Euler numbers and diametral paths in Fibonacci cubes, Lucas cubes and Alternate Lucas cubes

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Abstract

The diameter of a graph is the maximum distance between pairs of vertices in the graph. A pair of vertices whose distance is equal to its diameter are called diametrically opposite vertices. The collection of shortest paths between diametrically opposite vertices are referred as diametral paths. In this work, we enumerate the number of diametral paths for Fibonacci cubes, Lucas cubes and Alternate Lucas cubes. We present bijective proofs that show that these numbers are related to alternating permutations and are enumerated by Euler numbers.
Keywords: Shortest path, diametral path, Fibonacci cube, Lucas cube, Alternate Lucas cube, Euler number.
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1 Introduction

Given a connected graph $G = (V, E)$, one of the basic problem is to enumerate the number of shortest paths between pairs of vertices in G. The solution to this problem provides an important topological property of an interconnection network, in terms of its connectivity, fault-tolerance, communication expense [17] and has important applications such as for counting minimum (s, t)-cut in planar graphs and route guidance systems [3].

The so-called single-source shortest paths problem consists of finding the shortest paths between a given vertex and all other vertices in the graph. One can solve this problem by using the algorithms such as Breadth-First-Search for unweighted graphs or Dijkstra's algorithm [8]. Similarly, Dijkstra's algorithm can be used to solve the single-pair shortest paths problem in a weighted, directed graph with nonnegative weights.

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The process of finding all shortest paths between a pair of vertices in a graph is another problem. This can be considered a search for the most efficient routes through the graph. In [4], it is proved that finding the number of shortest paths in a general graph is NP-hard. For planar graphs with m vertices an oracle is presented in [2] to find the number of shortest paths for a given pair of vertices whose time complexity is $O(\sqrt{m})$ with $O(m^{1.5})$ space. This approach is improved in [11] and a new oracle for counting shortest paths in planar graphs is presented, where Voronoi diagrams are used to speed up the query time.

In the literature, the problem of enumerating the number of shortest paths have been considered for some special families of graphs. Explicit formulas have been obtained for the hexagonal network [10], the star graph [13], the (n,k)-star graph [5, 7] and the arrangement graph [6]. In an (n,k)-star graph the number of shortest paths is enumerated by counting the minimum factorizations of a permutation in terms of the transpositions corresponding to edges in that graph [7]. For the arrangement graph this number is enumerated by establishing a bijection between these shortest paths and a collection of ordered forests of certain bi-colored trees [6].

The distance $d(u,v)$ between two vertices $u, v \in V$ is the number of edges in a shortest path between u and v. The *diameter* of G is defined as the maximum distance between pairs of vertices in V and is denoted by $\mathrm{diam}(G)$.

A pair of vertices $u, v \in V$ with $d(u,v) = \mathrm{diam}(G)$, are called *diametrically opposite* vertices. The collection of shortest paths between diametrically opposite vertices are referred to as *diametral paths*. For a pair of diametrically opposite vertices $u, v \in V$ we let $c(u,v;G)$ denote the number of diametral paths from u to v in G.

As an example, for the n-dimensional hypercube Q_n the number of diametral paths between any diametrically opposite pair u and v can be enumerated by establishing a bijection between these shortest paths and the permutations on n symbols, so that

$$c(u,v;Q_n) = n!\,.$$

In this paper we enumerate the number of diametral paths for three special subgraphs of hypercube graphs, namely Fibonacci cubes [12], Lucas cubes [15] and Alternate Lucas cubes [9]. We present bijective proofs of our results. Surprisingly, these numbers are related to alternating permutations and are enumerated by Euler numbers.

2 Preliminaries

We let $[n] = \{1, 2, \ldots, n\}$. The n-dimensional hypercube Q_n is the graph defined on the vertex set B_n, where

$$B_n = \{b_1 b_2 \ldots b_n \mid b_i \in \{0,1\},\ i \in [n]\}\,.$$

Two vertices $u, v \in B_n$ are adjacent if and only if the Hamming distance $d(u,v) = 1$, that is, u and v differ in exactly one coordinate. For convenience, $Q_0 = K_1$. It is clear from the definition that $\mathrm{diam}(Q_n) = n$ and for any vertex $u \in B_n$ there exist a unique vertex $\bar{u} \in B_n$ such that $d(u,\bar{u}) = n$, where $\bar{u}$ denotes the complement of the binary string of u.

For $n \geq 1$, let

$$\mathcal{FB}_n = \{b_1 b_2 \dots b_n \in B_n \mid b_i \cdot b_{i+1} = 0,\ i \in [n-1]\}\,.$$

The n-dimensional Fibonacci cube Γ_n ($n \geq 1$) is an induced subgraph of Q_n with vertex set $\mathcal{FB}_n$. We take $\Gamma_0 = K_1$. Similarly, for $n \geq 1$, let

$$\mathcal{LB}_n = \{b_1 b_2 \dots b_n \in \mathcal{FB}_n \mid b_1 \cdot b_n = 0\}$$

and for $n \geq 3$,

$$\mathcal{ALB}_n = \{b_1 b_2 \dots b_n \in \mathcal{FB}_n \mid b_n \cdot b_{n-2} = 0\}\,.$$

The n-dimensional Lucas cube Λ_n and Alternate Lucas cube $\mathcal{L}_n$ are defined as the induced subgraphs of $\Gamma_n \subseteq Q_n$ and with sets $\mathcal{LB}_n$ and $\mathcal{ALB}_n$, respectively.

Q_n has a useful decomposition in which its vertex set is partitions into two sets $B_n = 0B_{n-1} \cup 1B_{n-1}$, where $0B_{n-1}$ denotes the vertices that start with a 0 and $1B_{n-1}$ denotes the vertices that start with a 1. Using this decomposition we can write

$$Q_n = 0Q_{n-1} + 1Q_{n-1}$$

where $0Q_{n-1}$ and $1Q_{n-1}$ denote the induced subgraphs of Q_n with vertex sets $0B_{n-1}$ and $1B_{n-1}$ respectively, and $+$ denotes the perfect matching between $0Q_{n-1}$ and $1Q_{n-1}$. Similarly, we have the following fundamental decompositions for Fibonacci cubes, Lucas cubes and Alternate Lucas cubes:

$$\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}\,,$$

where there is a perfect matching between $10\Gamma_{n-2}$ and $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$,

$$\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0\,,$$

where there is a perfect matching between $10\Gamma_{n-3}0$ and $00\Gamma_{n-3}0 \subset 0\Gamma_{n-1}$,

$$\mathcal{L}_n = 0\mathcal{L}_{n-1} + 10\mathcal{L}_{n-2}\,,$$

where there is a perfect matching between $10\mathcal{L}_{n-2}$ and $00\mathcal{L}_{n-2} \subset 0\mathcal{L}_{n-1}$.

2.1 Euler numbers

Following [18], a permutation $\sigma = \sigma_1 \sigma \dots \sigma_n$ of $[n]$ is *alternating* if $\sigma_1 > \sigma_2, < \sigma_3 > \sigma_4 < \cdots$. In other words, $\sigma_i < \sigma_{i+1}$ for i even and $a_i > a_{i+1}$ for i odd. σ is *reverse alternating* if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 \cdots$. Let E_n denote the number of alternating permutations of $[n]$ with $E_0 = 1$. These are known as the *Euler numbers*. The number of reverse alternating permutations of $[n]$ is also given by E_n.

By a result of Désiré André [1], we have

$$2E_{n+1} = \sum_{k=0}^{n} \binom{n}{k} E_k E_{n-k}\,,$$

and the exponential generating function of the sequence of Euler numbers is given by

$$\begin{aligned}\sum_{n\geq 0} E_n \frac{x^n}{n!} &= \sec x + \tan x \\ &= 1 + x + \frac{x^2}{2!} + 2\frac{x^3}{3!} + 5\frac{x^4}{4!} + 16\frac{x^5}{5!} + 61\frac{x^6}{6!} + \cdots\end{aligned}$$

First few terms of Euler numbers (sequence A000111 in the OEIS [16]) are $E_0 = 1$, $E_1 = 1$, $E_2 = 1$, $E_3 = 2$, $E_4 = 5$, $E_5 = 16$, $E_6 = 61$.

3 Calculation for the Fibonacci cubes

In this section we determine the number of diametral paths in Γ_n. Since $n = \mathrm{diam}(\Gamma_n)$, these paths are of length n. Γ_n is an induced subgraph of Q_n with vertex set $\mathcal{FB}_n$. We have the following easy result.

Proposition 1. *There is a unique pair of diametrically opposite vertices in Γ_n. They are*

(i) $u = (01)^{\frac{n}{2}}$ and $v = (10)^{\frac{n}{2}}$ if n is even,

(ii) $u = (01)^{\frac{n-1}{2}}0$ and $v = (10)^{\frac{n-1}{2}}1$ if n is odd.

Even though Γ_n is undirected, here we view the edges on each such path to be directed from u to v. By direct inspection we have $c(u, v; \Gamma_1) = 1$, $c(u, v; \Gamma_2) = 1$, $c(u, v; \Gamma_3) = 2$, $c(u, v; \Gamma_4) = 5$, $c(u, v; \Gamma_5) = 16$. For a path

$$u = s_0 \to s_1 \to \cdots \to s_n = v \ ,$$

each vertex s_{i+1} is obtained from the vertex s_i by flipping a 0 to a 1, or a 1 to a 0, with the proviso that no consecutive 1s appear in any s_i. We see in particular that $c(u, v; \Gamma_3) = 2$ as there are two paths of length 3 from u to v when $n = 3$ as shown in the Table 1.

Step	b_1	b_2	b_3
$v = s_3$	1	0	1
s_2	1	0	0
s_1	0	0	0
$u = s_0$	0	1	0

Step	b_1	b_2	b_3
$v = s_3$	1	0	1
s_2	0	0	1
s_1	0	0	0
$u = s_0$	0	1	0

Table 1: Two different paths from $u = 010$ to $v = 101$ in Γ_3.

Here we write u in the bottom most row. The ith step shows the string s_i after i edges on the path have been traversed. Note that in this representation, the path proceeds from bottom up and the row indices are increasing from bottom up as well.

By using this representation we give a bijective proof that the sequence of the numbers of diametral paths in Fibonacci cubes is precisely the sequence of Euler numbers.

Theorem 1. *Let $u, v \in \Gamma_n$ such that $d(u, v) = n$. Then for $n \geq 1$, we have*

$$c(u, v; \Gamma_n) = E_n \ ,$$

where E_n is the nth Euler number.

Proof. We give a bijection between paths of length n from u to v in $\mathcal{FB}_n$ and alternating permutations σ of $[n]$. The bijection is best communicated by an example. Suppose $n = 8$ and we are given the path from $u = 01010101$ to $v = 10101010$ whose steps are shown in Table 2.

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$v = s_8$	1	0	1	0	1	0	1	0
s_7	1	0	1	0	1	0	0	0
s_6	1	0	0	0	1	0	0	0
s_5	1	0	0	0	1	0	0	1
s_4	0	0	0	0	1	0	0	1
s_3	0	1	0	0	1	0	0	1
s_2	0	1	0	0	0	0	0	1
s_1	0	1	0	0	0	1	0	1
$u = s_0$	0	1	0	1	0	1	0	1

Table 2: A path from diametrically opposite vertices $u = (01)^4$ to $v = (10)^4$ in Γ_8.

As the first step, we mark the first appearance of 1 as we go up the table in every column with an odd index. In Table 3 these entries are circled.

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$v = s_8$	1	0	1	0	1	0	**1**	0
s_7	1	0	**1**	0	1	0	0	0
s_6	1	0	0	0	1	0	0	0
s_5	**1**	0	0	0	1	0	0	1
s_4	0	0	0	0	1	0	0	1
s_3	0	1	0	0	**1**	0	0	1
s_2	0	1	0	0	0	0	0	1
s_1	0	1	0	0	0	1	0	1
$u = s_0$	0	1	0	1	0	1	0	1

Table 3: First appearance of 1 as we go up in every column with an odd index is marked in the path from $u = (01)^4$ to $v = (10)^4$ in Γ_8.

Next, we mark the first appearance of 0 as we go up the table in every column with an even index. Circling these entries gives Table 4.

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
$v = s_8$	1	0	1	0	1	0	**1**	0
s_7	1	0	**1**	0	1	0	0	0
s_6	1	0	0	0	1	0	0	**0**
s_5	**1**	0	0	0	1	0	0	1
s_4	0	**0**	0	0	1	0	0	1
s_3	0	1	0	0	**1**	0	0	1
s_2	0	1	0	0	0	**0**	0	1
s_1	0	1	0	**0**	0	1	0	1
$u = s_0$	0	1	0	1	0	1	0	1

Table 4: First appearance of 0/1 as we go up in every column with an even/odd index is marked in the path from $u = (01)^4$ to $v = (10)^4$ in Γ_8.

After this we record the corresponding step number in each column. For instance column 1 gives 5, column 2 gives 4, etc. by reading the indices of the corresponding rows. The resulting alternating permutation is below:

$$5 \quad 4 \quad 7 \quad 1 \quad 3 \quad 2 \quad 8 \quad 6$$

These steps are reversible. Suppose this time that $n = 7$ and we are given the alternating permutation 3 1 6 4 7 2 5. We construct Table 5 in which the odd numbered columns 1, 3, 5, 7 are assigned the label 1 in the rows 3, 6, 7, 5, which are the entries in the odd positions of the given permutation. The even numbered columns 2, 4, 6 are assigned the label 0 in the rows 1,4, 2, which are the entries in the even indexed positions of the given permutation.

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$v = s_7$					**1**		
s_6			**1**				
s_5							**1**
s_4				**0**			
s_3	**1**						
s_2						**0**	
s_1		**0**					
$u = s_0$							

Table 5: First appearance of 0/1 in every column with an even/odd index in the path from $u = (01)^3 0$ to $v = (10)^3 1$ in Γ_7 corresponding to the alternating permutation 3 1 6 4 7 2 5.

Now we fill in the odd indexed columns of this matrix by 0, up to the marked 1 in the column, followed by 0s all the way up; and we fill the even indexed columns by 1 up to the marked 0 in the column, followed by 1s all the way up. This results in the path of length $n = 7$ from u to v shown in Table 6 corresponding to the alternating permutation 3 1 6 4 7 2 5.

Step	b_1	b_2	b_3	b_4	b_5	b_6	b_7
$v = s_7$	1	0	1	0	**1**	0	1
s_6	1	0	**1**	0	0	0	1
s_5	1	0	0	0	0	0	**1**
s_4	1	0	0	**0**	0	0	0
s_3	**1**	0	0	1	0	0	0
s_2	0	0	0	1	0	**0**	0
s_1	0	**0**	0	1	0	1	0
$u = s_0$	0	1	0	1	0	1	0

Table 6: The path from the diametrically opposite vertices $u = (01)^3 0$ to $v = (10)^3 1$ in Γ_7 corresponding to the alternating permutation 3 1 6 4 7 2 5.

Considering now the general case, we see that going from u to v in n steps, every bit in u has to change exactly once. This means that the row indices of the marked entries in the matrix in Table 4 is a permutation σ of $[n]$. Now consider an element $t = \sigma_i$ with odd i with $1 < i < n$. This means that in step t of the path, that is in s_t, the entry in the ith column goes from 0 to 1. But since all of the vertices that appear in the table as rows are Fibonacci strings. This means that in s_{t-1} the entries in columns $i-1$ and $i+1$ which are adjacent to the entry at column i must already be 0. Therefore these entries were flipped from 1 to 0 in earlier steps. It follows that $\sigma_i > \sigma_{i+1}$ and $\sigma_i > \sigma_{i-1}$. The two extreme cases with $i = 1$ and $i = n$ are handled the same way. Therefore σ is an alternating permutation. The other direction is proved similarly. □

4 Calculation for the Lucas cubes

It is shown in [15] that

$$\operatorname{diam}(\Lambda_n) = \begin{cases} n & \text{for } n \text{ even}, \\ n-1 & \text{for } n \text{ odd}. \end{cases}$$

We have

Proposition 2. *The number of diametrically opposite pair of vertices in Λ_n is 1 if n is even and n if n is odd. They are*

- $(01)^{\frac{n}{2}}$ *and* $(10)^{\frac{n}{2}}$ *if n is even,*
- *cyclic shifts of the pair* $0(01)^{\frac{n-1}{2}}$ *and* $0(10)^{\frac{n-1}{2}}$ *if n is odd.*

Remark 1. *Note that there is a typo in [15, Proposition 1]. For n odd, the number of pairs of vertices in Λ_n at distance equal to the diameter is n, not $n-1$.*

Similar to the proof of Theorem 1 we obtain the following result for Λ_n.

Theorem 2. *Let $u, v \in \Lambda_n$ such that $d(u,v) = \mathrm{diam}(\Lambda_n)$. Then for $n \geq 2$, we have*

$$c(u, v; \Lambda_n) = \begin{cases} \frac{n}{2} E_{n-1} & \text{for } n \text{ even}, \\ E_{n-1} & \text{for } n \text{ odd}. \end{cases}$$

Proof. Assume first that n is even. By Proposition 2 we only need to consider the vertices $u = (01)^{\frac{n}{2}}$ and $v = (10)^{\frac{n}{2}}$. Mimicking the bijective proof of Theorem 1, we arrive at permutations σ of $[n]$ satisfying $\sigma_i > \sigma_{i+1}$ for any odd index i with $1 \leq i < n$, $\sigma_i > \sigma_{i-1}$ for any odd index i with $1 < i \leq n$ and the extra condition $\sigma_1 > \sigma_n$, since in Λ_n we have $b_1 \cdot b_n = 0$. This last requirement on σ is easily verified by tracing the first appearance of a 1 in the first and the last columns of the table of paths that define the bijection for Γ_n. Therefore, σ must be a circular alternating permutation, and these were enumerated by Kreweras [14].

For n odd, assume that $u_1 = 0(01)^{\frac{n-1}{2}}$ and $v_1 = 0(10)^{\frac{n-1}{2}}$. Then we know that $u_1, v_1 \in 0\Gamma_{n-1}$ and since $\Lambda_n = 0\Gamma_{n-1} + 10\Gamma_{n-3}0$ we have

$$c(u_1, v_1; \Lambda_n) = c(u_1, v_1; 0\Gamma_{n-1}) = E_{n-1}\,.$$

Let u_i and v_i be the $i-1$ right cyclic shifts of the vertices u_1 and v_1 for $i = 2, \ldots, n$, respectively. Then for any shortest path P from u_1 to v_1, the $i-1$ right cyclic shifts of all the vertices in P gives a shortest path from u_i to v_i for all $i \in \{2, \ldots, n\}$, which completes the proof. $\square$

5 Calculation for the Alternate Lucas cubes

For any integer $n \geq 3$, it is shown in [9] that $\mathrm{diam}(\mathcal{L}_n) = n - 1$. We have

Proposition 3. *For any integer $n \geq 4$, the number of diametrically opposite pair of vertices in $\mathcal{L}_n$ is 4. For $n \geq 4$, they are*

(i) $u = 0^s(10)^k001$ *and* $v = 1^s(01)^k010$,

(ii) $u = 0^s(10)^k010$ *and* $v = 1^s(01)^k001$,

(iii) $u = 0^s(10)^k100$ *and* $v = 1^s(01)^k001$,

(iv) $u = 0^s(10)^k100$ *and* $v = 1^s(01)^k010$,

where $n = 2k + 3 + s$, k is a nonnegative integer and $s \in \{0, 1\}$.

Theorem 3. *Let $n = 2k + 3 + s$, k be a nonnegative integer and $s \in \{0, 1\}$. For $n \geq 4$, we have*

$$c(0^s(10)^k100, 1^s(01)^k001; \mathcal{L}_n) = c(0^s(10)^k100, 1^s(01)^k010; \mathcal{L}_n) = E_{n-1}$$

$$c(0^s(10)^k001, 1^s(01)^k010; \mathcal{L}_n) = c(0^s(10)^k010, 1^s(01)^k001; \mathcal{L}_n) = \binom{n-1}{2} E_{n-3}\,.$$

Proof. We sketch the proof. As in the proof of Theorem 1, we need to consider the permutations σ of $[n]$ satisfying extra conditions depending on the pair of vertices. We will give the proof for n even ($s = 1$) and only for the pairs $u = 0^s(10)^k100$ and $v = 1^s(01)^k001$ and $u = 0^s(10)^k001$ and $v = 1^s(01)^k010$. The other cases can be obtained similarly.

For the pair $u = 0(10)^k100$ and $v = 1(01)^k001$ as we consider the shortest paths we will not change the $(n-1)$st position since it is 0 for each vertex. Therefore we need to consider the permutations σ of $[n] \setminus \{n-1\}$ satisfying $\sigma_i > \sigma_{i+1}$ for any odd index i with $1 \leq i \leq n-3$, $\sigma_i > \sigma_{i-1}$ for any odd index i with $1 < i \leq n-3$ and $\sigma_n > \sigma_{n-2}$, since in $\mathcal{L}_n$ we have $b_{n-2} \cdot b_n = 0$. By setting $\tau_i = \sigma_i$ for $i = 1, \ldots, n-2$ and $\tau_{n-1} = \sigma_n$ we observe that τ is an alternating permutation of $[n-1]$.

Now consider the pair $u = 0(10)^k001$ and $v = 1(01)^k010$. In the shortest paths under consideration, we will not change the $(n-2)$nd position since it is 0 for each vertex. Therefore we need to consider the permutations σ of $[n] \setminus \{n-2\}$ satisfying $\sigma_i > \sigma_{i+1}$ for any odd index i with $1 \leq i < n-3$, $\sigma_i > \sigma_{i-1}$ for any odd index i with $1 < i \leq n-3$ and $\sigma_{n-1} > \sigma_n$. By setting $\tau_i = \sigma_i$ for $i = 1, \ldots, n-3$ we observe that τ is an alternating permutation of $[n-3]$ and we have $\binom{n-1}{2}$ different choices for σ_{n-1}, σ_n which gives the desired result. $\square$

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