

Solving systems of multi-term fractional PDEs: Invariant subspace approach.

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Abstract

In the present paper invariant subspace method has been extended for solving systems of multi-term fractional partial differential equations (FPDEs) involving both time and space fractional derivatives. Further the method has also been employed for solving multi-term fractional PDEs in $(1 + n)$ dimensions. A diverse set of examples is solved to illustrate the method.

Key Words and Phrases: time and space fractional partial differential equation in higher dimension, systems of fractional partial differential equation, exact solution, invariant subspace method.

1 Introduction

Fractional order differential equations (FODEs) are receiving increasing attention owing to their applicability to almost all branches of science and engineering. It has been established that fractional order partial differential equations (FPDEs) provide appropriate framework for description of anomalous and non-Brownian diffusion. They are more effective while formulating processes having memory effects as fractional derivatives are non-local in nature [4, 7, 18].

Hence solving FODEs, FPDEs, especially nonlinear ones, is a challenging task and currently an active area of research. In pursuance to this researchers have developed new

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numerical/ analytical methods for solving FPDEs such as Adomian decomposition method (ADM) [1], new iterative method (NIM) [3], iterative Laplace transform method [15], method of separation of variables, homogeneous balanced principle [24], Lie symmetry analysis method [17] and so on.

One of the analytical methods for solving PDEs is invariant subspace method developed by Galaktionov and Svirshchevskii [9]. Invariant subspace method was employed for solving time fractional PDEs by many authors [10,20]. Further Choudhary and Daftardar-Gejji [6] extended the method for FPDEs having both time and space fractional derivatives. In 2009 a classification of two-component nonlinear diffusion equations based on invariant subspace method was proposed [19]. Sahadevan *et al.* did extensive study of Lie symmetry analysis and invariant subspace method for deriving exact solutions of the coupled FPDEs with fractional time derivative [22].

In the present paper we develop invariant subspace method for finding analytic solutions of systems of multi-term FPDEs having both time and space fractional derivatives. Further the method is employed for solving FPDEs in (1+n) dimensions. In the proposed method system of FPDEs and FPDEs in higher dimensions are reduced to respective system of FODEs which can be solved by known methods. Invariant subspace method is also used to solve FPDEs with fractional differential operator involving mixed fractional partial derivatives.

The organization of the paper is as follows. Section 2, deals with preliminaries and notations. In Section 3, we develop theory of invariant subspace method for r-coupled FPDEs, which is followed by illustrative examples. In Section 4 we extend invariant subspace method for FPDEs in (1+n) dimensions and explain the method with a variety of illustrative examples. Concluding remarks are made in Section 5.

2 Preliminaries and Notations

In this section, we introduce notations, definitions and preliminaries which are used in the present article. For more details readers may refer to [8, 16, 18].

Definition 2.1. *The Riemann-Liouville (R-L) fractional integral of order $\alpha > 0$ of function f is defined as*

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau, \quad 0 < t \leq b.$$

Definition 2.2. Caputo fractional derivative of order $\alpha > 0$ of f is defined as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \begin{cases} I^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, & n-1 < \alpha < n, \\ f^{(n)}(t), & \alpha = n, n \in \mathbb{N}. \end{cases}$$

Definition 2.3. Riemann-Liouville (R-L) fractional derivative of order $\alpha > 0$ of f is defined as

$$\frac{{}^{RL}d^\alpha f(t)}{dt^\alpha} = \begin{cases} D^n I^{n-\alpha} f(t) = \frac{d^n}{dt^n} \left(\frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} dx \right), & n-1 < \alpha < n, \\ f^{(n)}(t), & \alpha = n, n \in \mathbb{N}. \end{cases}$$

R-L integral, Caputo derivative and R-L derivative satisfy the following properties for $[\alpha] = n, n \in \mathbb{N}$ [8]:

1. $I^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\gamma+\alpha}$, if $\gamma > -1, t > 0$.
2. $\frac{d^\alpha t^\gamma}{dt^\alpha} = \begin{cases} 0, & \text{if } \gamma \in \{0, 1, 2, \dots, n-1\}, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{if } \gamma \in \mathbb{N} \text{ and } \gamma \geq n, \text{ or } \gamma \notin \mathbb{N} \text{ and } \gamma > n-1. \end{cases}$
3. $\frac{{}^{RL}d^\alpha t^\gamma}{dt^\alpha} = \begin{cases} 0, & \text{if } \gamma > -1 \text{ and } \alpha - \gamma \in \{0, 1, \dots, n-1\}, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} t^{\gamma-\alpha}, & \text{if } \gamma > -1 \text{ and } \alpha - \gamma \notin \mathbb{N}. \end{cases}$
4. $I^\alpha \left(\frac{d^\alpha f(t)}{dt^\alpha} \right) = f(t) - \sum_{k=0}^{n-1} D^{(k)} f(0) \frac{t^k}{k!}$, $n-1 < \alpha < n, t > 0$.

Note: In the property (2) condition $\gamma > n-1$ is very crucial as

$$\frac{d^\alpha (t^{-\alpha})}{dt^\alpha},$$

is not defined in case of Caputo derivate for $0 < \alpha < 1$. In the literature many authors are mistakenly ignoring the underlying required condition $\gamma > n-1$ (here $\gamma = -\alpha > 0$ is required but $-1 < -\alpha < 0$). Therefore

$$\frac{d^\alpha (t^{-\alpha})}{dt^\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)} t^{-2\alpha},$$

is not valid in case of Caputo derivative, though it holds correct for R-L derivative:

$$\frac{{}^{RL}d^\alpha(t^{-\alpha})}{dt^\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{-2\alpha}, \quad \gamma = -\alpha > -1.$$

In the present work we denote fractional partial derivative $\frac{\partial^{k\gamma}}{\partial t^{k\gamma}}$ and $\frac{{}^{RL}\partial^{k\gamma}}{\partial t^{k\gamma}}$ (Caputo and RL partial derivative respectively) as sequential fractional partial derivative [16], viz.,

$$\frac{\partial^{k\gamma} f}{\partial t^{k\gamma}} = \underbrace{\frac{\partial^\gamma}{\partial t^\gamma} \frac{\partial^\gamma}{\partial t^\gamma} \cdots \frac{\partial^\gamma}{\partial t^\gamma}}_{k\text{-times}} f, \quad \frac{{}^{RL}\partial^{k\gamma} f}{\partial t^{k\gamma}} = \underbrace{\frac{{}^{RL}\partial^\gamma}{\partial t^\gamma} \frac{{}^{RL}\partial^\gamma}{\partial t^\gamma} \cdots \frac{{}^{RL}\partial^\gamma}{\partial t^\gamma}}_{k\text{-times}} f.$$

Definition 2.4. Two-parametric Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha > 0, \beta > 0.$$

The n -th order derivative of $E_{\alpha,\beta}(z)$ is given by

$$E_{\alpha,\beta}^{(n)}(z) = \frac{d^n}{dz^n} E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{(k+n)! z^k}{k! \Gamma(\alpha k + \alpha n + \beta)}, \quad n = 0, 1, 2, \dots$$

The α -th order Caputo derivative of $E_\alpha(at^\alpha)$ is

$$\frac{d^\alpha}{dt^\alpha} [E_\alpha(at^\alpha)] = aE_\alpha(at^\alpha), \quad \alpha > 0, a \in \mathbb{R}.$$

Generalized fractional trigonometric functions for $[\gamma] = n$ are defined as [5]

$$\begin{aligned} \cos_\gamma(\lambda t^\gamma) &= \operatorname{Re}[E_\gamma(i\lambda^\gamma)] = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k} t^{(2k)\gamma}}{\Gamma(2k\gamma + 1)}, \\ \sin_\gamma(\lambda t^\gamma) &= \operatorname{Im}[E_\gamma(i\lambda^\gamma)] = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{2k+1} t^{(2k+1)\gamma}}{\Gamma((2k+1)\gamma + 1)}. \end{aligned}$$

The fractional trigonometric functions satisfy the following properties

$$\frac{d^\alpha}{dt^\alpha} [\cos_\gamma(\lambda t^\gamma)] = -\lambda \sin_\gamma(\lambda t^\gamma), \quad \frac{d^\alpha}{dt^\alpha} [\sin_\gamma(\lambda t^\gamma)] = \lambda \cos_\gamma(\lambda t^\gamma).$$

Laplace transform of the Caputo derivative of order α is,

$$\mathcal{L} \left\{ \frac{d^\alpha f(t)}{dt^\alpha}; s \right\} = s^\alpha \hat{f}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad n-1 < \alpha \leq n, n \in \mathbb{N}, \operatorname{Re}(s) > 0,$$

where $\hat{f}(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t) dt, \quad s \in \mathbb{R}.$

Laplace transform of $\varepsilon_n(t, a; \alpha, \beta) := t^{\alpha n + \beta - 1} E_{\alpha, \beta}^{(n)}(\pm at^\alpha)$ has the form

$$\mathcal{L}\{\varepsilon_n(t, a; \alpha, \beta); s\} = \frac{n! s^{\alpha - \beta}}{(s^\alpha \mp a)^{n+1}}, \quad \text{Re}(s) > |a|^{\frac{1}{\alpha}}, \quad n = 0, 1, 2, \dots \quad (2.1)$$

3 System of FPDES

In this section we extend invariant subspace method for solving systems of FPDEs. We introduce the following notations:

Let $f = (f_1, f_2, \dots, f_r) = (f_1(t, x), f_2(t, x), \dots, f_r(t, x)) \in \mathbb{R}^r$, where $t > 0, x \in \mathbb{R}.$

$N^1[f] := (N_1^1[f], N_2^1[f], \dots, N_r^1[f]) \in \mathbb{R}^r$, where

$$N_p^1[f] := \hat{N}_p^1 \left[x, f_1, f_2, \dots, f_r, \frac{\partial^\beta f_1}{\partial x^\beta}, \dots, \frac{\partial^\beta f_r}{\partial x^\beta}, \dots, \frac{\partial^{k\beta} f_1}{\partial x^{k\beta}}, \dots, \frac{\partial^{k\beta} f_r}{\partial x^{k\beta}} \right], \quad 1 \leq p \leq r, \text{ and}$$

$N^2[f] = (N_1^2[f], N_2^2[f], \dots, N_r^2[f]) \in \mathbb{R}^r$, where

$$N_p^2[f] = \hat{N}_p^2 \left[x, f_1, \dots, f_r, \frac{\partial^\beta f_1}{\partial x^\beta}, \dots, \frac{\partial^\beta f_r}{\partial x^\beta}, \dots, \frac{\partial^{\beta+k-1} f_1}{\partial x^{\beta+k-1}}, \dots, \frac{\partial^{\beta+k-1} f_r}{\partial x^{\beta+k-1}} \right], \quad 1 \leq p \leq r, \quad k \in \mathbb{N},$$

are linear/ non-linear fractional differential operators. Let $F = (F_1, F_2, \dots, F_r) \in \mathbb{R}^r$ be such that

$$F_p = \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{\gamma(i,p)} f_p(t, x)}{\partial t^{\gamma(i,p)}}, \quad p = 1, \dots, r,$$

where $\gamma(i, p) = i\alpha_p$ or $\gamma(i, p) = \alpha_p + i - 1.$

In this article $\frac{\partial^{j\beta}(\cdot)}{\partial x^{j\beta}}$ and $\frac{\partial^{\beta+j-1}(\cdot)}{\partial x^{\beta+j-1}}, j = 1, \dots, k$ denote Caputo derivatives with respect to variable x and $\frac{\partial^{\gamma(i,p)} f_p(\cdot)}{\partial t^{\gamma(i,p)}}$ denotes Caputo or Riemann-Liouville derivative with respect to variable $t.$ $[\alpha_p] = s_p, [\beta] = s,$ where $s_p, s \in \mathbb{N}.$ Henceforth throughout the article $p = 1, \dots, r.$

We consider the system of coupled FPDEs as

$$F = N^l[f], \quad l = 1, 2. \quad (3.1)$$

Eq. (3.1) consists of 4-kinds of different systems for $p=1, 2, \dots, r,$ viz.,

$$\begin{aligned}
\bullet F_p &= \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{i\alpha_p} f_p(t,x)}{\partial t^{i\alpha_p}} = N_p^1[f], & \bullet F_p &= \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{i\alpha_p} f_p(t,x)}{\partial t^{i\alpha_p}} = N_p^2[f], \\
\bullet F_p &= \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{\alpha_p+i-1} f_p(t,x)}{\partial t^{\alpha_p+i-1}} = N_p^1[f], & \bullet F_p &= \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{\alpha_p+i-1} f_p(t,x)}{\partial t^{\alpha_p+i-1}} = N_p^2[f].
\end{aligned}$$

Further note that F is a function of either Caputo derivative or Riemann-Liouville derivative, whereas $N^l[f]$ always involves only Caputo fractional derivatives.

3.1 Invariant subspace method for systems of FPDEs

Let $I = I_1^{n_1} \times I_2^{n_2} \times \dots \times I_r^{n_r}$ represent a linear space where $I_p^{n_p}$ denotes an n_p dimensional linear subspace over \mathbb{R} spanned by n_p linearly independent functions $\{\phi_p^j(x)\}_{j=1}^{n_p}$, i.e.,

$$I_p^{n_p} = \mathcal{Q} \left\{ \phi_p^1(x), \phi_p^2(x), \dots, \phi_p^{n_p}(x) \right\} = \left\{ \sum_{j=1}^{n_p} k_{pj} \phi_p^j(x) \mid k_{pj} \in \mathbb{R}, j = 1, 2, \dots, n_p \right\}, \forall p.$$

I is said to be invariant with respect to vector differential operators N^l , $l = 1, 2$ if N^l satisfies the following condition $\forall p$.

$$N_p^l : I_1^{n_1} \times I_2^{n_2} \times \dots \times I_r^{n_r} \longrightarrow I_p^{n_p}, l = 1, 2.$$

Thus there exist expansion coefficients ψ_p^j ($j = 1, 2, \dots, n_p$) of $N_p^l[f]$ with respect to the basis functions $\{\phi_p^j(x)\}_{j=1}^{n_p}$ such that

$$N_p^l \left[\sum_{j=1}^{n_1} k_{1j} \phi_1^j(x), \dots, \sum_{j=1}^{n_r} k_{rj} \phi_r^j(x) \right] = \sum_{j=1}^{n_p} \psi_p^j(k_{11}, k_{12}, \dots, k_{1n_1}, \dots, k_{r1}, \dots, k_{rn_r}) \phi_p^j(x),$$

$$(k_{p1}, k_{p2}, \dots, k_{pn_p}) \in \mathbb{R}^{n_p}, \forall p, l = 1, 2.$$

Theorem 3.1. *If a finite dimensional linear subspace $I = I_1^{n_1} \times I_2^{n_2} \times \dots \times I_r^{n_r}$ is invariant under the fractional differential operators $N^l[F]$, $l = 1, 2$, then the system of FPDEs (3.1) has a solution of the form*

$$f_p(t, x) = \sum_{j=1}^{n_p} K_{pj}(t) \phi_p^j(x), \quad p = 1, 2, \dots, r, \quad (3.2)$$

where the coefficients $K_{pj}(t)$ satisfy the following system of FODEs

$$\sum_{i=1}^{m_p} \lambda_{pi} \frac{d^{\gamma(i,p)} K_{pj}(t)}{dt^{\gamma(i,p)}} = \psi_p^j(K_{11}(t), \dots, K_{1n_1}(t), \dots, K_{r1}(t), \dots, K_{rn_r}(t)), \quad j = 1, \dots, n_p, \quad (3.3)$$

where $\gamma = \gamma(i, p) = i\alpha_p$ or $\gamma = \gamma(i, p) = \alpha_p + i - 1$, $\forall p$.

Proof. Let $f_p(t, x) = \sum_{j=1}^{n_p} K_{pj}(t)\phi_p^j(x)$, $\forall p$.

Using the linearity property of fractional derivative we get

$$F_p = \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{\gamma(i,p)} f_p(t, x)}{\partial t^{\gamma(i,p)}} = \sum_{i=1}^{m_p} \lambda_{pi} \frac{\partial^{\gamma(i,p)}}{\partial t^{\gamma(i,p)}} \left[\sum_{j=1}^{n_p} K_{pj}(t)\phi_p^j(x) \right] = \sum_{j=1}^{n_p} \left[\sum_{i=1}^{m_p} \lambda_{pi} \frac{d^{\gamma(i,p)}}{dt^{\gamma(i,p)}} K_{pj}(t) \right] \phi_p^j(x),$$

$$\gamma = \gamma(i, p) = i\alpha_p \text{ or } \gamma = \gamma(i, p) = \alpha_p + i - 1, \forall p. \quad (3.4)$$

Given that the fractional differential operators $N^l[F]$, $l = 1, 2$ admit invariant subspace I , there exist basis functions $\phi_p^1(x), \phi_p^2(x), \dots, \phi_p^{n_p}(x)$ such that

$$N_p^l \left[\sum_{j=1}^{n_1} k_{1j} \phi_1^j(x), \dots, \sum_{j=1}^{n_r} k_{rj} \phi_r^j(x) \right] = \sum_{j=1}^{n_p} \psi_p^j(k_{11}, k_{12}, \dots, k_{1n_1}, \dots, k_{r1}, \dots, k_{rn_r}) \phi_p^j(x),$$

$$(k_{p1}, k_{p2}, \dots, k_{pn_p}) \in \mathbb{R}^{n_p}, \forall p, l = 1, 2, \quad (3.5)$$

where $\{\psi_p^j\}'s$ are expansion coefficients of $N_p[I] \in I_p^{n_p}$ corresponding to $\{\phi_p^j\}'s$. Hence in view of Eq. (3.2) and Eq. (3.5), we deduce

$$N_p^l[f] = N_p^l[f_1, f_2, \dots, f_r] = N_p^l \left[\sum_{j=1}^{n_1} K_{1j}(t)\phi_1^j(x), \sum_{j=1}^{n_2} K_{2j}(t)\phi_2^j(x), \dots, \sum_{j=1}^{n_r} K_{rj}(t)\phi_r^j(x) \right]$$

$$= \sum_{j=1}^{n_p} \psi_p^j(K_{11}(t), \dots, K_{1n_1}(t), \dots, K_{r1}(t), \dots, K_{rn_r}(t)) \phi_p^j(x), \forall p, l = 1, 2. \quad (3.6)$$

Substituting (3.4) and (3.6) in (3.1) we get

$$\sum_{j=1}^{n_p} \left[\sum_{i=1}^{m_p} \lambda_{pi} \frac{d^{\gamma(i,p)} K_{pj}(t)}{dt^{\gamma(i,p)}} - \psi_p^j(K_{11}(t), \dots, K_{1n_1}(t), \dots, K_{r1}(t), \dots, K_{rn_r}(t)) \right] \phi_p^j(x) = 0, \forall p. \quad (3.7)$$

From the Eq. (3.7) and using the fact that $\{\phi_p^j\}'s$ are linearly independent, the system of FPDEs (3.1) is reduced to the required system of FODEs (3.3). \square

3.2 Illustrative examples

3.2.1 System of fractional version of generalized Burger's equations

Consider the following coupled generalized nonlinear fractional Burger's equations for $t > 0$, $\alpha \in (0, 1) \setminus \{1/2\}$ and $\beta \in (0, 1]$.

$$\begin{aligned} \frac{{}^{RL}D^\alpha f}{Dt^\alpha} + a_0 \frac{\partial^{2\beta} f}{\partial x^{2\beta}} + a_1 f \left(\frac{\partial^\beta f}{\partial x^\beta} \right) + a_2 \left(f \frac{\partial^\beta g}{\partial x^\beta} + g \frac{\partial^\beta f}{\partial x^\beta} \right) &= 0, \\ \frac{{}^{RL}D^\alpha g}{Dt^\alpha} + b_0 \frac{\partial^{2\beta} g}{\partial x^{2\beta}} + b_1 g \left(\frac{\partial^\beta g}{\partial x^\beta} \right) + b_2 \left(f \frac{\partial^\beta g}{\partial x^\beta} + g \frac{\partial^\beta f}{\partial x^\beta} \right) &= 0, \end{aligned} \quad (3.8)$$

where a_0, a_1, a_2, b_0, b_1 and $b_2 (\neq 0)$ are arbitrary constants depending upon the system parameters such as Peclet number, Brownian diffusivity and Stokes velocity of particles due to gravity.

Comparing with the system of FPDE (3.1) we conclude that

$$\begin{aligned} N_1[f, g] &= -a_0 \frac{\partial^{2\beta} f}{\partial x^{2\beta}} - a_1 f \left(\frac{\partial^\beta f}{\partial x^\beta} \right) - a_2 \left(f \frac{\partial^\beta g}{\partial x^\beta} + g \frac{\partial^\beta f}{\partial x^\beta} \right), \\ N_2[f, g] &= -b_0 \frac{\partial^{2\beta} g}{\partial x^{2\beta}} - b_1 g \left(\frac{\partial^\beta g}{\partial x^\beta} \right) - b_2 \left(f \frac{\partial^\beta g}{\partial x^\beta} + g \frac{\partial^\beta f}{\partial x^\beta} \right), \end{aligned}$$

are the corresponding nonlinear fractional differential operators.

Observe that $I = I_1^2 \times I_2^2 = \mathcal{Q}\{1, x^\beta\} \times \mathcal{Q}\{1, x^\beta\}$ is invariant under the operator $N[f, g]$ as

$$\begin{aligned} N_1[k_1 + k_2 x^\beta, l_1 + l_2 x^\beta] &= -\Gamma(1 + \beta) \left[(a_1 k_1 k_2 + a_2 k_1 l_2 + a_2 l_1 k_2) - (a_1 k_2^2 + 2a_2 k_2 l_2) x^\beta \right] \in I_1^2, \\ N_2[k_1 + k_2 x^\beta, l_1 + l_2 x^\beta] &= -\Gamma(1 + \beta) \left[(b_1 l_1 l_2 + b_2 k_1 l_2 + b_2 l_1 k_2) - (b_1 l_2^2 + 2b_2 k_2 l_2) x^\beta \right] \in I_2^2. \end{aligned}$$

In view of Theorem 3.1, system (3.8) admits solution of the form

$$f(t, x) = K_1(t) + K_2(t)x^\beta, \quad g(t, x) = L_1(t) + L_2(t)x^\beta, \quad (3.9)$$

where $K_1(t), K_2(t), L_1(t)$ and $L_2(t)$ satisfy the following system of FODEs

$$\frac{{}^{RL}D^\alpha K_1(t)}{dt^\alpha} = -\Gamma(1 + \beta) [a_1 K_1(t)K_2(t) + a_2 K_1(t)L_2(t) + a_2 L_1(t)K_2(t)], \quad (3.10)$$

$$\frac{{}^{RL}D^\alpha K_2(t)}{dt^\alpha} = -\Gamma(1 + \beta) [a_1 K_2^2(t) + 2a_2 K_2(t)L_2(t)], \quad (3.11)$$

$$\frac{{}^{RL}D^\alpha L_1(t)}{dt^\alpha} = -\Gamma(1 + \beta) [b_1 L_1(t)L_2(t) + b_2 K_1(t)L_2(t) + b_2 L_1(t)K_2(t)], \quad (3.12)$$

$$\frac{{}^{RL}D^\alpha L_2(t)}{dt^\alpha} = -\Gamma(1 + \beta) [b_1 L_2^2(t) + 2b_2 K_2(t)L_2(t)]. \quad (3.13)$$

Solving Eq. (3.11) and Eq. (3.13), we obtain

$$L_2(t) = M_2 t^{-\alpha}, \quad K_2(t) = \frac{-b_1}{2b_2} M_2 t^{-\alpha} - \frac{\Gamma(1-\alpha)t^{-\alpha}}{2b_2\Gamma(1-2\alpha)\Gamma(1+\beta)}, \quad M_2 (\neq 0) \text{ is arbitrary.} \quad (3.14)$$

Using (3.14), and solving Eq. (3.10) and Eq. (3.12), we deduce the solution of the system of generalized fractional Burger's equations (3.8) as

$$f(t, x) = \frac{-M_1\Gamma(1-\alpha)t^{-\alpha}}{2b_2M_2\Gamma(1-2\alpha)\Gamma(1+\beta)} - \frac{b_1M_1t^{-\alpha}}{2b_2} + \left[\frac{-b_1M_2t^{-\alpha}}{2b_2} - \frac{\Gamma(1-\alpha)t^{-\alpha}}{2b_2\Gamma(1-2\alpha)\Gamma(1+\beta)} \right] x^\beta, \\ g(t, x) = M_1 t^{-\alpha} + [M_2 t^{-\alpha}] x^\beta, \text{ where } a_1, a_2, b_1, b_2, M_1, M_2 (\neq 0) \text{ are arbitrary.}$$

Note that in particular, for $\alpha = \beta = 1$, $a_1 = b_1 = -2$, $a_0 = b_0 = -1$ and $a_2 = b_2 = 1$ Eq. (3.8) becomes coupled Burger equation [23].

Now consider fractional version of coupled Burger equation.

$$\begin{aligned} \frac{RL\partial^\alpha f}{\partial t^\alpha} &= \frac{\partial^{2\beta} f}{\partial x^{2\beta}} + 2f \left(\frac{\partial^\beta f}{\partial x^\beta} \right) - \left(f \frac{\partial^\beta g}{\partial x^\beta} + g \frac{\partial^\beta f}{\partial x^\beta} \right), \\ \frac{RL\partial^\alpha g}{\partial t^\alpha} &= \frac{\partial^{2\beta} g}{\partial x^{2\beta}} + 2g \left(\frac{\partial^\beta g}{\partial x^\beta} \right) - \left(f \frac{\partial^\beta g}{\partial x^\beta} + g \frac{\partial^\beta f}{\partial x^\beta} \right), \end{aligned} \quad (3.15)$$

Using $a_1 = b_1 = -2$, $a_2 = b_2 = 1$ and Eq. (3.10) we find the value of M_2 as

$$M_2 = \frac{-\Gamma(1-\alpha)}{2\Gamma(1-2\alpha)\Gamma(1+\beta)}.$$

Thus

$$K_1(t) = 2M_1 t^{-\alpha}, \quad K_2(t) = \frac{-\Gamma(1-\alpha)t^{-\alpha}}{\Gamma(1-2\alpha)\Gamma(1+\beta)}, \quad L_1(t) = M_1 t^{-\alpha}, \quad L_2(t) = \frac{-\Gamma(1-\alpha)t^{-\alpha}}{2\Gamma(1-2\alpha)\Gamma(1+\beta)}. \quad (3.16)$$

From (3.9) and (3.16) we deduce the solution of fractional version of coupled Burger equations (3.15) as

$$f(t, x) = 2M_1 t^{-\alpha} + \left[\frac{-\Gamma(1-\alpha)t^{-\alpha}}{\Gamma(1-2\alpha)\Gamma(1+\beta)} \right] x^\beta, \quad g(t, x) = M_1 t^{-\alpha} + \left[\frac{-\Gamma(1-\alpha)t^{-\alpha}}{2\Gamma(1-2\alpha)\Gamma(1+\beta)} \right] x^\beta, \quad M_1 \in \mathbb{R}. \quad (3.17)$$

The solution (3.17) is depicted in Fig. 1.

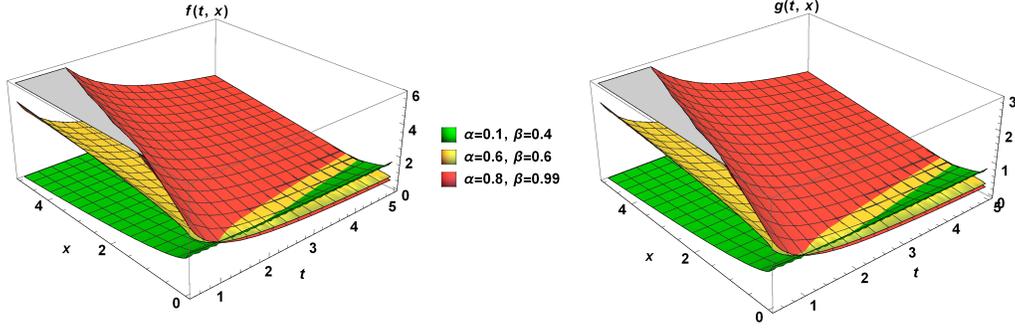


Figure 1: Plot of $f(t, x)$ and $g(t, x)$ for $M_1 = 1$, and different values of α and β .

3.2.2 Exact solution of coupled FPDEs

Consider the following system of nonlinear FPDEs for $t > 0, 0 < \alpha_1, \alpha_2, \beta \leq 1$.

$$\begin{aligned} \frac{\partial^{\alpha_1} f}{\partial t^{\alpha_1}} + \frac{\partial^{\alpha_1+1} f}{\partial t^{\alpha_1+1}} &= \frac{\partial^{2\beta} f}{\partial x^{2\beta}} + m_1 \left(g \frac{\partial^\beta g}{\partial x^\beta} \right) + a_1 m_1 g^2, \\ \frac{\partial^{\alpha_2} g}{\partial t^{\alpha_2}} + \frac{\partial^{\alpha_2+1} g}{\partial t^{\alpha_2+1}} &= \frac{\partial^{2\beta} g}{\partial x^{2\beta}} + n_1 \frac{\partial^{2\beta} f}{\partial x^{2\beta}} + a_2^2 n_1 f + n_2 g, \end{aligned} \quad (3.18)$$

where a_1, a_2, m_1, m_2, n_1 and n_2 are arbitrary constants.

In view of (3.1),

$$\begin{aligned} N_1[f, g] &= \frac{\partial^{2\beta} f}{\partial x^{2\beta}} + m_1 \left(g \frac{\partial^\beta g}{\partial x^\beta} \right) + a_1 m_1 g^2, \\ N_2[f, g] &= \frac{\partial^{2\beta} g}{\partial x^{2\beta}} + n_1 \frac{\partial^{2\beta} f}{\partial x^{2\beta}} + a_2^2 n_1 f + n_2 g. \end{aligned}$$

Clearly $I = I_1^2 \times I_2^1 = \mathcal{Q}\{\sin_\beta(a_2 x^\beta), \cos_\beta(a_2 x^\beta)\} \times \mathcal{Q}\{E_\beta(-a_1 x^\beta)\}$ is an invariant subspace under $N[f, g]$ as

$$\begin{aligned} N_1 \left[k_1 \sin_\beta(a_2 x^\beta) + k_2 \cos_\beta(a_2 x^\beta), l_1 E_\beta(-a_1 x^\beta) \right] &= -k_1 a_2^2 \sin_\beta(a_2 x^\beta) - k_2 a_2^2 \cos_\beta(a_2 x^\beta) \in I_1^2, \\ N_2 \left[k_1 \sin_\beta(a_2 x^\beta) + k_2 \cos_\beta(a_2 x^\beta), l_1 E_\beta(-a_1 x^\beta) \right] &= (a_1^2 + n_2) l_1 E_\beta(-a_1 x^\beta) \in I_2^1. \end{aligned}$$

Thus Theorem 3.1 implies that the system (3.18) admits solution of the form

$$f(t, x) = K_1(t) \sin_\beta(a_2 x^\beta) + K_2(t) \cos_\beta(a_2 x^\beta), \quad g(t, x) = L_1(t) E_\beta(-a_1 x^\beta), \quad (3.19)$$

where $K_1(t)$, $K_2(t)$ and $L_1(t)$ satisfy the system of FODEs

$$\frac{d^{\alpha_1} K_1(t)}{dt^{\alpha_1}} + \frac{d^{\alpha_1+1} K_1(t)}{dt^{\alpha_1+1}} = -a_2^2 K_1(t), \quad (3.20)$$

$$\frac{d^{\alpha_1} K_2(t)}{dt^{\alpha_1}} + \frac{d^{\alpha_1+1} K_2(t)}{dt^{\alpha_1+1}} = -a_2^2 K_2(t), \quad (3.21)$$

$$\frac{d^{\alpha_2} L_1(t)}{dt^{\alpha_2}} + \frac{d^{\alpha_2+1} L_1(t)}{dt^{\alpha_2+1}} = (a_1^2 + n_2) L_1(t) = a L_1(t), \quad a = (a_1^2 + n_2). \quad (3.22)$$

We apply Laplace transform to Eq. (3.20) and obtain

$$\begin{aligned} \tilde{K}_1(s) &= \frac{K_1(0)s^{\alpha_1}}{s^{\alpha_1} + s^{\alpha_1+1} + a_2^2} + \frac{[K_1(0) + K_1'(0)]s^{\alpha_1-1}}{s^{\alpha_1} + s^{\alpha_1+1} + a_2^2} \\ &= K_1(0) \sum_{m=0}^{\infty} \frac{(-1)^m a_2^{2m} s^{-\alpha_1 m}}{(s+1)^{m+1}} + [K_1(0) + K_1'(0)] \sum_{m=0}^{\infty} \frac{(-1)^m a_2^{2m} s^{-\alpha_1 m-1}}{(s+1)^{m+1}}, \quad \text{Re}(s) > 1. \end{aligned}$$

Taking inverse Laplace transform and using the relation (2.1)

$$K_1(t) = K_1(0) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_2^{2m} t^{(\alpha_1+1)m} E_{1,\alpha_1 m+1}^{(m)}(-t) + [K_1(0) + K_1'(0)] \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_2^{2m} t^{(\alpha_1+1)m+1} E_{1,\alpha_1 m+2}^{(m)}(-t).$$

Proceeding on similar lines we evaluate $K_2(t)$ and $L_2(t)$. Hence an exact solution of the system (3.18) is

$$\begin{aligned} f(t, x) &= \left[K_1(0) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_2^{2m} t^{(\alpha_1+1)m} E_{1,\alpha_1 m+1}^{(m)}(-t) + [K_1(0) + K_1'(0)] \right. \\ &\quad \left. \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_2^{2m} t^{(\alpha_1+1)m+1} E_{1,\alpha_1 m+2}^{(m)}(-t) \right] \sin_{\beta}(a_2 x^{\beta}) + \left[K_2(0) \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_2^{2m} t^{(\alpha_1+1)m} \right. \\ &\quad \left. \times E_{1,\alpha_1 m+1}^{(m)}(-t) + [K_2(0) + K_2'(0)] \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} a_2^{2m} t^{(\alpha_1+1)m+1} E_{1,\alpha_1 m+2}^{(m)}(-t) \right] \cos_{\beta}(a_2 x^{\beta}), \\ g(t, x) &= \left[L_1(0) \sum_{m=0}^{\infty} \frac{a^m}{m!} t^{(\alpha_2+1)m} E_{1,\alpha_2 m+1}^{(m)}(-t) + [L_1(0) + L_1'(0)] \right. \\ &\quad \left. \times \sum_{m=0}^{\infty} \frac{a^m}{m!} t^{(\alpha_2+1)m+1} E_{1,\alpha_2 m+2}^{(m)}(-t) \right] E_{\beta}(-a_1 x^{\beta}), \end{aligned}$$

where $a = a_1^2 + n_2$ and a_1, a_2, n_2 are arbitrary.

3.2.3 Coupled fractional Boussinesq equations

Fractional version of coupled Boussinesq equations along with initial conditions for $t > 0$, $0 < \alpha_1, \alpha_2, \beta \leq 1$ is

$$\frac{\partial^{\alpha_1} f}{\partial t^{\alpha_1}} = \frac{-\partial^\beta g}{\partial x^\beta} = N_1[f, g], \quad (3.23)$$

$$\frac{\partial^{\alpha_2} g}{\partial t^{\alpha_2}} = -m_1 \frac{\partial^\beta f}{\partial x^\beta} + 3f \left(\frac{\partial^\beta f}{\partial x^\beta} \right) + m_2 \frac{\partial^{3\beta} f}{\partial x^{3\beta}} = N_2[f, g], \quad (3.24)$$

$$f(0, x) = e + 2x^\beta, \quad g(0, x) = \frac{3}{2}, \quad (3.25)$$

where m_1, m_2 are arbitrary constants.

$I = I_1^2 \times I_2^2 = \mathcal{L}\{1, x^\beta\} \times \mathcal{L}\{1, x^\beta\}$ is invariant subspace w.r.t. the operator $N[f, g]$ as

$$N_1[k_1 + k_2 x^\beta, l_1 + l_2 x^\beta] = -\Gamma(1 + \beta)l_2 \in I_1^2,$$

$$N_2[k_1 + k_2 x^\beta, l_1 + l_2 x^\beta] = \Gamma(1 + \beta) [-m_1 k_2 + 3k_1 k_2] + 3\Gamma(1 + \beta)k_2^2 x^\beta \in I_2^2.$$

Hence using Theorem 3.1, system (3.23)-(3.24) has the following solution

$$f(t, x) = K_1(t) + K_2(t)x^\beta, \quad g(t, x) = L_1(t) + L_2(t)x^\beta, \quad (3.26)$$

where the unknowns functions $K_1(t), K_2(t), L_1(t)$ and $L_2(t)$ satisfy the following system of FODEs

$$\frac{d^{\alpha_1} K_1(t)}{dt^{\alpha_1}} = -\Gamma(1 + \beta)L_2(t), \quad (3.27)$$

$$\frac{d^{\alpha_1} K_2(t)}{dt^{\alpha_1}} = 0, \quad (3.28)$$

$$\frac{d^{\alpha_2} L_1(t)}{dt^{\alpha_2}} = \Gamma(1 + \beta) [-m_1 K_2(t) + 3K_1(t)K_2(t)], \quad (3.29)$$

$$\frac{d^{\alpha_2} L_2(t)}{dt^{\alpha_2}} = 3\Gamma(1 + \beta)K_2^2(t). \quad (3.30)$$

Eq. (3.28) implies that $K_2(t) = b$ (constant). Hence substituting value of $K_2(t)$ in Eq. (3.30) and performing fractional integration on both sides we obtain $L_2(t) = L_2(0) +$

$3b^2 \frac{\Gamma(1+\beta)t^{\alpha_2}}{\Gamma(1+\alpha_2)}$. Proceeding on similar lines solution (3.26) takes the following form

$$\begin{aligned}
 f(t, x) &= \left[a - d \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha_1)} t^{\alpha_1} - 3b^2 \frac{\Gamma(1+\beta)^2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} \right] + bx^\beta, \\
 g(t, x) &= \left[c - m_1 b \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha_2)} t^{\alpha_2} + 3ba \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha_2)} t^{\alpha_2} - 3bd \frac{\Gamma(1+\beta)^2 t^{\alpha_1+\alpha_2}}{\Gamma(1+\alpha_1+\alpha_2)} - 9b^3 \frac{\Gamma(1+\beta)^3 t^{\alpha_1+2\alpha_2}}{\Gamma(1+\alpha_1+2\alpha_2)} \right] \\
 &\quad + \left[d + 3b^2 \frac{\Gamma(1+\beta)t^{\alpha_2}}{\Gamma(1+\alpha_2)} \right] x^\beta, \quad a, b, c \text{ and } d \text{ are arbitrary.} \tag{3.31}
 \end{aligned}$$

Note: In particular for $\alpha_1 = \alpha_2 = \alpha$, and $\beta = 1$ the solution has the form

$$\begin{aligned}
 f(t, x) &= \left[a - \frac{dt^\alpha}{\Gamma(1+\alpha)} - \frac{3b^2 t^{2\alpha}}{\Gamma(1+2\alpha)} \right] + bx, \\
 g(t, x) &= \left[c - \frac{m_1 b t^\alpha}{\Gamma(1+\alpha)} + \frac{3bat^\alpha}{\Gamma(1+\alpha)} - \frac{3bd t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{9b^3 t^{3\alpha}}{\Gamma(1+3\alpha)} \right] + \left[d + \frac{3b^2 t^\alpha}{\Gamma(1+\alpha)} \right] x.
 \end{aligned}$$

This is a solution of the time fractional coupled Boussinesq equation obtained by Sahadevan *et. al* [21]. Exact solution of the system (3.23)-(3.24) along with the initial conditions (3.25) is

$$\begin{aligned}
 f(t, x) &= \left[e - \frac{12\Gamma(1+\beta)^2}{\Gamma(1+\alpha_1+\alpha_2)} t^{\alpha_1+\alpha_2} \right] + 2x^\beta, \\
 g(t, x) &= \left[\frac{3}{2} - \frac{2m_1\Gamma(1+\beta)}{\Gamma(1+\alpha_2)} t^{\alpha_2} + \frac{6e\Gamma(1+\beta)}{\Gamma(1+\alpha_2)} t^{\alpha_2} - \frac{72\Gamma(1+\beta)^3 t^{\alpha_1+2\alpha_2}}{\Gamma(1+\alpha_1+2\alpha_2)} \right] + \left[\frac{12\Gamma(1+\beta)}{\Gamma(1+\alpha_2)} t^{\alpha_2} \right] x^\beta. \tag{3.32}
 \end{aligned}$$

The solution (3.32) is plotted in Fig. 2.

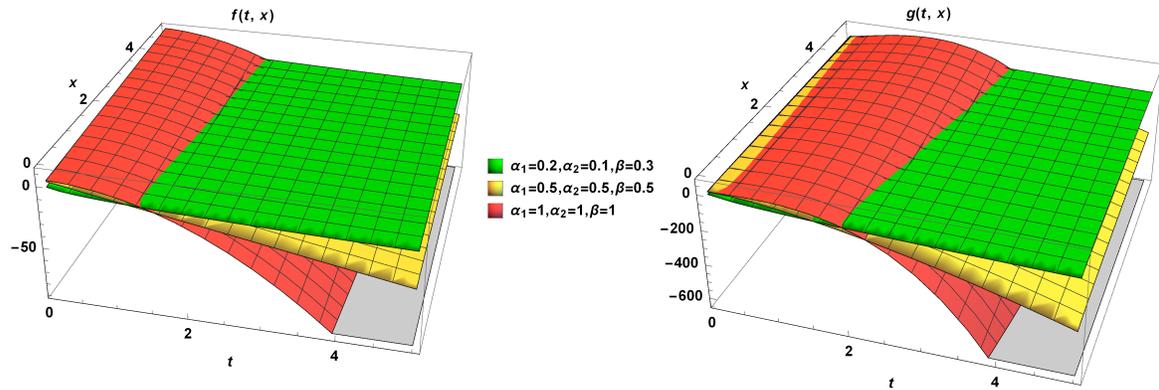


Figure 2: Plot of the solution (3.32) for various values of α_1 , α_2 and β .

3.2.4 Exact solution of fractional version of system of KdV type of equations

Consider fractional KdV type of fractional coupled equations for $t > 0$, $\alpha \in (0, 1] \setminus \{\frac{1}{2}\}$, $\beta \in (0, 1]$.

$$\begin{aligned}\frac{{}^{RL}\partial^\alpha f}{\partial t^\alpha} &= a_1 \left(f \frac{\partial^\beta f}{\partial x^\beta} \right) + a_2 \left(g \frac{\partial^\beta g}{\partial x^\beta} \right) + a_3 \frac{\partial^{3\beta} f}{\partial x^{3\beta}}, \\ \frac{{}^{RL}\partial^\alpha g}{\partial t^\alpha} &= b_1 \left(f \frac{\partial^\beta g}{\partial x^\beta} \right) + b_2 \left(g \frac{\partial^\beta f}{\partial x^\beta} \right) + b_3 \frac{\partial^{3\beta} g}{\partial x^{3\beta}}.\end{aligned}\quad (3.33)$$

Here a_1, a_2, a_3, b_1, b_2 and b_3 are arbitrary constants such that $b > a_1$ and $a_2 \geq 0$. Note that for $\alpha = 1 = \beta$, and

- When a_1, a_2, a_3, b_1, b_3 are arbitrary constants and $b_2 = 0$ then the coupled system (3.33) is the coupled KdV system given in ref. [22].
- When $a_1 = a_2 = b_1 = b_2 = 6$, and $a_3 = 1 = b_3$, then the system (3.33) is well-known complex coupled KdV system studied in [13].
- When $a_1 = 6a$, $a_2 = 2b$, $a_3 = a$, $b_1 = -3$, $b_2 = 0$ and $b_3 = -1$, (a, b are arbitrary), then the system (3.33) reduces to Hirota-Satsuma (HS)-KdV system proposed by Hirota and Satsuma in 1981 to model interactions of two long waves with different dispersion relations [12].
- When $a_1 = 6, a_2 = 2, a_3 = 1, b_1 = b_2 = 2$ and $b_3 = 0$, then the KdV type system (3.33) is treated as Ito type coupled KdV system [14].

Further note when $g = 0$, and $a_1 = 6, a_2 = 0, a_3 = -1$ the fractional KdV system (3.33) reduces to fractional KdV equation studied by Choudhary and Daftardar-Gejji [6].

In system (3.33)

$$\begin{aligned}N_1[f, g] &= a_1 \left(f \frac{\partial^\beta f}{\partial x^\beta} \right) + a_2 \left(g \frac{\partial^\beta g}{\partial x^\beta} \right) + a_3 \frac{\partial^{3\beta} f}{\partial x^{3\beta}}, \\ N_2[f, g] &= b_1 \left(f \frac{\partial^\beta g}{\partial x^\beta} \right) + b_2 \left(g \frac{\partial^\beta f}{\partial x^\beta} \right) + b_3 \frac{\partial^{3\beta} g}{\partial x^{3\beta}}.\end{aligned}$$

Note that $I = I_1^2 \times I_2^2 = \mathcal{Q}\{1, x^\beta\} \times \mathcal{Q}\{1, x^\beta\}$ is invariant subspace with respect to the operator $N[f, g]$ since

$$\begin{aligned}N_1[k_1 + k_2 x^\beta, l_1 + l_2 x^\beta] &= \Gamma(1 + \beta) [a_1 k_1 k_2 + a_2 l_1 l_2] + \Gamma(1 + \beta) [a_1 k_2^2 + a_2 l_2^2] x^\beta \in I_1^2, \\ N_2[k_1 + k_2 x^\beta, l_1 + l_2 x^\beta] &= \Gamma(1 + \beta) [b_1 k_1 l_2 + b_2 l_1 k_2] + \Gamma(1 + \beta) [(b_1 + b_2) k_2 l_2] x^\beta \in I_2^2.\end{aligned}$$

Hence the system (3.33) admits solution of the form

$$f(t, x) = K_1(t) + K_2(t)x^\beta, g(t, x) = L_1(t) + L_2(t)x^\beta, \quad (3.34)$$

where the unknown functions $K_1(t), K_2(t), L_1(t)$ and $L_2(t)$ satisfy the following system of FODEs.

$$\frac{{}^{RL}d^\alpha K_1(t)}{dt^\alpha} = \Gamma(1 + \beta) [a_1 K_1(t)K_2(t) + a_2 L_1(t)L_2(t)], \quad (3.35)$$

$$\frac{{}^{RL}d^\alpha K_2(t)}{dt^\alpha} = \Gamma(1 + \beta)[a_1 K_2^2(t) + a_2 L_2^2(t)], \quad (3.36)$$

$$\frac{{}^{RL}d^\alpha L_1(t)}{dt^\alpha} = \Gamma(1 + \beta)[b_1 K_1(t)L_2(t) + b_2 L_1(t)K_2(t)], \quad (3.37)$$

$$\frac{{}^{RL}d^\alpha L_2(t)}{dt^\alpha} = \Gamma(1 + \beta)[bK_2(t)L_2(t)], \quad b = (b_1 + b_2). \quad (3.38)$$

Solving system (3.35)-(3.38), we deduce the following solution of system (3.33):

For $\alpha = 1$,

$$f(t, x) = \frac{-\sqrt{a_2}M_1}{\sqrt{b-a_1}t} - \left(\frac{1}{b\Gamma(1+\beta)t} \right) x^\beta,$$

$$g(t, x) = \frac{M_1}{t} + \left(\frac{\sqrt{b-a_1}}{b\sqrt{a_2}\Gamma(1+\beta)t} \right) x^\beta,$$

For $\alpha \in (0, 1) \setminus \{\frac{1}{2}\}$,

$$f(t, x) = \frac{\sqrt{a_2}}{\sqrt{b-a_1}} M_1 t^{-\alpha} + \left[\frac{\Gamma(1-\alpha)t^{-\alpha}}{b\Gamma(1-2\alpha)\Gamma(1+\beta)} \right] x^\beta,$$

$$g(t, x) = M_1 t^{-\alpha} + \left[\frac{\sqrt{b-a_1}\Gamma(1-\alpha)t^{-\alpha}}{b\sqrt{a_2}\Gamma(1-2\alpha)\Gamma(1+\beta)} \right] x^\beta,$$

where M_1, a_1, a_2, b_1 and b_2 are arbitrary such that $b = b_1 + b_2, b > a_1, a_2 \geq 0$.

Solution of the system (3.33) for $\alpha \in (0, 1] \setminus \{\frac{1}{2}\}, \beta \in (0, 1]$ is plotted in Fig. 3.

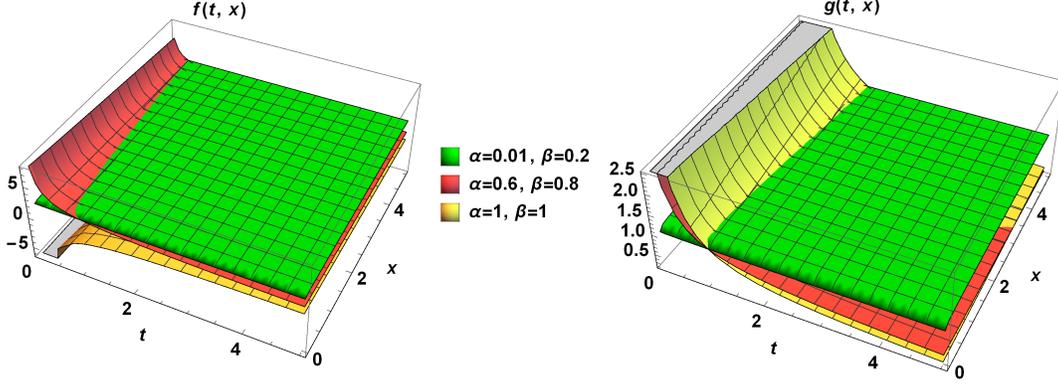


Figure 3: Plot of the solution of the system (3.33) for $M_1 = 1, a_1 = 2, a_2 = 4, b = 3$.

4 FPDEs in (1+n) dimension

In this section we consider higher dimensional FPDEs of the form

$$\sum_{i=1}^r \lambda_i \frac{\partial^{\gamma(i)} f(t, \bar{x})}{\partial t^{\gamma(i)}} = N^l[f(t, \bar{x})], \quad (4.1)$$

where $\bar{x} = (x_1, x_2, \dots, x_n)$, $l = 1, 2$, $\gamma(i) = i\alpha$ or $\gamma(i) = \alpha + i - 1$.

$N^1[f(t, \bar{x})] = \hat{N}^1 \left[\bar{x}, f, \frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}}, \dots, \frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}}, \frac{\partial^{2\beta_1} f}{\partial x_1^{2\beta_1}}, \dots, \frac{\partial^{k\beta_{n-1}} f}{\partial x_{n-1}^{k\beta_{n-1}}}, \frac{\partial^{k\beta_n} f}{\partial x_n^{k\beta_n}} \right]$, and

$N^2[f(t, \bar{x})] = \hat{N}^2 \left[\bar{x}, f, \frac{\partial^{\beta_1} f}{\partial x_1^{\beta_1}}, \dots, \frac{\partial^{\beta_n} f}{\partial x_n^{\beta_n}}, \frac{\partial^{\beta_1+1} f}{\partial x_1^{\beta_1+1}}, \dots, \frac{\partial^{\beta_{n-1}+k-1} f}{\partial x_{n-1}^{\beta_{n-1}+k-1}}, \frac{\partial^{\beta_n+k-1} f}{\partial x_n^{\beta_n+k-1}} \right]$ are nonlinear fractional differential operators in higher dimensions. Here $\frac{\partial^{\gamma(i)} f(\cdot)}{\partial t^{\gamma(i)}}$ is Caputo (or Riemann-Liouville) derivative with respect to t . $\frac{\partial^{j\beta_i} f(\cdot)}{\partial x_i^{j\beta_i}}$ and $\frac{\partial^{\beta_i+j-1} f(\cdot)}{\partial x_i^{\beta_i+j-1}}$, $i = 1, \dots, n$, $j = 1, \dots, k$ ($k \in \mathbb{N}$) are Caputo derivatives with respect to variable x_i . $[\alpha] = s, [\beta_i] = s_i$, where $s, s_i \in \mathbb{N}, i = 1, \dots, k$ and $\lambda_i \in \mathbb{R}$.

4.1 Invariant subspace method FPDEs in higher dimensions

Let I^m denote the m -dimensional linear space over \mathbb{R} spanned by m linearly independent basis functions $\{\phi_j(x_1, x_2, \dots, x_n) : j = 1, \dots, m\}$, i.e.,

$$I^m = \mathcal{L}\{\phi_1(\bar{x}), \phi_2(\bar{x}), \dots, \phi_m(\bar{x})\} = \left\{ \sum_{j=1}^m k_j \phi_j(\bar{x}) \mid k_j \in \mathbb{R}, i = 1, \dots, m \right\}.$$

A finite dimensional linear space I^m is said to be invariant under fractional differential operators $N^l[f(t, \bar{x})]$ ($l = 1, 2$), if $N^l[f] \in I^m, \forall f \in I^m$.

Theorem 4.1. *If a finite dimensional linear space I^m is invariant under the operators $N^l[f(t, \bar{x})]$, $l = 1, 2$, then FPDE (4.1) has a solution of the form*

$$f(t, \bar{x}) = \sum_{j=1}^m K_j(t)\phi_j(\bar{x}), \quad (4.2)$$

where the coefficient $\{K_j\}$'s satisfy the following system of FODEs

$$\sum_{i=1}^r \lambda_i \frac{d^{\gamma(i)} K_j(t)}{dt^{\gamma(i)}} = \psi_j(K_1(t), K_2(t), \dots, K_m(t)), \quad j = 1, \dots, m. \quad (4.3)$$

Here $\gamma(i) = i\alpha$ or $\gamma(i) = \alpha + i - 1$, and $\{\psi_j\}$'s are the expansion coefficients of $N^l[f(t, \bar{x})]$, $l = 1, 2$ with respect to basis function $\{\phi_j\}$'s of I^m .

Proof. Using linearity of fractional derivatives and Eq. (4.2), L.H.S of FPDE (4.1) reduces to

$$\sum_{i=1}^r \lambda_i \frac{\partial^{\gamma(i)} f(t, \bar{x})}{\partial t^{\gamma(i)}} = \sum_{i=1}^r \lambda_i \frac{\partial^{\gamma(i)}}{\partial t^{\gamma(i)}} \left(\sum_{j=1}^m K_j(t)\phi_j(\bar{x}) \right) = \sum_{j=1}^m \left[\sum_{i=1}^r \lambda_i \frac{d^{\gamma(i)} K_j(t)}{dt^{\gamma(i)}} \right] \phi_j(\bar{x}). \quad (4.4)$$

Further as I^m is an invariant space under the operator $N^l[f]$, there exist m linearly independent functions $\phi_1(\bar{x}), \phi_2(\bar{x}), \dots, \phi_m(\bar{x})$ such that

$$N^l \left[\sum_{j=1}^m k_j \phi_j(\bar{x}) \right] = \sum_{j=1}^m \psi_j(k_1, k_2, \dots, k_m) \phi_j(\bar{x}), \quad \text{for } k_j \in \mathbb{R}, l = 1, 2, \quad (4.5)$$

where $\{\psi_j\}$'s are expansion coefficients of $N^l[f] \in I^m$ with respect to the basis $\{\phi_j\}_{j=1}^m$.

In view of Eq. (4.2) and Eq. (4.5)

$$N^l[f(t, \bar{x})] = N^l \left[\sum_{j=1}^m K_j(t)\phi_j(\bar{x}) \right] = \sum_{j=1}^m \psi_j(K_1(t), \dots, K_m(t))\phi_j(\bar{x}), \quad l = 1, 2. \quad (4.6)$$

Substituting Eq. (4.4) and Eq. (4.6) in Eq. (4.1), we get

$$\sum_{j=1}^m \left[\sum_{i=1}^r \lambda_i \frac{d^{\gamma(i)} K_j(t)}{dt^{\gamma(i)}} - \psi_j(K_1(t), K_2(t), \dots, K_m(t)) \right] \phi_j(\bar{x}) = 0. \quad (4.7)$$

Using Eq. (4.7) and the fact that $\{\phi_j\}$'s are basis functions, we get the following system of FODEs

$$\sum_{i=1}^r \lambda_i \frac{d^{\gamma(i)} K_j(t)}{dt^{\gamma(i)}} = \psi_j(K_1(t), K_2(t), \dots, K_m(t)), \quad j = 1, \dots, m,$$

where $\gamma(i) = i\alpha$ or $\gamma(i) = \alpha + i - 1$. □

4.2 Illustrative examples for FPDEs in higher dimensions

4.2.1 Fractional dispersive KdV equation in (1+n) dimensions.

Consider the following linear fractional dispersive KdV equation

$$\frac{\partial^\alpha f}{\partial t^\alpha} + \frac{\partial^{3\beta_1} f}{\partial x_1^{3\beta_1}} + \frac{\partial^{3\beta_2} f}{\partial x_2^{3\beta_2}} + \cdots + \frac{\partial^{3\beta_n} f}{\partial x_n^{3\beta_n}} = 0, \quad t > 0, \quad \beta_1, \beta_2, \dots, \beta_n \in (0, 1]. \quad (4.8)$$

In view of FPDE (4.1), we note that $N[f] = -\frac{\partial^{3\beta_1} f}{\partial x_1^{3\beta_1}} - \cdots - \frac{\partial^{3\beta_n} f}{\partial x_n^{3\beta_n}}$. Observe that when

$$I^{2n} = \mathfrak{Q}\{\cos_{\beta_1}(\lambda_1 x_1^{\beta_1}), \sin_{\beta_1}(\lambda_1 x_1^{\beta_1}), \cos_{\beta_2}(\lambda_2 x_2^{\beta_2}), \sin_{\beta_2}(\lambda_2 x_2^{\beta_2}), \dots, \cos_{\beta_n}(\lambda_n x_n^{\beta_n}), \sin_{\beta_n}(\lambda_n x_n^{\beta_n})\},$$

$$N \left[\sum_{i=1}^n \left(k_{i1} \cos_{\beta_i}(\lambda_i x_i^{\beta_i}) + k_{i2} \sin_{\beta_i}(\lambda_i x_i^{\beta_i}) \right) \right] = \sum_{i=1}^n \left(-\lambda_i^3 k_{i1} \sin_{\beta_i}(\lambda_i x_i^{\beta_i}) + \lambda_i^3 k_{i2} \cos_{\beta_i}(\lambda_i x_i^{\beta_i}) \right) \in I^{2n}.$$

Hence I^{2n} is an invariant subspace of fractional operator $N[f]$. Hence Theorem (4.1) implies an exact solution of the form

$$f(t, \bar{x}) = \sum_{i=1}^n \left(k_{i1}(t) \cos_{\beta_i}(\lambda_i x_i^{\beta_i}) + k_{i2}(t) \sin_{\beta_i}(\lambda_i x_i^{\beta_i}) \right), \quad \lambda_i, \quad i = 1, \dots, n \text{ are distinct.} \quad (4.9)$$

where $K_{i1}(t)$ and $K_{i2}(t)$, $i = 1, \dots, n$ are the unknown functions to be determined by solving the following system of FODEs:

$$\frac{d^\alpha K_{11}(t)}{dt^\alpha} = \lambda_1^3 K_{12}(t), \quad (4.10)$$

$$\frac{d^\alpha K_{12}(t)}{dt^\alpha} = -\lambda_1^3 K_{11}(t), \quad (4.11)$$

$$\frac{d^\alpha K_{21}(t)}{dt^\alpha} = \lambda_2^3 K_{22}(t), \quad (4.12)$$

$$\frac{d^\alpha K_{22}(t)}{dt^\alpha} = -\lambda_2^3 K_{21}(t), \quad (4.13)$$

$$\frac{d^\alpha K_{n1}(t)}{dt^\alpha} = \lambda_2^3 K_{n2}(t), \quad (4.14)$$

$$\frac{d^\alpha K_{n2}(t)}{dt^\alpha} = -\lambda_2^3 K_{n1}(t). \quad (4.15)$$

After solving the system (4.10) – (4.15), we get

$$K_{i1}(t) = a_i \sin_\alpha(\lambda_i^3 t^\alpha), \quad K_{i2}(t) = a_i \cos_\alpha(\lambda_i^3 t^\alpha). \quad (4.16)$$

From (4.9) and (4.16) we find the following exact solution of system (4.8) as

$$f(t, \bar{x}) = \sum_{i=1}^n \left(a_i \sin_\alpha(\lambda_i^3 t^\alpha) \cos_{\beta_i}(\lambda_i x_i^{\beta_i}) + a_i \cos_\alpha(\lambda_i^3 t^\alpha) \sin_{\beta_i}(\lambda_i x_i^{\beta_i}) \right), \quad (4.17)$$

where a_i and λ_i are arbitrary constants for $i = 1, 2, \dots, n$.
The solution (4.17) for $n = 2$ is depicted in Fig. 4.

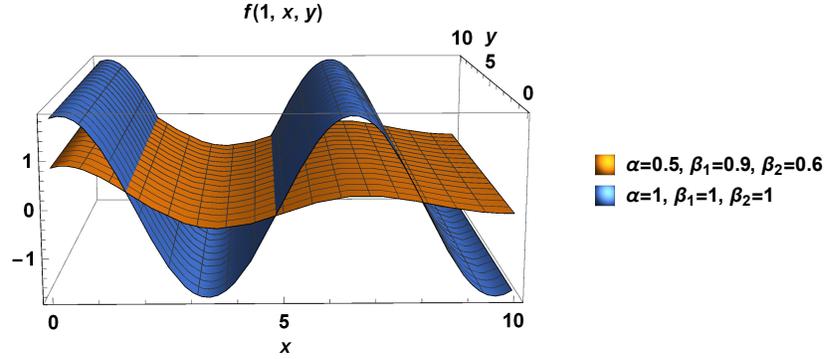


Figure 4: Plot of $f(t, \bar{x})$ of Eq. (4.8) for $n = 2, a_1 = a_2 = \lambda_1 = 1$ and $\lambda_2 = 2$ at $t = 1$.

Note: The fractional dispersion KdV equation admits another invariant subspace $I^n = \mathcal{Q}\{E_{\beta_1}(\lambda_1 x_1^{\beta_1}), E_{\beta_2}(\lambda_2 x_2^{\beta_2}), \dots, E_{\beta_n}(\lambda_n x_n^{\beta_n})\}$, leading to following distinct solution:

$$f(t, \bar{x}) = \sum_{i=1}^n \left[a_i E_{\alpha}(\lambda_i^3 t^{\alpha}) \right] E_{\beta_i}(\lambda_i x_i^{\beta_i}), \quad a_i \in \mathbb{R} \text{ for } i = 1, \dots, n.$$

4.2.2 Fractional version of (1+2) dimensional population model

We discuss two dimensional nonlinear FPDE for the population density f .

$$\frac{\partial^{\alpha} f}{\partial t^{\alpha}} = \frac{\partial^{\beta}}{\partial x^{\beta}} \left(\frac{\partial^{\beta} f^2}{\partial x^{\beta}} \right) + \frac{\partial^{\gamma}}{\partial y^{\gamma}} \left(\frac{\partial^{\gamma} f^2}{\partial y^{\gamma}} \right) + \psi(f), \quad \alpha, \beta, \gamma \in (0, 1]. \quad (4.18)$$

Note that for $\alpha = \beta = \gamma = 1$ and

- $\psi(f) = cf, c \in \mathbb{R}$, the population model (4.18) follows Malthusian law [11],
- $\psi(f) = f(c_1 - c_2 f), c_1, c_2 \in \mathbb{R}$, Eq. (4.18) satisfies Verhulst law [11].

Here we consider $\psi(f) = cf$. Hence $N[f(t, x, y)] = \frac{\partial^{\beta}}{\partial x^{\beta}} \left(\frac{\partial^{\beta} f^2}{\partial x^{\beta}} \right) + \frac{\partial^{\gamma}}{\partial y^{\gamma}} \left(\frac{\partial^{\gamma} f^2}{\partial y^{\gamma}} \right) + cf$ is the fractional nonlinear operator. $I^3 = \mathcal{Q}\{1, x^{\beta}, y^{\gamma}\}$ is invariant under $N[f]$ since

$$N[k_1 + k_2 x^{\beta} + k_3 y^{\gamma}] = ck_1 + \Gamma(2\beta + 1)k_2^2 + \Gamma(2\gamma + 1)k_3^2 + ck_2 x^{\beta} + ck_3 y^{\gamma} \in I^3.$$

In view of Theorem 4.1, solution of the equation under consideration (3.18) is

$$f(t, x, y) = K_1(t) + K_2(t)x^\beta + K_3(t)y^\gamma, \quad (4.19)$$

where $K_1(t)$, $K_2(t)$ and $K_3(t)$ satisfy the following set of equations:

$$\frac{d^\alpha K_1(t)}{dt^\alpha} = cK_1(t) + \Gamma(2\beta + 1)K_2(t)^2 + \Gamma(2\gamma + 1)K_3(t)^2, \quad (4.20)$$

$$\frac{d^\alpha K_2(t)}{dt^\alpha} = cK_2(t), \quad (4.21)$$

$$\frac{d^\alpha K_3(t)}{dt^\alpha} = cK_3(t). \quad (4.22)$$

Solving Eq. (4.21) and Eq. (4.22) using Laplace transform technique, we obtain

$$K_2(t) = a_2 E_\alpha(ct^\alpha), \quad K_3(t) = a_3 E_\alpha(ct^\alpha), \quad a_2, a_3 \text{ are arbitrary.}$$

Substituting the obtained values of $K_2(t)$ and $K_3(t)$ in Eq. (4.20), we get

$$K_1(t) = cK_1(t) + A [E_\alpha(ct^\alpha)]^2, \quad A = a_2^2 \Gamma(2\beta + 1) + a_3^2 \Gamma(2\gamma + 1). \quad (4.23)$$

We apply NIM [3] to solve Eq. (4.23). Applying I^α to both sides of Eq. (4.23), we obtain the following integral equation

$$K_1(t) = a_1 + AI^\alpha [E_\alpha(ct^\alpha)]^2 + M[K_1(t)], \quad \text{where } M[K_1(t)] = cI^\alpha K_1(t).$$

Let

$$\begin{aligned} K_1^0(t) &= a_1 + AI^\alpha [E_\alpha(ct^\alpha)]^2, \\ K_1^1(t) &= M[K_1^0(t)] = \frac{a_1 c t^\alpha}{\Gamma(\alpha + 1)} + AcI^{2\alpha} [E_\alpha(ct^\alpha)]^2, \\ &\vdots \\ K_1^n(t) &= M[K_1^{n-1}(t)] = \frac{a_1 c^n t^{n\alpha}}{\Gamma(\alpha + n)} + Ac^n I^{(n+1)\alpha} [E_\alpha(ct^\alpha)]^2. \end{aligned}$$

Hence

$$K_1(t) = \sum_{m=0}^{\infty} K_1^m(t) = a_1 E_\alpha(ct^\alpha) + A \sum_{m=0}^{\infty} c^m I^{(m+1)\alpha} [E_\alpha(ct^\alpha)]^2. \quad (4.24)$$

Using (4.23) and (4.24), solution (4.19) takes the following form:

$$f(t, x, y) = a_1 E_\alpha(ct^\alpha) + A \sum_{m=0}^{\infty} c^m I^{(m+1)\alpha} [E_\alpha(ct^\alpha)]^2 + [a_2 E_\alpha(ct^\alpha)] x^\beta + [a_3 E_\alpha(ct^\alpha)] y^\gamma.$$

4.2.3 Exact solution of fractional scale wave equation in (1+2) dimension

Consider fractional version of scale wave equation

$$\frac{\partial^\beta}{\partial x^\beta} \left(\frac{\partial^\beta f}{\partial x^\beta} \right) + \frac{\partial^\gamma}{\partial y^\gamma} \left(\frac{\partial^\gamma f}{\partial y^\gamma} \right) - a \frac{\partial^\alpha f}{\partial t^\alpha} - \frac{\partial^{\alpha+1} f}{\partial t^{\alpha+1}} = 0, \quad t > 0, \quad \alpha, \beta, \gamma \in (0, 1]. \quad (4.25)$$

Comparing with the equation (4.1) we note that

$$N[f] = \frac{\partial^\beta}{\partial x^\beta} \left(\frac{\partial^\beta f}{\partial x^\beta} \right) + \frac{\partial^\gamma}{\partial y^\gamma} \left(\frac{\partial^\gamma f}{\partial y^\gamma} \right).$$

Observe that $I^2 = \mathcal{L}\{E_\beta(\lambda_1 x^\beta), E_\gamma(-\lambda_2 y^\gamma)\}$ is one of the required invariant subspaces, since

$$N[k_1 E_\beta(\lambda_1 x^\beta) + k_2 E_\gamma(-\lambda_2 y^\gamma)] = \lambda_1^2 k_1 E_\beta(\lambda_1 x^\beta) + \lambda_2^2 k_2 E_\gamma(-\lambda_2 y^\gamma) \in I^2.$$

Hence $f(t, x, y) = K_1(t)E_\beta(\lambda_1 x^\beta) + K_2(t)E_\gamma(-\lambda_2 y^\gamma)$ is a solution of the fractional scale wave equation (4.25), where $K_1(t)$ and $K_2(t)$ are the unknown functions that satisfy following system of FODEs.

$$a \frac{d^\alpha K_1(t)}{dt^\alpha} + \frac{d^{\alpha+1} K_1(t)}{dt^{\alpha+1}} = \lambda_1^2 K_1(t), \quad (4.26)$$

$$a \frac{d^\alpha K_2(t)}{dt^\alpha} + \frac{d^{\alpha+1} K_2(t)}{dt^{\alpha+1}} = \lambda_2^2 K_2(t), \quad (4.27)$$

Using Laplace transform technique to Eqs. (4.26)-(4.27), we deduce the exact solution of fractional scale wave equation (4.25) as

$$f(t, x, y) = \left[b_1 \sum_{m=0}^{\infty} \frac{\lambda_2^{2m}}{m!} t^{(\alpha+1)m} E_{1,\alpha m+1}^{(m)}(-at) + (b_1 + b_2) \sum_{m=0}^{\infty} \frac{\lambda_2^{2m}}{m!} t^{(\alpha+1)m+1} E_{1,\alpha m+2}^{(m)}(-at) \right] E_\beta(\lambda_1 x^\beta) \\ + \left[a_1 \sum_{m=0}^{\infty} \frac{\lambda_1^{2m}}{m!} t^{(\alpha+1)m} E_{1,\alpha m+1}^{(m)}(-at) + (a_1 + a_2) \sum_{m=0}^{\infty} \frac{\lambda_1^{2m}}{m!} t^{(\alpha+1)m+1} E_{1,\alpha m+2}^{(m)}(-at) \right] E_\gamma(-\lambda_2 y^\gamma),$$

where a_1, a_2, b_1 and $b_2 \in \mathbb{R}$.

4.2.4 Solutions of fractional order Boussinesq equation

Consider the IVP for fractional order Boussinesq equation where $t > 0$, $\alpha, \beta \in (0, 1]$.

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{\partial^\beta}{\partial x^\beta} \left((rf + s) \frac{\partial^\beta (rf + s)}{\partial x^\beta} \right) + \frac{\partial^\beta}{\partial y^\beta} \left((rf + s) \frac{\partial^\beta (rf + s)}{\partial y^\beta} \right) = N[f], \quad (4.28)$$

$$f(0, x, y) = \frac{9}{5} + e^2 y^\gamma, \quad (4.29)$$

where r, s are arbitrary. For $\alpha = \beta = \gamma = 1$, Eq. (4.28) is a two dimensional heat and mass transfer equation with temperature dependent diffusion coefficient [25].

We choose $I^3 = \mathcal{L}\{1, x^\beta, y^\gamma\}$ which satisfies the invariant subspace property as follows:

$$N[k_1 + k_2 x^\beta + k_3 y^\gamma] = r^2 [\Gamma(\beta + 1)^2 k_2^2 + \Gamma(\gamma + 1)^2 k_3^2] \in I^3.$$

In view of Theorem (4.1), an exact solution of Eq. (4.28) has the form

$$f(t, x, y) = K_1(t) + K_2(t)x^\beta + K_3(t)y^\gamma,$$

where the unknown functions $K_1(t), K_2(t)$ and $K_3(t)$ are to be evaluated by solving the following system:

$$\frac{d^\alpha K_1(t)}{dt^\alpha} = r^2 [\Gamma(\beta + 1)^2 K_2(t)^2 + \Gamma(\gamma + 1)^2 K_3(t)^2], \quad (4.30)$$

$$\frac{d^\alpha K_2(t)}{dt^\alpha} = 0, \quad (4.31)$$

$$\frac{d^\alpha K_3(t)}{dt^\alpha} = 0. \quad (4.32)$$

Clearly $K_2(t) = a_2$ and $K_3(t) = a_3$, where a_1, a_2 are constants. Eq. (4.30) becomes

$$\frac{d^\alpha K_1(t)}{dt^\alpha} = r^2 [a_2^2 \Gamma(\beta + 1)^2 + a_3^2 \Gamma(\gamma + 1)^2]. \quad (4.33)$$

Solving Eq. (4.33), we get $K_1(t) = a_1 + \frac{r^2 [a_2^2 \Gamma(\beta + 1)^2 + a_3^2 \Gamma(\gamma + 1)^2] t^\alpha}{\Gamma(\alpha + 1)}$. Hence exact solution of IVP (4.28)-(4.29) is

$$f(t, x, y) = \frac{9}{5} + \frac{e^4 r^2 \Gamma(\gamma + 1)^2}{\Gamma(\alpha + 1)} t^\alpha + e^2 y^\gamma. \quad (4.34)$$

The solution (4.34) is plotted in Fig. 5.

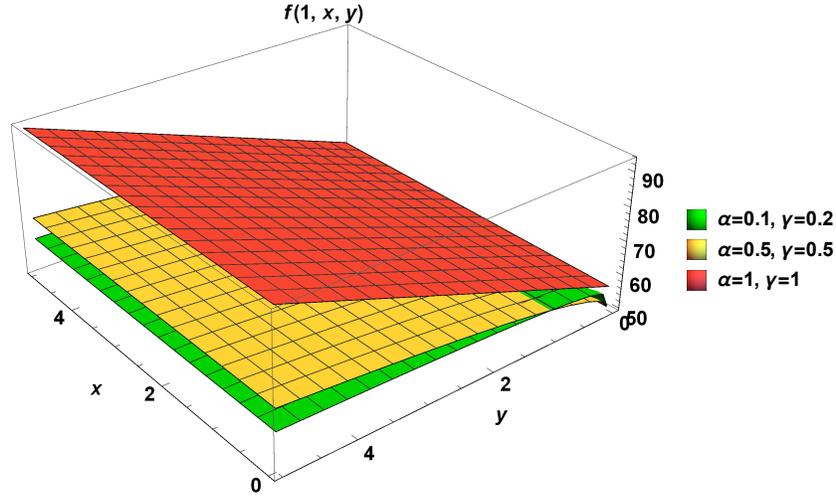


Figure 5: Plot of solution of the IVP (4.28)-(4.29) for different values of α and γ at $t = 1$.

Note: It can be verified that the Eq. (4.28) also admits another invariant subspace $I^4 = \mathcal{Q}\{1, x^{2\beta}, y^{2\gamma}, x^\beta y^\gamma\}$. Thus proceeding on similar lines as previous examples another exact solution corresponding to I^4 can be found.

4.2.5 Exact solution of fractional diffusion like PDE in (1+2) dimensions

Consider fractional PDE with initial condition as follows

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \frac{1}{2} \left(x^{2\beta} \frac{\partial^{2\gamma} f}{\partial y^{2\gamma}} + y^{2\gamma} \frac{\partial^{2\beta} f}{\partial x^{2\beta}} \right), \quad t > 0, \quad \alpha, \beta, \gamma \in (0, 1], \quad (4.35)$$

$$f(0, x, y) = y^{2\gamma}. \quad (4.36)$$

Here $I^3 = \mathcal{Q}\{1, x^{2\beta}, y^{2\gamma}\}$ is an invariant subspace for $N[f] = \frac{1}{2} \left(x^{2\beta} \frac{\partial^{2\gamma} f}{\partial y^{2\gamma}} + y^{2\gamma} \frac{\partial^{2\beta} f}{\partial x^{2\beta}} \right)$ as

$$N[k_1 + k_2 x^{2\beta} + k_3 y^{2\gamma}] = \frac{\Gamma(2\gamma + 1)}{2} k_3 x^{2\beta} + \frac{\Gamma(2\beta + 1)}{2} k_2 y^{2\gamma} \in I^3.$$

Since criteria of Theorem (4.1) is satisfied, an exact solution of Eq. (4.35) is of the form

$$f(t, x, y) = K_1(t) + K_2(t)x^{2\beta} + K_3(t)y^{2\gamma}, \quad (4.37)$$

where the unknown functions $K_1(t)$, $K_2(t)$ and $K_3(t)$ satisfy following system of FODEs:

$$\frac{d^\alpha K_1(t)}{dt^\alpha} = 0, \quad (4.38)$$

$$\frac{d^\alpha K_2(t)}{dt^\alpha} = \frac{\Gamma(2\gamma + 1)}{2} K_3(t), \quad (4.39)$$

$$\frac{d^\alpha K_3(t)}{dt^\alpha} = \frac{\Gamma(2\beta + 1)}{2} K_2(t). \quad (4.40)$$

Clearly from (4.39)-(4.40), we deduce

$$\frac{d^\alpha}{dt^\alpha} \left(\frac{d^\alpha K_2(t)}{dt^\alpha} \right) = \lambda K_2(t), \quad \lambda = \lambda_1 \lambda_2, \quad \lambda_1 = \frac{\Gamma(2\gamma + 1)}{2}, \quad \lambda_2 = \frac{\Gamma(2\beta + 1)}{2}. \quad (4.41)$$

Taking Laplace transform of Eq. (4.41), we get

$$s^\alpha \mathcal{L}(D_t^\alpha K_2(t); s) - s^{\alpha-1} (D_t^\alpha K_2(0)) = \lambda \tilde{K}_2(s)$$

$$\tilde{K}_2(s) = b_1 \frac{s^{2\alpha-1}}{s^{2\alpha} - \lambda} + b_2 \frac{s^{\alpha-1}}{s^{2\alpha} - \lambda}, \quad \text{where } b_1 = K_2(0), \quad b_2 = D^\alpha K_2(0).$$

Performing inverse Laplace transform, we obtain $K_2(t) = b_1 E_{2\alpha,1}(\lambda t^{2\alpha}) + b_2 t^\alpha E_{2\alpha,\alpha+1}(\lambda t^{2\alpha})$, where b_1 and b_2 are arbitrary. Substituting value of $K_2(t)$ in Eq. (4.40) and solving the same, we get $K_3(t) = c + \lambda_2 b_1 t^\alpha E_{2\alpha,\alpha+1}(\lambda_2 t^{2\alpha}) + \lambda_2 b_2 t^{2\alpha} E_{2\alpha,2\alpha+1}(\lambda_2 t^{2\alpha})$.

Hence we get an exact solution of Eq. (4.35) as

$$f(t, x, y) = a + \left[b_1 E_{2\alpha,1}(\lambda t^{2\alpha}) + b_2 t^\alpha E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}) \right] x^{2\beta} + \left[c + \lambda_2 b_1 t^\alpha E_{2\alpha,\alpha+1}(\lambda_2 t^{2\alpha}) \right. \\ \left. + \lambda_2 b_2 t^{2\alpha} E_{2\alpha,2\alpha+1}(\lambda_2 t^{2\alpha}) \right] y^{2\gamma}, \quad c = b_2, \quad \text{and } a, b_1, b_2 \in \mathbb{R}. \quad (4.42)$$

Using initial condition (4.36), solution (4.42) reduces to

$$f(t, x, y) = \left[t^\alpha E_{2\alpha,\alpha+1}(\lambda t^{2\alpha}) \right] x^{2\beta} + \left[1 + \lambda_2 t^{2\alpha} E_{2\alpha,2\alpha+1}(\lambda_2 t^{2\alpha}) \right] y^{2\gamma}. \quad (4.43)$$

Note: For $\alpha = \beta = \gamma = 1$, $c = b_2$ and under an initial condition $f(0, x, y) = y^2$, solution (4.42) reduces to

$$f(t, x, y) = (\sinh t)x^2 + (\cosh t)y^2. \quad (4.44)$$

Solution (4.44) coincides with the solution for the heat like equation $\frac{\partial f}{\partial t} = \frac{1}{2} \left(x^2 \frac{\partial^2 f}{\partial y^2} + y^2 \frac{\partial^2 f}{\partial x^2} \right)$, obtained by NIM [2].

The solution (4.43) is depicted in Fig. 6.

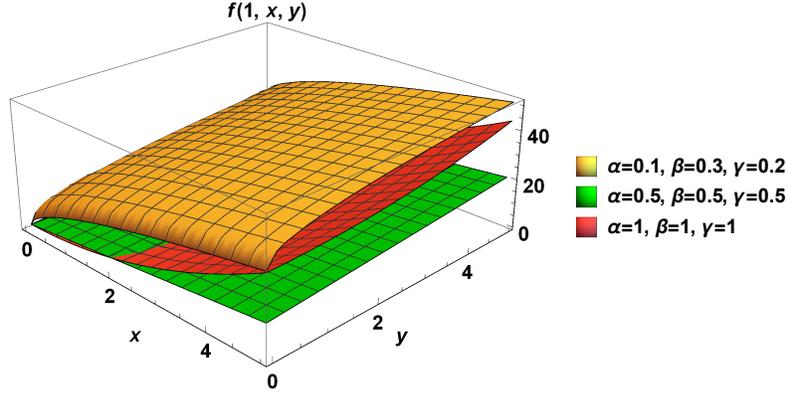


Figure 6: 3D-Plot of solution of the IVP (4.35)-(4.36) at $t = 1$.

5 Fractional differential operators with mixed partial derivatives

It should be noted that invariant subspace method can also be employed for solving FPDEs with fractional differential operators $N^l[f]$ ($l = 1, 2$) involving mixed fractional partial derivatives. Analysis of such FPDEs can be done on the similar lines as done in Sec. (3) and Sec. (4). We illustrate the method by solving an example.

Consider the following system of nonlinear FPDEs for $t > 0, 0 < \alpha_1, \alpha_2, \beta, \gamma \leq 1$.

$$\begin{aligned} \frac{\partial^{\alpha_1} f}{\partial t^{\alpha_1}} &= \frac{\partial^\gamma}{\partial t^\gamma} \left(\frac{\partial^{2\beta} f}{\partial x^{2\beta}} \right) + m_1 \left(g \frac{\partial^\beta g}{\partial x^\beta} \right) + a_1 m_1 g^2, \\ \frac{\partial^{\alpha_2} g}{\partial t^{\alpha_2}} &= \frac{\partial^\gamma}{\partial t^\gamma} \left(\frac{\partial^{2\beta} g}{\partial x^{2\beta}} \right) + n_1 \frac{\partial^\beta}{\partial x^\beta} \left(\frac{\partial^\beta f}{\partial x^\beta} \right) - a_2^2 n_1 f + n_2 g, \end{aligned} \quad (5.1)$$

where $\gamma < \alpha_1, \gamma < \alpha_2$, and a_1, a_2, m_1, m_2, n_1 and n_2 are arbitrary constants.

Observe that $I = I_1^2 \times I_2^1 = \mathfrak{L}\{E_\beta(a_2 x^\beta), E_\beta(-a_2 x^\beta)\} \times \mathfrak{L}\{E_\beta(-a_1 x^\beta)\}$, is an invariant subspace corresponding to the given fractional differential operator. Hence the system (5.1) admits solution of the form

$$f(t, x) = K_1(t)E_\beta(a_2 x^\beta) + K_2 E_\beta(-a_2 x^\beta), \quad g(t, x) = L_1(t)E_\beta(-a_1 x^\beta), \quad (5.2)$$

such that

$$\frac{d^{\alpha_1} K_1(t)}{dt^{\alpha_1}} = a_2^2 \frac{d^\gamma K_1(t)}{dt^\gamma}, \quad (5.3)$$

$$\frac{d^{\alpha_1} K_2(t)}{dt^{\alpha_1}} = a_2^2 \frac{d^\gamma K_2(t)}{dt^\gamma}, \quad (5.4)$$

$$\frac{d^{\alpha_2} L_1(t)}{dt^{\alpha_2}} = a_1^2 \frac{d^\gamma L_1(t)}{dt^\gamma} + n_2 L_1(t). \quad (5.5)$$

Solving the system of FODEs (5.3)-(5.5) and substituting the values of $K_1(t)$, $K_2(t)$ and $L_1(t)$ in Eq. (5.2) we get an exact solution of the system (5.1) as

$$\begin{aligned} f(t, x) &= \left[b_1 \sum_{m=0}^{\infty} \left(\frac{a_2^{2m} t^{(\alpha_1-\gamma)m}}{\Gamma((\alpha_1-\gamma)m+1)} - \frac{a_2^{2m+2} t^{(\alpha_1-\gamma)(m+1)}}{\Gamma((\alpha_1-\gamma)(m+1)+1)} \right) \right] E_\beta(a_2 x^\beta) \\ &\quad + \left[b_2 \sum_{m=0}^{\infty} \left(\frac{a_2^{2m} t^{(\alpha_1-\gamma)m}}{\Gamma((\alpha_1-\gamma)m+1)} - \frac{a_2^{2m+2} t^{(\alpha_1-\gamma)(m+1)}}{\Gamma((\alpha_1-\gamma)(m+1)+1)} \right) \right] E_\beta(-a_2 x^\beta), \\ g(t, x) &= \left[c_1 \sum_{m=0}^{\infty} \left(\frac{n_2^m}{m!} t^{\alpha_2 m} E_{\alpha_2-\gamma, \gamma m+1}^{(m)}(a_1^2 t^{\alpha_2-\gamma}) - \frac{a_1^2 n_2^m}{m!} t^{2\alpha_2-2\gamma} E_{\alpha_2-\gamma, \alpha_2+\gamma(m-1)+1}^{(m)}(a_1^2 t^{\alpha_2-\gamma}) \right) \right] E_\beta(-a_1 x^\beta), \end{aligned}$$

where b_1, b_2, c_1 are arbitrary constants.

6 Conclusions and future scope

Present article extends invariant subspace method for solving nonlinear systems of FPDEs involving both time and space fractional derivatives. In this method system of FPDEs are reduced to systems of FODEs which can be further solved by existing methods. The proposed method has been illustrated by finding exact solutions of various systems, *viz.*, system of generalized fractional Burger's equations, coupled fractional Boussinesq equations, fictionalized system of KdV type of equations. Further we demonstrate how invariant subspace method can be employed for FPDEs in (1+n) dimension. The effectiveness of this method is illustrated by finding closed form solutions for fractional dispersive KdV equation in (1+n) dimensions, fractional population model, fractional scale wave equation, fractional order Bossinesq equation and fractional diffusion like PDE in (1+2) dimensions. We have modelled equations using RL as well as Caputo derivatives and considered multi-term expressions in time. Invariant subspace method is also used to find unique solutions along with initial conditions.

We observe that (1+1) dimensional FPDEs admit more than one invariant subspaces, each of which yields different exact solution [6]. Similarly FPDEs in higher dimensions

admit more than one invariant subspaces. The solutions obtained can be expressed in terms of Mittag-Leffler functions, fractional trigonometric functions etc. We demonstrate that invariant subspace method is very effective tool in finding exact solutions of wide class of linear and non linear systems of FPDEs and FPDEs in higher dimensions. Further we have also employed invariant subspace method for solving FPDEs with fractional differential operator involving mixed fractional partial derivatives.

Due to lack of composition rule and chain rule, we have severe limitations in finding invariant subspaces corresponding to fractional operator using existing algorithms developed for ordinary PDEs. We have found invariant subspaces for the fractional operators by trial and error method. Developing proper theory and algorithms for finding all sets of invariant subspaces for fractional operators is an open area to explore. Similarly, suitable theory may be developed for finding maximum dimension of invariant subspaces.

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