# On normalized Laplacian eigenvalues of power graphs associated to finite cyclic groups

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Abstract. For a simple connected graph G of order n, the normalized Laplacian is a square matrix of order n, defined as  $\mathcal{L}(G) = D(G)^{-\frac{1}{2}}L(G)D(G)^{-\frac{1}{2}}$ , where  $D(G)^{-\frac{1}{2}}$  is the diagonal matrix whose i-th diagonal entry is  $\frac{1}{\sqrt{d_i}}$ . In this article, we find the normalized Laplacian eigenvalues of the joined union of regular graphs in terms of the adjacency eigenvalues and the eigenvalues of quotient matrix associated with graph G. For a finite group G, the power graph  $\mathcal{P}(G)$  of a group G is defined as the simple graph in which two distinct vertices are joined by an edge if and only if one is the power of other. As a consequence of the joined union of graphs, we investigate the normalized Laplacian eigenvalues of power graphs of finite cyclic group  $\mathbb{Z}_n$ .

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#### 1 Introduction

A simple graph is denoted by G(V(G), E(G)), where  $V(G) = \{v_1, v_2, \dots, v_n\}$  is its vertex set and E(G) is its edge set. The *order* of G is n = |V(G)| and *size* is m = |E(G)|. The *neighborhood* 

of a vertex v in G, denoted by N(v), is the set of all those vertices of G which are adjacent to v. The degree  $d_G(v)$  (or  $d_v$ ) of a vertex v is the number of vertices in G that are incident to v. The adjacency matrix, denoted by A(G), is defined by

$$A(G) = \begin{cases} 1 & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise,} \end{cases}$$

where  $v_i \sim v_j$  denotes  $v_i$  is adjacent to  $v_j$  in G. The eigenvalues of A(G) are denoted by  $\lambda_i$  and are called the adjacency eigenvalues of G. Let  $D(G) = diag(d_1, d_2, \ldots, d_n)$  be the diagonal matrix of vertex degrees  $d_i = d_G(v_i)$ ,  $i = 1, 2, \ldots, n$ , associated to G. The real symmetric and positive semi-definite matrix L(G) = D(G) - A(G) is the Laplacian matrix and its eigenvalues are known as Laplacian eigenvalues of G. More literature about adjacency and Laplacian matrix can be found in [11].

The normalized Laplacian matrix of a graph G, denoted by  $\mathcal{L}(G)$ , is defined as

$$\mathcal{L}(G) = \begin{cases} 1 & \text{if } v_i = v_j \text{ and } v_i \neq 0, \\ \frac{-1}{\sqrt{d_{v_i} d_{v_j}}} & \text{if } v_i \sim v_j, \\ 0 & \text{otherwise.} \end{cases}$$

This matrix was introduced by Chung [7] to study the random walks of G and is equivalently defined as  $\mathcal{L}(G) = D^{-\frac{1}{2}}L(G)D^{-\frac{1}{2}}$ , where  $D^{-\frac{1}{2}}$  is the diagonal matrix whose i-th diagonal entry is  $\frac{1}{\sqrt{d_i}}$ . Clearly,  $\mathcal{L}(G)$  is real symmetric and positive semi-definite matrix. Its eigenvalues are real and are known as normalized Laplacian eigenvalues. We denote normalized Laplacian eigenvalues by  $\lambda_i(\mathcal{L})$  and order them as  $0 = \lambda_1(\mathcal{L}) \leq \lambda_2(\mathcal{L}) \leq \cdots \leq \lambda_n(\mathcal{L}) = 2$ . In certain situations, normalized Laplacian is a natural tool that works better than adjacency and Laplacian matrices. More literature about  $\mathcal{L}(G)$  can be seen in [5, 12, 13, 25] and the references therein.

As usual, we denote the complete graph, the bipartite graphs and the cycle graph by  $K_n$ ,  $K_{a,b}$ ,  $C_n$ , respectively. For other notations and terminology, we refer to [11, 22].

The rest of the paper is organized as follows. In Section 2, we obtain the normalized Laplacian eigenvalues of the joined union of regular graphs  $G_1, G_2, \ldots, G_n$  in terms of their adjacency eigenvalues and the eigenvalues of the quotient matrix associated with the joined union. In Section 3, we discuss the normalized Laplacian eigenvalues of the power graphs of the finite cyclic groups  $\mathbb{Z}_n$ .

## 2 Normalized Laplacian eigenvalues of the joined union of graphs

Consider the matrix

$$M = \begin{pmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,s} \\ m_{2,1} & m_{2,2} & \cdots & m_{2,s} \\ \vdots & \vdots & \ddots & \vdots \\ m_{s,1} & m_{s,2} & \cdots & m_{s,s} \end{pmatrix}_{n \times n},$$

whose rows and columns are partitioned according to a partition  $P = \{P_1, P_2, \dots, P_m\}$  of the set  $X = \{1, 2, \dots, n\}$ . The quotient matrix  $\mathcal{Q}$  of the matrix M is the  $s \times s$  matrix whose entries are the average row sums of the blocks  $m_{i,j}$ . The partition P is said to be equitable if each block  $m_{i,j}$  of M has constant row (and column) sum and in this case the matrix  $\mathcal{Q}$  is called as equitable quotient matrix. In general, the eigenvalues of  $\mathcal{Q}$  interlace the eigenvalues of M. In case the partition is equitable, we have following lemma.

**Lemma 2.1** [3, 11] If the partition P of X of matrix M is equitable, then each eigenvalue of Q is an eigenvalue of M.

Let G(V, E) be a graph of order n and  $G_i(V_i, E_i)$  be graphs of order  $n_i$ , where i = 1, ..., n. The *joined union* [24]  $G[G_1, ..., G_n]$  is the graph H(W, F) with

$$W = \bigcup_{i=1}^{n} V_i$$
 and  $F = \bigcup_{i=1}^{n} E_i \bigcup \left(\bigcup_{\{v_i, v_j\} \in E} V_i \times V_j\right).$ 

Equivalently, the joined union  $G[G_1, \ldots, G_n]$  is obtained by joining edges from each vertex of  $G_i$  to every vertex of  $G_j$  whenever  $v_i \sim v_j$  in G. Thus, the usual join  $G_1 \nabla G_2$  is a particular case of the joined union  $K_2[G_1, G_2]$ .

In [26], the authors discussed the normalized Laplacian eigenvalues of  $G[G_1, G_2, \ldots, G_n]$  in terms of the normalized Laplacian eigenvalues of  $G_i$ 's and the eigenvalues of another matrix using the technique of Cardosa et. al. [10]. Using a different approach, we will discuss the normalized Laplacian eigenvalues of  $G[G_1, G_2, \ldots, G_n]$  in terms of the adjacency eigenvalues of the graphs  $G_1, G_2, \ldots, G_n$  and the eigenvalues of the quotient matrix, where each of the  $G_i$  is an  $r_i$  regular graph.

**Theorem 2.2** Let G be a graph of order n and size m. Let  $G_i$  be  $r_i$  regular graphs of order  $n_i$  having adjacency eigenvalues  $\lambda_{i1} = r_i \geq \lambda_{i2} \geq \ldots \geq \lambda_{in_i}$ , where  $i = 1, 2, \ldots, n$ . Then the normalized Laplacian eigenvalues of the graph  $G[G_1, \ldots, G_n]$  are given by

$$1 - \frac{1}{r_i + \alpha_i} \lambda_{ik}(G_i)$$
, for  $i = 1, ..., n$  and  $k = 2, 3, ..., n_i$ ,

where  $\alpha_i = \sum_{v_j \in N_G(v_i)} n_i$  is the sum of the orders of the graphs  $G_j$ ,  $j \neq i$  which correspond to the neighbours of vertex  $v_i \in G$ . The remaining n eigenvalues are given by the equitable quotient matrix M of (2.2).

**Proof.** Let  $V(G) = \{v_1, \ldots, v_n\}$  be the vertex set of G and let  $V(G_i) = \{v_{i1}, \ldots, v_{in_i}\}$  be the vertex set of the graph  $G_i$ , for  $i = 1, 2, \ldots, n$ . Let  $H = G[G_1, \ldots, G_n]$  be the joined union of  $r_i$  regular graphs  $G_i$ , for  $i = 1, 2, \ldots, n$ . It is clear that the order of H is  $N = \sum_{i=1}^n n_i$ . Since degree of each vertex  $v_{ij} \in V(H)$ , is degree inside  $G_i$  and the sum of orders of  $G_j$ 's,  $j \neq i$ , which correspond to the neighbours of the vertex  $v_i$  in G, where  $1 \leq i \leq n$  and  $1 \leq j \leq n_i$ , therefore, for each  $v_{ij} \in V(G_i)$ , we have

$$d_H(v_{ij}) = r_i + \sum_{v_j \in N_G(v_i)} n_j = r_i + \alpha_i,$$
(2.1)

where  $\alpha_i = \sum_{v_j \in N_G(v_i)} n_j$ . Under suitable labelling of the vertices in H, the normalized Laplacian matrix of H can be written as

$$\mathcal{L}(H) = \begin{pmatrix} g_1 & a(v_1, v_2) & \dots & a(v_1, v_n) \\ a(v_2, v_1) & g_2 & \dots & a(v_2, v_n) \\ \vdots & \vdots & \ddots & \vdots \\ a(v_n, v_1) & a(v_n, v_2) & \dots & g_n \end{pmatrix},$$

where, for i = 1, 2, ..., n,

$$g_i = I_{n_i} - \frac{1}{r_i + \alpha_i} A(G_i) \text{ and } a(v_i, v_j) = \begin{cases} \frac{1}{\sqrt{(\alpha_i + r_i)(\alpha_j + r_j)}} J_{n_i \times n_j}, & \text{if } v_i \sim v_j \text{ in } G \\ \mathbf{0}_{n_i \times n_j}, & \text{otherwise.} \end{cases}$$

 $A(G_i)$  is the adjacency matrix of  $G_i$ ,  $J_{n_i \times n_j}$  is the matrix having all entries 1,  $\mathbf{0}_{n_i \times n_j}$  is the zero matrix of order  $n_i \times n_j$  and  $I_{n_i}$  is the identity matrix of order  $n_i$ .

As each  $G_i$  is an  $r_i$  regular graph, so the all one vector  $e_{n_i} = (\underbrace{1, 1, \dots, 1}_{n_i})^T$  is the eigenvector

of the adjacency matrix  $A(G_i)$  corresponding to the eigenvalue  $r_i$  and all other eigenvectors are orthogonal to  $e_{n_i}$ . Let  $\lambda_{ik}$ ,  $2 \leq k \leq n_i$ , be any eigenvalue of  $A(G_i)$  with the corresponding eigenvector  $X = (x_{i1}, x_{i2}, \dots, x_{in_i})^T$  satisfying  $e_{n_i}^T X = 0$ . Clearly, the column vector X can be regarded as a function defined on  $V(G_i)$  assigning the vertex  $v_{ij}$  to  $x_{ij}$ , that is,  $X(v_{ij}) = x_{ij}$  for i = 1, 2, ..., n and  $j = 1, 2, ..., n_i$ . Now, consider the vector  $Y = (y_1, y_2, ..., y_n)^T$ , where

$$y_j = \begin{cases} x_{ij} & \text{if } v_{ij} \in V(G_i) \\ 0 & \text{otherwise.} \end{cases}$$

Since,  $e_{n_i}^T X = 0$  and coordinates of the vector Y corresponding to vertices in  $\bigcup_{j \neq i} V_j$  of H are zeros, we have

$$\mathcal{L}(H)Y = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ X - \frac{1}{r_i + \alpha_i} \lambda_{ik} X \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \left(1 - \frac{1}{r_i + \alpha_i} \lambda_{ik}\right) Y.$$

This shows that Y is an eigenvector of  $\mathcal{L}(H)$  corresponding to the eigenvalue  $1 - \frac{1}{r_i + \alpha_i} \lambda_{ik}$ , for every eigenvalue  $\lambda_{ik}$ ,  $2 \leq k \leq n_i$ , of  $A(G_i)$ . In this way, we have obtained  $\sum_{i=1}^{n} n_i - n = N - n$  eigenvalues. The remaining n normalized Laplacian eigenvalues of H are the eigenvalues of the equitable quotient matrix

$$M = \begin{pmatrix} \frac{\alpha_1}{\alpha_1 + r_1} & \frac{-n_2 a_{12}}{\sqrt{(r_1 + \alpha_1)(r_2 + \alpha_2)}} & \dots & \frac{-n_n a_{1n}}{\sqrt{(r_1 + \alpha_1)(r_n + \alpha_n)}} \\ \frac{-n_1 a_{21}}{\sqrt{(r_2 + \alpha_2)(r_1 + \alpha_1)}} & \frac{\alpha_2}{\alpha_2 + r_2} & \dots & \frac{-n_n a_{2n}}{\sqrt{(r_2 + \alpha_2)(r_n + \alpha_n)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-n_1 a_{n1}}{\sqrt{(r_n + \alpha_n)(r_1 + \alpha_1)}} & \frac{-n_2 a_{n2}}{\sqrt{(r_n + \alpha_n)(r_2 + \alpha_2)}} & \dots & \frac{\alpha_n}{\alpha_n + r_n} \end{pmatrix}, \quad (2.2)$$

where, for  $i \neq j$ ,

$$a_{ij} = \begin{cases} 1, & v_i \sim v_j \\ 0, & \text{otherwise.} \end{cases}$$

The next observation is a consequence of Theorem 2.2 and gives the normalized Laplacian eigenvalues of  $K_{n_1,n_2,...,n_p}$ .

Corollary 2.3 The normalized Laplacian eigenvalues of the complete p-partite graph  $K_{n_1,n_2,...,n_p} = K_p[\overline{K}_{n_1},\overline{K}_{n_2},...,\overline{K}_{n_p}]$  with  $N = \sum_{i=1}^p n_i$  consists of the eigenvalue 1 with multiplicity N-p and the remaining p eigenvalues are given by the matrix

$$\begin{pmatrix}
1 & \frac{-n_2}{\sqrt{\alpha_1 \alpha_2}} & \dots & \frac{-n_p}{\sqrt{\alpha_1 \alpha_p}} \\
\frac{-n_1}{\sqrt{\alpha_2 \alpha_1}} & 1 & \dots & \frac{-n_p}{\sqrt{\alpha_2 \alpha_p}} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{-n_1}{\sqrt{\alpha_p \alpha_2}} & \frac{-n_2}{\sqrt{\alpha_p \alpha_2}} & \dots & 1
\end{pmatrix}.$$

**Proof.** This follows from Theorem 2.2, by taking  $G_i = \overline{K}_i$  and  $\lambda_{ik}(G_i) = 0$  for each i and each k.

In particular, if partite sets are of equal size, say  $n_1 = n_2 = \cdots = n_p = t$ , then we have the following observation.

Corollary 2.4 Let  $G = K_{t,t,...,t}$  be a complete p-partite graph of order N = pt. Then the normalized Laplacian eigenvalues of G consists of the eigenvalue 1 with multiplicity N - p, the eigenvalue  $\frac{p}{p-1}$  with multiplicity p-1 and the simple eigenvalue 0.

**Proof.** By Theorem 2.2, we have  $\alpha_i = t(p-1)$ , for i = 1, 2, ..., p. Also, by Corollary 2.3, we see that 1 is an eigenvalue with multiplicity pt - p and other eigenvalues are given by

$$M_{p} = \begin{pmatrix} 1 & \frac{-1}{p-1} & \dots & \frac{-1}{p-1} \\ \frac{-1}{p-1} & 1 & \dots & \frac{-1}{p-1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-1}{p-1} & \frac{-1}{p-1} & \dots & 1 \end{pmatrix}.$$

By simple calculations, we see that the normalized Laplacian eigenvalues of matrix  $M_p$  consists of the eigenvalue  $\frac{p}{p-1}$  with multiplicity p-1 and the simple eigenvalue 0.

Another consequence of Theorem 2.2, gives the normalized Laplacian eigenvalues of the join of two regular graphs.

Corollary 2.5 Let  $G_i$  be an  $r_i$  regular graph of order  $n_i$  for i = 1, 2. Let  $\lambda_{ik}, 2 \le k \le n_i, i = 1, 2$  be the adjacency eigenvalues of  $G_i$ . Then the normalized Laplacian eigenvalues of  $G = G_1 \nabla G_2$  consists of the eigenvalue  $1 - \frac{1}{r_1 + n_2} \lambda_{1k} A(G_1), k = 2, \ldots, n_1$ , the eigenvalues  $1 - \frac{1}{r_2 + n_1} \lambda_{2k} A(G_1), k = 2, \ldots, n_2$  and the remaining two eigenvalues are given by the quotient

$$\begin{pmatrix}
\frac{n_2}{r_1 + n_2} & \frac{-n_2}{\sqrt{(r_1 + n_2)(r_2 + n_1)}} \\
\frac{-n_1}{\sqrt{(r_1 + n_2)(r_2 + n_1)}} & \frac{n_1}{r_2 + n_1}
\end{pmatrix}.$$
(2.3)

Since  $G_1$  and  $G_2$  are regular graphs, we observe that the two eigenvalues of matrix (2.3) are the largest and the smallest normalized Laplacian eigenvalue of  $G = G_1 \nabla G_2$ .

**Proposition 2.6** The largest and the smallest normalized Laplacian eigenvalues of  $G_1 \nabla G_2$  are the eigenvalues of the matrix (2.3).

Proposition 2.7 (i) The normalized Laplacian eigenvalues of the complete bipartite graph  $K_{a,b} = K_a \nabla K_b$  are

$$\{0,1^{[a+b-2]},2\}$$
.

(ii) The normalized Laplacian eigenvalues of the complete split graph  $CS_{\omega,n-\omega} = K_{\omega} \nabla \overline{K}_{n-\omega}$ , with clique number  $\omega$  and independence number  $n-\omega$  are given by

$$\left\{0, \left(\frac{n}{n-1}\right)^{[\omega-1]}, \frac{2n-\omega+1}{n-1}\right\}.$$

- (iii) The normalized Laplacian eigenvalues of the cone graph  $C_{a,b} = C_a \nabla \overline{K}_b$  consists of the eigenvalues  $1 \frac{1}{2+b} 2\cos\left(\frac{2\pi k}{n}\right)$ , where  $k = 2, \ldots, n-1$ , the simple eigenvalues 0 and  $\frac{2b+2}{b+2}$ .
- (iv) The normalized Laplacian eigenvalues of the wheel graph  $W_n = C_{n-1} \nabla K_1$  consists of the eigenvalues  $1 \frac{1}{3}2\cos\left(\frac{2\pi k}{m}\right)$ , where  $k = 2, \ldots, n-2$ , and the simple eigenvalues  $\left\{0, \frac{4}{3}\right\}$ .

**Proof.** (i). This follows from Corollary 2.5, by taking  $n_1 = a, n_2 = b, r_1 = r_2 = 0$  and  $\lambda_{1k} = 0$ , for k = 2, ..., a and  $\lambda_{2k} = 0$  for each k = 2, ..., b.

(ii). We recall that the adjacency spectrum of  $K_{\omega}$  is  $\{\omega - 1, -1^{(\omega - 1)}\}$ . Now, the result follows from Corollary 2.5 by taking  $n_1 = \omega, n_2 = n - \omega, r_1 = \omega - 1, r_2 = 0, \lambda_{1k} = -1$ , for  $k = 2, \ldots, \omega$ 

and  $\lambda_{2k} = 0$  for  $k = 2, 3, ..., n - \omega$ .

(iii). Since adjacency spectrum of  $C_n$  is  $\left\{2\cos\left(\frac{2\pi k}{n}\right): k=1,2,\ldots,n\right\}$ , by taking  $n_1=a,n_2=b,r_1=2,r_2=0$  and  $\lambda_{1k}=2\cos\left(\frac{2\pi i}{m}\right)$  for  $k=2,3,\ldots,a-1$  and  $\lambda_{2,k}$  for  $k=2,\ldots,b-1$  in Corollary 2.5, we get the required eigenvalues.

(iv). This is a special case of part (iii) with 
$$a = n - 1$$
 and  $b = 1$ .

A friendship graph  $F_n$  is a graph of order 2n+1, obtained by joining  $K_1$  with n copies of  $K_2$ , that is,  $F_n = K_1 \nabla(nK_2) = K_{1,n}[K_1,\underbrace{K_2,K_2,\ldots,K_2}_n]$ , where  $K_1$  corresponds to the root vertex (vertex of degree greater than one) in  $K_{1,n}$ . In particular, replacing some of  $K_2$ 's by  $K_1$ 's in  $F_n$  we get a firefly type graph, denoted by  $F_{p,n-p}$  and written as

$$F_{p,n-p} = K_{1,n}[K_1, \underbrace{K_1, K_1, \dots, K_1}_{p} \underbrace{K_2, K_2, \dots, K_2}_{n-p}].$$

A generalized or multi-step wheel network  $W_{a,b}$  is a graph derived from a copies of  $C_b$  and  $K_1$ , in such a way that all the vertices of each  $C_b$  are adjacent to  $K_1$ . Its order is ab+1 and can be written as  $W_{a,b} = K_1 \nabla(aC_b) = K_{1,a}[K_1, \underbrace{C_b, \ldots, C_b}]$ .

The normalized Laplacian eigenvalues of the friendship graph  $F_n$ , the firefly type graph  $F_{p,n-p}$  and  $W_{a,b}$  are given by the following.

**Proposition 2.8** (i) The normalized Laplacian eigenvalues of  $F_n$  are

$$\left\{0, \left(\frac{1}{2}\right)^{n-1}, \left(\frac{3}{2}\right)^{n+1}\right\}.$$

(ii) The normalized Laplacian eigenvalues of  $F_{p,n-p}$  are

$$\left\{0, \left(\frac{1}{2}\right)^{[n-p-1]}, 1^{[p-1]}, \left(\frac{3}{2}\right)^{[n-p]}, \frac{5\sqrt{2n-p} \pm \sqrt{2n+7p}}{4\sqrt{2n-p}}\right\}.$$

(iii) The normalized Laplacian eigenvalues of  $W_{a,b}$  consists of the eigenvalues 0, the eigenvalue  $\frac{4}{3}$  and the eigenvalues  $1 - \frac{2}{3}\cos\left(\frac{2\pi k}{b}\right)$ , for  $k = 2, \ldots, b$ .

**Proof.** (i). By Theorem 2.2 and the definition of  $F_n$ , we have

$$\alpha_1 = 2n, \alpha_2 = \dots = \alpha_{n+1} = 1$$
 and  $r_1 = 0, r_2 = \dots = r_{n+1} = 1$ .

So, by Theorem 2.2, we see that  $\frac{3}{2}$  is the normalized Laplacian eigenvalues of  $F_n$  with multiplicity n. The remaining eigenvalues are given by the block matrix

$$\begin{pmatrix}
1 & \frac{-1}{\sqrt{n}} & \dots & \frac{-1}{\sqrt{n}} & \frac{-1}{\sqrt{n}} \\
\frac{-1}{2\sqrt{n}} & \frac{1}{2} & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{-1}{2\sqrt{n}} & 0 & \dots & \frac{1}{2} & 0 \\
\frac{-1}{2\sqrt{n}} & 0 & \dots & 0 & \frac{1}{2}
\end{pmatrix}.$$
(2.4)

Clearly,  $\frac{1}{2}$  is the normalized laplacian eigenvalue of (2.4) with multiplicity n-1 and the remaining two eigenvalues of block matrix (2.4) are given by the quotient matrix

$$\begin{pmatrix} 1 & \frac{-n}{\sqrt{n}} \\ \frac{-1}{2\sqrt{n}} & \frac{1}{2} \end{pmatrix}.$$

(ii). Since  $\alpha_1 = p + 2(n-p) = 2n - p$  and  $\alpha_2 = \cdots = \alpha_{2n+1-p} = 1$ , so by Theorem 2.2, with  $r_1 = \cdots = r_{p+1} = 0$ ,  $r_{p+2} = \cdots = r_{2n+1-p} = 1$ , we see that  $\frac{3}{2}$  is the normalized Laplacian eigenvalue of  $F_{p,n-p}$  with multiplicity n-p. The other normalized Laplacian eigenvalues of  $F_{p,n-p}$  are given by the block matrix

$$\begin{pmatrix}
1 & \frac{-1}{\sqrt{2n-p}} & \cdots & \frac{-1}{\sqrt{2n-p}} & \frac{-2}{\sqrt{2(2n-p)}} & \cdots & \frac{-2}{\sqrt{2(2n-p)}} \\
\frac{-1}{\sqrt{2n-p}} & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
\frac{-1}{\sqrt{2n-p}} & 0 & \cdots & 1 & 0 & \cdots & 0 \\
\frac{-2}{\sqrt{2(2n-p)}} & 0 & \cdots & 0 & \frac{1}{2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
\frac{-2}{\sqrt{2(2n-p)}} & 0 & \cdots & 0 & 0 & \cdots & \frac{1}{2}
\end{pmatrix} . (2.5)$$

By simple calculations, 1 and  $\frac{1}{2}$  are the normalized Laplacian eigenvalues of (2.5) and the

remaining eigenvalues of block matrix (2.5) are given by the quotient matrix

$$\begin{pmatrix}
1 & \frac{-p}{\sqrt{2n-p}} & \frac{-2(n-p)}{\sqrt{2(2n-p)}} \\
\frac{-1}{\sqrt{2n-p}} & 1 & 0 \\
\frac{-1}{\sqrt{2(2n-p)}} & 0 & \frac{1}{2}
\end{pmatrix}.$$
(2.6)

Now, it is easy to see that 0 and  $\frac{5\sqrt{2n-p}\pm\sqrt{2n+7p}}{4\sqrt{2n-p}}$  are the normalized Laplacian eigenvalues of quotient matrix (2.6).

(iii). As in part (iii) of Proposition 2.7, we see that  $1 - \frac{2}{3}\cos\left(\frac{2\pi k}{b}\right)$ , for k = 2, ..., b. are the normalized Laplacian eigenvalues of  $W_{a,b}$ . The other eigenvalues are given by the block matrix

$$\begin{pmatrix}
1 & \frac{-b}{\sqrt{3ab}} & \dots & \frac{-b}{\sqrt{3ab}} & \frac{-b}{\sqrt{3ab}} \\
\frac{-1}{\sqrt{3ab}} & \frac{1}{3} & \dots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{-1}{\sqrt{3ab}} & 0 & \dots & \frac{1}{3} & 0 \\
\frac{-1}{\sqrt{3ab}} & 0 & \dots & 0 & \frac{1}{3}
\end{pmatrix}$$

Now, as in part (i),  $\left\{0, \left(\frac{1}{3}\right)^{a-1}, \frac{4}{3}\right\}$  are the remaining normalized Laplacian eigenvalues of  $W_{a,b}$ .

## 3 Normalized Laplacian eigenvalues of the power graphs of cyclic group $\mathbb{Z}_n$

In this section, we consider the power graphs of finite cyclic group  $\mathbb{Z}_n$ . As an application to Theorem 2.2 and its consequences obtained in Section 2, we determine the normalized Laplacian eigenvalues of power graph of  $\mathbb{Z}_n$ .

All groups are assumed to be finite and every cyclic group of order n is taken as isomorphic copy of integral additive modulo group  $\mathbb{Z}_n$  with identity denoted by 0. Let  $\mathcal{G}$  be a finite group of order n with identity e. The power graph of group  $\mathcal{G}$ , denoted by  $\mathcal{P}(\mathcal{G})$ , is the simple graph with vertex set as the elements of group  $\mathcal{G}$  and two distinct vertices  $x, y \in \mathcal{G}$  are adjacent if and

only if one is the positive power of the other, that is,  $x^i = y$  or  $y^j = x$ , for positive integers i, j with  $2 \le i, j \le n$ . These graphs were introduced in [16], see also [8]. Such graphs have valuable applications and are related to automata theory [17], besides being useful in characterizing finite groups. We let  $U_n^* = \{x \in \mathbb{Z}_n : (x, n) = 1\} \cup \{0\}$ , where (x, n) denotes greatest common divisor of x and n. Our other group theory notations are standard and can be taken from [20]. More work on power graphs can be seen in [1,4,8,9,18] and the references therein.

The adjacency spectrum, the Laplacian and the signless Laplacian spectrum of power graphs of finite cyclic and dihedral groups have been investigated in [2, 6, 15, 19, 21]. The normalized Laplacian eigenvalues of power graphs of certain finite groups were studied in [14].

Let n be a positive integer and d divides n, written as d|n. The divisor d is the proper divisor of n, if 1 < d < n. Let  $\mathbb{G}_n$  be a simple graph with vertex set as the proper divisor set  $\{d_i : 1, n \neq d_i | n, 1 \leq i \leq t\}$  and edge set  $\{d_i d_j : d_i | d_j, 1 \leq i < j \leq t\}$ , for  $1 \leq i < j \leq t$ . If the canonical decomposition of n is  $n = p_1^{n_1} p_2^{n_2} \dots p_r^{n_r}$ , where  $r, n_1, n_2, \dots, n_r$  are positive integers and  $p_1, p_2, \dots, p_r$  are distinct prime numbers, then the number of divisors of n are  $\prod_{i=1}^r (n_i + 1)$ . So the order of graph  $\mathbb{G}_n$  is  $|V(\mathbb{G}_n)| = \prod_{i=1}^r (n_i + 1) - 2$ . Also,  $\mathbb{G}_n$  is a connected graph [23], provided n is neither a prime power nor the product of two distinct primes. In [18],  $\mathbb{G}_n$  is used as the underlying graph for studying the power graph of finite cyclic group  $\mathbb{Z}_n$  and it has been shown that for each proper divisor  $d_i$  of n,  $\mathcal{P}(\mathbb{Z}_n)$  has a complete subgraph of order  $\phi(d_i)$ .

The following theorem shows that  $\frac{n}{n-1}$  is always the normalized Laplacian eigenvalue of the power graph  $\mathcal{P}(\mathbb{Z}_n)$ .

**Theorem 3.1** Let  $\mathbb{Z}_n$  be a finite cyclic group of order  $n \geq 3$ . Then  $\frac{n}{n-1}$  is normalized Laplacian eigenvalue of  $\mathcal{P}(\mathbb{Z}_n)$  with multiplicity at least  $\phi(n)$ .

**Proof.** Let  $\mathbb{Z}_n$  be the cyclic group of order  $n \geq 3$ . Then the identity 0 and invertible elements of the group  $\mathbb{Z}_n$  in the power graph  $\mathcal{P}(\mathbb{Z}_n)$  are adjacent to every other vertex in  $\mathcal{P}(\mathbb{Z}_n)$ . Since it is well known that the number of invertible elements of  $\mathbb{Z}_n$  are  $\phi(n)$  in number, so the induced power graph  $\mathcal{P}(U_n^*)$  is the complete graph  $K_{\phi(n)+1}$ . Thus, by Theorem 3.5, we see that  $\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathcal{P}(\mathbb{Z}_n \setminus U_n^*)$ . By applying Corollary 2.5, we get

$$1 - \frac{1}{r_1 + \alpha_1}(-1) = 1 + \frac{1}{\phi(n) + n - \phi(n) - 1} = \frac{n}{n - 1}$$

as the normalized Laplacian eigenvalue with multiplicity at least  $\phi(n)$ , since  $\frac{n}{n-1}$  can also be the normalized Laplacian eigenvalue of quotient matrix (2.3).

If  $n = p^z$ , where p is prime and z is a positive integer, then we have following observation.

Corollary 3.2 If  $n = p^z$ , where p is prime and z is a positive integer, then the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$  are  $\left\{\left(\frac{n}{n-1}\right)^{[(n-1)]}, 0\right\}$ .

**Proof.** If  $n = p^z$ , where p is prime and z is a positive integer, then as shown in [8],  $\mathcal{P}(\mathbb{Z}_n)$  is isomorphic to the complete graph  $K_n$  and hence the result follows.

The next observation gives the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$ , when n is the product of two primes.

Corollary 3.3 Let n = pq be the product of two distinct primes. Then the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$  are  $\left\{0, \left(\frac{n}{n-1}\right)^{[\phi(n)]}, \left(1+\frac{1}{q\phi(p)}\right)^{[\phi(p)-1]}, \left(1+\frac{1}{p\phi(q)}\right)^{[\phi(q)-1]}\right\}$  and the zeros of polynomial

$$p(x) = x \left( x^2 - x \left( \frac{\phi(n) + 1}{q\phi(p)} + \frac{\phi(p) + \phi(q)}{q\phi(p) + \phi(q)} + \frac{\phi(n) + 1}{p\phi(q)} \right) + \frac{(\phi(n) + 1)\phi(p)}{p\phi(q)(q\phi(p) + \phi(q))} + \frac{(\phi(n) + 1)^2}{n\phi(n)} + \frac{(\phi(n) + 1)\phi(q)}{q\phi(p)(q\phi(p) + \phi(q))} \right).$$

**Proof.** If n = pq, where p and q, (p < q) are primes, then  $\mathcal{P}(\mathbb{Z}_n)$  [9] can be written as

$$\mathcal{P}(\mathbb{Z}_n) = (K_{\phi(p)} \cup K_{\phi(q)}) \nabla K_{\phi(n)+1} = P_3[K_{\phi(p)}, K_{\phi(pq)+1}, K_{\phi(q)}].$$

By Theorem 3.1,  $\frac{n}{n-1}$  is the normalized Laplacian eigenvalue with multiplicity  $\phi(n)$ . Again, by Theorems 2.2 and 3.6, we see that  $\frac{1}{q\phi(p)}$  and  $\frac{1}{q\phi(p)}$  are the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$  with multiplicity  $\phi(p) - 1$  and  $\phi(q) - 1$  respectively. The remaining three normalized Laplacian eigenvalues are given by the following matrix

$$\begin{pmatrix} \frac{\phi(n)+1}{q\phi(p)} & \frac{-(\phi(n)+1)}{\sqrt{q\phi(p)(q\phi(p)+\phi(q))}} & 0\\ \frac{-\phi(p)}{\sqrt{q\phi(p)(q\phi(p)+\phi(q))}} & \frac{\phi(p)+\phi(q)}{q\phi(p)+\phi(q)} & \frac{-\phi(q)}{\sqrt{p\phi(q)(q\phi(p)+\phi(q))}} \\ 0 & \frac{-(\phi(n)+1)}{\sqrt{p\phi(q)(q\phi(p)+\phi(q))}} & \frac{\phi(n)+1}{p\phi(q)} \end{pmatrix}.$$

By Corollaries 3.2 and 3.3, we have the following proposition.

**Proposition 3.4** Equality holds in Theorem (3.1), if n is some prime or product of two primes.

The following theorem [18] shows that the power graph of a finite cyclic group  $\mathbb{Z}_n$  can be written as the joined union each of whose components are cliques.

**Theorem 3.5** If  $\mathbb{Z}_n$  is a finite cyclic group of order  $n \geq 3$ , then the power graph  $\mathcal{P}(\mathbb{Z}_n)$  is given by

$$\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathbb{G}_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}],$$

where  $\mathbb{G}_n$  is the graph of order t defined above.

Using Theorem 2.2 and its consequences, we can compute the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$  in terms of the adjacency spectrum of  $K_{\omega}$  and zeros of the characteristic polynomial of the auxiliary matrix.

We form a connected graph  $H = K_1 \nabla \mathbb{G}_n$  which is of diameter at most two if  $\mathbb{G}_n$  is not complete, otherwise its diameter is 1. In the following result, we compute the normalized Laplacian eigenvalues of the power graph of  $\mathbb{Z}_n$  by using Theorems 2.2 and 3.5.

**Theorem 3.6** The normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$  are

$$\left\{ \left( \frac{n}{n-1} \right)^{(\phi(n))}, \left( \frac{\phi(d_1) + \alpha_2}{d_1 + \alpha_2 - 1} \right)^{[\phi(d_1) - 1]}, \dots, \left( \frac{\phi(d_t) + \alpha_{r+1}}{d_t + \alpha_{t+1} - 1} \right)^{[\phi(d_t) - 1]} \right\}$$

and the t+1 eigenvalues of the following matrix  $\mathcal{M}$ 

$$\mathcal{M} = \begin{pmatrix} \frac{n - 1 - \phi(n)}{n - 1} & \frac{-\phi(d_1)a_{12}}{\sqrt{(\phi(n) + \alpha_1)(r_2 + \alpha_2)}} & \cdots & \frac{-\phi(d_t)a_{1(t+1)}}{\sqrt{(\phi(n) + \alpha_1)(r_{t+1} + \alpha_{t+1})}} \\ \frac{-\phi(d_1)a_{21}}{\sqrt{(r_2 + \alpha_2)(\phi(n) + \alpha_1)}} & \frac{\alpha_2}{\alpha_2 + r_2} & \cdots & \frac{-\phi(d_t)a_{2(t+1)}}{\sqrt{(r_2 + \alpha_2)(r_{t+1} + \alpha_{t+1})}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-\phi(d_1)a_{(t+1)1}}{\sqrt{(r_n + \alpha_{t+1})(\phi(n) + \alpha_1)}} & \frac{-\phi(d_2)a_{(t+1)2}}{\sqrt{(r_n + \alpha_{t+1})(r_2 + \alpha_2)}} & \cdots & \frac{\alpha_{t+1}}{\alpha_{t+1} + r_{t+1}} \end{pmatrix},$$

$$(3.7)$$

where, for  $i \neq j$ ,

$$a_{ij} = \begin{cases} 1, & v_i \sim v_j \\ 0, & v_i \nsim v_j \end{cases}$$

and  $r_i = \phi(d_i) - 1$ , for i = 2, ..., t + 1.

**Proof.** Let  $\mathbb{Z}_n$  be a finite cyclic group of order n. Since the identity element 0 and the  $\phi(n)$  generators of the group  $\mathbb{Z}_n$  are adjacent to every other vertex of  $\mathcal{P}(\mathbb{Z}_n)$ , therefore, by Theorem 3.5, we have

$$\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathbb{G}_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}] = H[K_{\phi(n)+1}, K_{\phi(d_1)}, K_{\phi(d_2)}, \dots, K_{\phi(d_t)}],$$

where  $H = K_1 \nabla \mathbb{G}_n$  is the graph with vertex set  $\{v_1, \ldots, v_{t+1}\}$ . Taking  $G_1 = K_{\phi(n)+1}$  and  $G_i = K_{\phi(d_{i-1})}$ , for  $i = 2, \ldots, t+1$ , in Theorem 2.2 and using the fact that the adjacency spectrum of  $K_{\omega}$  consists of the eigenvalue  $\omega - 1$  with multiplicity 1 and the eigenvalue -1 with multiplicity  $\omega - 1$ , it follows that

$$1 - \frac{1}{r_1 + \alpha_1} \lambda_{1k} A(G_1) = 1 - \frac{1}{r_1 + \alpha_1} (-1) = 1 + \frac{1}{\phi(n) + n - \phi(n) - 1} = \frac{n}{n - 1}$$

is a normalized eigenvalue of  $\mathcal{P}(\mathbb{Z}_n)$  with multiplicity  $\phi(n)$ . Note that we have used the fact that vertex  $v_1$  of graph H is adjacent to every other vertex of H and  $\alpha_1 = \sum_{d|n,d\neq 1,n} \phi(d) = \sum_{d|n,d\neq 1,n} \phi(d)$ 

$$n-1-\phi(n)$$
, as  $\sum_{d\mid s}\phi(d)=s$ . Similarly, we can show that  $\frac{\phi(d_1)+\alpha_2}{\phi(d_1)+\alpha_2-1},\ldots,\frac{\phi(d_t)+\alpha_{t+1}}{\phi(d_t)+\alpha_{t+1}-1}$  are the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_n)$  with multiplicities  $\phi(d_1)-1,\ldots,\phi(d_t)-1$ , respectively. The remaining normalized Laplacian eigenvalues are the eigenvalues of the quotient matrix  $\mathcal{M}$  given by (3.7).

From Theorem 3.6, it is clear that all the normalized Laplacian eigenvalues of the power graph  $\mathcal{P}(\mathbb{Z}_n)$  are completely determined except the t+1 eigenvalues, which are the eigenvalues of the matrix  $\mathcal{M}$  in Equation (3.7). Further, it is also clear that the matrix  $\mathcal{M}$  depends upon the structure of the graph  $\mathbb{G}_n$ , which is not known in general. However, if we give some particular value to n, then it may be possible to know the structure of graph  $\mathbb{G}_n$  and hence about the matrix  $\mathcal{M}$ . This information may be helpful to determine the t+1 remaining normalized Laplacian eigenvalues of the power graph  $\mathcal{P}(\mathbb{Z}_n)$ .

We discuss some particular cases of Theorem 3.6.

Now, let n = pqr, where p, q, r with p < q < r are primes. From the definition of  $\mathbb{G}_n$ , the vertex set and edge set of  $\mathbb{G}_n$  are  $\{p, q, r, pq, pr, qr\}$  and  $\{(p, pq), (p, pr), (q, pq), (q, qr), (r, pr), (r, qr)\}$  respectively, and is shown in Figure (1). Let  $H = K_1 \nabla \mathbb{G}_n$ . Then

$$\mathcal{P}(\mathbb{Z}_n) = H[K_{\phi(n)+1}, K_{\phi(p)}, K_{\phi(q)}, K_{\phi(r)}, K_{\phi(pq)}, K_{\phi(pr)}, K_{\phi(qr)}].$$

By Theorem 2.2, we have

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) = (n - \phi(n) - 1, \phi(n) + 1 + \phi(pq) + \phi(pr), \phi(n) + 1 + \phi(pq) + \phi(qr)),$$

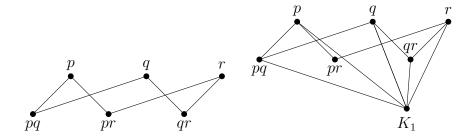


Figure 1: Divisor graph  $\mathbb{G}_{pqr}$  and  $H = K_1 \nabla \mathbb{G}_{pqr}$ .

$$\phi(n) + 1 + \phi(pr) + \phi(qr), \phi(n) + 1 + \phi(p) + \phi(q), \phi(n) + 1 + \phi(p) + \phi(r), \phi(n) + 1 + \phi(q) + \phi(r))$$
and 
$$(\alpha_1 + r_1, \alpha_2 + r_2, \alpha_3 + r_3, \alpha_4 + r_4, \alpha_5 + r_5, \alpha_6 + r_6, \alpha_7 + r_7) = (n - 1, \phi(n) + \phi(p) + \phi(pq) + \phi(pr), \phi(n) + \phi(qr) + \phi(qr), \phi(n) + \phi(r) + \phi(qr), \phi(n) + \phi(qr) + \phi(qr), \phi(n) + \phi$$

Also, by Theorem 3.1,  $\frac{n}{n-1}$  is the normalized Laplacian eigenvalue with multiplicity  $\phi(n)$ . Using the above information and Theorem 3.6, the second distinct normalized Laplacian eigenvalue is  $1 + \frac{1}{r_2 + \alpha_2} = 1 + \frac{1}{\phi(n) + \phi(p) + \phi(pq) + \phi(pr)}$  with multiplicity  $\phi(p) - 1$ . In a similar way, we see that the other eigenvalues are

$$1 + \frac{1}{\phi(n) + \phi(q) + \phi(pq) + \phi(qr)}, 1 + \frac{1}{\phi(n) + \phi(r) + \phi(pr) + \phi(qr)}, 1 + \frac{1}{\phi(n) + \phi(pq) + \phi(q) + \phi(q)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(pr) + \phi(p) + \phi(p)}, 1 + \frac{1}{\phi(n) + \phi(p)}, 1 + \frac{1}{\phi(n)}, 1 + \frac{1}{\phi(n) +$$

with multiplicities  $\phi(q) - 1$ ,  $\phi(r) - 1$ ,  $\phi(pq) - 1$ ,  $\phi(pr) - 1$ ,  $\phi(qr) - 1$ , respectively. The remaining 7 eigenvalues are given by the following matrix

$$\begin{pmatrix} z_1 & -\phi(p)c_{12} & -\phi(q)c_{13} & -\phi(r)c_{14} & -\phi(pq)c_{15} & -\phi(pr)c_{16} & -\phi(qr)c_{17} \\ (\phi(n)+1)c_{21} & z_2 & 0 & 0 & -\phi(pq)c_{25} & -\phi(pr)c_{26} & 0 \\ (\phi(n)+1)c_{31} & 0 & z_3 & 0 & -\phi(pq)c_{35} & 0 & -\phi(qr)c_{37} \\ (\phi(n)+1)c_{41} & 0 & 0 & z_4 & 0 & -\phi(pr)c_{46} & -\phi(qr)c_{47} \\ (\phi(n)+1)c_{51} & -\phi(p)c_{25} & -\phi(q)c_{35} & 0 & z_5 & 0 & 0 \\ (\phi(n)+1)c_{61} & -\phi(p)c_{26} & 0 & -\phi(r)c_{64} & 0 & z_6 & 0 \\ (\phi(n)+1)c_{71} & 0 & -\phi(q)c_{75} & -\phi(r)c_{74} & 0 & 0 & z_7 \end{pmatrix},$$

where,

$$z_1 = \frac{n - \phi(n) - 1}{n - 1}, \ z_2 = \frac{\phi(n) + 1 + \phi(pq) + \phi(pr)}{\phi(n) + \phi(p) + \phi(pq) + \phi(pr)}, \ z_3 = \frac{\phi(n) + 1 + \phi(pq) + \phi(qr)}{\phi(n) + \phi(q) + \phi(pq) + \phi(qr)},$$

$$z_{4} = \frac{\phi(n) + 1 + \phi(pr) + \phi(qr)}{\phi(n) + \phi(r) + \phi(pr) + \phi(qr)}, \quad z_{5} = \frac{\phi(n) + 1 + \phi(p) + \phi(q)}{\phi(n) + \phi(pq) + \phi(p) + \phi(q)}$$

$$z_{6} = \frac{\phi(n) + 1 + \phi(p) + \phi(r)}{\phi(n) + \phi(pr) + \phi(p) + \phi(r)}, \quad z_{7} = \frac{\phi(n) + 1 + \phi(q) + \phi(r)}{\phi(n) + \phi(qr) + \phi(q) + \phi(r)}$$
and 
$$c_{ij} = c_{ji} = \frac{1}{\sqrt{(r_{i} + \alpha_{i})(r_{j} + \alpha_{j})}}.$$

Next, we discuss the normalized Laplacian eigenvalues of the finite cyclic group  $\mathbb{Z}_n$ , with  $n = p^{n_1}q^{n_2}$ , where p < q are primes and  $n_1 \leq n_2$  are positive integers. We consider the case when both  $n_1$  and  $n_2$  are even, and the case when they are odd can be discussed similarly.

**Theorem 3.7** Let  $\mathcal{P}(\mathbb{Z}_{p^{n_1}q^{n_2}})$  be the power graph of the finite cyclic group  $\mathbb{Z}_{p^{n_1}q^{n_2}}$  of order  $n = p^{n_1}q^{n_2}$ , where p < q are primes and  $n_1 = 2m_1 \le n_2 = 2m_2$  are even positive integers. Then the normalized Laplacian eigenvalues of  $\mathcal{P}(\mathbb{Z}_{p^{n_1}q^{n_2}})$  consists of the eigenvalues

$$\left(\frac{n}{n-1}\right)^{[\phi(n)]}, \left(\frac{n-q^{n_2}+1}{n-q^{n_2}}\right)^{[\phi(p)-1]}, \\ \vdots \\ \left(\frac{p^{m_1-1}+q^{n_2}(p^{n_1}-p^{m_1-1})}{p^{m_1-1}+q^{n_2}(p^{n_1}-p^{m_1-1})}-1\right)^{[\phi(p^{m_1})-1]}, \\ \vdots \\ \left(\frac{p^{n_1-1}+q^{n_2}\phi(p^{n_1})}{p^{n_1-1}+q^{n_2}\phi(p^{n_1})-1}\right)^{[\phi(p^{n_1})-1]}, \left(\frac{n-p^{n_1}+1}{n-p^{n_1}}\right)^{[\phi(q)-1]}, \\ \vdots \\ \left(\frac{q^{m_2-1}+p^{n_1}(q^{n_2}-q^{m_2-1})}{q^{m_2-1}+p^{n_1}(q^{n_2}-q^{m_2-1})-1}\right)^{[\phi(q^{m_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2-1}+p^{n_1}\phi(q^{n_2})}{q^{n_2-1}+p^{n_1}\phi(q^{n_2})-1}\right)^{[\phi(q^{n_2})-1]}, \left(\frac{\phi(p)+\phi(q)+(q^{n_2}-1)(p^{n_1}-1)+1}{\phi(p)+\phi(q)+(q^{n_2}-1)(p^{n_1}-1)}\right)^{[\phi(pq)-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{m_2}-q^{m_2-1}(p^{n_1}-p)}{q^{n_2}(p^{n_1}-1)+q^{m_2}-q^{m_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{m_2})-1]}, \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{m_2}-q^{m_2-1}(p^{n_1}-p)}{q^{n_2}(p^{n_1}-1)+q^{m_2}-q^{m_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{m_2})-1]}, \\ \vdots \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{m_2-1}(p^{n_1}-p)}{q^{n_2}(p^{n_1}-1)+q^{m_2}-q^{m_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{m_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{m_2-1}(p^{n_1}-p)}{q^{n_2}(p^{n_1}-1)+q^{m_2}-q^{m_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{m_2-1}(p^{n_1}-p)}{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{m_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1]}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1}, \\ \vdots \\ \left(\frac{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}{q^{n_2}(p^{n_1}-1)+q^{n_2}-q^{n_2-1}(p^{n_1}-p)-1}\right)^{[\phi(pq^{n_2})-1}\right)^{[\phi(pq^{n_2})-1}$$

$$\left( \frac{pq^{n_2} + \phi(p^{n_1})(q^{n_2} - q)}{pq^{n_2} + \phi(p^{n_1})(q^{n_2} - q) - 1} \right)^{[\phi(pq^{n_2}) - 1]},$$

$$\vdots$$

$$\left( \frac{p^{m_1} + p^{n_1}(q^{n_2} - 1) - p^{m_1 - 1}(q^{n_2} - q)}{p^{m_1} + p^{n_1}(q^{n_2} - 1) - p^{m_1 - 1}(q^{n_2} - q) - 1} \right)^{[\phi(p^{m_1}q) - 1]},$$

$$\vdots$$

$$\left( \frac{n + p^{m_1}q^{m_2} + p^{m_1 - 1}q^{m_2 - 1} - \phi(p^{m_1}q^{m_2}) - p^{n_1}q^{m_2 - 1} - p^{m_1 - 1}q^{n_2}}{n + p^{m_1}q^{m_2} + p^{m_1 - 1}q^{m_2 - 1} - \phi(p^{m_1}q^{m_2}) - p^{n_1}q^{m_2 - 1} - p^{m_1 - 1}q^{n_2} - 1} \right)^{[\phi(p^{m_1}q^{m_2}) - 1]},$$

$$\vdots$$

$$\left( \frac{p^{m_1}q^{n_2} + \phi(q^{n_2})(p^{n_1} - p^{m_1})}{p^{m_1}q^{n_2} + \phi(q^{n_2})(p^{n_1} - p^{m_1}) - 1} \right)^{[\phi(p^{n_1}q^{n_2}) - 1]},$$

$$\vdots$$

$$\left( \frac{p^{n_1}q + \phi(p^{n_1})(q^{n_2} - q)}{p^{n_1}q + \phi(p^{n_1})(q^{n_2} - q^{m_1})} \right)^{[\phi(p^{n_1}q^{m_2}) - 1]},$$

$$\vdots$$

$$\left( \frac{p^{n_2}q^{m_1} + \phi(p^{n_1})(q^{n_2} - q^{m_1})}{p^{n_2}q^{m_1} + \phi(p^{n_1})(q^{n_2} - q^{m_1}) - 1} \right)^{[\phi(p^{n_1}q^{m_2}) - 1]},$$

$$\vdots$$

$$\left( \frac{p^{n_1}q^{n_2 - 1} + \phi(n)}{p^{n_1}q^{n_2 - 1} + \phi(n) - 1} \right)^{[\phi(p^{n_1}q^{n_2} - 1) - 1]},$$

and the remaining eigenvalues are given by matrix (3.7).

**Proof.** Suppose that  $n = p^{n_1}q^{n_2}$ , where  $n_1 = 2m_1$  and  $n_2 = 2m_2$  are even with  $n_1 \le n_2$  and  $m_1$  and  $m_2$  are positive integers. Since the total number of divisors of n are  $(n_1 + 1)(n_2 + 1)$ , so the order of  $\mathbb{G}_{p^{n_1}q^{n_2}}$  is  $(n_1 + 1)(n_2 + 1) - 2$ . The proper divisor set of n is

$$D(n) = \left\{ p, p^2, \cdots, p^{m_1}, \dots, p^{n_1}, q, q^2, \dots, q^{m_2}, \dots, q^{n_2}, pq, pq^2, \dots, pq^{m_2}, \dots, pq^{n_2}, \dots, p^{m_1}q, p^{m_1}q^2, \dots, p^{m_1}q^{m_2}, \dots, p^{m_1}q^{m_2}, \dots, p^{m_1}q^{n_2}, \dots, p^{m_1$$

By the definition of graph  $\mathbb{G}_n$ , we see that p is not adjacent to  $p, q, q^2, \dots, q^{m_2}, \dots, q^{n_2}$ . So we write adjacency of vertices in terms of iterations and avoid divisors outside the set D(n). Thus, we observe that

$$p \sim p^i, p^j q^k$$
, for  $i = 2, 3, \dots, n_1, j = 1, 2, \dots, n_1, k = 1, 2, \dots, n_2$ 

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p^{m_1} \sim p^i, p^j q^k, for i = 1, 2, \dots, n_1, i \neq m_1, j = m_1, \dots, n_1, k = 1, 2, \dots, n_2
     p^{n_1} \sim p^i, p^{n_1}q^j, for i = 1, 2, \dots, n_1 - 1, j = 1, 2, \dots, n_2 - 1,
        q \sim q^i, p^j q^k, for i = 2, 3, \dots, n_2, j = 1, 2, \dots, n_1, k = 1, 2, \dots, n_2
     a^{m_2} \sim q^i, p^j q^k, for i = 1, 2, 3, \dots, n_1, i \neq m_2, j = 1, 2, 3, \dots, n_1, k = m_2, \dots, n_2
     q^{n_2} \sim q^i, p^j q^{n_2}, \text{ for } i = 1, 2, 3, \dots, n_2 - 1, \ j = 1, 2, 3, \dots, n_1 - 1,
      pq \sim p, q, p^i q^j, for i = 1, 2, 3, \dots, n_1, j = 1, 2, 3, \dots, n_2,
   pq^{m_2} \sim p, q^i, pq^j, p^kq^k for i = 1, 2, 3, \dots, m_2, j = 1, 2, 3, \dots, n_2, j \neq m_2, k = 2, 3, \dots, n_1,
          l=m_2,\ldots,n_2,
    pq^{n_2} \sim p, q^i, pq^j, p^kq^{n_2}, \text{ for } i = 1, 2, \dots, n_2, \ j = 1, 2, \dots, n_2 - 1, \ k = 2, 3, \dots, n_1 - 1,
   p^{m_1}q \sim p^i, q, p^{m_1}q^j, p^kq, p^lq^m \text{ for } i = 1, 2, 3, \dots, m_1, \ j = 2, 3, \dots, n_2, \ k = 1, 2, \dots, m_1 - 1,
          l = m_1 + 1, \dots, n_1, m = 1, 2, \dots, n_2,
p^{m_1}q^{m_2} \sim p^i, q^j, p^kq^l for i = 1, 2, \dots, m_1, \ j = 1, 2, \dots, m_2, \ k = 1, 2, \dots, n_1, \ l = 1, 2, \dots, n_2
p^{m_1}q^{n_2} \sim p^i, q^j, p^kq^{n_2}, p^iq^j for i = 1, 2, \dots, m_1, \ j = 1, 2, \dots, n_2, \ k = m_1 + 1, m_1 + 2, \dots, n_1 - 1,
   p^{n_1}q \sim p^i, q, p^jq, p^nq^k \text{ for } i = 1, 2, \dots, n_1, \ j = 1, 2, \dots, n_1 - 1, \ k = 2, 3 \dots, n_2 - 1,
p^{n_1}q^{m_2} \sim p^i, q^j, p^{n_1}q^k, p^iq^j for i = 1, 2, \dots, n_1, \ j = 1, 2, \dots, m_2, \ k = m_2 + 1, m_2 + 2 \dots, n_2 - 1,
```

$$p^{n_1}q^{n_2-1} \sim p^i, q^j, p^iq^j$$
 for  $i = 1, 2, \dots, n_1, j = 1, 2, \dots, n_2 - 1$ .

Therefore, by Theorem 3.5, we have

$$\mathcal{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \nabla \mathbb{G}_n[K_{\phi(p)}, \dots, K_{\phi(p^{m_1})}, \dots, K_{\phi(p^{n_1})}, K_{\phi(q)}, \dots, K_{\phi(q^{m_2})}, \dots, K_{\phi(q^{n_2})}, K_{\phi(pq)}, \dots, K_{\phi(pq^{n_2})}, \dots, K_{\phi(pq^{n_2})}, \dots, K_{\phi(p^{m_1}q)}, \dots, K_{\phi(p^{m_1}q^{m_2})}, \dots, K_{\phi(p^{n_1}q^{n_2})}, \dots, K_{\phi(p^{n_1}q^{n_2})}, \dots, K_{\phi(p^{n_1}q^{n_2})}, \dots, K_{\phi(p^{n_1}q^{n_2})}, \dots, K_{\phi(p^{n_1}q^{n_2})}].$$

Now, by using Theorem 2.2, we calculate the values of  $\alpha_i$ 's and  $r_i + \alpha_i = r_i'$ 's. We recall some number theory identities, like  $\phi(xy) = \phi(x)\phi(y)$ , provided that (x,y) = 1,  $\sum_{i=1}^k \phi(p^i) = p^k - 1$  and  $\sum_{d|s} \phi(d) = s$ . Using this information and definition of  $\alpha_i$ 's, we have

$$\alpha_1 = \sum_{1,n \neq d \mid n} \phi(d) = n - 1 - \phi(n)$$

and

$$\alpha_{2} = \phi(p^{2}) + \dots + \phi(p^{m_{1}}) + \dots + \phi(p^{n_{1}}) + \phi(pq) + \dots + \phi(pq^{m_{2}}) + \dots + \phi(pq^{n_{2}}) + \dots + \phi(pq^{n_{2}}) + \dots + \phi(p^{m_{1}}q^{m_{2}}) + \dots + \phi(p^{m_{1}}q^{n_{2}}) + \dots + \phi(p^{n_{1}}q^{n_{2}}) + \dots + \phi(p^{n_{1}}$$

Proceeding in the same way as above, other  $\alpha_i$ 's are

$$\alpha_{3} = q^{n_{2}}(p^{n_{1}} - p) + p - \phi(p^{2}),$$

$$\vdots$$

$$\alpha_{m_{1}+1} = p^{m_{1}-1} + q^{n_{2}}(p^{n_{1}} - p^{m_{1}-1}) - \phi(p^{m_{1}}),$$

$$\vdots$$

$$\alpha_{n_{1}+1} = p^{n_{1}-1} + \phi(p^{n_{1}})(q^{n_{2}-1} - 1), \ \alpha_{n_{1}+2} = n - \phi(q) - p^{n_{1}} + 1,$$

$$\vdots$$

$$\alpha_{n_{1}+m_{1}+1} = q^{m_{2}-1} + p^{n_{1}}(q^{n_{2}} - q^{m_{2}-1}) - \phi(q^{m_{2}}),$$

$$\vdots \\ \alpha_{n_1+n_2+1} = q^{n_2-1} + \phi(q^{n_2})(p^{n_1}-1), \\ \alpha_{n_1+n_2+2} = \phi(p) + \phi(q) + 1 - \phi(pq) + (q^{n_2}-1)(p^{n_1}-1), \\ \vdots \\ \alpha_{n_1+n_2+m_1+1} = q^{n_2}(p^{n_1}-1) + q^{m_2} - q^{m_2-1}(p^{n_1}-p) - \phi(pq^{m_2}), \\ \vdots \\ \alpha_{2n_1+n_2+1} = pq^{n_2} - \phi(pq^{n_2}) + \phi(p^{n_1})(q^{n_2}-q), \\ \vdots \\ \alpha_{m_1n_2+n_1+2} = p^{m_1} - \phi(p^{m_1}q) + p^{n_1}(q^{n_2}-1) - p^{m_1-1}(q^{n_1}-q), \\ \vdots \\ \alpha_{m_1n_2+n_1+2} = p^{m_1} - \phi(p^{m_1}q) + p^{n_1}q^{m_2} - 2\phi(p^{m_1}q^{m_2}) - q^{n_2}p^{m_1-1} - p^{n_2}q^{m_2-1}, \\ \vdots \\ \alpha_{m_1n_2+n_1+1} = n + p^{m_1-1}q^{m_2-1} + p^{m_1}q^{m_2} - 2\phi(p^{m_1}q^{m_2}) - q^{n_2}p^{m_1-1} - p^{n_2}q^{m_2-1}, \\ \vdots \\ \alpha_{(m_1+1)n_2+n_1+1} = p^{m_1}q^{n_2} - \phi(p^{m_1}q^{n_2}) + \phi(q^{n_2})(p^{n_1}-p^{m_1}), \\ \vdots \\ \alpha_{n_1n_2+n_1+m_1+1} = p^{n_1}q^{n_2} + \phi(p^{n_1})(q^{n_2}-q) - \phi(p^{n_1}q), \\ \vdots \\ \alpha_{(n_1+1)n_2+n_1} = p^{n_1}q^{n_2-1} + \phi(n) - \phi(p^{n_1}q^{n_2-1}). \\ \text{Also, value of } r_i + \alpha_i = r_i' \text{s are given by} \\ r_1' = n - 1, \ r_2' = n - q^{n_2}, \\ \vdots \\ r_{m_1+1}' = p^{m_1-1} + q^{n_2}(p^{n_1}-p^{m_1-1}) - 1, \\ \vdots \\ \vdots \\ r_{m_1+1}' = p^{m_1-1} + q^{n_2}(p^{n_1}-p^{m_1-1}) - 1, \\ \vdots \\ \vdots \\ \vdots \\ n_1' = n - 1, \ n_1' = n - 1, \ n_2' = n - q^{n_2}, \\ \vdots \\ n_1' = n_1' = n - 1, \ n_2' = n - q^{n_2}, \\ \vdots \\ n_1' = n_1' = n - 1, \ n_1' = n - 1$$

 $r'_{n_1+1} = p^{n_1-1} + \phi(p^{n_1})q^{n_2} - 1, \ r'_{n_1+2} = n - p^{n_1},$ 

 $r_{n_1+m_1+1}' = q^{m_2-1} + p^{n_1}(q^{n_2} - q^{m_2-1}) - 1,$ 

$$\begin{array}{c} \vdots \\ r^{'}_{n_1+n_2+1} = q^{n_2-1} + \phi(q^{n_2})p^{n_1} - 1, \ r^{'}_{n_1+n_2+2} = \phi(p) + \phi(q) + (q^{n_2}-1)(p^{n_1}-1), \\ \vdots \\ r^{'}_{n_1+n_2+m_1+1} = q^{n_2}(p^{n_1}-1) + q^{m_2} - q^{m_2-1}(p^{n_1}-p) - 1, \\ \vdots \\ r^{'}_{2n_1+n_2+1} = pq^{n_2} + \phi(p^{n_1})(q^{n_2}-q) - 1, \\ \vdots \\ r^{'}_{m_1n_2+n_1+2} = p^{m_1} + p^{n_1}(q^{n_2}-1) - p^{m_1-1}(q^{n_1}-q) - 1, \\ \vdots \\ r^{'}_{m_1N_2+N_1+m_1+1} = n + p^{m_1-1}q^{m_2-1} + p^{m_1}q^{m_2} - \phi(p^{m_1}q^{m_2}) - q^{n_2}p^{m_1-1} - p^{n_2}q^{m_2-1} - 1, \\ \vdots \\ r^{'}_{(m_1+1)n_2+n_1+1} = p^{m_1}q^{n_2} + \phi(q^{n_2})(p^{n_1}-p^{m_1}) - 1, \\ \vdots \\ r^{'}_{n_1n_2+n_1+2} = p^{n_1}q + \phi(p^{n_1})(q^{n_2}-q) - 1, \\ \vdots \\ r^{'}_{n_1n_2+n_1+m_1+1} = p^{n_1}q^{m_2} + \phi(p^{n_1})(q^{n_2}-q^{m_2}) - 1, \\ \vdots \\ r^{'}_{n_1n_2+n_1+m_1+1} = p^{n_1}q^{n_2-1} + \phi(n) - 1. \end{array}$$

We note that each of  $G_i = K_i$  and by Theorems 3.1 and 2.2, we get the desired eigenvalues as in the statement. By substituting the values of  $\alpha_i$ 's,  $r'_i$ 's and using the adjacency relations, the remaining normalized Laplacian eigenvalues are the eigenvalues of matrix (3.7).

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