 (length multiple of $d+1$ and width $\geqslant d$). So $W \leqslant \frac{d+1}{2}b(v)$ for all these nodes.

We distinguish three cases according to the node position: in all the cases the conditions insure that the cycle is inside the grid and that its length is a multiple of $d+1$ and its width $\geqslant d$.

1. $y \leqslant d$ (and so $x \geqslant d$). Let q' be the smallest integer $\geqslant d$ such that $q' + x \equiv 0$ $(mod\ (d+1))$; so $q' \leqslant 2d \leqslant q$. We use the cycle (note that this cycle contains the vertical axis):
$$(0,0) — (x,0) — (x,y) — (x,q') — (0,q') — (0,0).$$
2. $y \geqslant d$ and $x \leqslant 2d$. Let p' be the smallest integer $\geqslant 2d$ such that $p' + y \equiv 0$ $(mod\ (d+1))$; so $p' \leqslant 3d \leqslant p$. We use the cycle (note that this cycle contains the horizontal axis):
$$(0,0) — (p',0) — (p',y)) — (x,y) — (0,y) — (0,0).$$

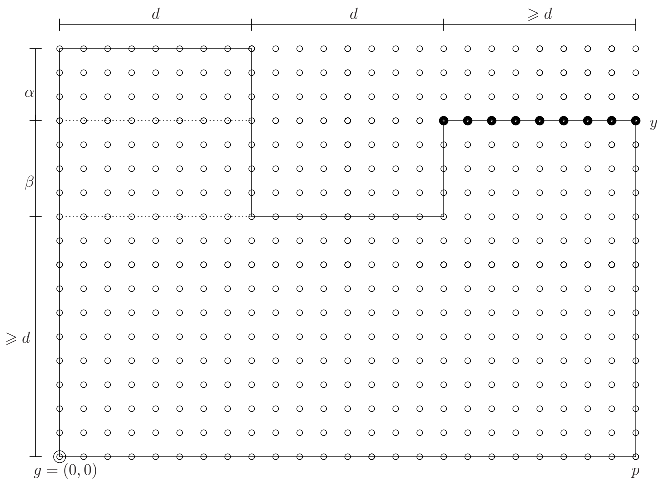


Figure 22: Routing method for a node v with a cycle of width d ($d = 8$, $x \geqslant 2d = 16$, $y = 14$, $p = 24 \geqslant 3d$, $q = 17 \geqslant 2d$).

3. $y \geqslant d$ and $x \geqslant 2d$. We use the cycle C:
   $(0,0)$ — $(p,0)$ — (p,y) — $(2d,y)$ — $(2d,y-\beta)$ — $(d,y-\beta)$ — $(d,y+\alpha)$ — $(0,y+\alpha)$ — $(0,0)$.
   In this case, the cycle has a detour as presented in Figure 22. To obtain a cycle of width d, the variable β has to respect the constraint $\beta \leqslant y - d$ and, to respect the grid size, $\alpha \leqslant q - y$. The length of the cycle is $|C| = 2(p + y + \alpha + \beta)$. Let γ be the smallest integer such that $p + y + \gamma \equiv 0 \ (mod\ (d+1))$, therefore $0 \leqslant \gamma \leqslant d$. The cycle has a length multiple of $d + 1$ if we choose $\alpha + \beta = \gamma$, with α and β satisfying the constraints. Such a possible choice is:

   - if $q - y \geqslant \gamma$, then choose $\alpha = \gamma$ and $\beta = 0$.
   - if $q - y \leqslant \gamma$, then choose $\alpha = q - y$ and $\beta = \gamma - \alpha = \gamma - q + y$. Here, as $\gamma \leqslant d$ and $q \geqslant 2d$, then $\gamma - q \leqslant -d$ and so $\beta \leqslant y - d$ as wanted.

   For the example in Figure 22, we have: $d = 8$, $x \geqslant 2d = 16$, $y = 14$, $p = 24 \geqslant 3d$, $q = 17 \geqslant 2d$). So, $\gamma = 7$ and the constraints are $\beta \leqslant y - d = 6$ and $\alpha \leqslant q - y = 3$. We are in the second subcase and so, we choose $\alpha = 3$ and $\beta = 4$.

■

In the case $d = 2$, we get the following Proposition using Proposition 2 for the lower bound and Theorem 10 for the upper bound (as in this case Z_{Ext} consists of the vertices $v \notin \{(0,1),(1,0),(1,1)\}$).

Proposition 17 *For the grid with g in the corner in the distance-2 model ($d = 2$)*

$$W_{min} = b(0,1) + b(1,0) + 2b(1,1) + \frac{3}{2} \sum_{v \notin \{(0,1),(1,0),(1,1)\}} b(v).$$

The next Theorem shows that the lower bound $\frac{d+1}{2}$ is also attained for the vertices $v = (x,y)$ with $\lfloor \frac{d}{2} \rfloor + 1 \leqslant x \leqslant d - 1$ and $\lfloor \frac{d}{2} \rfloor + 1 \leqslant y \leqslant d - 1$ (vertices in Z_E, see Figure 21).

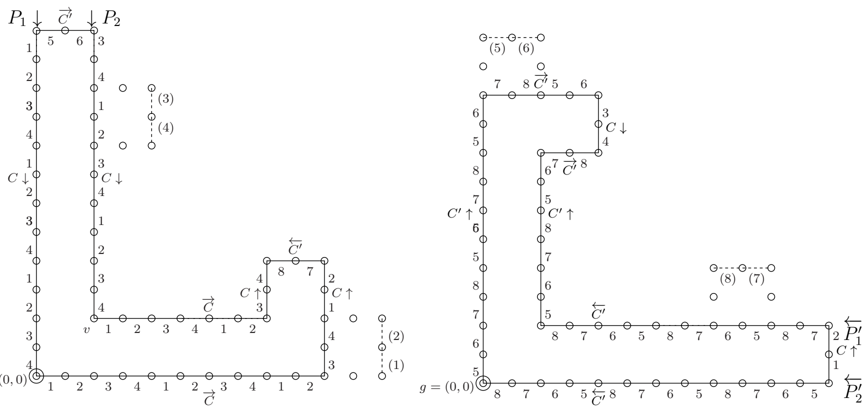
Theorem 11 *Let G be a 2-dimensional grid $p \times q$ with $p, q \geqslant 6k$ if $d = 2k - 1$ or $p, q \geqslant 7k$ if $d = 2k$ and gateway g in the corner. If $v = (x,y) \in Z_E$, then $W_{min} = \frac{d+1}{2} b(v)$ for the single routing of the demand $b(v)$.*

Proof: We exhibit 2 interference free $2(d+1)$-labeled cycles containing v and g (that is 4 paths from v to g each one carrying a flow of $\frac{b(v)}{4}$). For that we use an ordered set of $2(d+1)$ labels partitioned into two disjoint ordered sets $C = (1, 2, \ldots, d+1)$ and $C' = (d+2, \ldots, 2(d+1))$. Therefore, $W \geqslant \frac{d+1}{2} b(v)$ by Corollary 4 and $W_{min} = \frac{d+1}{2} b(v)$ by Theorems 4 and 5.

We first give the solution for d odd ($d = 2k-1$). The first cycle consists of

- $P_1 = (0, 6k) — (0, 0) — (5k, 0) — (5k, 2k)$,
- $P_2 = (k, 6k) — (k, 2k-2) — P(x, y) — (2k-2, k) — (4k, k) — (4k, 2k)$ where $P(x, y)$ denotes a shortest path from $(k, 2k-2)$ to $(2k-2, k)$ going through vertex (x, y),
- plus the two segments $(0, 6k) — (k, 6k)$ and $(5k, 2k) — (4k, 2k)$.

We C-label P_1, with label 1 given to the edge $(0, 6k)(0, 6k-1)$ (or $(0, 0)(1, 0)$). We also C-label P_2, with label $k+1$ for the edge $(k, 6k)(k, 6k-1)$. By Proposition 15 there is no interference. Then we we use a C'-labeling for the segment $(0, 6k) — (k, 6k)$ with label $2k+1$ for the edge $(0, 6k)(1, 6k)$ and a C'-labeling for the segment $(5k, 2k) — (4k, 2k)$ with label $3k+1$ for the edge $(5k, 2k)(5k-1, 2k)$.



(a) First cycle. (b) Second cycle.

Figure 23: Example of routing with 2 cycles for $v \in Z_E$ (d =3, k=2).

The second cycle is obtained by symmetry through the first diagonal as follows.

- $P'_1 = (6k, 0) — (0, 0) — (0, 5k) — (2k, 5k)$ C'-labeled with label $2k+1$ for the edge $(6k, 0)(6k-1, 0)$,
- $P'_2 = (6k, k) — (2k-2, k) — P(x, y) — (k, 2k-2) — (k, 4k) — (2k, 4k)$ C'-labeled with label $3k+1$ for the edge $(6k, k)(6k-1, k)$,
- plus the two segments $(6k, 0) — (6k, k)$ C-labeled with label 1 for the edge $(6k, 0)(6k, 1)$ and the segment $(2k, 5k) — (2k, 4k)$ C-labeled with label $k+1$ for the edge $(2k, 5k)(2k, 5k-1)$.

The construction has been done in order to avoid interference. The reader can see an example for $d = 3$ ($k = 2$), $C = (1, 2, 3, 4)$ and $C' = (5, 6, 7, 8)$ in Figure 23. To show that cycles are interference free we have furthermore indicated in the first figure in parenthesis the C-labels used in the second cycle and similarly in the second figure the C'-labels used in the first cycle.

The solution for d even is obtained similarly. The first cycle consists of

- $P_1 = (0,7k)$ —$(0,0)$ —$(6k-1,0)$ —$(6k-1,2k+2)$ C-labeled with label 1 for the edge $(0,7k)(0,7k-1)$),
- $P_2 = (k+1,7k)$—$(k+1,2k-1)$—$P(x,y)$—$(2k-1,k+1)$—$(4k-1,k+1)$—$(4k-1,2k+2)$ C-labeled with label $k+2$ for the edge $(k+1,7k)(k+1,7k-1)$ where $P(x,y)$ denotes a shortest path from $(k+1,2k-1)$ to $(2k-1,k+1)$ going through vertex (x,y),
- plus the two segments $(0,7k)$ —$(k+1,7k)$ C'-labeled with label $4k-4$ if $k \geqslant 3$ (9 for $k=2$) for the edge $(0,7k)(1,7k)$ and the segment $(6k-1,2k+2)$ —$(4k-1,2k+2)$, C'-labeled with label $3k+4$ for the edge $(6k-1,2k+2)(6k-2,2k+2)$.

The second cycle is obtained by symmetry to the first diagonal. ■

Theorem 12 *Let G be a 2-dimensional grid $p \times q$ with $p,q \geqslant 6k$ and let d be odd. If $v=(x,y) \in Z_C$, then $W_{min} = (\frac{d+1}{2} + 2k - x - y)b(v)$ for the single routing of the demand $b(v)$.*

Proof: In that case v^* belongs to Z_E. We can first route the demand of $v \in Z_C$ to v^* via a shortest path in $d(v,v^*) = 2k-(x+y)$ rounds and then apply Theorem 11 for v^* to route the demand in $\frac{d+1}{2}b(v)$ rounds. ■

7 Upper bounds for grids with simultaneous source routings

7.1 Gateway in the middle: routing the demand of a combination of nodes

In this article, we present a routing strategy which enables to route simultaneously the same flow (less than or equal to the smallest demand) from 2 (for d odd) or 4 vertices (for d even).

7.1.1 Case d odd

This case is solved for the demand of a single node with a cycle (see Proposition 16). That is, we can attain the lower bound for the problem, but not necessarily with *integer round weights.* Here, we present a solution which deals with this requirement. In this solution, we pair the vertices and find for each pair of vertices two interference free $(d+1)$-labeled paths connecting each vertex to g. The vertices can be in the same region, in two adjacent regions, or in two opposite regions of the grid.

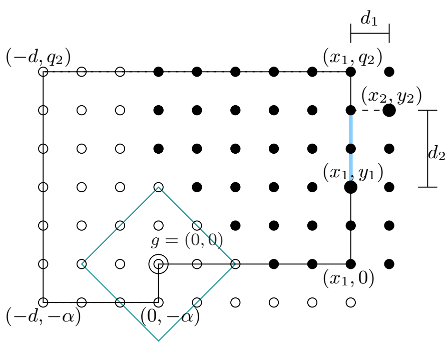
Theorem 13 *Let d be odd ($d = 2k-1$) and let G be a 2-dimensional grid with $\min(p_1,p_2,q_1,q_2) \geqslant d+1$, and with the gateway g in the middle. For any pair of vertices v_1 and v_2 not in K_0, there exist 2 paths that can be interference free $(d+1)$-labeled, one path from the node v_1 to g and the other path from v_2 to g.*

Proof: To prove that, we use the splitting in 4 regions (see Definition 12) and distinguish 3 cases:

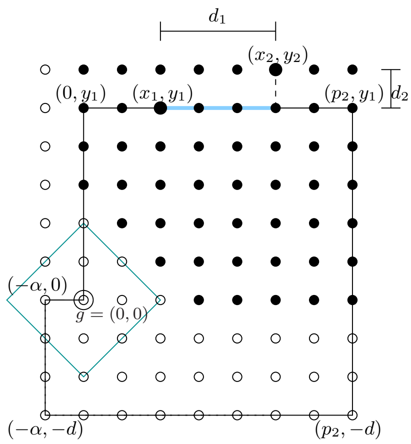
1. The two nodes are in opposite regions R_A and R_C (or R_B and R_D). Let $v_1 = (x_1,y_1)$ with $x_1 \geqslant 0$, $y_1 > 0$; and $v_2 = (x_2,y_2)$ with $x_2 \leqslant 0$, $y_2 < 0$. In that, we use Corollary 7 as $d(v_1,v_2) = d(v_1,g) + d(g,v_2)$. The shortest paths are for example (x_1,y_1)—$(0,y_1)$—$(0,0)$ and $(0,0)$ —$(0,y_2)$ —(x_2,y_2).
2. The two nodes $v_1 = (x_1,y_1)$ and $v_2 = (x_2,y_2)$ are in the same region (we suppose it is R_A).

- Subcase 1: In this case $x_1 = x_2$ (or $y_1 = y_2$), so we can use the cycle presented in the proof of Proposition 16 which contains the column x_1 and so both v_1 and v_2. Then it suffices to take the disjoint paths on the cycle joining v_1 and v_2 to the gateway which are $(d+1)$-labeled. The case $y_1 = y_2$ can be done by interchanging the role of the coordinates.
- Subcase 2: Otherwise, we can suppose w.l.o.g that $x_1 < x_2$. We distinguish two cases in accordance with the values of $d_1 = x_2 - x_1$ and $d_2 = |y_2 - y_1|$.
- Subcase 2.1: $d_1 \leqslant d_2$: we use the following strategy; we consider the cycle obtained in the proof of Proposition 16 for the node v_1. We use one part of the cycle as the path from g to v_1. We use the other part as a path from g to (x_1, y_2) and add the horizontal subpath from (x_1, y_2) to (x_2, y_2) of length d_1. More precisely the two paths are
  - if $y_1 < y_2$: $(0,0)$—$(x_1, 0)$—v_1 and v_2—(x_1, y_2)—(x_1, q_2)—$(-d, q_2)$—$(-d, -\alpha)$—$(0, -\alpha)$ —$(0, 0)$
  - if $y_1 > y_2$: $(0,0)$—$(x_1, 0)$—(x_1, y_2)—v_2 and v_1—(x_1, q_2)—$(-d, q_2)$—$(-d, -\alpha)$—$(0, -\alpha)$ —$(0, 0)$

  Doing so, we have not used the subpath of the cycle between (x_1, y_1) and (x_1, y_2) of length d_2. We can use the labels of this subpath to label without interference the subpath from (x_1, y_2) to (x_2, y_2) (see Figure 24(a) for an example).
- Subcase 2.2: $d_1 \geqslant d_2$: in that case, we use a cycle containing the horizontal line y_1 if $y_1 < y_2$ (see Figure 24(b)), or the line y_2 if $y_1 > y_2$. We delete the part between v_1 and (x_2, y_1) (resp. (x_1, y_2) and v_2) and use the labels of its edges to label the added path from (x_2, y_1) to v_2 (resp. (x_1, y_2) to v_1). More precisely the two paths are
  - if $y_1 < y_2$: $(0,0)$—$(0, y_1)$—v_1 and v_2—(x_2, y_1)—(p_2, y_1)—$(p_2, -d)$—$(-\alpha, -d)$—$(-\alpha, 0)$ —$(0, 0)$
  - if $y_1 > y_2$: $(0,0)$—$(0, y_2)$—(x_1, y_2)—v_1 and v_2—(p_2, y_2)—$(p_2, -d)$—$(-\alpha, -d)$—$(-\alpha, 0)$ —$(0, 0)$



(a) Paths for the case $d_1 \leqslant d_2$.



(b) Paths for the case $d_1 \geqslant d_2$.

Figure 24: Subcase 2.1 ($x_1 < x_2$ and $y_1 < y_2$). The labels on the gray line are removed from the cycle and used to label the dotted line.

3. The two nodes are in adjacent regions like R_A and R_B (or R_B and R_C,or R_C and R_D, or R_D and R_A). Let $v_1 = (x_1, y_1) \in R_A$; and $v_2 = (x_2, y_2) \in R_B$.

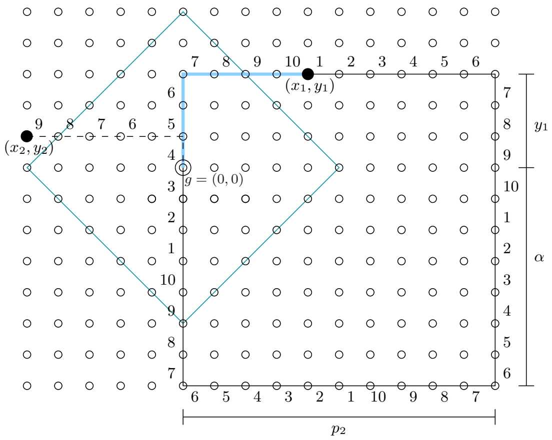


Figure 25: Routing strategy with $d(v_1, g) \geqslant d(v_2, g)$ $(d = 9)$.

We suppose that $d(v_1, g) \geqslant d(v_2, g)$ as in Figure 25; otherwise, we invert the routings of the two nodes. The node v_2 (with the smallest distance to the gateway) uses the shortest path $P_2 = (x_2, y_2) — (0, y_2) — (0, 0)$.

If $x_1 \geqslant d$ or $y_1 = 0$, the node v_1 also uses a shortest path, $P_1 = (x_1, y_1) — (x_1, 0) — (0, 0)$ and we can interference free label $P_1 \cup P_2$ with $d + 1$ labels. Otherwise, we use a longer path $P_1 = (0, 0) — (0, -\alpha) — (d + 1, -\alpha) — (d + 1, y_1) — (x_1, y_1)$.

P_1 is a part of the cycle $C = (0, 0) — (0, -\alpha) — (d + 1, -\alpha) — (d + 1, y_1) — (x_1, y_1) — (0, y_1) — (0, 0)$. The length is $|C| = 2(d + 1) + 2y_1 + 2\alpha$. Let us choose α as the smallest integer such that $y_1 + \alpha \equiv 0$ $(mod\ d + 1)$. As $y_1 \geqslant 1$, then $0 \leqslant \alpha \leqslant d$. With this choice, the cycle is in the grid (as we have by hypothesis $q_1 \geqslant d + 1$) and has length multiple of $d + 1$. To insure its width is $\geqslant d$, we need to verify that $y_1 + \alpha \geqslant d$; that is clearly satisfied if $y_1 \geqslant d$, but also if $y_1 \leqslant d$ as in this case $\alpha = d + 1 - y_1$. Therefore the cycle C can be $d + 1$-labeled and we can use the labels of the unused subpath $(0, 0) — (0, y_1) — (x_1, y_1)$ to label the path from g to v_2 of length smaller by hypothesis $(d(v_1, g) \geqslant d(v_2, g))$.

■

Theorem 14 *Let d be odd $(d = 2k - 1)$ and let G be a 2-dimensional grid with gateway g in the middle, and with $\min(p_1, p_2, q_1, q_2) \geqslant d + 1$. If $\sum_{v \notin V_{K_0}} b(v)$ is even, then $W_{min} = S_0 = \sum_{v \in V_{K_0}} d(v, g)b(v) + \frac{d+1}{2} \sum_{v \notin V_{K_0}} b(v)$ is solution for IRWP.*

Proof: For vertices in K_0, we use Corollary 5. For $v \notin V_{K_0}$, if $b(v)$ is even, we send the demand in $\frac{d+1}{2} b(v)$ rounds by Proposition 16. If $b(v)$ is odd, we send a flow of $b(v) - 1$ using $\frac{d+1}{2}(b(v) - 1)$ rounds by Proposition 16. An even number of vertices remains with a demand of 1, as the total demand $\sum_{v \notin V_{K_0}} b(v)$ is even. Then, by Theorem 13, these nodes can be grouped two by two and the demand of each pair of nodes can be sent using $d + 1$ rounds. ■

7.1.2 Case d even

When d is even, we have a lower bound of $k + \frac{1}{4} b(v) = 4\frac{k+1}{4} b(v)$, $\forall v \notin V_{K_0}$ by Proposition 3. To reach a global bound, we exhibit for any set of 4 vertices (one in each region) 4 pairwise interference free $(4k + 1)$ labeled paths and then apply Corollary 4. It is depicted in Figure 26 for $d = 10$.

Proposition 18 *Let d be even, $d = 2k$. Let G be a 2-dimensional grid with the gateway in the middle and with* $\min(p_1, p_2, q_1, q_2) \geqslant k+1$. *For any set of 4 nodes $v_A \in R_A$, $v_B \in R_B$, $v_C \in R_C$, $v_D \in R_D$, there exist 4 pairwise interference free $(4k+1)$-labeled from v_A,v_B,v_C,v_D to the gateway.*

Proof: We use a set of $4k+1$ labels $a_1, \ldots, a_k$,$b_1, \ldots, b_k$,$c_1, \ldots, c_k$,$d_1, \ldots, d_k$ and ϵ. We give a path for each vertex in R_A and the labels associated to the edges used by these paths. The paths for the other regions R_B, R_C, R_D are obtained by rotation. The labels associated to the edges in the other regions are obtained by applying the bijection ω where $\omega(a_i) = b_i$, $\omega(b_i) = c_i$, $\omega(c_i) = d_i$, $\omega(d_i) = a_i$ and $\omega(\epsilon) = \epsilon$. So $\omega^2(a_i) = c_i$ and $\omega^3(a_i) = d_i$. If an edge e of R_A is labeled l, the rotated edge $\rho(e)$ in R_B is labeled $\omega(l)$, the edge $\rho^2(e)$ in R_C $\omega^2(l)$ and the edge $\rho^3(e)$ in R_D $\omega^3(l)$. We use three different types of paths from $v_A \in R_A$ to g according to the position of vertex v_A:

- Paths of type 1: for $v_A = (0, y)$ (on the vertical axis $x = 0$ and $y \geqslant 1$), we use the vertical path $(0, y) — (0, 0)$;
- Paths of type 2: for $v_A = (x, y)$ with $x > 0$ and $y \geqslant k+1$, we use a shortest path going first horizontally and then vertically $(x, y) — (0, y) — (0, 0)$.
- Paths of type 3: for $v_A = (x, y)$ with $x > 0$ and $1 \leqslant y \leqslant k$, we do not use a shortest path. The path goes first vertically till $(x, k+1)$ then horizontally till $(0, k+1)$ and vertically till g. So the path is $(x, y) — (x, k+1) — (0, k+1) — (0, 0)$.

We associate one of the $4k+1$ labels to each edge in these paths. In fact, we associate a chain of labels to each set of consecutive edges in a path. Let the chain A^+ represent the sequence of labels $a_1, a_2, \ldots, a_k$ (in this order) and A^-, the inverted chain $a_k, a_{k+1}, \ldots, a_1$. We define similarly B^+,B^-,C^+,C^-,D^+,D^-. To label the whole path, we use concatenations of these chains (for example, A^+, ϵ, C^- represents the sequence $a_1, a_2, \ldots, a_k, \epsilon, c_k, c_{k-1}, \ldots, c_1$) and to go further, we use subsequent repetitions of concatenations. The main idea consists in doing a labeling which gives the labels A^+ (resp. B^+,C^+,D^+) to the edges of the cross in g on the axis in R_A (resp. R_B, R_C, R_D) starting from g. The 4 end edges of the cross get all the same label ϵ. The other labels are given to avoid interferences. Notice that the paths defined above use only 3 types of edges:

- Edges of type 1: edges on the vertical axis $(0, y)(0, y+1)$. We label them starting from vertex $(0, 0)$ using repetitions of the sequence A^+, ϵ, C^-. Doing so, for $1 \leqslant i \leqslant k$ and $\lambda \geqslant 0$, edges $(0, i-1+\lambda(d+1))(0, i+\lambda(d+1))$ are labeled a_i; edges $(0, -i+(\lambda+1)(d+1))(0, -i+1+(\lambda+1)(d+1))$ are labeled c_i. Edges $(0, k+\lambda(d+1))(0, k+1+\lambda(d+1))$ are labeled ϵ for $\lambda \geqslant 0$.
- Edges of type 2: edges on the horizontal lines $(x, y)(x+1, y)$ with $x \geqslant 0$ and $y \geqslant k+1$. We label them starting from the vertex $(0, y)$ and using the repetition of the sequence B^-, D^+, ϵ. So for $1 \leqslant i \leqslant k$ and $\lambda \geqslant 0$, edges $(k-i+\lambda(d+1), y)(k+1-i+\lambda(d+1), y)$ are labeled b_i and edges $(k-1+i+\lambda(d+1), y)(k+i+\lambda(d+1), y)$ are labeled d_i. Edges $(2k+\lambda(d+1), y)(2k+1+\lambda(d+1), y)$ are labeled ϵ for $\lambda \geqslant 0$.
- Edges of type 3: the remaining vertical edges $(x, y)(x, y+1)$ with $x > 0$ and $1 \leqslant y \leqslant k$ ((x, y) not in V_{K_0}). We label them using the chain C^-. If $x \geqslant k$, we label the edges starting from vertex $(x, 1)$ using the chain C^-. So for $x \geqslant k$, $1 \leqslant i \leqslant k$, edges $(x, k+1-i)(x, k+2-i)$ are labeled c_i. If $1 \leqslant x \leqslant k-1$, we start from vertex $(x, k+1-x)$ using part of C^-. So only for $k+1-x \leqslant i \leqslant k$, edges $(x, 2k+1-x-i)(x, 2k+2-x-i)$ are labeled c_i.

For the vertex $\rho(v_A) \in R_B$ (resp. $\rho^2(v_A) \in R_C$ and $\rho^3(v_A) \in R_D$), we use the rotated path $\rho(P_{v_A})$ (resp. $\rho^2(P_{v_A}), \rho^3(P_{v_A})$) where P_{v_A} corresponds to the path from v_A to g described above. Notice that each path use only edges of its region, for example a path P_{v_A} use only edges of R_A. Recall also that we label a path using bijection ω (if an edge e of R_A is labeled l, $\rho^i(e)$ is labeled $\omega^i(l)$).

Now consider any 4 vertices v_A,v_B,v_C,v_D one in each region. We have to show that the 4 paths associated are pairwise interference free. So we have to show that two edges with the same label are at distance d. If the two edges are on the same path, that follows from the fact we are using repetitions of chains of length $d+1$.

For example, we use repetitions of the chain A^+, ϵ, C^- in a path of type 1. It is also the case for a path of type 2 where we use first repetitions of chain A^+, ϵ, C^- on the vertical subpath then B^+, D^-, ϵ on the horizontal line. Note that the edges labeled ϵ on the horizontal line are at distance $\geqslant d$ from the vertical axis and so do not interfere with the edges also labeled ϵ in the vertical subpath. For a path of type 3, we use A^+, ϵ (reaching $(0, k+1)$) followed by repetitions of B^-, D^+, ϵ and then part of the chain C^+. So here again, there is no interference.

Now consider two edges on two different paths. If both are labeled ϵ, they are at distance $\geqslant d$, the nearest ones being these of the cross centered in g which are at distance exactly d (and form an independent set in the conflict graph). For the other labels, due to the symmetry (rotation and bijection ω), it suffices to prove that two edges labeled a_i are at distance $\geqslant d$.

The edges labeled a_i in each of the 4 paths are of the following form:

- Form A: those in region R_A, $(0, i-1+\lambda_1(d+1))(0, i+\lambda_1(d+1))$, $\lambda_1 \geqslant 0$;
- Form B: those in region R_B obtained by rotation of edges (x, y) labeled d_i in region R_A as $a_i = \omega(d_i)$. They are of the form $(-y, k-1+i+\lambda_2(d+1))(-y, k+i+\lambda_2(d+1))$ with $y \geqslant k+1$, $\lambda_2 \geqslant 0$;
- Form C or C': those in region R_C obtained by symmetry ρ^2 of an edge (x, y) labeled c_i in R_A. They are of the form C (if the original edge is on the vertical axis) $(0, i-(\lambda_3+1)(d+1))(0, i-1-(\lambda_3+1)(d+1))$, $\lambda_3 \geqslant 0$;
  Form C' $(-x, i-k-1)(-x, i-k-2)$ if $x \geqslant k$ and $(-x, x+i-2k-1)(-x, x+i-2k-2)$ if $1 \leqslant x \leqslant k-1$ $(k+1-x \leqslant i \leqslant k)$;
- Form D: those in region R_D obtained by rotation ρ^3 of an edge (x, y) labeled b_i in R_A. They are of the form $(y, i-k-\lambda_4(d+1))(y, i-1-k-\lambda_4(d+1))$ with $y \geqslant k+1$, $\lambda_4 \geqslant 0$.

Now we compute the distance between two edges in two different paths, so in two different regions and so of two different forms, and verify that in all the cases the distance is $\geqslant d$.

- A and B: if $\lambda_2 \geqslant \lambda_1$, then distance $= y+k-1+(\lambda_2-\lambda_1)(d+1) \geqslant y+k-1 \geqslant 2k = d$, as $y \geqslant k+1$;
  if $\lambda_2 < \lambda_1$, then distance $= y+k+(\lambda_1-\lambda_2-1)(d+1) \geqslant y+k-1 \geqslant 2k+1 = d$, as $y \geqslant k+1$;
- A and C: distance $= d+(\lambda_1+\lambda_3)(d+1) \geqslant d$;
- A and C': if $x \geqslant k$, distance $= x+k+\lambda_1(d+1) \geqslant 2k$ as $(x \geqslant k)$;
  if $1 \leqslant x \leqslant k-1$, distance $= 2k+\lambda_1(d+1) \geqslant 2k$;
- A and D: distance $= y+k-1+(\lambda_1+\lambda_4)(d+1) \geqslant y+k-1 \geqslant 2k$, as $(y \geqslant k+1)$;

- B and C: distance $= y + 3k + (\lambda_2 + \lambda_3)(d + 1) \geqslant 4k + 1 \geqslant 2k$, as $(y \geqslant k + 1)$;
- B and C': if $x \geqslant k$, distance $= |y - x| + 2k + \lambda_2(d + 1) \geqslant 2k$;
  if $(1 \leqslant x \leqslant k - 1)$, distance $= y - 2x + 3k + \lambda_2(d + 1) \geqslant 2k + 3$;
- B and D: distance $\geqslant d$ (the labels are separated by K_0);
- C and D: if $\lambda_3 \geqslant \lambda_4$, then distance $= y - k + d + 1 + (\lambda_3 - \lambda_4)(d+1) \geqslant 2k+2$, as $y \geqslant k+1$;
  if $\lambda_3 < \lambda_4$, then distance $= y - k - 1 + (\lambda_4 - \lambda_3 - 1)(d + 1) \geqslant 2k$, as $y \geqslant k + 1$;
- C' and D: if $x \geqslant k$, distance $\geqslant y + x \geqslant 2k + 1$;
  if $1 \leqslant x \leqslant k - 1$, then if $\lambda_4 = 0$ distance $= y + k \geqslant 2k + 1$ and $\lambda_4 > 0$ distance $= y + 2x + k - 1 + (\lambda_4 - 1)(d + 1) \geqslant 2k + 2$.

■

Theorem 15 *Let d be even and G be a 2-dimensional grid with the gateway in the middle. If the regions are balanced ($\sum_{v\in R_A} b(v) = \sum_{v\in R_B} b(v) = \sum_{v\in R_C} b(v) = \sum_{v\in R_D} b(v)$) then $W_{min} = \sum_{v\in V_{K_0}} d(v, g)b(v) + \frac{d}{2}\sum_{v\notin V_{K_0}} b(v) + \frac{1}{4}\sum_{v\notin V_{K_0}} b(v)$. Furthermore, if the $b(v)$ are integers, W_{min} is also a solution to IRWP.*

Proof: A lower bound has been presented in Proposition 3. For the vertices in V_{K_0}, we apply Corollary 5. For the other vertices, we group them 4 by 4 and apply Theorem 9, that works because the regions are balanced. If the $b(v)$ are integers, we send an integer flow on each path. ■

Theorem 16 *Let d be even ($d = 2k$) and G be a ($2p + 1 \times 2p + 1$)-dimensional grid with the gateway in the middle, $p \geqslant k + 1$. Let the demand be uniform ($b(v) = b, \forall v$), $W_{min} = b[(k + \frac{1}{4})(N - 1) - \frac{k(k+1)(4k-1)}{6}]$.*

Proof: It follows from Proposition 10 and the fact that the 4 regions have, in this case, the same number of vertices. ■

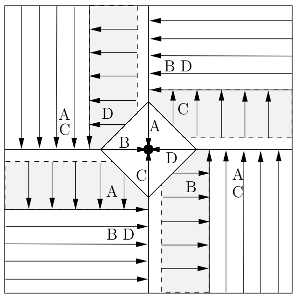
Note that the lower bound of $S_0 + \frac{1}{4}\sum_{v\notin K_0} b(v)$ is attained in many other cases. Indeed for the vertices in Z_α ($\alpha = A, B, C, D$) (see Figure 21) we can do a single routing and so it suffices to be able to group the vertex in the subregions $X_\alpha \cup Y_\alpha$ by groups of 4. That is possible if $max_\alpha \sum_{v\in X_\alpha\cup Y_\alpha} b(v) \leqslant min_\alpha \sum_{v\in Z_\alpha} b(v)$ ($\alpha = A, B, C, D$). Indeed suppose the maximum is attained for $\alpha = A$, we group 4 by 4 the vertices of $X_A \cup Y_A$ with the same number of vertices in R_B,R_C,R_D using all vertices of $X_B \cup Y_B$, $X_C \cup Y_C$, $X_D \cup Y_D$. For the remaining vertices (which are all in Z_α) we use Theorem 9. In particular, we have equality in Proposition 10 for any ($p \times q$)-grid (with p and q large enough) with uniform demand.

7.2 Gateway in the corner: routing the demand of a combination of nodes

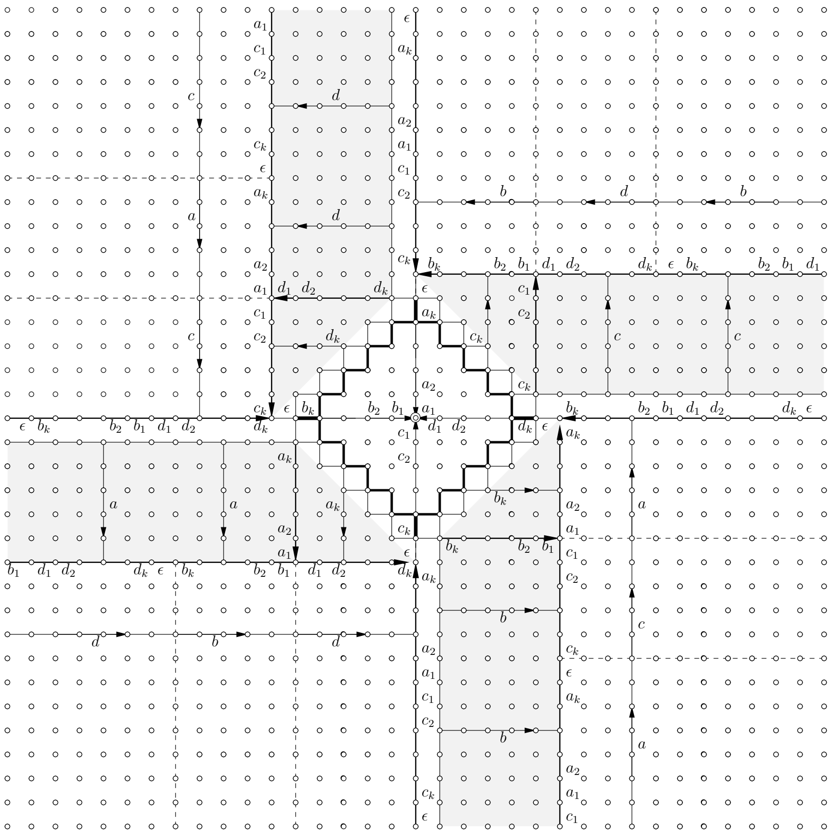
Note that for some nodes the lower bound lb(v) cannot be attained via a single routing because these nodes are too near and also in some cases the lower bound for the demand in a single node is strictly greater than lb(v) like in the example at the end of Subsection 3.2. However, we can route the demand of such nodes together (sharing rounds) with the demand of some other nodes. That can be done as soon as the demands are somewhat balanced in particular it is the case when the demand is uniform where we can obtain the following theorem.

Theorem 17 *Let G be a 2-dimensional grid $p \times q$ with $p, q \geqslant 4d$, and gateway g in the corner. The lower bounds given in Propositions 11 and 12 are attained.*

We do not give the proof which is very tedious and uses many subcases. The idea consists in pairing the vertices according the subregions to which they belong. We have to define more regions like Z_A, Z'_A, Z_B, Z'_B, Z_D, Z'_D (see Figure 21). To illustrate the idea, we show how to pair the vertices of Z_A and Z'_A where Z_A consists of the vertices (x, y) such that $x > k$, $y \leqslant \frac{k}{2}$ and $x \leqslant y + \frac{d}{2}$ and Z'_A is obtained by symmetry through the first diagonal. More precisely we route together $v = (x, y)$ via the path (x, y) — $(x, 0)$ — $(0, 0)$ and $v' = (y, d + 1 + y - x)$ via the path $(y, d+1+y-x)$ — $(0, d+1+y-x)$ — $(0, 0)$. Note that $\mathrm{lb}(v) = d+1-d(v, v^*) = d+1+y-x$ and $\mathrm{lb}(v') = d + 1 + x' - y' = x$; therefore $\mathrm{lb}(v) + \mathrm{lb}(v') = d + 1 + y$. We use the $d + 1 + y$ labels as follows. We label the path $(x, 0)$ — $(0, 0)$ — $(0, d + 1 + y - x)$ with a sequence of d+1 labels and then use the remaining y labels in increasing order for the subpath $(x, 0)$ — (x, y) and in decreasing order for the subpath $(0, d + 1 + y - x)$ — $(y, d + 1 + y - x)$. One can check that the edges with the same label are at distance $\geqslant d$. In the example of Figure 27, $d = 11$, $v = (8, 3)$ and $v' = (3, 7)$.



(a) The regions start and finish labels.



(b) Labels re-usability with $d = 10$.

Figure 26: Routing 4 nodes (one in each region) with 4 paths.

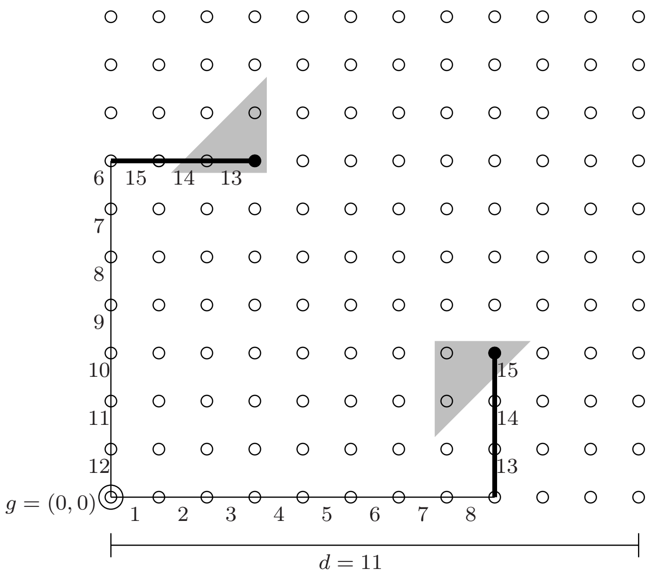


Figure 27: Example for Z_A and Z'_A with d odd.

8 Conclusion

We studied the problem of finding the minimum number of rounds needed to gather information in a radio mesh access network where interference constraints are present. We presented tools to obtain lower and upper bounds on general graphs giving several tight results. For the asymmetrical interference, a 4-approximation is given in [24]. We show that, considering symmetrical interference, we can obtain a 2-approximation for d odd and a $\frac{2}{d}+2 \leqslant 3$-approximation for d even. We then apply our tools to give precise values for the grids; in particular we give closed formulae when the demand is uniform.

Table 2 summarizes the known results and our main contributions presented according the following categories:

- **Problem**: *RWP* or *IRWP* (that assumes integer round weights);
- **Demand**: *Any*, *Even* (when $\sum_{v \notin V_{K_0}} b(v)$ is even), *Uniform* (i.e. every node has the same demand), *Balanced* (when the "partitions" contain the same amount of demands); or from a specific zone Z.
- **Interference**: *Asymmetrical interference model* (see Section 1.2) or d (any, even or odd) for the *distance-d interference model*.
- **graph**: *General*, *Grid-M* (a grid with the gateway in the middle) or *Grid-C* (a grid with the gateway in the corner).

Table 2: Results.

Problem	**Demand**	**Interference**	**Graph**	**Complexity**	**Reference**
RWP	Any	Asymmetrical	General	4-approx.	[24]
IRWP	Any	d odd	General	2-approx.	Theorem 1
IRWP	Any	d even	General	$2/d+2$-approx.	Theorem 1
RWP	Any	d odd	Grid-M	Polynomial	Theorem 6
IRWP	Even	d odd	Grid-M	Polynomial	Theorem 14
RWP	Z_A Any	d even	Grid-M	Polynomial	Theorem 9
IRWP	Balanced	d even	Grid-M	Polynomial	Theorem 15
RWP	Uniform	d any	Grid-M	Closed Form.	Theorems 7, 16
RWP	Z_{Ext}, Z_E Any	d any	Grid-C	Polynomial	Theorem 10
RWP	Z_C Any	d odd	Grid-C	Polynomial	Theorem 10
RWP	Uniform	d any	Grid-C	Closed Form.	Theorem 17

An attractive challenge is to consider multiple gateways. Our methods can be applied if they are far enough and evenly distributed; but if the gateways are near the problem becomes very difficult.

References

[1] I. Akyildiz, W. Su, Y. Sankarasubramaniam, and E. Cayirci. Wireless sensor networks: a survey. *Computer networks, Elsevier*, 38(4):393–422, 2002.

[2] H. Balakrishnan, C.L. Barrett, V.A. Kumar, M.V. Marathe, and S. Thite. The distance-2 matching problem and its relationship to the MAC-layer capacity of ad hoc wireless networks, *IEEE Journal on Selected Areas in Communications*, 22(6):1069–1079,2004.

[3] J-C. Bermond, R. Correa, and M-L. Yu. Optimal gathering protocols on paths under interference constraints. *Discrete Mathematics*, 309:5574–5587, 2009

[4] J.-C. Bermond, J. Galtier, R. Klasing, N. Morales, and S. Pérennes. Hardness and approximation of gathering in static radio networks. *Parallel Processing Letters*, 16(2):165–183, 2006.

[5] J-C. Bermond, L. Gargano, S.Pérennes, A. Rescigno, and U. Vaccaro Optimal Time Data Gathering in Wireless Networks with Multidirectional Antennas. *Theoretical Computer Science*, 509:122–139, 2013.

[6] J-C. Bermond, L. Gargano, and A. Rescigno. Gathering with minimum delay in sensor tree networks. *Journal of interconnection Networks*, 11:1–33, 2010.

[7] J-C. Bermond, R. Klasing, N. Morales, S. Pérennes, and P. Reyes. Gathering radio messages in the path. *Discrete Mathematics, Algorithms and Applications*, 5(1):1–28, 2013.

[8] J-C. Bermond, B. Li, N. Nisse, H. Rivano, and M-L. Yu. Data gathering and personalized broadcasting in radio grids with interference. *Theoretical Computer Science*, 562:453–475, 2015

[9] J-C. Bermond, D. Mazauric, V. Misra, and P. Nain. A distributed scheduling algorithm for wireless networks with constant overhead and arbitrary binary interference. *SIGMETRICS 2010, Columbia University, New york, US*,ACM, 2010.

[10] J-C. Bermond and J. Peters. Optimal gathering in radio grids with interference. *Theoretical Computer Science*, 457:10–26, 2012.

[11] J-C. Bermond, and M-L. Yu. Optimal gathering algorithms in multi-hop radio tree networks with interferences *Ad Hoc and Sensor Wireless Networks*, 9:109–128, 2010

[12] P. Bertin, J-F. Bresse, and B.L. Sage. Accès haut débit en zone rurale: une solution adhoc. *France Telecom R&D*, 22:16–18, 2005.

[13] V. Bonifaci, R. Klasing, P. Korteweg, L. Stougie, and A. Marchetti-Spaccamela. Data Gathering in Wireless Networks chapter in A. Koster and X. Munoz, editors, *Graphs and Algorithms in Communication Networks*, pages 357–377. Springer Monograph, 2009.

[14] V. Bonifaci, P. Korteweg, A. Marchetti-Spaccamela, and L. Stougie. An approximation algorithm for the wireless gathering problem. *Operations Research Letters*, 36(5):605–608, 2008.

[15] L.X. Bui, S. Sanghavi, and R. Srikant. Distributed link scheduling with constant overhead. *IEEE/ACM Transactions on Networking*, 17(5):1467–1480, 2009.

[16] M. Campêlo, C. Huiban, and R. Sampaio. On the complexity of the flow coloring problem. *Discrete Applied Mathematics*, 197:75–92, 2015.

[17] L. Chen, S.H. Low, and J.C. Doyle. Joint congestion control and media access control design for ad hoc wireless networks. *INFOCOM 2005*. Proceedings IEEE 3:2212–2222, 2005.

[18] A. Eryilmaz, A.E. Ozdaglar, D. Shah, and E. Modiano. Distributed cross-layer algorithms for the optimal control of multihop wireless networks. *IEEE/ACM Transactions on Networking*, 18(2):638–651, 2010.

[19] C. Florens, M. Franceschetti, and R.J. McEliece. Lower bounds on data collection time in sensory networks. *IEEE Journal on Selected Areas in Communications*, 22(6):1110–1120, 2004.

[20] L. Gargano and A. Rescigno. Optimal fast data gathering in sensor networks. In *Discrete Applied Mathematics*, 157:1858–1872, 2009.

[21] C. Gomes and G. Huiban. Multiobjective Analysis in Wireless Mesh Networks. IEEE International Symposium on Modeling, Analysis, and Simulation of Computer and Telecommunication Systems. *IEEEXplore Digital Library*, 103–108, 2007.

[22] A. Gupta, X. Lin, and R. Srikant Low complexity distributed link scheduling algorithms for wireless networks. In *INFOCOM 2007*, pages 1631–1639, 2007.

[23] B. Hajek and G. Sasaki. Link scheduling in polynomial time. *IEEE Transactions on Information Theory*, 34(5):910–917, 2006.

[24] R. Klasing, N. Morales, and S. Pérennes. On the complexity of bandwidth allocation in radio networks. *Theoretical Computer Science*, 406(3):225–239, 2008.

[25] R. Klasing, Z. Lotker, A. Navarra, and S. Pérennes. From balls and bins to points and vertices . *Algorithmic Operations Research* , 4(2):133–143, 2009.

[26] P. Korteweg Online algorithms for gathering networks. *PhD Thesis* Eindhoven technical university, 2008.

[27] V. S. Anil Kumar, M.V. Marathe, S. Parthasarathy, and A Srinivasan. End-to-end packet-scheduling in wireless ad-hoc networks. In *SODA '04: proceedings of the fifteenth annual ACM-SIAM symposium on Discrete algorithms*, pages 1021–1030, Philadelphia, PA, USA, 2004.

[28] K. Menger. Zur allgemeinen Kurventheorie. *Fund. Math.*, 10:96–115, 1927.

[29] R. Mazumdar, G. Sharma, and N. Shroff. Maximum weighted matching with interference constraints. *FAWN, Pisa, Italy*, pages 70–74, 2006.

[30] Y. Revah and M. Segal. Improved bounds for data-gathering time in sensor networks. *Computer Communications*, 31(17):4026–4034, 2008.

[31] L.J. Stockmeyer and V.V. Vazirani. NP-completeness of some generalizations of the maximum matching problem. *Information Processing Letters*, Elsevier, 15(1):14–19, 1982.

[32] P.J. Wan. Multiflows in multihop wireless networks. In *MobiHoc '09: proceedings of the tenth ACM international symposium on Mobile ad hoc networking and computing*, pages 85–94, New York, NY, USA, 2009.

[33] Y. Wang, W. Wang, X.-Y. Li, and W.-Z. Song. Interference-aware joint routing and tdma link scheduling for static wireless networks. *IEEE Transactions on Parallel and Distributed Systems*, 19(12):1709–1726, 2008.