# Enumeration of Self-Dual Cyclic Codes of some Specific Lengths over Finite Fields 

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#### Abstract

Self-dual cyclic codes form an important class of linear codes. It has been shown that there exists a self-dual cyclic code of length $n$ over a finite field if and only if $n$ and the field characteristic are even. The enumeration of such codes has been given under both the Euclidean and Hermitian products. However, in each case, the formula for selfdual cyclic codes of length $n$ over a finite field contains a characteristic function which is not easily computed. In this paper, we focus on more efficient ways to enumerate self-dual cyclic codes of lengths $2^{\nu} p^{r}$ and $2^{\nu} p^{r} q^{s}$, where $\nu, r$, and $s$ are positive integers. Some number theoretical tools are established. Based on these results, alternative formulas and efficient algorithms to determine the number of self-dual cyclic codes of such lengths are provided.


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## 1 Introduction

Self-dual cyclic codes constitute an important class of linear codes due to their rich algebraic structures, their fascinating links to other objects, and their wide

[^0]applications. Such codes have been extensively studied for both theoretical and practical reasons (see [2], 4], [5], [6], 2], [11] and references therein). Some major results on Euclidean self-dual cyclic codes have been discussed in [4] and [6]. The complete characterization and enumeration of such codes have been established in [4]. These results have been generalized to the Hermitian case in [5]. Some related results on the enumeration of self-dual cyclic codes over finite chain rings can be found in [3].

For a prime power $q$, denote by $\mathbb{F}_{q}$ the finite field of $q$ elements. A linear code $C$ of length $n$ over $\mathbb{F}_{q}$ is defined to be a subspace of the $\mathbb{F}_{q}$-vector space $\mathbb{F}_{q}^{n}$. The Euclidean dual of a linear code $C$ is defined to be

$$
C^{\perp_{\mathrm{E}}}=\left\{\boldsymbol{v} \in \mathbb{F}_{q}^{n} \mid\langle\boldsymbol{v}, \boldsymbol{c}\rangle_{\mathrm{E}}=0 \text { for all } \boldsymbol{c} \in C\right\}
$$

where $\langle\boldsymbol{v}, \boldsymbol{u}\rangle_{\mathrm{E}}:=\sum_{i=1}^{n} v_{i} u_{i}$ is the Euclidean inner product between $\boldsymbol{v}=$ $\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ and $\boldsymbol{u}=\left(u_{1}, u_{2}, \ldots, u_{n}\right)$ in $\mathbb{F}_{q}^{n}$. In the case where $q$ is a square, the Hermitian dual of a linear code $C$ can be defined as well and it is defined to be

$$
C^{\perp_{\mathrm{H}}}=\left\{\boldsymbol{v} \in \mathbb{F}_{q}^{n} \mid\langle\boldsymbol{v}, \boldsymbol{c}\rangle_{\mathrm{H}}=0 \text { for all } \boldsymbol{c} \in C\right\}
$$

where $\langle\boldsymbol{v}, \boldsymbol{u}\rangle_{\mathrm{H}}:=\sum_{i=1}^{n} v_{i} u_{i}^{\sqrt{q}}$ is the Hermitian inner product between $\boldsymbol{v}$ and $\boldsymbol{u}$ in $\mathbb{F}_{q}^{n}$. A code $C$ is said to be Euclidean self-dual (resp. Hermitian self-dual) if $C=C^{\perp_{\mathrm{E}}}$ (resp., $C=C^{\perp_{\mathrm{H}}}$ ).

A linear code $C$ is said to be cyclic if it is invariant under the right cyclic shift. It is well known that every cyclic code of length $n$ over $\mathbb{F}_{q}$ can be view as an (isomorphic) ideal in the principal ideal ring $\mathbb{F}_{q}[x] /\left\langle x^{n}-1\right\rangle$ uniquely generated by a monic divisor $g(x)$ of $x^{n}-1$. Such the polynomial is called the generator polynomial of $C$. For a polynomial $f(x)=\sum_{i=0}^{k} f_{i} x^{i}$ of degree $k$ in $\mathbb{F}_{q}[x]$ with $f_{0} \neq 0$, the reciprocal polynomial of $f(x)$ is defined to be $f^{*}(x):=f_{0}^{-1} x^{k} \sum_{i=0}^{k} f_{i}(1 / x)^{i}$. If $q$ is a square, the conjugate reciprocal polynomial of $f(x)$ is defined to be $f^{\dagger}(x):=f_{0}^{-\sqrt{q}} x^{k} \sum_{i=0}^{k} f_{i}^{\sqrt{q}}(1 / x)^{i}$. In [4] and [5], it has been shown that a cyclic code of length $n$ with the generator polynomial $g(x)$ is Euclidean self-dual (resp., Hermitian self-dual) if and only if $g(x)=h^{*}(x)$ (resp., $g(x)=h^{\dagger}(x)$ ), where $h(x)=\frac{x^{n}-1}{g(x)}$. Based on this results, the following characterizations for the existence of self-dual cyclic codes were obtained in [4] and [5].

Theorem 1.1 ( [4]). There exists a Euclidean self-dual cyclic code of length $n$ over $\mathbb{F}_{q}$ if and only if $n$ and $q$ are even.

Theorem 1.2 ( 5 ). If $q$ is a square, then there exists an Hermitian self-dual cyclic code of length $n$ over $\mathbb{F}_{q}$ if and only if $n$ and $q$ are even.

From the above characterizations, it suffices to study Euclidean (resp., Hermitian) self-dual cyclic codes of even length $n=n^{\prime} 2^{\nu}$ over $\mathbb{F}_{2^{l}}$ (resp., $\mathbb{F}_{2^{2 l}}$ ), where $n^{\prime}$ is odd and $\nu$ is a positive integer.

The following functions are keys for determining the number of self-dual cyclic codes. Let $\mathbb{O}$ denote the set of odd positive integers. For each positive integer $l$, let $\chi_{l}: \mathbb{O} \rightarrow\{0,1\}$ and $\lambda_{l}: \mathbb{O} \rightarrow\{0,1\}$ be functions defined by

$$
\chi_{l}(j)= \begin{cases}0 & \text { if there exists an integer } s \geq 1 \text { such that } j \mid\left(2^{l s}+1\right)  \tag{1}\\ 1 & \text { otherwise }\end{cases}
$$

and
$\lambda_{l}(j)= \begin{cases}0 & \text { if there exists an odd integer } s \geq 1 \text { such that } j \mid\left(2^{l s}+1\right), \\ 1 & \text { otherwise. }\end{cases}$
For coprime positive integers $i$ and $j$, let $\operatorname{ord}_{j}(i)$ denote the multiplicative order of $i$ modulo $j$.

The formulas for the number of Euclidean self-dual cyclic codes of length $n$ over $\mathbb{F}_{2^{l}}$ and the number of Hermitian self-dual cyclic codes of length $n$ over $\mathbb{F}_{2^{2 l}}$ were given in [4] and [5] as follows.

Theorem 1.3 ( [4, Theorem 3]). Let l be a positive integer and let $n=n^{\prime} 2^{\nu}$ be a positive integer such that $n^{\prime} \geq 1$ is odd and $\nu \geq 1$. Then the number of Euclidean self-dual cyclic codes of length $n$ over $\mathbb{F}_{2^{l}}$ is

$$
\begin{equation*}
\left(2^{\nu}+1\right)^{\frac{1}{2} \sum_{d \mid n^{\prime}} \chi_{l}(d) \frac{\phi(d)}{\operatorname{ord}_{d}\left(2^{l}\right)}} . \tag{3}
\end{equation*}
$$

Theorem 1.4 ( [5, Corollary 3.7]). Let $l$ be a positive integer and let $n=n^{\prime} 2^{\nu}$ be a positive integer such that $n^{\prime} \geq 1$ is odd and $\nu \geq 1$. Then the number of Hermitian self-dual cyclic codes of length $n$ over $\mathbb{F}_{2^{2 l}}$ is

$$
\begin{equation*}
\left(2^{\nu}+1\right)^{\frac{1}{2} \sum_{d \mid n^{\prime}} \lambda_{l}(d) \frac{\phi(d)}{\operatorname{orrd}_{d}\left(2^{2 l}\right)}} . \tag{4}
\end{equation*}
$$

From the above theorems, the difficulty is to compute

$$
\begin{equation*}
t\left(n^{\prime}, l\right):=\frac{1}{2} \sum_{d \mid n^{\prime}} \chi_{l}(d) \frac{\phi(d)}{\operatorname{ord}_{d}\left(2^{l}\right)} \tag{5}
\end{equation*}
$$

and

$$
\begin{equation*}
\tau\left(n^{\prime}, l\right):=\frac{1}{2} \sum_{d \mid n^{\prime}} \lambda_{l}(d) \frac{\phi(d)}{\operatorname{ord}_{d}\left(2^{2 l}\right)} \tag{6}
\end{equation*}
$$

We can see that the formulas contain the functions $\operatorname{ord}_{d}, \chi_{l}$ and $\lambda_{l}$ which are possible but not easy to determined. In this paper, we focus on Euclidean and Hermitian self-dual cyclic codes of some specific lengths and aim to reduce the complexity in computing $t\left(n^{\prime}, l\right)$ and $\tau\left(n^{\prime}, l\right)$ in (5) and (6), respectively. Precisely, we focus on the enumeration of self-dual cyclic codes of length $2^{\nu} p^{r}$ and $2^{\nu} p^{r} q^{s}$ with respect to the Euclidean and Hermitian inner products, where $p$ and $q$ are distinct odd primes and $\nu, r$, and $s$ are positive integers.

After this introduction, some number theoretical tools and an efficient algorithm to determine the number of Euclidean self-dual cyclic codes of length $2^{\nu} p^{r}$ and $2^{\nu} p^{r} q^{s}$ over $\mathbb{F}_{2^{l}}$ are given in Section 2. The analogous results for the Hermitian case are given in Section 3.

## 2 Euclidean Self-Dual Cyclic Codes

In this section, number theoretical tools and efficient algorithms for determining the formula for Euclidean self-dual cyclic codes of length $2^{\nu} p^{r}$ and $2^{\nu} p^{r} q^{s}$ over $\mathbb{F}_{2^{l}}$ in Theorem 1.3 are given. From Theorem 1.3 , it is sufficient to focus on the value of $t\left(n^{\prime}, l\right)$ in (5), where $n^{\prime} \in\left\{p^{r}, p^{r} q^{s}\right\}$.

### 2.1 Number Theoretical Results

In order to give an efficient way to compute the number of Euclidean self-dual cyclic codes, we begin with the following number theoretical results.

For a prime $p$ and integers $i \geq 0$ and $j \geq 1$, we say that $p^{i}$ exactly divides $j$, denoted by $p^{i} \| j$, if $p^{i}$ divides $j$ but $p^{i+1}$ does not divide $j$.

Lemma 2.1. Let $p$ be an odd prime and let $l$ be a positive integer. Let $\gamma$ and $i$ be the integers such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$ and $2^{i} \| l$. Then one of the following statements holds.

1. $i<\gamma$ if and only if $2^{\gamma-i} \| \operatorname{ord}_{p}\left(2^{l}\right)$.
2. $i \geq \gamma$ if and only if $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd.

Proof. We first recall that

$$
\begin{equation*}
\operatorname{ord}_{p}(2)=\operatorname{gcd}\left(\operatorname{ord}_{p}(2), l\right) \operatorname{ord}_{p}\left(2^{l}\right) \tag{7}
\end{equation*}
$$

Let $j$ be a nonnegative integer such that $2^{j} \| \operatorname{ord}_{p}\left(2^{l}\right)$. By considering the highest power of two that appear in (7), we have $j=\gamma-\min (i, \gamma)$. This completes the proof.

Corollary 2.2. Let $p$ be an odd prime and let $l$ be a positive integer. Let $\gamma$ be the integer such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$. Then the following statements holds.

1. $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd if and only if $2^{\gamma} \mid l$.
2. If $\gamma \geq 1$, then $2 \| \operatorname{ord}_{p}\left(2^{l}\right)$ if and only if $2^{\gamma-1} \| l$.
3. If $\gamma \geq 1$, then $4 \mid \operatorname{ord}_{p}\left(2^{l}\right)$ if and only if $2^{\gamma-1} \nmid l$.

Proof. Let $i$ be an integer such that $2^{i} \| l$. Then the first part is immediately deduced from Lemma 2.1 since $2^{i} \| l$.

If $2^{\gamma-1} \| l$, then $i=\gamma-1<\gamma$. Again by Lemma 2.1, $2 \| \operatorname{ord}_{p}\left(2^{l}\right)$. Conversely, if $2 \| \operatorname{ord}_{p}\left(2^{l}\right)$ which implies that $\operatorname{ord}_{p}\left(2^{l}\right)$ is even, then by Lemma 2.1 we deduced that $i<\gamma$ and hence $\gamma-i=1$. Thus $2^{\gamma-1}| | l$. The second part is proved.

If $2^{\gamma-1} \nmid l$, then $\gamma-1>i$ which means $\gamma-i \geq 2$ and $\gamma>i$. Thus by Lemma 2.1, $2^{2} \mid \operatorname{ord}_{p}\left(2^{l}\right)$. Conversely, if $4 \mid \operatorname{ord}_{p} 2^{l}$, then $\gamma-i \geq 2$ and this implies that $2^{\gamma-1} \nmid l$. This completes the third part.

For an odd integer $d>1$, necessary and sufficient conditions for $\chi_{l}(d)$ to be zero were determined in [7] in terms of $\operatorname{ord}_{p}\left(2^{l}\right)$, where $p$ is a prime divisor of $d$.

Lemma 2.3 ( [7, Theorem 1]). Let $d>1$ be an odd integer and let $l$ be $a$ positive integer. Then $\chi_{l}(d)=0$ if and only if there exists $e \geq 1$ such that $2^{e} \| \operatorname{ord}_{p}\left(2^{l}\right)$ for every prime $p$ dividing $d$.

Generally, for any positive integer $l$ and any odd integer $d>1$, the value of $\chi_{l}(d)$ can be obtained by considering a parity of $\operatorname{ord}_{p}\left(2^{l}\right)$ for each prime divisor $p$ of $d$. It is easy to see that the following corollary holds.

Corollary 2.4. Let $p$ be an odd prime and let $l$ be a positive integer. Then the following statements hold.

1. $\chi_{l}(p)=1$ if and only if $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd.
2. $\chi_{l}(p)=0$ if and only if $\operatorname{ord}_{p}\left(2^{l}\right)$ is even.
3. $\chi_{l}\left(p^{i}\right)=\chi_{l}(p)$ for all positive integers $i$.

The values $\operatorname{ord}_{p^{i}}\left(2^{l}\right)$ for all $1 \leq i \leq r$ play a vital role in determining $t\left(p^{r}, l\right)$. Here, we simplify $\operatorname{ord}_{p^{i}}\left(2^{l}\right)$ in terms of $\operatorname{ord}_{p}\left(2^{l}\right)$.

Lemma 2.5. Let $p$ be an odd prime and let $l$ and $i$ be positive integers. If $\alpha$ is the largest integer such that $p \nmid \operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$, then

$$
\operatorname{ord}_{p^{i}}\left(2^{l}\right)= \begin{cases}\operatorname{ord}_{p}\left(2^{l}\right) & \text { if } i \leq \alpha, \\ p^{i-\alpha} \operatorname{ord}_{p}\left(2^{l}\right) & \text { if } \alpha<i .\end{cases}
$$

In particular, if 2 is a primitive root modulo $p^{2}$, then

$$
\operatorname{ord}_{p^{i}}\left(2^{l}\right)=\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), l\right)}
$$

for all positive integers $i$.
Proof. The first part of this lemma follows from [8, Theorem 3.6].
Next, we assume that 2 is a primitive root modulo $p^{2}$. Then 2 is a primitive root modulo $p$ and 2 is a primitive root modulo $p^{i}$ for $j \geq 2$ by the Primitive Element Theorem. In other words, $\operatorname{ord}_{p^{i}}\left(2^{l}\right)=\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), l\right)}$ for all positive integers $i$.

Corollary 2.6. Let $p$ be an odd prime and let $l$ and $i$ be positive integers. If $p^{i} \| \operatorname{ord}_{p^{r}}\left(2^{l}\right)$, then $\alpha=r-i$ is the largest integer such that $p \nmid \operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$.

Proof. By Lemma [2.5, it is easy to see that for any $1 \leq j \leq \alpha, \operatorname{ord}_{p^{i}}\left(2^{l}\right)$ is not divisible by $p$ and for any $i \geq 1, p^{i} \| \operatorname{ord}_{p^{\alpha+i}}\left(2^{l}\right)$.

By the assumption, we have $p^{i} \| \operatorname{ord}_{p^{r}}\left(2^{l}\right)$. Thus $r=\alpha+i$ and hence the largest integer such that $p \nmid \operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$ is $r-i$ as desired.

From Corollary 2.6, the integer $\alpha$ can be computed. Hence, for each $1 \leq$ $i \leq r, \operatorname{ord}_{p^{i}}\left(2^{l}\right)$ follows from Lemma 2.5,

### 2.2 Euclidean Self-Dual Cyclic Codes of Length $2^{\nu} p^{r}$

In this subsection, an alternative and simplified formula for Euclidean self-dual cyclic codes of length $2^{\nu} p^{r}$ over $\mathbb{F}_{2^{l}}$ is given based on the number theoretical tools given in Subsection 2.1. An efficient algorithm to compute the number of such self-dual codes is provided as well.

Theorem 2.7. Let $p$ be an odd prime and let land $r$ be positive integers. Let $\alpha$ be the largest integer such that $p \nmid \operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$ and let $\gamma$ be the integer such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$. Then

$$
t\left(p^{r}, l\right)= \begin{cases}\frac{\left.\operatorname{gcd}^{\operatorname{cord}}(2), l\right)}{2 \operatorname{ord}_{p}(2)}\left(p^{\alpha}-1+(p-1)(r-\alpha) p^{\alpha-1}\right) & \text { if } 2^{\gamma} \mid l  \tag{8}\\ 0 & \text { if } 2^{\gamma} \nmid l\end{cases}
$$

In particular, if 2 is a primitive root modulo $p^{2}$, then for any positive integer $r$

$$
t\left(p^{r}, l\right)= \begin{cases}\frac{1}{2}\left(\sum_{i=1}^{r} \operatorname{gcd}\left(p^{i-1}(p-1), l\right)\right) & \text { if } 2^{\gamma} \mid l  \tag{9}\\ 0 & \text { if } 2^{\gamma} \nmid l .\end{cases}
$$

Proof. Assume that $2^{\gamma} \nmid l$. Then $\operatorname{ord}_{p}\left(2^{l}\right)$ is even by Corollary 2.2. By Corollary 2.4. we have $\chi_{l}\left(p^{i}\right)=0$ for all $1 \leq i \leq r$. It follows that $t\left(p^{r}, l\right)=0$.

Next, assume that $2^{\gamma} \mid l$. By Corollary [2.2, it follows that $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd. Hence, by Corollary 2.4, $\chi_{l}\left(p^{i}\right)=1$ for all $1 \leq i \leq r$. From (5) and Lemma 2.5, it can be concluded that

$$
\begin{align*}
t\left(p^{r}, l\right) & =\frac{1}{2} \sum_{i=0}^{r} \chi_{l}\left(p^{i}\right) \frac{\phi\left(p^{i}\right)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}  \tag{10}\\
& =\frac{1}{2} \sum_{i=1}^{r} \frac{p^{i-1}(p-1)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}  \tag{11}\\
& =\frac{1}{2}\left(\sum_{i=1}^{\alpha} \frac{\left.p^{i-1}(p-1)\right)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}+\sum_{i=\alpha+1}^{r} \frac{p^{i-1}(p-1)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}\right)  \tag{12}\\
& =\frac{1}{2}\left(\sum_{i=1}^{\alpha} \frac{p^{i-1}(p-1)}{\operatorname{ord}_{p}\left(2^{l}\right)}+\sum_{i=\alpha+1}^{r} \frac{p^{i-1}(p-1)}{p^{i-\alpha} \operatorname{ord}_{p}\left(2^{l}\right)}\right)  \tag{13}\\
& =\frac{p-1}{2 \operatorname{ord}_{p}\left(2^{l}\right)}\left(\sum_{i=1}^{\alpha} p^{i-1}+\sum_{i=\alpha+1}^{r} p^{\alpha-1}\right)  \tag{14}\\
& =\frac{(p-1) \operatorname{gcd}^{2}\left(\operatorname{ord}_{p}(2), l\right)}{2 \operatorname{ord}_{p}(2)}\left(\sum_{i=1}^{\alpha} p^{i-1}+(r-\alpha) p^{\alpha-1}\right)  \tag{15}\\
& \left.=\frac{(p-1){\operatorname{gcd}\left(\operatorname{ord}_{p}(2), l\right)}_{2 \operatorname{ord}_{p}(2)}^{p^{\alpha}-1}}{p-1}+(r-\alpha) p^{\alpha-1}\right)  \tag{16}\\
& =\frac{\operatorname{gcd}\left(\operatorname{ord}_{p}(2), l\right)}{2 \operatorname{ord}_{p}(2)}\left(p^{\alpha}-1+(p-1)(r-\alpha) p^{\alpha-1}\right) . \tag{17}
\end{align*}
$$

Finally, assume that 2 is a primitive root modulo $p^{2}$. By Lemma 2.5, we
have $\operatorname{ord}_{p^{i}}\left(2^{l}\right)=\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), l\right)}$ for all integers $1 \leq i \leq r$. Hence,

$$
\begin{align*}
t\left(p^{r}, l\right) & =\frac{1}{2} \sum_{i=0}^{r} \chi_{l}\left(p^{i}\right) \frac{\phi\left(p^{i}\right)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}  \tag{18}\\
& =\frac{1}{2} \sum_{i=1}^{r} \frac{p^{i-1}(p-1)}{\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), l\right)}}  \tag{19}\\
& =\frac{1}{2} \sum_{i=1}^{r} \operatorname{gcd}\left(p^{i-1}(p-1), l\right) \tag{20}
\end{align*}
$$

as desired.
From the above theorem, we obtain the following corollary.
Corollary 2.8. If $p$ is an odd prime and $l, r$ are positive integers satisfying $l$ is not divisible by $p, \operatorname{ord}_{p}\left(2^{l}\right)$ is odd and 2 is a primitive root modulo $p^{2}$, then $t\left(p^{r}, l\right)=\frac{r}{2} \operatorname{gcd}(p-1, l)$.

Example 2.9. Let $p=11, l=4$ and $r=3$. We can see that 2 is a primitive root modulo $11^{2}$ and $\operatorname{ord}_{11}\left(2^{4}\right)=5$ is odd. Thus $t\left(11^{3}, 4\right)=\frac{3}{2} \operatorname{gcd}(10,4)=3$.

The results discussed above can be summarized in Algorithm 1 .
To compute $t\left(p^{r}, l\right)$ directly from (5), we need to compute $\chi_{l}\left(p^{i}\right), \phi\left(p^{i}\right)$, and $\operatorname{ord}_{p^{i}}\left(2^{l}\right)$ for all $0 \leq i \leq r$. It is not difficult to see that Algorithm 1 can reduce some complexity since it requires to compute only $\operatorname{ord}_{p}(2), \operatorname{ord}_{p^{2}}(2), \operatorname{ord}_{p^{r}}\left(2^{l}\right)$ and some basic expressions in (8) or (9).

### 2.3 Euclidean Self-Dual Cyclic Codes of Length $2^{\nu} p^{r} q^{s}$

In this subsection, an alternative and simplified formula for Euclidean self-dual cyclic codes of length $2^{\nu} p^{r} q^{s}$ over $\mathbb{F}_{2^{l}}$ is given as well as an efficient algorithm to compute the number of such self-dual codes.

The following lemma is a key to simplified the formula of $t\left(p^{r} q^{s}, l\right)$.
Lemma 2.10. Let $p$ and $q$ be distinct odd primes and let $l$ be a positive integer. Let $\gamma$ and $\beta$ be the integers such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$ and $2^{\beta} \| \operatorname{ord}_{q}(2)$, respectively. Then the following statements hold.

1. $\chi_{l}(p q)=1$ if and only if one of the following statements holds.
(a) $\chi_{l}(p)=1$ or $\chi_{l}(q)=1$.

For an odd prime $p$ and positive integers $l$ and $r$, do the following steps.

1. Compute $\operatorname{ord}_{p}(2)$.
2. Determine $\gamma$ such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$.
2.1 If $2^{\gamma} \nmid l$, then $t\left(p^{r}, l\right)=0$ by (8). Done.
2.2 If $2^{\gamma} \mid l$, then compute $\operatorname{ord}_{p^{2}}(2)$.
2.2.1 If $\operatorname{ord}_{p^{2}}(2)=p(p-1)$, then evaluate (9). Done.
2.2.2 If $\operatorname{ord}_{p^{2}}(2) \neq p(p-1)$, then do the following steps.
i) Compute $\operatorname{ord}_{p^{r}}\left(2^{l}\right)$.
ii) Determine the largest integer $\alpha$ such that $p \nmid$ $\operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$ (by Corollary 2.6).
iii) Evaluate (8). Done.

Figure 1: Steps in Computing $t\left(p^{r}, l\right)$
(b) $\chi_{l}(p)=0=\chi_{l}(q)$ and $\gamma \neq \beta$.
2. $\chi_{l}(p q)=1$ if and only if $\chi_{l}(p)=0=\chi_{l}(q)$ and $\gamma=\beta$.
3. $\chi_{l}\left(p^{i} q^{j}\right)=\chi(p q)$ for all positive integers $i$ and $j$.

Proof. To prove the first part, let $\gamma^{\prime}$ and $\beta^{\prime}$ be the integers such that $2^{\gamma^{\prime}} \| \operatorname{ord}_{p}\left(2^{l}\right)$ and $2^{\beta^{\prime}} \| \operatorname{ord}_{q}\left(2^{l}\right)$, respectively. Assume that $\chi_{l}(p q)=1$. By Lemma 2.3, it follows that 1$)$ either $\operatorname{ord}_{p}\left(2^{l}\right)$ or $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd, or 2$) \operatorname{ord}_{p}\left(2^{l}\right)$ and $\operatorname{ord}_{p}\left(2^{l}\right)$ are even and $\gamma^{\prime} \neq \beta^{\prime}$ by Corollary [2.2. The former implies that $\chi_{l}(p)=1$ or $\chi_{l}(q)=1$. The latter implies that $\chi_{l}(p)=0=\chi_{l}(q)$ and $\gamma^{\prime} \neq \beta^{\prime}$. Since $\operatorname{ord}_{p}\left(2^{l}\right)$ and $\operatorname{ord}_{p}\left(2^{l}\right)$ are even, Lemma2.1]implies that $\gamma=\gamma^{\prime}-i$ and $\beta=\beta^{\prime}-i$. Thus $\gamma \neq \beta$.

Conversely, assume that the statement (a) or (b) holds. If $\chi_{l}(p)=1$ or $\chi_{l}(q)=1$, then $\chi_{l}(p q)=1$ by Lemma 2.3. Assume that $\chi_{l}(p)=0=\chi_{l}(q)$ and $\gamma \neq \beta$. Since $\chi_{l}(p)=0=\chi_{l}(q)$, we have $\gamma^{\prime}>0$ and $\beta^{\prime}>0$ by Corollary 2.4. Since $\operatorname{ord}_{p}\left(2^{l}\right)$ and $\operatorname{ord}_{p}\left(2^{l}\right)$ are even, Lemma 2.1 implies that $\gamma^{\prime}=\gamma-i$ and $\beta^{\prime}=\beta-i$. Thus $\gamma^{\prime} \neq \beta^{\prime}$. Therefore, $\chi_{l}(p q)=1$ as desired.

It is not difficult to see that the second part and the first part are equivalent and the third one follows from Lemma 2.3.

A simplified formula for $t\left(p^{r} q^{s}, l\right)$ is given as follows.
Theorem 2.11. Let $p$ and $q$ be distinct odd primes and let $r, s$, and $l$ be positive integers. Then
$t\left(p^{r} q^{s}, l\right)=\chi_{l}(p) t\left(p^{r}, l\right)+\chi_{l}(q) t\left(q^{s}, l\right)+\chi_{l}(p q) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), \operatorname{ord}_{q^{j}}\left(2^{l}\right)\right)}$.
Proof. We note that $\operatorname{ord}_{p^{i} q^{j}}\left(2^{l}\right)=\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), \operatorname{ord}_{q^{j}}\left(2^{l}\right)\right)$. Using (5) and Lemma 2.10, the result follows.

The next corollary follows from Corollary [2.2, Lemma 2.10 and Theorem 2.11.

Corollary 2.12. Let $p$ and $q$ be distinct odd primes and let $r, s$, and $l$ be positive integers. Let $\gamma$ and $\beta$ be the integers such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$ and $2^{\beta} \| \operatorname{ord}_{q}(2)$, respectively. Then one of the following statements holds.

1. If $2^{\gamma} \mid l$ and $2^{\beta} \mid l$, then

$$
t\left(p^{r} q^{s}, l\right)=t\left(p^{r}, l\right)+t\left(q^{s}, l\right)+\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), \operatorname{ord}_{q^{j}}\left(2^{l}\right)\right)}
$$

2. If $2^{\gamma} \mid l$ and $2^{\beta} \nmid l$, then

$$
t\left(p^{r} q^{s}, l\right)=t\left(p^{r}, l\right)+\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), \operatorname{ord}_{q^{j}}\left(2^{l}\right)\right)} .
$$

3. If $2^{\gamma} \nmid l$ and $2^{\beta} \mid l$, then

$$
t\left(p^{r} q^{s}, l\right)=t\left(q^{s}, l\right)+\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), \operatorname{ord}_{q^{j}}\left(2^{l}\right)\right)} .
$$

4. If $2^{\gamma} \nmid l$ and $2^{\beta} \nmid l$ and $\gamma \neq \beta$, then

$$
t\left(p^{r} q^{s}, l\right)=\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), \operatorname{ord}_{q^{j}}\left(2^{l}\right)\right)} .
$$

5. If $2^{\gamma} \nmid l$ and $2^{\beta} \nmid l$ but $\gamma=\beta$, then

$$
t\left(p^{r} q^{s}, l\right)=0 .
$$

It is not difficult to see that the complexity in Corollary 2.12 is lower than a direct computation in (5).

## 3 Hermitian Self-Dual Cyclic Codes

In this section, we focus on the enumeration of Hermitian self-dual cyclic codes of lengths $2^{\nu} p^{r}$ and $2^{\nu} p^{r} q^{s}$ over $\mathbb{F}_{2^{2 l}}$, where $p$ and $q$ are distinct odd primes and $\nu, r$, and $s$ are positive integers. A simplification of the formula for $\tau\left(n^{\prime}, l\right)$ is established for all $n^{\prime} \in\left\{p^{r}, p^{r} q^{s}\right\}$.

### 3.1 Number Theoretical Results

Properties of $\lambda_{l}$ and $\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)$ used in the enumeration of Hermitian self-dual cyclic codes are discussed.

Lemma 3.1 ( [5, Theorem 4.1]). Let $j>1$ be an odd integer and let $l$ be a positive integer. Then $\lambda_{l}(j)=0$ if and only if $2 \| \operatorname{ord}_{p}\left(2^{l}\right)$ for every prime $p$ dividing $j$.

The next corollary follows immediately from Lemma 3.1.
Corollary 3.2. Let $p$ be an odd prime and let $l$ be a positive integer. Then the following statements hold.

1. $\lambda_{l}(p)=1$ if and only if $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd or $4 \mid \operatorname{ord}_{p}\left(2^{l}\right)$.
2. $\lambda_{l}(p)=0$ if and only if $2 \| \operatorname{ord}_{p}\left(2^{l}\right)$.
3. $\lambda_{l}\left(p^{i}\right)=\lambda_{l}(p)$ for all positive integers $i$.

Next, we determine $\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)$.
Lemma 3.3. Let $p$ be an odd prime and let $l$ and $i$ be positive integers. If $\alpha$ is the largest integer such that $p \nmid \operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$, then one of the following statements holds.

1. If $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd, then

$$
\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)=\operatorname{ord}_{p^{i}}\left(2^{l}\right)= \begin{cases}\operatorname{ord}_{p}\left(2^{l}\right) & \text { if } i \leq \alpha, \\ p^{i-\alpha} \operatorname{ord}_{p}\left(2^{l}\right) & \text { if } \alpha<i\end{cases}
$$

2. If $\operatorname{ord}_{p}\left(2^{l}\right)$ is even, then

$$
\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)=\frac{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}{2}= \begin{cases}\frac{\operatorname{ord}_{p}\left(2^{l}\right)}{2} & \text { if } i \leq \alpha, \\ \frac{p^{i-\alpha} \operatorname{ord}_{p}\left(2^{l}\right)}{2} & \text { if } \alpha<i .\end{cases}
$$

In particular, if 2 is a primitive root modulo $p^{2}$, then

$$
\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)=\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), 2 l\right)}
$$

for all positive integers $i$.
Proof. From Lemma 2.5, $\operatorname{ord}_{p}\left(2^{l}\right)$ and $\operatorname{ord}_{p^{i}}\left(2^{l}\right)$ have the same parity for all positive integers $i$. Since $\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)=\frac{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}{\operatorname{gcd}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), 2\right)}$ and

$$
\operatorname{gcd}\left(\operatorname{ord}_{p^{i}}\left(2^{l}\right), 2\right)= \begin{cases}1 & \text { if } \operatorname{ord}_{p}\left(2^{l}\right) \text { is odd } \\ 2 & \text { if } \operatorname{ord}_{p}\left(2^{l}\right) \text { is even }\end{cases}
$$

the results follow from Lemma 2.4.

### 3.2 Hermitian Self-Dual Cyclic Codes of Length $2^{\nu} p^{r}$

In this subsection, an explicit formula for the number of Hermitian self-dual cyclic codes of length $2^{\nu} p^{r}$ over $\mathbb{F}_{2^{2 l}}$ is given together with an efficient algorithm to compute the number of such self-dual codes.

Theorem 3.4. Let $p$ be an odd prime and $r$ be a positive integer. Let $\alpha$ be the largest positive integer such that $p \nmid \operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$ and let $\gamma$ be the integer such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$. Then

$$
\tau\left(p^{r}, l\right)= \begin{cases}\frac{\operatorname{gcd}^{2}\left(\operatorname{ord}_{p}(2), l\right)}{2 \operatorname{ord}_{p}(2)}\left(p^{\alpha}-1+(p-1)(r-\alpha) p^{\alpha-1}\right) & \text { if } 2^{\gamma} \mid l  \tag{21}\\ \frac{\operatorname{gcd}^{2}\left(2 d_{p}(2), l\right)}{\operatorname{ord}_{p}(2)}\left(p^{\alpha}-1+(p-1)(r-\alpha) p^{\alpha-1}\right) & \text { if } 2^{\gamma-1} \nmid l \\ 0 & \text { if } 2^{\gamma-1} \mid l .\end{cases}
$$

In particular, if 2 is a primitive root modulo $p^{2}$, then for any positive integer $r$

$$
\tau\left(p^{r}, l\right)= \begin{cases}\frac{1}{2}\left(\sum_{i=1}^{r} \operatorname{gcd}\left(p^{i-1}(p-1), 2 l\right)\right) & \text { if } 2^{\gamma} \mid l \text { or } 2^{\gamma-1} \nmid l  \tag{22}\\ 0 & \text { if } 2^{\gamma-1}| | l\end{cases}
$$

Proof. From Corollary 2.2, $2^{\gamma-1}| | l$ if and only if $2 \| \operatorname{ord}_{p}\left(2^{l}\right)$ which is equivalent to $\chi_{l}(p)=0$ by Corollary 3.2, In this case, $\lambda_{l}\left(p^{i}\right)=0$ for all $1 \leq i \leq r$. Equivalently, $\tau\left(p^{r}, l\right)=0$ if and only if $2^{\gamma-1} \mid l$.

Assume that $2^{\gamma} \mid l$ or $2^{\gamma-1} \nmid l$. By Corollary [2.2, it follows that $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd or $4 \mid \operatorname{ord}_{p}\left(2^{l}\right)$. Consider the following two cases.
Case 1: $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd. From the definition of $\tau$ in (6) and Lemma 3.3, we have

$$
\begin{align*}
\tau\left(p^{r}, l\right) & =\frac{1}{2} \sum_{i=0}^{r} \lambda_{l}\left(p^{i}\right) \frac{\phi\left(p^{i}\right)}{\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)}  \tag{23}\\
& =\frac{1}{2} \sum_{i=1}^{r} \frac{p^{i-1}(p-1)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}  \tag{24}\\
& =\frac{\operatorname{gcd}\left(\operatorname{ord}_{p}(2), l\right)}{2 \operatorname{ord}_{p}(2)}\left(p^{\alpha}-1+(p-1)(r-\alpha) p^{\alpha-1}\right) . \tag{25}
\end{align*}
$$

Case 2: $4 \mid \operatorname{ord}_{p}\left(2^{l}\right)$. From (6) and Lemma 3.3, it follows that

$$
\begin{align*}
\tau\left(p^{r}, l\right) & =\frac{1}{2} \sum_{i=0}^{r} \lambda_{l}\left(p^{i}\right) \frac{\phi\left(p^{i}\right)}{\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)}  \tag{26}\\
& =\frac{1}{2} \sum_{i=1}^{r} \frac{2 p^{i-1}(p-1)}{\operatorname{ord}_{p^{i}}\left(2^{l}\right)}  \tag{27}\\
& =\frac{\operatorname{gcd}^{\left(\operatorname{ord}_{p}(2), l\right)}}{\operatorname{ord}_{p}(2)}\left(p^{\alpha}-1+(p-1)(r-\alpha) p^{\alpha-1}\right) \tag{28}
\end{align*}
$$

Finally, assume further that 2 is a primitive root modulo $p^{2}$. By Lemma 2.5, we have $\operatorname{ord}_{p^{i}}\left(2^{l}\right)=\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), l\right)}$ for all $1 \leq i \leq r$. Hence,

$$
\begin{align*}
\tau\left(p^{r}, l\right) & =\frac{1}{2} \sum_{i=0}^{r} \chi_{l}\left(p^{i}\right) \frac{\phi\left(p^{i}\right)}{\operatorname{ord}_{p^{i}}\left(2^{2 l}\right)}  \tag{29}\\
& =\frac{1}{2} \sum_{i=1}^{r} \frac{p^{i-1}(p-1)}{\frac{p^{i-1}(p-1)}{\operatorname{gcd}\left(p^{i-1}(p-1), 2 l\right)}}  \tag{30}\\
& =\frac{1}{2} \sum_{i=1}^{r} \operatorname{gcd}\left(p^{i-1}(p-1), 2 l\right) \tag{31}
\end{align*}
$$

as desired.
Corollary 3.5. Let $l$ be a positive integer and $p$ be an odd prime satisfying $l$ is not divisible by $p$. If 2 is a primitive root modulo $p^{2}$ and $\operatorname{ord}_{p}\left(2^{l}\right)$ is odd or $\operatorname{ord}_{p}\left(2^{l}\right)$ is divisible by 4 , then $\tau\left(p^{r}, l\right)=r \operatorname{gcd}\left(\frac{p-1}{2}, l\right)$ for all positive integers $r$.

Example 3.6. Let $p=11, l=4$ and $r=3$. We can see that 2 is a primitive root modulo $11^{2}$ and $\operatorname{ord}_{11}\left(2^{4}\right)=5$ is odd. Thus $\tau\left(11^{3}, 4\right)=3 \operatorname{gcd}(5,4)=3$.

The results on $\tau\left(p^{r}, l\right)$ discussed above can be summarized in Algorithm 2.

For an odd prime $p$ and positive integers $l$ and $r$, do the following steps.

1. Compute $\operatorname{ord}_{p}(2)$.
2. Determine $\gamma$ such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$.
2.1 If $2^{\gamma-1} \mid l l$, then $\tau\left(p^{r}, l\right)=0$ by (21). Done.
2.2 If $2^{\gamma} \mid l$ or $2^{\gamma-1} \nmid l$, then compute ord $_{p^{2}}(2)$.
2.2.1 If $\operatorname{ord}_{p^{2}}(2)=p(p-1)$, then evaluate (22). Done.
2.2.2 If ord $p_{p^{2}}(2) \neq p(p-1)$, then do the following steps.
i) Compute $\operatorname{ord}_{p^{r}}\left(2^{l}\right)$.
ii) Determine the largest integer $\alpha$ such that $p \nmid$ $\operatorname{ord}_{p^{\alpha}}\left(2^{l}\right)$ (by Corollary 2.6).
iii) Evaluate (21). Done.

Figure 2: Steps in Computing $\tau\left(p^{r}, l\right)$
Similar to the Euclidean case, a direct computation of $\tau\left(p^{r}, l\right)$ from (5) requires the values of $\chi_{l}\left(p^{i}\right), \phi\left(p^{i}\right)$ and $\operatorname{ord}_{p^{i}}\left(2^{l}\right)$ for all $0 \leq i \leq r$. The complexity can be reduced using Algorithm 2 since it requires only $\operatorname{ord}_{p}(2), \operatorname{ord}_{p^{2}}(2)$, $\operatorname{ord}_{p^{r}}\left(2^{l}\right)$ and some basic expressions in (22) or (21).

From Theorems 1.3 and 1.4, a Euclidean self-dual cyclic code over $\mathbb{F}_{2^{l}}$ exists for all integers $l$ but an Hermitian self-dual cyclic code over $\mathbb{F}_{2^{l}}$ exists if and only if $l$ is even. Over $\mathbb{F}_{2^{2 l}}$, both the Euclidean and Hermitian self-dual cyclic codes always exist. The numbers of self-dual codes in the two families can be compared in terms of $t\left(p^{r}, 2 l\right)$ and $\tau\left(p^{r}, l\right)$ as follows.

Proposition 3.7. Let $p$ be an odd prime and $r$ be positive integers. Let $\gamma$ be the integer such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$. Then one of the following holds

1. If $2^{\gamma} \mid l$, then $\tau\left(p^{r}, l\right)=t\left(p^{r}, 2 l\right)$.
2. If $2^{\gamma-1} \mid l l$, then $0=\tau\left(p^{r}, l\right)<t\left(p^{r}, 2 l\right)$.
3. If $2^{\gamma-1} \nmid l$, then $\tau\left(p^{r}, l\right)>t\left(p^{r}, 2 l\right)=0$.

Proof. For the first part, it suffices to show that $\operatorname{gcd}\left(\operatorname{ord}_{p}(2), l\right)=\operatorname{gcd}\left(\operatorname{ord}_{p}(2), 2 l\right)$. Since $2^{\gamma} \| \operatorname{ord}_{p}(2)$ and $2^{\gamma} \mid l$, we have $\operatorname{gcd}\left(\operatorname{ord}_{p}(2), l\right)=\operatorname{gcd}\left(\operatorname{ord}_{p}(2), 2 l\right)$ as desired.

For the rest of the theorem, it follows easily from Theorems 2.7 and 3.4.

### 3.3 Hermitian Self-Dual Cyclic Codes of Length $2^{\nu} p^{r} q^{s}$

From Lemma 3.1, we have the following lemma.
Lemma 3.8. Let $p$ and $q$ be distinct odd primes and let l be a positive integer. Then the following statements hold.

1. $\lambda_{l}(p q)=1$ if and only if $\lambda_{l}(p)=1$ or $\lambda_{l}(q)=1$.
2. $\lambda_{l}\left(p^{i} q^{j}\right)=\lambda(p q)$ for all positive integers $i$ and $j$.

Theorem 3.9. Let $p$ and $q$ be distinct odd primes and let $r, s$, and $l$ be positive integers. Then

$$
\tau\left(p^{r} q^{s}, l\right)=\lambda_{l}(p) \tau\left(p^{r}, l\right)+\lambda_{l}(q) \tau\left(q^{s}, l\right)+\lambda_{l}(p q) \sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{2 l}\right), \operatorname{ord}_{q^{j}}\left(2^{2 l}\right)\right)} .
$$

Proof. We note that $\operatorname{ord}_{p^{i} q^{j}}\left(2^{2 l}\right)=\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{2 l}\right), \operatorname{ord}_{q^{j}}\left(2^{2 l}\right)\right)$. The theorem follows from (6) and Lemma 3.8.

The next corollary follows from Corollary 2.2, Lemma 3.8 and Theorem 3.9 .

Corollary 3.10. Let $p$ and $q$ be distinct odd primes and let $r$, $s$, and $l$ be positive integers. Let $\gamma$ and $\beta$ be the integers such that $2^{\gamma} \| \operatorname{ord}_{p}(2)$ and $2^{\beta} \| \operatorname{ord}_{q}(2)$, respectively. Then one of the following statements holds.

1. If $2^{\gamma} \mid l$ or $2^{\gamma-1} \nmid l$, and $2^{\beta} \mid l$ or $2^{\beta-1} \nmid l$, then

$$
\tau\left(p^{r} q^{s}, l\right)=\tau\left(p^{r}, l\right)+\tau\left(q^{s}, l\right)+\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{2 l}\right), \operatorname{ord}_{q^{j}}\left(2^{2 l}\right)\right)}
$$

2. If $2^{\gamma} \mid l$ or $2^{\gamma-1} \nmid l$, and $2^{\beta-1}| | l$, then

$$
\tau\left(p^{r} q^{s}, l\right)=\tau\left(p^{r}, l\right)+\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{2 l}\right), \operatorname{ord}_{q^{j}}\left(2^{2 l}\right)\right)} .
$$

3. If $2^{\gamma-1}| | l$, and $2^{\beta} \mid l$ or $2^{\beta-1} \nmid l$, then

$$
\tau\left(p^{r} q^{s}, l\right)=\tau\left(q^{s}, l\right)+\sum_{i=1}^{r} \sum_{j=1}^{s} \frac{\phi\left(p^{i} q^{j}\right)}{\operatorname{lcm}\left(\operatorname{ord}_{p^{i}}\left(2^{2 l}\right), \operatorname{ord}_{q^{j}}\left(2^{2 l}\right)\right)} .
$$

4. If $2^{\gamma-1}| | l$ and $2^{\beta-1}| | l$, then

$$
\tau\left(p^{r} q^{s}, l\right)=0
$$

Similar to the Euclidean case, the complexity of the direct computation of $\tau\left(p^{r}, l\right)$ in (6) can be reduced using Corollary 2.12.

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