On the integer $\{k\}$ -domination number of circulant graphs

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Abstract

Let G = (V, E) be a simple undirected graph. G is a circulant graph defined on $V = \mathbb{Z}_n$ with difference set $D \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ provided two vertices i and j in \mathbb{Z}_n are adjacent if and only if $\min\{|i-j|, n-|i-j|\} \in D$. For convenience, we use G(n; D) to denote such a circulant graph.

A function $f: V(G) \to \mathbb{N} \cup \{0\}$ is an integer $\{k\}$ -domination function if for each $v \in V(G)$, $\sum_{u \in N_G[v]} f(u) \ge k$. By considering all $\{k\}$ -domination functions f, the minimum value of $\sum_{v \in V(G)} f(v)$ is the $\{k\}$ -domination number of G, denoted by $\gamma_k(G)$. In this paper, we prove that if D = $\{1, 2, \ldots, t\}, 1 \le t \le \frac{n-1}{2}$, then the integer $\{k\}$ -domination number of G(n; D) is $\lceil \frac{kn}{2t+1} \rceil$.

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1 Introduction and preliminaries

The study of domination number of a graph G has been around for quite a long time. Due to its importance in applications, there are various versions of extension study, see [6] for reference.

The idea of integer $\{k\}$ -domination was proposed by Domke et al. in [3]. It can be dealt as a labeling problem. The vertices of the graph G are labeled by integers in $\mathbb{N} \cup \{0\}$ such that for each vertex v, the total (sum) values in its closed neighborhood $N_G[v]$ must be at least k. The problem is asking for finding the minimum total value labeled on G. Finally, we say that $f: V(G) \to \mathbb{N} \cup \{0\}$

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is an integer $\{k\}$ -domination function if for each $v \in V(G)$, $\sum_{u \in N_G[v]} f(u) \ge k$. Among all such functions f, the minimum value of $\sum_{v \in V(G)} f(v)$ is called the integer $\{k\}$ -domination number of G, denoted by $\gamma_k(G)$.

It is not difficult to see that the original domination number of a graph G, $\gamma(G)$, can be recognized as $\gamma_1(G)$ since the vertices with label "1" gives a dominating set. For more information about domination problem, the readers may refer to [2, 4, 5, 6, 8]. Hence, the integer $\{k\}$ -domination problem is also an NP-hard problem. So far, results obtained are all on special classes of graphs, see [1, 3, 7, 9].

In this paper, we shall consider the class of circulant graph G = G(n; D)where $D = \{1, 2, ..., t\}, 1 \le t \le \frac{n-1}{2}$, i.e., $V(G) = \mathbb{Z}_n$ and two vertices i and j are adjacent if and only if $d(i, j) := \min\{|i - j|, n - |i - j|\} \in D$. Since $D = \{1, 2, ..., t\}, G(n; D)$ is exactly the power graph C_n^t where C_n is a cycle of order n.

The following results are obtained by Lin [10]. For clearness, we also outline its proof in which basic linear algebra is applied.

Proposition 1.1 ([10]). Let G be the circulant graph G(n; D) where $D = \{1, 2, \ldots, t\}$. Then, $\gamma_k(G) \ge \lceil \frac{kn}{2t+1} \rceil$.

Proof. Let A be the adjacency matrix of G and f be an $\{k\}$ -domination function of G. Let $\mathbf{1}_n$ denote the all 1 column vector of length n. Then, we have

$$(2t+1)\sum_{v\in V(G)}f(v) = (f(v_1), f(v_2), \dots, f(v_n))(A+I_n)\mathbf{1}_n \ge \mathbf{1}_n^T \cdot k \cdot \mathbf{1}_n = nk,$$

which implies the inequality.

By the aid of an algorithm, Lin was able to show the following.

Proposition 1.2 ([10]). For $t \leq 5$, $\gamma_k(G(n; \{1, 2, ..., t\})) = \lceil \frac{kn}{2t+1} \rceil$.

But, for larger t, it remains unsettled. Our main result of this paper shows that the equality holds for all $1 \le t \le \frac{n-1}{2}$.

2 The main result

By Proposition 1.1, in order to determine $\gamma_k(G)$, it suffices to show that $\gamma_k(G) \leq \lceil \frac{nk}{2t+1} \rceil$. That is, we need a proper distribution of values for $f(v_1), f(v_2), \ldots, f(v_n)$ such that for each v_i , $\sum_{u \in N_G[v_i]} f(u) \geq k$ and $\sum_{i=1}^n f(v_i) \leq \lceil \frac{nk}{2t+1} \rceil$. Since we are dealing with circulant graphs, $\sum_{u \in N_G[v_i]} f(u)$ is in fact the sum of 2t + 1 consecutive labels assigned to the circle $C_n = (v_1, v_2, \ldots, v_n)$. Therefore, we turn our focus on providing suitable labels to meet the condition.

For example, let n = 8 and t = 2. Then, the following labeling of (v_1, v_2, \ldots, v_8) , $(x_4, x_3, x_1, x_2, x_0, x_4, x_3, x_1)$ will satisfy the requirement, where $x_i = \lfloor \frac{k+i}{5} \rfloor$, i = 0, 1, 2, 3, 4.

We are considering $1 \le t \le \frac{n-1}{2}$ in what follows. First, we need an estimaton of the sum of rational numbers which take its floor or ceiling values.

Lemma 2.1. For positive integers a, b and nonnegative integer k, we have the following.

- (1) $\lfloor \frac{k+\lfloor x \rfloor}{a} \rfloor = \lfloor \frac{k+x}{a} \rfloor$ for any real number x,
- (2) $k = \sum_{i=0}^{a-1} \lfloor \frac{k+i}{a} \rfloor$, and
- (3) $\left\lceil \frac{ak}{b} \right\rceil = \sum_{i=1}^{a} \left\lfloor \frac{k + \left\lceil ib/a \right\rceil 1}{b} \right\rfloor.$

Proof. (1) and (2) are easy to check, we prove (3).

$$\begin{bmatrix} \frac{ak}{b} \end{bmatrix} = \sum_{i=0}^{a-1} \lfloor \frac{\lfloor ak/b \rfloor + i}{a} \rfloor$$

$$= \sum_{i=0}^{a-1} \lfloor \frac{\lfloor (ak+b-1)/b \rfloor + i}{a} \rfloor = \sum_{i=0}^{a-1} \lfloor \frac{(ak+b-1)/b+i}{a} \rfloor$$

$$= \sum_{i=0}^{a-1} \lfloor \frac{k + ((i+1)b-1)/a}{b} \rfloor = \sum_{i=0}^{a-1} \lfloor \frac{k + \lfloor ((i+1)b-1)/a \rfloor}{b} \rfloor$$

$$= \sum_{i=0}^{a-1} \lfloor \frac{k + \lceil ((i+1)b-1-a+1)/a \rceil}{b} \rfloor = \sum_{i=0}^{a-1} \lfloor \frac{k + \lceil (i+1)b/a \rceil - 1}{b} \rfloor.$$

Since the variables a, b and k are all integers, the uniqueness of the formula in Lemma 2.1(3) can be confirmed.

Corollary 2.2. If integers $0 \le s_0 \le s_1 \le \cdots \le s_{a-1} < b$ satisfy

$$\lceil \frac{ak}{b} \rceil = \sum_{i=0}^{a-1} \lfloor \frac{k+s_i}{b} \rfloor$$

for positive integers a, b and nonnegative integer k, then $s_i = \lceil (i+1)b/a \rceil - 1$ for i = 0, 1, ..., a - 1.

According to the s_i 's given above, we split $[b] := \{0, 1, \ldots, b-1\}$ into subintervals with maximal elements s_i 's. For positive integers a < b, let [b] be partitioned into a subsets such that

$$S_i = \left\{ \lceil \frac{bi}{a} \rceil, \lceil \frac{bi}{a} \rceil + 1, \dots, \lceil \frac{b(i+1)}{a} \rceil - 1 \right\}$$

for i = 0, 1, ..., a - 1. It is clear that $||S_i| - |S_j|| \le 1$ for all i, j. We analyze the subsets containing more elements in the following.

Lemma 2.3. Let q and r be the quotient and remainder of b divided by a, respectively. Then the cardinality $|S_i| = q + 1$ if $i = \lfloor \frac{aj}{r} \rfloor$ for j = 0, 1, ..., r - 1.

Proof. By definition, $|S_i| = \lceil (i+1)b/a \rceil - \lceil ib/a \rceil = q + \lceil (i+1)r/a \rceil - \lceil ir/a \rceil$. Therefore, $|S_i| = q + 1$ if and only if there exists some integer $0 \le j \le r - 1$ such that $ir/a \le j < (i+1)r/a$. The above inequality can be rewritten as $i \le ja/r < i + 1$, and hence $i = \lfloor ja/r \rfloor$.

Example 2.4. Let a = 3, b = 8, and r = 2 be the remainder of b divided by a. Then

$$\lceil \frac{3k}{8} \rceil = \sum_{i=1}^{3} \lfloor \frac{k + \lceil 8i/3 \rceil - 1}{8} \rfloor = \lfloor \frac{k+2}{8} \rfloor + \lfloor \frac{k+5}{8} \rfloor + \lfloor \frac{k+7}{8} \rfloor, \qquad (1)$$

and $[8] = \{0, 1, \dots, 7\}$ can be partitioned into 3 subsets such that

 $S_0 = \{0, 1, 2\}, S_1 = \{3, 4, 5\} and S_2 = \{6, 7\},$

where the subsets numbered with $\lfloor \frac{aj}{r} \rfloor = 0$ and 1 as j = 0 and 1, respectively, have more than 1 elements. Note that the maximal elements 2, 5, 7 of subsets S_i 's are the integers in (1) that construct $\lceil \frac{3k}{8} \rceil$.

Additionally, we need a result of the comparison between two sequences. For two real finite non-decreasing sequences $A = (a_i), A' = (a'_i)$ of the same length n, we say that $A \leq A'$ if $a_i \leq a'_i$ for i = 0, 1, ..., n - 1.

Lemma 2.5. Let A and A' be two subsequences of a real finite non-decreasing sequence B which have equal length 0 < |A| = |A'| < |B|. Then $A \le A'$ if and only if $B \setminus A' \le B \setminus A$.

Proof. Because of the symmetry, we prove $A \leq A'$ implies $B \setminus A' \leq B \setminus A$ by induction on |A| in the following. It is clearly true when |A| = 1. Suppose the statement is correct for |A| < m < |B|. Assume that $A = (a_i)_{i=0}^{m-1}$ and $A' = (a'_i)_{i=0}^{m-1}$ satisfying $A \leq A'$. From induction hypothesis, $B \setminus (a'_i)_{i=1}^{m-1} \leq B \setminus (a_i)_{i=1}^{m-1}$. It is clear that the non-decreasing sequence obtained by exchanging an entry a of the original sequence into $\tilde{a} \geq a$ (and inserting \tilde{a} to the appropriate position) is not less than the original sequence. Thus, we have

$$B \setminus A' \le B \setminus A \le B \setminus A,$$

where \hat{A} is obtained from A by deleting a_0 and adding a'_0 . The result follows.

Now, we are ready to find the desired integer $\{k\}$ -domination function f. Let $[a] := \{0, 1, \ldots, a - 1\}$ for each positive integer a. For a sequence A of length a, let the entries of A indexed by [a] and A(i) be the *i*-th entry of A. For $0 \le i < j \le a$, the subsequence $A[i : j] := [A(i), A(i + 1), \ldots, A(j - 1)]$. If A is a permutation of [a], then the complement of A is a sequence \overline{A} of length a defined as $\overline{A}(i) = a - 1 - A(i)$ for $0 \le i \le a - 1$. The concatenation $A \circ B$ of two sequences A and B of lengths a and b, respectively, is a sequence of length a+b obtained by attaching B to A defined as $(A \circ B)(i) = A(i)$ if $0 \le i \le a-1$ and $(A \circ B)(j) = B(j-a)$ if $a \le j \le a+b-1$.

Let A be a permutation of [a] and thus a sequence of length a. For positive integers a < b, we call B the extension sequence of the pair (A, b) if B is a permutation of [b] satisfying B(i) < B(j) if and only if $A(i_0) < A(j_0)$ or $A(i_0) = A(j_0)$ with i < j, where i_0 and j_0 are the remainders of i and j divided by a, respectively. For example, when (a, b) = (3, 7) and A = [0, 1, 2], the extension sequence of (A, b) is B = [0, 3, 5, 1, 4, 6, 2], which is attained by extending A to the sequence [0, 1, 2, 0, 1, 2, 0] of length 7 and renumbering it with $0, 1, \ldots, 6$.

A permutation A of [a] is said to be *nice* corresponding to some b > a with $a \nmid b$ if

$$A(i) < A(i+r) \quad \text{for } 0 \le i \le a-r-1 \tag{2}$$

and

$$A(j) < A(j - a + r)$$
 for $a - r \le j \le a - d - 1$, (3)

where r is the remainder of b divided by a and d = gcd(a, b). Note that if r = d then the condition (3) can be ignored. For example, [1,3,0,2,4] is nice corresponding to 8 (or any larger integer congruent to 3 modulo 5) and [4,1,6,3,0,5,2,7] is nice corresponding to 13.

The following properties will carry out the recursive constructions.

Proposition 2.6. Suppose that R is a nice permutation of [r] corresponding to some a > r with $r \nmid a$. Let \overline{R} be the complement of R. Then the extension sequence of (\overline{R}, a) is also nice corresponding to some b > a with $b \equiv r \pmod{a}$.

Proof. Let A be the extension sequence of (\overline{R}, a) . Note that A(i) < A(i+r) for $0 \le i \le a-r-1$ can be verified directly by the definition of extension sequences. Assume that s is the remainder of a divided by r. It's left to consider the case $a-r \le j \le a-d-1$ where r is the remainder of b divided by a and $d = \gcd(a, b)$. Assume that s is the remainder of a divided by r. By Euclidean algorithm, $s \ge d$.

Case 1: $a - r \le j \le a - s - 1$.

In order to show A(j) < A(j - a + r), we observe that the remainders of j and j - a + r divided by r are j' + s and j', respectively, where j' = j - a + r. Moreover, since R is nice, we have $\overline{R}(i + s) < \overline{R}(i)$ for $0 \le i \le r - s - 1$. The result is straightforward by the definition of extension sequences.

Case 2: $a - s \le j \le a - d - 1$.

In this case, the remainders of j and j-a+r divided by r become j'-r+s and j', respectively, where j' = j-a+r. Once again, since R is nice, $\overline{R}(i-r+s) < \overline{R}(i)$ for $r-s \le i \le r-d-1$. We have the proof.

Proposition 2.7. Let positive integers a < b with r > 0 the remainder of b divided by a and R a permutation of [r] with complement \overline{R} . If the extension sequence A of (R, a) satisfies

$$\lceil \frac{rk}{a} \rceil = \sum_{i=a-r}^{a-1} \lfloor \frac{k+A(i)}{a} \rfloor,$$

then the extension sequence B of (A', b) satisfies

$$\lceil \frac{ak}{b} \rceil = \sum_{i=b-a}^{b-1} \lfloor \frac{k+B(i)}{b} \rfloor$$

where A' is the extension sequence of (\overline{R}, a) .

Proof. By Corollary 2.2,

$$\{A(i) \mid a - r \le i \le a - 1\} = \left\{ \lceil \frac{aj}{r} \rceil - 1 \mid 1 \le j \le r \right\}.$$

Let q and s be the quotient and remainder of a divided by r, respectively. Claim that the set of A'[0:r] equals the set of a-1-A[a-r:a-1]. It is clear for s=0. If s>0, we have

$$A(i) = a - 1 - A'(a - s + i)$$
 for $i = 0, 1, \dots, s - 1$,

since entries in A that larger than A(i) become smaller than A'(a-s+i) in A', and vice versa. Therefore, the set of A[s+jr:s+(j+1)r] equals to the set of a-1-A'[a-s-(j+1)r:a-s-jr] for $j=0,1,\ldots,q$. The claim follows by taking j=q-1. Moreover, since

$$a - 1 - \left(\left\lceil \frac{ai}{r} \right\rceil - 1 \right) = a + \left\lfloor \frac{-ai}{r} \right\rfloor = \left\lfloor \frac{a(r-i)}{r} \right\rfloor$$

for $1 \leq i \leq r$, we have

$$\{A'(i) \mid 0 \le i \le r-1\} = \left\{ \lfloor \frac{aj}{r} \rfloor \mid 0 \le j \le r-1 \right\}$$

which exactly indicates the indices of subsets defined in Lemma 2.3. Hence the set of B[b - a : b] gives the maximal elements in each of the subsets $S_0, S_1, \ldots, S_{a-1}$, and this fact completes the proof.

For each pair of positive integers (a, b) with a < b, define two codes C_1 and C_2 as follows. If a divides b, then

$$C_1(a,b) := [0,1,\ldots,a-1]$$

If the remainder r of b divided by a is positive, then

$$C_1(a,b) :=$$
 the extension sequence of $(\overline{C_1(r,a)},a)$

where $\overline{C_1}(r, a)$ is the complement of $C_1(r, a)$. Now C_2 can be constructed subsequently. Let $C_2(a, b)$ be the extension sequence of $(C_1(a, b), b)$. It is clear that $C_1(a, b)$ and $C_2(a, b)$ are permutations of [a] and [b], respectively. Suppose that each entry α in $C_2(a, b)$ is corresponding to $\lfloor \frac{k+\alpha}{b} \rfloor$. Then the following result can be obtained by proving that

$$C := B[b-a:b] \circ \underbrace{B \circ B \circ \cdots \circ B}_{q}$$

is a feasible distribution of the circulant graph G, where b = 2t + 1, n = qb + a, and $B = C_2(a, b)$.

Theorem 2.8.

$$\gamma_k(G) = \lceil \frac{kn}{2t+1} \rceil.$$

Proof. By Proposition 1.1, it suffices to show that $\gamma_k \leq \lceil \frac{kn}{2t+1} \rceil$. Let $G = G(n; \{1, 2, \ldots, t\}), n = qb + a$ and b = 2t + 1. First, we construct B[b - a : b]. If a divides b such that $b = \ell a$, then

$$B[b-a:b] = [\ell - 1, 2\ell - 1, \dots, a\ell - 1]$$

which collects the numbers $\lceil ib/a \rceil - 1$ for $1 \leq i \leq a$ given in Lemma 2.1. Therefore, any substring of C of length a is not larger than B[b - a : b]. By Lemma 2.5, every length b string of $B \circ B[b - a : b]$ or $B[b - a : b] \circ B$ is not less than $[0, 1, \ldots, b - 1]$, so does C. Furthermore, the sequence $[0, 1, \ldots, b - 1]$ is of sum

$$\sum_{i=0}^{b-1} \lfloor \frac{k+i}{b} \rfloor = k,$$

which confirms the case for a divides b.

On the other hand, let $a > \gcd(a, b)$ and r be the remainder of b divided by a. Since the initial case is examined above, by Proposition 2.7, B[b - a : b]collects the elements $\{ \lceil ib/a \rceil - 1 \}_{i=1}^{a}$. For the initial case, if r divides a then $C_1(r, a) = [0, 1, \ldots, r-1]$ and it is easy to check that $C_1(a, b)$ is nice. Moreover, by Proposition 2.6, $C_1(a, b)$ is always nice, and hence

$$C[i:i+a] \le B[b-a:b] \text{ for } 0 \le i \le a-d-1,$$

where d = gcd(a, b). Moreover, we also have $C[i : i + a] \leq B[b - a : b]$ for $a - d \leq i \leq qb - 1$ immediately from the construction of C. The result follows. \Box

Example 2.9. Assume that G(n; D) is a circulant graph on n = 8 vertices with $D = \{1, 2\}$ (i.e., t = 2). Let b = 2t + 1 = 5 and a = 3 be the remainder of n divided by b. First of all, we obtain $C_1(3,5)$ by the process of Euclidean algorithm. Since the initial condition $C_1(1,2) = [0], C_1(2,3) = [0,1]$ is directly

the extension code of [0]. Next, the complement of $C_1(2,3)$ is $\overline{C_1(2,3)} = [1,0]$. Thus, $C_1(3,5) = [1,0,2]$, the extension code of ([1,0],3). Hence,

$$C_2(3,5) = [2,0,4,3,1],$$

the extension code of $(C_1(3,5),5)$. Attach the last 3 entries in front of $C_2(3,5)$, we attain the distribution [4,3,1,2,0,4,3,1]. Then the circular sequence $f(v): v \in V(G)$ is given by

$$(\lfloor \frac{k+4}{5} \rfloor, \lfloor \frac{k+3}{5} \rfloor, \lfloor \frac{k+1}{5} \rfloor, \lfloor \frac{k+2}{5} \rfloor, \lfloor \frac{k}{5} \rfloor, \lfloor \frac{k+4}{5} \rfloor, \lfloor \frac{k+3}{5} \rfloor, \lfloor \frac{k+1}{5} \rfloor)$$

which satisfies $\sum_{v \in V(G)} f(v) = \lceil 8k/5 \rceil$ and $\sum_{u \in N_G[v]} f(u) \ge k$ for each $v \in V(G)$.

3 Concluding remark

We remark finally that the construction of code C_2 can be obtained by giving an algorithm with inputs a and b.

Data: Positive integers a < b. **Result:** $C_2(a, b)$. $C_1(a,b)$ if a = gcd(a, b) then | return [0, 1, ..., a - 1]; end else | $r \leftarrow$ the remainder of b divided by a; $R \leftarrow C_1(r, a)$; return the extension sequence of (\overline{R}, a) ; end Main(a,b) return the extension sequence of $(C_1(a, b), b)$;

References

- B. Brešar, M. A. Henning and S. Klavžar, On integer domination in graphs and vizing-like problems, *Taiwanese J. Math.* 10(2006) 1317-1328.
- [2] T. T. Chelvam and S. Mutharasu, Bounds for domination parameter in circulant graphs, Advanced Studies in Contemporary Mathematics (Kyungshang) 22(4)(2012) 525-529.
- [3] G. Domke, S. T. Hedetniemi, R. C. Laskar and G. Fricke, Relationships between integer and fractional parameters of graphs, *Graph Theory, Com*binatorics, and applications 1(1991) 371-387.
- [4] D. Gonçalves, A. Pinlou, M. Rao and S. Thomassé, The domination number of grids, SIAM J. Discrete Math. 25(3)(2011) 1443-1453.

- [5] R. M. Gray, Toeplitz and circulant matrix: a review, Fundation and Trends in Communication Theory 2(2006) 155-239.
- [6] T. W. Haynes and S. T. Hedetniemi, P. J. Slater, Domination in graphs: advanced topics, Marcel Deliker, New York (1998).
- [7] X. Hou and Y. Lu, On the {k}-domination number of Cartesian products of graphs, *Discrete Math.* 309(2009) 3413-3419.
- [8] N. John and S. Suen, Graph products and integer domination, *Discrete Math.* 313(2013) 217-224.
- [9] Y.-T. Kuan, A study of integer domination number, M. S. Thesis, National Chian Tung University (2017).
- [10] X. Lin, Integer $\{k\}$ -domination number of circulant graphs, M. S. thesis, National Chiao Tung University (2018).