# On the integer $\{k\}$-domination number of circulant graphs 

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#### Abstract

Let $G=(V, E)$ be a simple undirected graph. $G$ is a circulant graph defined on $V=\mathbb{Z}_{n}$ with difference set $D \subseteq\left\{1,2, \ldots,\left\lfloor\frac{n}{2}\right\rfloor\right\}$ provided two vertices $i$ and $j$ in $\mathbb{Z}_{n}$ are adjacent if and only if $\min \{|i-j|, n-|i-j|\} \in D$. For convenience, we use $G(n ; D)$ to denote such a circulant graph.

A function $f: V(G) \rightarrow \mathbb{N} \cup\{0\}$ is an integer $\{k\}$-domination function if for each $v \in V(G), \sum_{u \in N_{G}[v]} f(u) \geq k$. By considering all $\{k\}$-domination functions $f$, the minimum value of $\sum_{v \in V(G)} f(v)$ is the $\{k\}$-domination number of $G$, denoted by $\gamma_{k}(G)$. In this paper, we prove that if $D=$ $\{1,2, \ldots, t\}, 1 \leq t \leq \frac{n-1}{2}$, then the integer $\{k\}$-domination number of $G(n ; D)$ is $\left\lceil\frac{k n}{2 t+1}\right\rceil$.


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## 1 Introduction and preliminaries

The study of domination number of a graph $G$ has been around for quite a long time. Due to its importance in applications, there are various versions of extension study, see [6] for reference.

The idea of integer $\{k\}$-domination was proposed by Domke et al. in 3]. It can be dealt as a labeling problem. The vertices of the graph $G$ are labeled by integers in $\mathbb{N} \cup\{0\}$ such that for each vertex $v$, the total (sum) values in its closed neighborhood $N_{G}[v]$ must be at least $k$. The problem is asking for finding the minimum total value labeled on $G$. Finally, we say that $f: V(G) \rightarrow \mathbb{N} \cup\{0\}$

[^0]is an integer $\{k\}$-domination function if for each $v \in V(G), \sum_{u \in N_{G}[v]} f(u) \geq k$. Among all such functions $f$, the minimum value of $\sum_{v \in V(G)} f(v)$ is called the integer $\{k\}$-domination number of $G$, denoted by $\gamma_{k}(G)$.

It is not difficult to see that the original domination number of a graph $G, \gamma(G)$, can be recognized as $\gamma_{1}(G)$ since the vertices with label "1" gives a dominating set. For more information about domination problem, the readers may refer to [2, 4, [5, 6, 8, Hence, the integer $\{k\}$-domination problem is also an NP-hard problem. So far, results obtained are all on special classes of graphs, see [1] 3, 7, 9].

In this paper, we shall consider the class of circulant graph $G=G(n ; D)$ where $D=\{1,2, \ldots, t\}, 1 \leq t \leq \frac{n-1}{2}$, i.e., $V(G)=\mathbb{Z}_{n}$ and two vertices $i$ and $j$ are adjacent if and only if $d(i, j):=\min \{|i-j|, n-|i-j|\} \in D$. Since $D=\{1,2, \ldots, t\}, G(n ; D)$ is exactly the power graph $C_{n}^{t}$ where $C_{n}$ is a cycle of order $n$.

The following results are obtained by Lin 10. For clearness, we also outline its proof in which basic linear algebra is applied.

Proposition 1.1 ( 10$]$ ). Let $G$ be the circulant graph $G(n ; D)$ where $D=$ $\{1,2, \ldots, t\}$. Then, $\gamma_{k}(G) \geq\left\lceil\frac{k n}{2 t+1}\right\rceil$.
Proof. Let $A$ be the adjacency matrix of $G$ and $f$ be an $\{k\}$-domination function of $G$. Let $\mathbf{1}_{n}$ denote the all 1 column vector of length $n$. Then, we have

$$
(2 t+1) \sum_{v \in V(G)} f(v)=\left(f\left(v_{1}\right), f\left(v_{2}\right), \ldots, f\left(v_{n}\right)\right)\left(A+I_{n}\right) \mathbf{1}_{n} \geq \mathbf{1}_{n}^{T} \cdot k \cdot \mathbf{1}_{n}=n k,
$$

which implies the inequality.
By the aid of an algorithm, Lin was able to show the following.
Proposition 1.2 (10]). For $t \leq 5, \gamma_{k}(G(n ;\{1,2, \ldots, t\}))=\left\lceil\frac{k n}{2 t+1}\right\rceil$.
But, for larger $t$, it remains unsettled. Our main result of this paper shows that the equality holds for all $1 \leq t \leq \frac{n-1}{2}$.

## 2 The main result

By Proposition 1.1 in order to determine $\gamma_{k}(G)$, it suffices to show that $\gamma_{k}(G) \leq$ $\left\lceil\frac{n k}{2 t+1}\right\rceil$. That is, we need a proper distribution of values for $f\left(v_{1}\right), f\left(v_{2}\right), \ldots, f\left(v_{n}\right)$ such that for each $v_{i}, \sum_{u \in N_{G}\left[v_{i}\right]} f(u) \geq k$ and $\sum_{i=1}^{n} f\left(v_{i}\right) \leq\left\lceil\frac{n k}{2 t+1}\right\rceil$. Since we are dealing with circulant graphs, $\sum_{u \in N_{G}\left[v_{i}\right]} f(u)$ is in fact the sum of $2 t+1$ consecutive labels assigned to the circle $C_{n}=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$. Therefore, we turn our focus on providing suitable labels to meet the condition.

For example, let $n=8$ and $t=2$. Then, the following labeling of $\left(v_{1}, v_{2}, \ldots, v_{8}\right)$, $\left(x_{4}, x_{3}, x_{1}, x_{2}, x_{0}, x_{4}, x_{3}, x_{1}\right)$ will satisfy the requirement, where $x_{i}=\left\lfloor\frac{k+i}{5}\right\rfloor$, $i=0,1,2,3,4$.

We are considering $1 \leq t \leq \frac{n-1}{2}$ in what follows. First, we need an estimaton of the sum of rational numbers which take its floor or ceiling values.

Lemma 2.1. For positive integers $a, b$ and nonnegative integer $k$, we have the following.

$$
\begin{aligned}
& \text { (1) }\left\lfloor\frac{k+\lfloor x\rfloor}{a}\right\rfloor=\left\lfloor\frac{k+x}{a}\right\rfloor \text { for any real number } x \text {, } \\
& \text { (2) } k=\sum_{i=0}^{a-1}\left\lfloor\frac{k+i}{a}\right\rfloor \text {, and } \\
& \text { (3) }\left\lceil\frac{a k}{b}\right\rceil=\sum_{i=1}^{a}\left\lfloor\frac{k+\lceil i b / a\rceil-1}{b}\right\rfloor
\end{aligned}
$$

Proof. (1) and (2) are easy to check, we prove (3).

$$
\begin{aligned}
\left\lceil\frac{a k}{b}\right\rceil & =\sum_{i=0}^{a-1}\left\lfloor\frac{\lceil a k / b\rceil+i}{a}\right\rfloor \\
& =\sum_{i=0}^{a-1}\left\lfloor\frac{\lfloor(a k+b-1) / b\rfloor+i}{a}\right\rfloor=\sum_{i=0}^{a-1}\left\lfloor\frac{(a k+b-1) / b+i}{a}\right\rfloor \\
& =\sum_{i=0}^{a-1}\left\lfloor\frac{k+((i+1) b-1) / a}{b}\right\rfloor=\sum_{i=0}^{a-1}\left\lfloor\frac{k+\lfloor((i+1) b-1) / a\rfloor}{b}\right\rfloor \\
& =\sum_{i=0}^{a-1}\left\lfloor\frac{k+\lceil((i+1) b-1-a+1) / a\rceil}{b}\right\rfloor=\sum_{i=0}^{a-1}\left\lfloor\frac{k+\lceil(i+1) b / a\rceil-1}{b}\right\rfloor .
\end{aligned}
$$

Since the variables $a, b$ and $k$ are all integers, the uniqueness of the formula in Lemma 2.1(3) can be confirmed.

Corollary 2.2. If integers $0 \leq s_{0} \leq s_{1} \leq \cdots \leq s_{a-1}<b$ satisfy

$$
\left\lceil\frac{a k}{b}\right\rceil=\sum_{i=0}^{a-1}\left\lfloor\frac{k+s_{i}}{b}\right\rfloor
$$

for positive integers $a, b$ and nonnegative integer $k$, then $s_{i}=\lceil(i+1) b / a\rceil-1$ for $i=0,1, \ldots, a-1$.

According to the $s_{i}$ 's given above, we split $[b]:=\{0,1, \ldots, b-1\}$ into subintervals with maximal elements $s_{i}$ 's. For positive integers $a<b$, let [b] be partitioned into $a$ subsets such that

$$
S_{i}=\left\{\left\lceil\frac{b i}{a}\right\rceil,\left\lceil\frac{b i}{a}\right\rceil+1, \ldots,\left\lceil\frac{b(i+1)}{a}\right\rceil-1\right\}
$$

for $i=0,1, \ldots, a-1$. It is clear that $\| S_{i}\left|-\left|S_{j}\right|\right| \leq 1$ for all $i, j$. We analyze the subsets containing more elements in the following.

Lemma 2.3. Let $q$ and $r$ be the quotient and remainder of $b$ divided by $a$, respectively. Then the cardinality $\left|S_{i}\right|=q+1$ if $i=\left\lfloor\frac{a j}{r}\right\rfloor$ for $j=0,1, \ldots, r-1$.

Proof. By definition, $\left|S_{i}\right|=\lceil(i+1) b / a\rceil-\lceil i b / a\rceil=q+\lceil(i+1) r / a\rceil-\lceil i r / a\rceil$. Therefore, $\left|S_{i}\right|=q+1$ if and only if there exists some integer $0 \leq j \leq r-1$ such that $i r / a \leq j<(i+1) r / a$. The above inequality can be rewritten as $i \leq j a / r<i+1$, and hence $i=\lfloor j a / r\rfloor$.

Example 2.4. Let $a=3, b=8$, and $r=2$ be the remainder of $b$ divided by $a$. Then

$$
\begin{equation*}
\left\lceil\frac{3 k}{8}\right\rceil=\sum_{i=1}^{3}\left\lfloor\frac{k+\lceil 8 i / 3\rceil-1}{8}\right\rfloor=\left\lfloor\frac{k+2}{8}\right\rfloor+\left\lfloor\frac{k+5}{8}\right\rfloor+\left\lfloor\frac{k+7}{8}\right\rfloor \tag{1}
\end{equation*}
$$

and $[8]=\{0,1, \ldots, 7\}$ can be partitioned into 3 subsets such that

$$
S_{0}=\{0,1,2\}, S_{1}=\{3,4,5\} \text { and } S_{2}=\{6,7\}
$$

where the subsets numbered with $\left\lfloor\frac{a j}{r}\right\rfloor=0$ and 1 as $j=0$ and 1 , respectively, have more than 1 elements. Note that the maximal elements $2,5,7$ of subsets $S_{i}$ 's are the integers in (11) that construct $\left\lceil\frac{3 k}{8}\right\rceil$.

Additionally, we need a result of the comparison between two sequences. For two real finite non-decreasing sequences $A=\left(a_{i}\right), A^{\prime}=\left(a_{i}^{\prime}\right)$ of the same length $n$, we say that $A \leq A^{\prime}$ if $a_{i} \leq a_{i}^{\prime}$ for $i=0,1, \ldots, n-1$.

Lemma 2.5. Let $A$ and $A^{\prime}$ be two subsequences of a real finite non-decreasing sequence $B$ which have equal length $0<|A|=\left|A^{\prime}\right|<|B|$. Then $A \leq A^{\prime}$ if and only if $B \backslash A^{\prime} \leq B \backslash A$.

Proof. Because of the symmetry, we prove $A \leq A^{\prime}$ implies $B \backslash A^{\prime} \leq B \backslash A$ by induction on $|A|$ in the following. It is clearly true when $|A|=1$. Suppose the statement is correct for $|A|<m<|B|$. Assume that $A=\left(a_{i}\right)_{i=0}^{m-1}$ and $A^{\prime}=\left(a_{i}^{\prime}\right)_{i=0}^{m-1}$ satisfying $A \leq A^{\prime}$. From induction hypothesis, $B \backslash\left(a_{i}^{\prime}\right)_{i=1}^{m-1} \leq$ $B \backslash\left(a_{i}\right)_{i=1}^{m-1}$. It is clear that the non-decreasing sequence obtained by exchanging an entry $a$ of the original sequence into $\tilde{a} \geq a$ (and inserting $\tilde{a}$ to the appropriate position) is not less than the original sequence. Thus, we have

$$
B \backslash A^{\prime} \leq B \backslash \widetilde{A} \leq B \backslash A
$$

where $\widetilde{A}$ is obtained from $A$ by deleting $a_{0}$ and adding $a_{0}^{\prime}$. The result follows.

Now, we are ready to find the desired integer $\{k\}$-domination function $f$. Let $[a]:=\{0,1, \ldots, a-1\}$ for each positive integer $a$. For a sequence $A$ of length $a$, let the entries of $A$ indexed by $[a]$ and $A(i)$ be the $i$-th entry of $A$. For $0 \leq i<j \leq a$, the subsequence $A[i: j]:=[A(i), A(i+1), \ldots, A(j-1)]$. If $A$ is a permutation of $[a]$, then the complement of $A$ is a sequence $\bar{A}$ of length $a$ defined as $\bar{A}(i)=a-1-A(i)$ for $0 \leq i \leq a-1$. The concatenation $A \circ B$ of two sequences $A$ and $B$ of lengths $a$ and $b$, respectively, is a sequence of length
$a+b$ obtained by attaching $B$ to $A$ defined as $(A \circ B)(i)=A(i)$ if $0 \leq i \leq a-1$ and $(A \circ B)(j)=B(j-a)$ if $a \leq j \leq a+b-1$.

Let $A$ be a permutation of $[a]$ and thus a sequence of length $a$. For positive integers $a<b$, we call $B$ the extension sequence of the pair $(A, b)$ if $B$ is a permutation of $[b]$ satisfying $B(i)<B(j)$ if and only if $A\left(i_{0}\right)<A\left(j_{0}\right)$ or $A\left(i_{0}\right)=A\left(j_{0}\right)$ with $i<j$, where $i_{0}$ and $j_{0}$ are the remainders of $i$ and $j$ divided by $a$, respectively. For example, when $(a, b)=(3,7)$ and $A=[0,1,2]$, the extension sequence of $(A, b)$ is $B=[0,3,5,1,4,6,2]$, which is attained by extending $A$ to the sequence $[0,1,2,0,1,2,0]$ of length 7 and renumbering it with $0,1, \ldots, 6$.

A permutation $A$ of $[a]$ is said to be nice corresponding to some $b>a$ with $a \nmid b$ if

$$
\begin{equation*}
A(i)<A(i+r) \text { for } 0 \leq i \leq a-r-1 \tag{2}
\end{equation*}
$$

and

$$
\begin{equation*}
A(j)<A(j-a+r) \text { for } a-r \leq j \leq a-d-1 \tag{3}
\end{equation*}
$$

where $r$ is the remainder of $b$ divided by $a$ and $d=\operatorname{gcd}(a, b)$. Note that if $r=d$ then the condition (3) can be ignored. For example, $[1,3,0,2,4]$ is nice corresponding to 8 (or any larger integer congruent to 3 modulo 5) and $[4,1,6,3,0,5,2,7]$ is nice corresponding to 13 .

The following properties will carry out the recursive constructions.
Proposition 2.6. Suppose that $R$ is a nice permutation of $[r]$ corresponding to some $a>r$ with $r \nmid a$. Let $\bar{R}$ be the complement of $R$. Then the extension sequence of $(\bar{R}, a)$ is also nice corresponding to some $b>a$ with $b \equiv r(\bmod a)$.

Proof. Let $A$ be the extension sequence of $(\bar{R}, a)$. Note that $A(i)<A(i+r)$ for $0 \leq i \leq a-r-1$ can be verified directly by the definition of extension sequences. Assume that $s$ is the remainder of $a$ divided by $r$. It's left to consider the case $a-r \leq j \leq a-d-1$ where $r$ is the remainder of $b$ divided by $a$ and $d=\operatorname{gcd}(a, b)$. Assume that $s$ is the remainder of $a$ divided by $r$. By Euclidean algorithm, $s \geq d$.
Case 1: $a-r \leq j \leq a-s-1$.
In order to show $A(j)<A(j-a+r)$, we observe that the remainders of $j$ and $j-a+r$ divided by $r$ are $j^{\prime}+s$ and $j^{\prime}$, respectively, where $j^{\prime}=j-a+r$. Moreover, since $R$ is nice, we have $\bar{R}(i+s)<\bar{R}(i)$ for $0 \leq i \leq r-s-1$. The result is straightforward by the definition of extension sequences.
Case 2: $a-s \leq j \leq a-d-1$.
In this case, the remainders of $j$ and $j-a+r$ divided by $r$ become $j^{\prime}-r+s$ and $j^{\prime}$, respectively, where $j^{\prime}=j-a+r$. Once again, since $R$ is nice, $\bar{R}(i-r+s)<\bar{R}(i)$ for $r-s \leq i \leq r-d-1$. We have the proof.

Proposition 2.7. Let positive integers $a<b$ with $r>0$ the remainder of $b$ divided by $a$ and $R$ a permutation of $[r]$ with complement $\bar{R}$. If the extension sequence $A$ of $(R, a)$ satisfies

$$
\left\lceil\frac{r k}{a}\right\rceil=\sum_{i=a-r}^{a-1}\left\lfloor\frac{k+A(i)}{a}\right\rfloor
$$

then the extension sequence $B$ of $\left(A^{\prime}, b\right)$ satisfies

$$
\left\lceil\frac{a k}{b}\right\rceil=\sum_{i=b-a}^{b-1}\left\lfloor\frac{k+B(i)}{b}\right\rfloor
$$

where $A^{\prime}$ is the extension sequence of $(\bar{R}, a)$.
Proof. By Corollary 2.2,

$$
\{A(i) \mid a-r \leq i \leq a-1\}=\left\{\left.\left\lceil\frac{a j}{r}\right\rceil-1 \right\rvert\, 1 \leq j \leq r\right\}
$$

Let $q$ and $s$ be the quotient and remainder of $a$ divided by $r$, respectively. Claim that the set of $A^{\prime}[0: r]$ equals the set of $a-1-A[a-r: a-1]$. It is clear for $s=0$. If $s>0$, we have

$$
A(i)=a-1-A^{\prime}(a-s+i) \quad \text { for } i=0,1, \ldots, s-1
$$

since entries in $A$ that larger than $A(i)$ become smaller than $A^{\prime}(a-s+i)$ in $A^{\prime}$, and vice versa. Therefore, the set of $A[s+j r: s+(j+1) r]$ equals to the set of $a-1-A^{\prime}[a-s-(j+1) r: a-s-j r]$ for $j=0,1, \ldots, q$. The claim follows by taking $j=q-1$. Moreover, since

$$
a-1-\left(\left\lceil\frac{a i}{r}\right\rceil-1\right)=a+\left\lfloor\frac{-a i}{r}\right\rfloor=\left\lfloor\frac{a(r-i)}{r}\right\rfloor
$$

for $1 \leq i \leq r$, we have

$$
\left\{A^{\prime}(i) \mid 0 \leq i \leq r-1\right\}=\left\{\left.\left\lfloor\frac{a j}{r}\right\rfloor \right\rvert\, 0 \leq j \leq r-1\right\}
$$

which exactly indicates the indices of subsets defined in Lemma 2.3. Hence the set of $B[b-a: b]$ gives the maximal elements in each of the subsets $S_{0}, S_{1}, \ldots, S_{a-1}$, and this fact completes the proof.

For each pair of positive integers $(a, b)$ with $a<b$, define two codes $C_{1}$ and $C_{2}$ as follows. If $a$ divides $b$, then

$$
C_{1}(a, b):=[0,1, \ldots, a-1] .
$$

If the remainder $r$ of $b$ divided by $a$ is positive, then

$$
C_{1}(a, b):=\text { the extension sequence of }\left(\overline{C_{1}(r, a)}, a\right)
$$

where $\overline{C_{1}}(r, a)$ is the complement of $C_{1}(r, a)$. Now $C_{2}$ can be constructed subsequently. Let $C_{2}(a, b)$ be the extension sequence of $\left(C_{1}(a, b), b\right)$. It is clear that $C_{1}(a, b)$ and $C_{2}(a, b)$ are permutations of $[a]$ and $[b]$, respectively. Suppose that each entry $\alpha$ in $C_{2}(a, b)$ is corresponding to $\left\lfloor\frac{k+\alpha}{b}\right\rfloor$. Then the following result can be obtained by proving that

$$
C:=B[b-a: b] \circ \underbrace{B \circ B \circ \cdots \circ B}_{q}
$$

is a feasible distribution of the circulant graph $G$, where $b=2 t+1, n=q b+a$, and $B=C_{2}(a, b)$.

Theorem 2.8.

$$
\gamma_{k}(G)=\left\lceil\frac{k n}{2 t+1}\right\rceil
$$

Proof. By Proposition [1.1 it suffices to show that $\gamma_{k} \leq\left\lceil\frac{k n}{2 t+1}\right\rceil$. Let $G=$ $G(n ;\{1,2, \ldots, t\}), n=q b+a$ and $b=2 t+1$. First, we construct $B[b-a: b]$. If $a$ divides $b$ such that $b=\ell a$, then

$$
B[b-a: b]=[\ell-1,2 \ell-1, \ldots, a \ell-1]
$$

which collects the numbers $\lceil i b / a\rceil-1$ for $1 \leq i \leq a$ given in Lemma 2.1 Therefore, any substring of $C$ of length $a$ is not larger than $B[b-a: b]$. By Lemma 2.5, every length $b$ string of $B \circ B[b-a: b]$ or $B[b-a: b] \circ B$ is not less than $[0,1, \ldots, b-1]$, so does $C$. Furthermore, the sequence $[0,1, \ldots, b-1]$ is of sum

$$
\sum_{i=0}^{b-1}\left\lfloor\frac{k+i}{b}\right\rfloor=k
$$

which confirms the case for $a$ divides $b$.
On the other hand, let $a>\operatorname{gcd}(a, b)$ and $r$ be the remainder of $b$ divided by $a$. Since the initial case is examined above, by Proposition [2.7 $B[b-a: b]$ collects the elements $\{\lceil i b / a\rceil-1\}_{i=1}^{a}$. For the initial case, if $r$ divides $a$ then $C_{1}(r, a)=[0,1, \ldots, r-1]$ and it is easy to check that $C_{1}(a, b)$ is nice. Moreover, by Proposition 2.6, $C_{1}(a, b)$ is always nice, and hence

$$
C[i: i+a] \leq B[b-a: b] \quad \text { for } 0 \leq i \leq a-d-1
$$

where $d=\operatorname{gcd}(a, b)$. Moreover, we also have $C[i: i+a] \leq B[b-a: b]$ for $a-d \leq i \leq q b-1$ immediately from the construction of $C$. The result follows.

Example 2.9. Assume that $G(n ; D)$ is a circulant graph on $n=8$ vertices with $D=\{1,2\}$ (i.e., $t=2$ ). Let $b=2 t+1=5$ and $a=3$ be the remainder of $n$ divided by $b$. First of all, we obtain $C_{1}(3,5)$ by the process of Euclidean algorithm. Since the initial condition $C_{1}(1,2)=[0], C_{1}(2,3)=[0,1]$ is directly
the extension code of [0]. Next, the complement of $C_{1}(2,3)$ is $\overline{C_{1}(2,3)}=[1,0]$. Thus, $C_{1}(3,5)=[1,0,2]$, the extension code of $([1,0], 3)$. Hence,

$$
C_{2}(3,5)=[2,0,4,3,1],
$$

the extension code of $\left(C_{1}(3,5), 5\right)$. Attach the last 3 entries in front of $C_{2}(3,5)$, we attain the distribution $[4,3,1,2,0,4,3,1]$. Then the circular sequence $f(v): v \in$ $V(G)$ is given by

$$
\left(\left\lfloor\frac{k+4}{5}\right\rfloor,\left\lfloor\frac{k+3}{5}\right\rfloor,\left\lfloor\frac{k+1}{5}\right\rfloor,\left\lfloor\frac{k+2}{5}\right\rfloor,\left\lfloor\frac{k}{5}\right\rfloor,\left\lfloor\frac{k+4}{5}\right\rfloor,\left\lfloor\frac{k+3}{5}\right\rfloor,\left\lfloor\frac{k+1}{5}\right\rfloor\right)
$$

which satisfies $\sum_{v \in V(G)} f(v)=\lceil 8 k / 5\rceil$ and $\sum_{u \in N_{G}[v]} f(u) \geq k$ for each $v \in$ $V(G)$.

## 3 Concluding remark

We remark finally that the construction of code $C_{2}$ can be obtained by giving an algorithm with inputs $a$ and $b$.

```
Data: Positive integers \(a<b\).
Result: \(C_{2}(a, b)\).
\(C_{1}(a, b)\) if \(a=\operatorname{gcd}(a, b)\) then
    | return \([0,1, \ldots, a-1]\);
end
else
    \(r \leftarrow\) the remainder of \(b\) divided by \(a\);
    \(R \leftarrow C_{1}(r, a)\);
    return the extension sequence of \((\bar{R}, a)\);
end
Main \((a, b)\) return the extension sequence of \(\left(C_{1}(a, b), b\right)\);
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