Symmetry breaking in planar and maximal outerplanar graphs

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Abstract

The distinguishing number (index) D(G) (D'(G)) of a graph G is the least integer d such that G has a vertex (edge) labeling with d labels that is preserved only by a trivial automorphism. In this paper we consider the maximal outerplanar graphs (MOP graphs) and show that MOP graphs, except K_3 , can be distinguished by at most two vertex (edge) labels. We also compute the distinguishing number and the distinguishing index of Halin and Mycielskian graphs.

Keywords: distinguishing number; distinguishing index; Mycielskian graph; Halin graph; maximal outerplanar graph.

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1 Introduction

Let G = (V, E) be a simple, connected and undirected graph, and let $\operatorname{Aut}(G)$ be its automorphism group. A labeling of G, $\phi : V \to \{1, 2, \ldots, r\}$, is said to be *r*distinguishing, if no non-trivial automorphism of G preserves all of the vertex labels. The point of the labels on the vertices is to destroy the symmetries of the graph, that is, to make the automorphism group of the labeled graph trivial. Formally, ϕ is *r*-distinguishing if for every non-identity $\sigma \in \operatorname{Aut}(G)$, there exists x in V such that $\phi(x) \neq \phi(\sigma(x))$. The distinguishing number of a graph G is defined by

 $D(G) = \min\{r \mid G \text{ has a labeling that is } r \text{-distinguishing}\}.$

This number has defined in [1]. The distinguishing index D'(G) of a graph G is the least number d such that G has an edge labeling with d labels that is preserved only by the identity automorphism of G. The distinguishing edge labeling was first

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defined by Kalinowski and Pilśniak [6] for graphs. Obviously, this invariant is not defined for graphs having K_2 as a connected component. If a graph has no nontrivial automorphisms, its distinguishing number is 1. In other words, D(G) = 1 for the asymmetric graphs. The other extreme, D(G) = |V(G)|, occurs if and only if $G = K_n$. The distinguishing index of some examples of graphs was exhibited. For instance, $D(P_n) = D'(P_n) = 2$ for every $n \ge 3$, and $D(C_n) = D'(C_n) = 3$ for n = 3, 4, 5, $D(C_n) = D'(C_n) = 2$ for $n \ge 6$. It is easy to see that the value |D(G) - D'(G)| can be large. For example $D'(K_{p,p}) = 2$ and $D(K_{p,p}) = p + 1$, for $p \ge 4$.

A maximal outerplanar graph is an outerplanar graph that cannot have any additional edges added to it while preserving outerplanarity. Every maximal outerplanar graph with n vertices has exactly 2n - 3 edges, and every bounded face of a maximal outerplanar graph is a triangle. Every maximal outerplanar graph satisfies a stronger condition than Hamiltonicity: it is *pancyclic*, meaning that for every vertex v and every k in the range from three to the number of vertices in the graph, there is a length k cycle containing v. A cycle of this length may be found by repeatedly removing a triangle that is connected to the rest of the graph by a single edge, such that the removed vertex is not v, until the outer face of the remaining graph has length k. In [2], Arvind et al. designed efficient algorithms for computing the distinguishing numbers of trees and planar graphs. Arvind et al. proved that the distinguishing number of a planar graph can be computed in time polynomial in the size of the graph, [3]. Fijavž et al. showed that every 3-connected planar graph is 5-distinguishing colorable except $K_{2,2,2}$ and $C_6 + \overline{K_2}$, [5]. They also proved that every 3-connected bipartite planar graph is 3-distinguishing colorable except Q_3 and $R(Q_3)$.

A Halin graph is a type of planar graph, constructed by connecting the leaves of a tree into a cycle. The tree must have at least four vertices, none of which has exactly two neighbors; it should be drawn in the plane so none of its edges cross (this is called planar embedding), and the cycle connects the leaves in their clockwise ordering in this embedding. Thus, the cycle forms the outer face of the Halin graph, with the tree inside it, see Figure 1. Every Halin graph is a Hamiltonian graph, and every edge of the graph belongs to a Hamiltonian cycle. Moreover, any Halin graph remains Hamiltonian after deletion of any vertex [4].

The Mycielskian or Mycielski graph of an undirected graph is a larger graph formed from it by a construction of Jan Mycielski (1955) [7]. The construction preserves the property of being triangle-free but increases the chromatic number; by applying the construction repeatedly to a triangle-free starting graph, Mycielski showed that there exist triangle-free graphs with arbitrary large chromatic number. Let the *n* vertices of the given graph *G* be $v_0, v_1, ..., v_{n-1}$. The Mycielski graph $\mu(G)$ of *G* contains *G* itself as an isomorphic subgraph, together with n + 1 additional vertices: a vertex u_i corresponding to each vertex v_i of *G*, and another vertex *w*. Each vertex u_i is connected by an edge to *w*, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge $v_i v_j$ of *G*, the Mycielski graph includes two edges, $u_i v_j$ and $v_i u_j$. The illustration shows Mycielski's construction as applied to a 5-vertex cycle graph with vertices v_i for $0 \le i \le 4$. The resulting Mycielskian is the *Grötzsch graph*, an 11-vertex graph with 20 edges, see Figure 2. Thus, if *G* has *n* vertices and *m* edges,

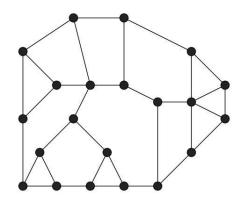


Figure 1: Examples of a Halin graph.

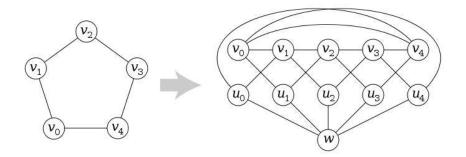


Figure 2: The Grötzsch graph as the Mycielskian of a 5-cycle graph.

 $\mu(G)$ has 2n+1 vertices and 3m+n edges.

We study the distinguishing number and the distinguishing index of planar and maximal outerplanar graphs in Section 2. Then we study symmetry breaking for the Halin graphs which are planar graphs, in Section 3. Finally, in Section 4, we obtain the distinguishing number and the distinguishing index of Mycielski graphs.

2 Maximal outerplanar graph

In this section we study symmetry breaking for planar and maximal outerplanar graphs. We first consider planar graphs.

2.1 Planar graphs

Theorem 2.1 If G is a connected graph with clique number four and the maximum degree $\Delta \geq 5$, then $D(G) \leq \Delta - 1$.

Proof. Let v_0, v_1, v_2, v_3 be vertices of an induced subgraph K_4 in G. We label the vertex v_0 with label $\Delta - 1$, and vertices v_1, v_2 and v_3 with labels 1, 2 and 3, respectively. Next we label the remaining unlabeled vertices in $N_G(v_0)$ with different labels in the set $\{1, 2, \ldots, \Delta - 2\} \setminus \{3\}$. Now we consider the BFS tree T of G with root v_0 , hence $N_G(v_0) = N_T(v_0)$. We label the vertex v_0 and each vertex in $N_T(v_0)$, exactly the same as G. For labeling of the remaining vertices of T, we use the following mathematical induction.

Let x be a labeled vertex of T at distance i from v_0 where $i \ge 1$. Then, we label the adjacent vertices to x which are at distance i + 1 of v_0 , with different labels in the set $\{1, 2, \ldots, \Delta - 1\}$.

We now consider this labeling for vertices of G. If there is an induced subgraph K_4 in G with vertices of labels exactly the same labels as the induced subgraph $v_0v_1v_2v_3$, i.e., labels 1,2,3 and $\Delta - 1$, except the induced subgraph $v_0v_1v_2v_3$, then we change the labels of vertices of this induced subgraph by permuting of labels of vertices in $N_T(x)$ where x is some of vertices of that induced subgraph K_4 . So without loss of generality we can assume that the induced subgraph K_4 with vertices v_0, v_1, v_2, v_3 is the only induced subgraph K_4 in G with label set $\{1, 2, 3, \Delta - 1\}$. Hence this induced subgraph is fixed, pointwise, under each automorphism of G preserving the labeling. Thus each automorphism of G preserving the labeling, fixes the vertex v_0 , each vertex of $N_G(v_0)$, and so preserves the distance of v_0 . We show that all vertices at distance $i, i \geq 1$, from v_0 are fixed under each automorphism of G preserving the labeling, by induction on i.

Let x be vertex at distance $i, i \geq 1$, of v_0 which is fixed under each automorphism of G preserving the labeling. Since the label of vertices adjacent to x in G which are at distance i + 1 from v_0 are distinct, so all of them are fixed, too. This argument can be used for all vertices at distance i from v_0 , and so we conclude that all vertices at distance i + 1 from v_0 are fixed. Therefore the identity automorphism is the only automorphism of G preserving the labeling and so $D(G) \leq \Delta - 1$.

The condition $\Delta(G) \geq 5$ is necessary in Theorem 2.1. For instance, see Figure 3 for a graph with $\Delta(G) = \chi(G) = D(G) = 4$. It is known that a connected planar graph with chromatic number $\chi(G) = 4$ has clique number 4, so the following result is an immediate consequence of Theorem 2.1.

Corollary 2.2 Let G be a connected planar graph with chromatic number $\chi(G) = 4$ and maximum degree $\Delta \geq 5$. Then $D(G) \leq \Delta - 1$.

2.2 Maximal outerplanar graphs

A maximal outerplanar graph, or briefly a MOP graph, is a graph that is isomorphic to a triangularization of a polygon. All MOPs can be constructed according to the following recursive rule (see, for instance, [9]): (i) the triangle, K_3 , is a MOP, and (ii) a MOP with n + 1 vertices can be obtained from some MOP M with n vertices $(n \ge 3)$, by adding a new vertex adjacent to two consecutive vertices on the Hamiltonian cycle of M. All MOPs with up to seven vertices are shown in Figure 4.

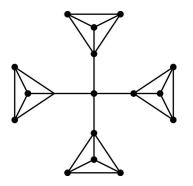


Figure 3: A graph with $\Delta(G) = \chi(G) = D(G) = 4$.

Theorem 2.3 If G is a connected graph of order $n \ge 6$ with exactly one Hamiltonian cycle, then $D(G) \le 2$.

Proof. Let $V(G) = \{v_1, \ldots, v_n\}$. Since there is only one cycle, say C, with consecutive vertices v_1, \ldots, v_n , so the restriction of any automorphism of G to C is an automorphism of C. We label the vertices of G with two labels 1 and 2 such that the cycle C with consecutive vertices v_1, \ldots, v_n has been distinguishingly. This labeling is a distinguishing labeling for G, Since if f is an automorphism of G preserving this labeling, then we can consider f as an automorphism of C preserving its labeling, and so f is the identity automorphism, since we labeled C distinguishingly.

Corollary 2.4 If G is a maximal outerplanar graph, then $D(G) \leq 2$, except for K_3 .

Proof. It is known that every maximal outerplanar graph has exactly one Hamiltonian cycle. By Figure 4, it can be seen that the distinguishing number of maximal outerplanar graphs with seven or less vertices is at most two, except for K_3 . Hence the result is a direct consequence of Theorem 2.3.

Theorem 2.5 If G is a maximal outerplanar graph, then $D'(G) \leq 2$, except for K_3 .

Proof. By Figure 4, it can be seen that the distinguishing index of maximal outerplanar graphs with seven or less vertices is at most two, except for K_3 . Hence the result is a direct consequence of Theorem 3.5.

Corollary 2.4 and Theorem 2.5 are true for 2-connected outerplanar graphs, too, because these graphs have exactly one Hamiltonian cycle.

3 Halin graphs

A Halin graph is constructed by connecting the leaves of a tree into a cycle. The tree must have at least four vertices, none of which has exactly two neighbors; it should be

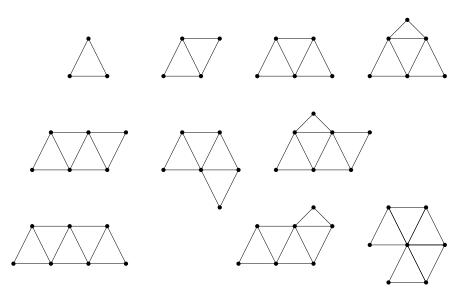


Figure 4: Maximal outerplanar graph with seven or less vertices.

drawn in the plane so that none of its edges cross, and the cycle connects the leaves in their clockwise ordering in this embedding. Every Halin graph is a Hamiltonian graph. We start with the following lemma.

Lemma 3.1 Let T be a tree of order at least four with no vertex of degree two, and G be its Halin graph. If f is an automorphism of G fixing the vertices of the cycle induced on the set of the leaves of T, say C, then f is the identity automorphism of G.

Proof. Let f fixes the vertices of C, but moves a vertex $x \in V(T) \setminus V(C)$ to y. Since f fixes the vertices of C, so we can consider f as an automorphism of T. On the other hand, since degree of x in T is at least 3, so the set of the nearest vertices of degree 1 to x in T is different from the set of the nearest vertices of degree 1 to y in T. Now since f maps x to y, so f moves some vertices of C which is a contradiction.

Theorem 3.2 If T is a tree of order at least four with no vertex of degree two, and G be its Halin graph, then $D(G) \leq 4$. The equality holds for $K_{1,3}$.

Proof. We label the vertices of the cycle induced on the set of the leaves of T, say C, distinguishingly with three labels 1,2, and 3. Next we label the remaining unlabeled vertices of G with a new label 4. This labeling is distinguishing. In fact, if f is an automorphism of G preserving the labeling, then the restriction of f to vertices of C is an automorphism of C. Since the cycle C has been labeled distinguishingly, so this restriction is the identity, and thus f is the identity automorphism of G, by Lemma 3.1.

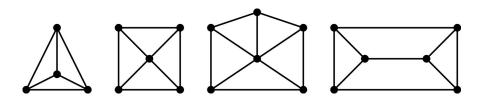


Figure 5: All Halin graphs of with six or less vertices.

Theorem 3.3 Let T be a tree of order at least four with no vertex of degree two, and G be its Halin graph. If T has no vertex of degree 3, then $D(G) \leq 3$. The equality holds for wheels W_4 and W_5 .

Proof. Since T has no vertex of degree 3, so the only vertices of degree 3 in G is the vertices of the cycle inuced on the set of leaves of T, say C. Hence the automorphisms of G map the vertices of C to itself, setwise. Thus the automorphism group of G is a subgroup of the automorphism group of T. Now, we label the vertices of the cycle induced on the set of the leaves of T, say C, distinguishingly with three labels 1,2, and 3. Next we label the remaining unlabeled vertices of G with an arbitrary label, say 1. Then this labeling is distinguishing. In fact, if f is an automorphism of G preserving the labeling, then since the restriction of f to vertices of C is an automorphism of C and the cycle C has been labeled distinguishingly, so this restriction is the identity, and thus f is the identity automorphism of G, by Lemma 3.1.

Theorem 3.4 Let T be a tree of order at least four with no vertex of degree two, and G be its Halin graph. If T has no vertex of degree 3 and the number of leaves of T is at least 6, then $D(G) \leq 2$.

Proof. It is known that $D(C_n) = 2$ for any $n \ge 6$. Thus if we label the vertices of the cycle induced on the set of the leaves of T, say C, distinguishingly with two labels 1 and 2, and next we label the remaining unlabeled vertices of G with an arbitrary label, say 1, then this labeling is distinguishing, by a similar argument as proof of Theorem 3.2.

We recall that a *traceable graph* is a graph that possesses a Hamiltonian path.

Theorem 3.5 [8] If G is a traceable graph of order $n \ge 7$, then $D'(G) \le 2$.

The assumption $n \ge 7$ is substantial in this theorem, because for example $D'(K_{3,3}) = 3$.

Theorem 3.6 The distinguishing index of Halin graphs is at most two, except for K_4 .

Proof. It can be seen that the distinguishing index of Halin graphs of order at most 6 is 2 except for K_4 , see Figure 5. For the Halin graphs of order at least 7, the result follows from Theorem 3.5.

4 Mycielski graphs

Let the *n* vertices of the given graph *G* be $v_0, v_1, ..., v_{n-1}$. The Mycielski graph $\mu(G)$ of *G* contains *G* itself as an isomorphic subgraph, together with n + 1 additional vertices: a vertex u_i corresponding to each vertex v_i of *G*, and another vertex *w*. Each vertex u_i is connected by an edge to *w*, so that these vertices form a subgraph in the form of a star $K_{1,n}$. In addition, for each edge $v_i v_j$ of *G*, the Mycielski graph includes two edges, $u_i v_j$ and $v_i u_j$. By above notations of vertices of Mycielski graph $\mu(G)$ we have the following result. We recall that graphs with no pairs of vertices with the same open neighborhoods are called *R*-thin.

Proposition 4.1 Let G be an R-thin graph. If f is an automorphism of $\mu(G)$ such that f(w) = w, then the restriction of f to each of sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ is isomorphic to an automorphism of G.

Proof. If f(w) = w, then since f preserves adjacency relation, so f maps the sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$ to itself, setwise. By the same reasoning the restriction of f to $\{v_1, \ldots, v_n\}$ is an automorphism of G. In sequel, we want to show that the restriction of f to the vertices $\{u_1, \ldots, u_n\}$ is isomorphic to the restriction of f to $\{v_1, \ldots, v_n\}$. For this purpose we suppose that $f(u_i) = u_{\sigma(i)}$ and $f(v_i) = v_{\tau(i)}$ where τ and σ are permutations of $\{1, \ldots, n\}$ and $1 \leq i \leq n$, and show that $\tau = \sigma$. Let v_k be an arbitrary vertex of G and $N_G(v_k) = \{v_{k_1}, \ldots, v_{k_t}\}$. Thus $N_{\mu(G)}(u_k) \setminus \{w\} = \{v_{\tau(k_1)}, \ldots, v_{\tau(k_t)}\}$. On the other hand $N_G(v_{\tau(k)}) = \{v_{\tau(k_1)}, \ldots, v_{\tau(k_t)}\}$. Therefore $v_{\tau(k)}$ and $v_{\sigma(k)}$ have the same open neighbor in G. Since G is an R-thin graph, so $\tau(k) = \sigma(k)$. Since k is arbitrary, so we have the result.

Theorem 4.2 If G is an R-thin graph of order $n \ge 2$, then $D(\mu(G)) \le D(G) + 1$.

Proof. We present a distinguishing vertex labeling of $\mu(G)$ with D(G) + 1 labels. Let $c: V(G) \to \{1, \ldots, D(G)\}$ be a distinguishing labeling of G. We extend c to a (D(G) + 1)-labeling \overline{c} of $\mu(G)$ with $\overline{c}(v_i) = c(v_i)$, $\overline{c}(u_i) = c(v_i)$ for $1 \leq i \leq D(G)$, and $\overline{c}(w) = 0$. We claim that \overline{c} is a distinguishing labeling of $\mu(G)$. In fact if f is an automorphism of $\mu(G)$ preserving the labeling \overline{c} , then f(w) = w, and so the restriction of f to vertices v_1, \ldots, v_n is an automorphism of G preserving the labeling c. Thus the restriction of f to the vertices of G is the identity, and so f is the identity automorphism of $\mu(G)$ by Proposition 4.1.

Theorem 4.3 If G is an R-thin graph of order $n \ge 3$ with no connected component K_2 , then $D'(\mu(G)) \le D'(G) + 1$.

Proof. Let $c': E(G) \to \{1, \ldots, D'(G)\}$ be a distinguishing edge labeling of G. We define a (D'(G) + 1)-labeling $\overline{c'}$ of $\mu(G)$ with $\overline{c'}(wu_i) = 0$ for $1 \le i \le D(G)$, $\overline{c'}(u_i v_j) = c'(v_i v_j)$ for $i, j \in \{1, \ldots, n\}$ for which $v_i v_j \in E(G)$, and $\overline{c'}(e) = c'(e)$ for all $e \in E(G)$. The mapping $\overline{c'}$ is a distinguishing labeling of $\mu(G)$. In fact if f is an automorphism of

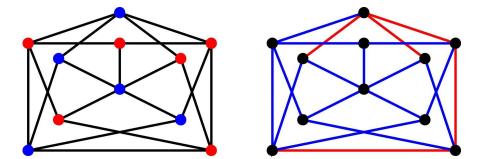


Figure 6: A 2-distinguishing labeling of vertices and edges of M_4 .

 $\mu(G)$ preserving the labeling $\overline{c'}$, then f(w) = w, and so the restriction of f to vertices v_1, \ldots, v_n is an automorphism of G preserving the labeling c'. Thus the restriction of f to the vertices of G is the identity, and so f is the identity automorphism of $\mu(G)$ by Proposition 4.1.

Applying the Mycielskian repeatedly, starting with a graph with a single edge, produces a sequence of graphs $M_i = \mu(M_{i-1})$, also sometimes called the Mycielski graphs. The first few graphs in this sequence are the graph $M_2 = K_2$ with two vertices connected by an edge, the cycle graph $M_3 = C_5$, and the Grötzsch graph with 11 vertices and 20 edges. It is clear that M_i is an *R*-thin graph if and only if M_{i-1} is an *R*-thin graph. Since M_2 is an *R*-thin graph, so M_i 's are *R*-thin graphs for any $i \geq 2$.

Theorem 4.4 For any $i \ge 5$, $D(M_i) \le D(M_{i-1})$ and $D'(M_i) \le D'(M_{i-1})$.

Proof. The degree of vertex w in M_i is $|V(M_{i-1})|$, and w is the only vertex with this degree, so w is fixed under each automorphism of M_i . We first show that $D(M_i) \leq D(M_{i-1})$. We label the vertices of isomorphic subgraph M_{i-1} in M_i , i.e., v's, distinguishingly with $D(M_{i-1})$ labels, and label the corresponding vertices to the vertices M_{i-1} in M_i , i.e., u's, exactly the same as v's and label w with an arbitrary label. Then by the same reasoning as Theorem 4.2 we conclude that this labeling is distinguishing. Therefore $D(M_i) \leq D(M_{i-1})$.

For the second part of Theorem, we label the incident edges to w with an arbitrary label, say 1, and label the remaining edges of M_i exactly the same as Theorem 4.3. Then by the same reasoning we conclude that this labeling is distinguishing. Therefore $D'(M_i) \leq D'(M_{i-1})$.

Corollary 4.5 For $i \ge 4$, $D(M_i) = D'(M_i) = 2$. In particular, $D(M_3) = D'(M_3) = 3$ and $D(M_2) = 2$.

Proof. It is easy to see that $D(M_2) = 2$ and $D(M_3) = D'(M_3) = 3$. In Figure 6 we presented a 2-distinguishing labeling of vertices and edges of M_4 . Now the result follows from Theorem 4.4 and the induction on *i*.

We end this paper by the following conjecture:

Conjecture 4.6 Let G be a connected graph of order $n \ge 3$. Then $D(\mu(G)) \le D(G)$ and $D'(\mu(G)) \le D'(G)$, except for a finite number of graphs.

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