# DISTRIBUTIVE BIRACKS AND SOLUTIONS OF THE YANG-BAXTER EQUATION 

PŘEMYSL JEDLIČKA, AGATA PILITOWSKA, AND ANNA ZAMOJSKA-DZIENIO


#### Abstract

We investigate a class of non-involutive solutions of the Yang-Baxter equation which generalize derived (self-distributive) solutions. In particular, we study generalized multipermutation solutions in this class. We show that the Yang-Baxter (permutation) groups of such solutions are nilpotent. We formulate the results in the language of biracks which allows us to apply universal algebra tools.


## 1. Introduction

The Yang-Baxter equation is a fundamental equation occurring in integrable models in statistical mechanics and quantum field theory [17]. Let $V$ be a vector space. A solution of the Yang-Baxter equation is a linear mapping $r: V \otimes V \rightarrow V \otimes V$ such that

$$
(i d \otimes r)(r \otimes i d)(i d \otimes r)=(r \otimes i d)(i d \otimes r)(r \otimes i d) .
$$

Since description of all possible solutions seems to be extremely difficult, Drinfeld 4 introduced the following simplification.

Let $X$ be a basis of the space $V$ and let $\sigma: X^{2} \rightarrow X$ and $\tau: X^{2} \rightarrow X$ be two mappings. We say that $(X, \sigma, \tau)$ is a set-theoretic solution of the Yang-Baxter equation if the mapping $x \otimes y \mapsto$ $\sigma(x, y) \otimes \tau(x, y)$ extends to a solution of the Yang-Baxter equation. It means that $r: X^{2} \rightarrow X^{2}$, where $r=(\sigma, \tau)$ satisfies the braid relation:

$$
\begin{equation*}
(i d \times r)(r \times i d)(i d \times r)=(r \times i d)(i d \times r)(r \times i d) . \tag{1.1}
\end{equation*}
$$

A solution is called non-degenerate if the mappings $\sigma(x,-)=\sigma_{x}: X \rightarrow X$ and $\tau(-, y)=\tau_{y}$ : $X \rightarrow X$ are bijections, for all $x, y \in X$. A solution $(X, \sigma, \tau)$ is involutive if $r^{2}=\mathrm{id}_{X^{2}}$, and it is square free if $r(x, x)=(x, x)$, for every $x \in X$.

In [6] Section 3.2] Etingof, Schedler and Soloviev introduced, for each involutive solution $(X, \sigma, \tau)$, the equivalence relation $\sim$ on the set $X$ : for each $x, y \in X$

$$
x \sim y \quad \Leftrightarrow \quad \sigma_{x}=\sigma_{y} .
$$

They showed that the quotient set $X / \sim$ can be again endowed with a structure of an involutive solution. This does not work for non-involutive solutions. In [15] we showed that in the noninvolutive case the role similar to the relation $\sim$ is played by the relation $\approx$ defined on the set $X$ as follows: for each $x, y \in X$

$$
x \approx y \quad \Leftrightarrow \quad \sigma_{x}=\sigma_{y} \quad \text { and } \quad \tau_{x}=\tau_{y} .
$$

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We call a solution obtained on the set $X / \approx$, the retraction of the solution $X$ and denote it by $\operatorname{Ret}(X)$. A solution $X$ is said to be a multipermutation solution of level $k$, if $k$ is the smallest integer such that $\left|\operatorname{Ret}^{k}(X)\right|=1$.

Two types of solutions are particularly well studied: involutive solutions and derived solutions, i.e. those with all $\sigma_{x}$ or all $\tau_{y}$ being the identity mappings. In 19 Lebed and Vendramin have thoroughly investigated injective solutions, which generalize involutive ones. In this paper, we focus on generalization of derived solutions given by distributive solutions.

The Yang-Baxter group of a solution $(X, \sigma, \tau)$ is the group generated by all bijections $\sigma_{x}$ and $\tau_{x}$, for $x \in X$. There were several results for involutive solutions connecting properties of the Yang-Baxter group and multipermutation level of the solution [1, 2, 10, 11, 20, 22, 23].

Let now $(X, \sigma, \tau)$ be a derived solution such that all $\sigma_{x}=\operatorname{id}_{X}$. Then $\tau_{y} \tau_{x}=\tau_{\tau_{y}(x)} \tau_{y}$ for all $x, y \in X$. Moreover, this condition holds for each element $\eta$ of the Yang-Baxter group, i.e. $\eta \tau_{x}=\tau_{\eta(x)} \eta$ for all $x \in X$. For derived solutions with all $\tau_{x}=\mathrm{id}_{X}$, one obtains the dual situation. Here, we consider solutions which are not-necessarily derived, but for each element $\eta$ in their Yang-Baxter group one has: for each $x \in X$

$$
\begin{equation*}
\eta \sigma_{x}=\sigma_{\eta(x)} \eta \quad \text { and } \quad \eta \tau_{x}=\tau_{\eta(x)} \eta . \tag{1.2}
\end{equation*}
$$

These are called distributive solutions. Condition (1.2) means that each $\eta$ is actually an automorphism of a solution (see [12, Section 2]).

In [16] we described the involutive distributive solutions. They are always multipermutation solutions of level 2 and their (involutive) Yang-Baxter groups are always abelian [16, Theorem 7.6]. In this paper we focus on non-involutive case. The situation is now more complex.

## Main Theorem.

Let $(X, \sigma, \tau)$ be a non-degenerate distributive solution of Yang-Baxter equation and let $k>1$. Then $X$ is a multipermutation solution of level at most $k$ if and only if the Yang-Baxter group of $X$ is nilpotent of class at most $k-1$.

This theorem cannot be generalized for non-distributive solutions as there exist on one hand involutive solutions that are multipermutation but their Yang-Baxter groups are not nilpotent [23, Remark 7] and on the other hand there exist involutive solutions that are not multipermutation but their Yang-Baxter groups are nilpotent [23, Remark 6].

It is known (see e.g. [8, 24, 3]) that there is a one-to-one correspondence between solutions of the Yang-Baxter equation and biracks $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ - structures which satisfy some additional conditions (2.1)-(2.5). This fact allows us to prove the Main Theorem using the language of biracks (Theorem 4.6).

In particular, the correspondence exists between derived solutions and racks [7]. For each right rack $\left(X, *, \_{*}\right)$ its right multiplication group $\operatorname{RMlt}(X)=\left\langle R_{x}: X \rightarrow X ; a \mapsto a * x \mid x \in X\right\rangle$ is a subgroup of the automorphism group $\operatorname{Aut}(X)$. The similar property occurs for a left rack, and the reason for that in both cases, is a one-sided (self)distributivity. Therefore, we investigate biracks with at least one (self)distributive operation.

The paper is organized as follows: in Section 2 we characterize distributive biracks. We also give examples (Examples 2.6 and 2.12) of non-involutive distributive biracks which are not derived ones. Section 3 is devoted to the quotient of distributive biracks by the relation $\approx$ as well as by the relation $\sim$, that turns out to be congruence as well, in the distributive case (Theorem 3.4). We also show (Lemma (3.5) that the quotient birack by the congruence $\sim$ is always idempotent and derived. The last Section 4 contains the main result of the paper (Theorem 4.6). We prove there that a distributive birack is of multipermutation level $m$, for $m \geq 2$, if and only if its permutation group is nilpotent of class at most $m-1$.

## 2. Distributive biracks

As we already mentioned there is a one-to-one correspondence between solutions of the YangBaxter equation and algebraic structures called biracks, which naturally appear in low-dimensional topology [8, 5]. In [18] Kauffman introduced the virtual knot theory - biracks play there a similar role as racks in classical knot theory. However, still not much is known about their structure and properties.

The following equational definition of a birack was given first by Stanovský in [24] (see also [8).
Definition 2.1. A structure $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ with four binary operations is called a birack, if the following holds for any $x, y, z \in X$ :

$$
\begin{array}{r}
x \circ(x \backslash y)=y=x \backslash(x \circ y), \\
(y / \bullet x) \bullet x=y=(y \bullet x) / \bullet x, \\
x \circ(y \circ z)=(x \circ y) \circ((x \bullet y) \circ z), \\
(x \circ y) \bullet((x \bullet y) \circ z)=(x \bullet(y \circ z)) \circ(y \bullet z), \\
(x \bullet y) \bullet z=(x \bullet(y \circ z)) \bullet(y \bullet z) . \tag{2.5}
\end{array}
$$

Example 2.2 (Lyubashenko, see [4). Let $X$ be a non-empty set and let $f, g: X \rightarrow X$ be two bijections such that $f g=g f$. Define four binary operations: $x \circ y=f(y), x \backslash \circ y=f^{-1}(y)$, $x \bullet y=g(x)$ and $x / \bullet y=g^{-1}(x)$. Then $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is a birack called permutational. If $f=g=\mathrm{id}$, the birack is called a projection one.

Conditions (2.1) and (2.2) mean that $\left(X, \circ, \_{\circ}\right)$ is a left quasigroup and $(X, \bullet, / \bullet)$ is a right quasigroup. Condition (2.1) simply means that all left translations $L_{x}: X \rightarrow X$ by $x$

$$
L_{x}(a)=x \circ a,
$$

are bijections, with $L_{x}^{-1}(a)=x \backslash_{\circ} a$. Equivalently, that for every $x, y \in X$, the equation $x \circ u=y$ has the unique solution $u=L_{x}^{-1}(y)$ in $X$. Similarly, Condition (2.2) gives that all right translations $\mathbf{R}_{x}: X \rightarrow X$ by $x ; \mathbf{R}_{x}(a)=a \bullet x$, are bijections with $\mathbf{R}_{x}^{-1}(a)=a / \bullet x$.

The left multiplication group of a birack $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is the permutation group generated by left translations, i.e. the group $\operatorname{LMlt}(X)=\left\langle L_{x}: x \in X\right\rangle$. Similarly, one defines the right multiplication group of $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ as the permutation group generated by right translations, i.e. the group $\operatorname{RMlt}(X)=\left\langle\mathbf{R}_{x}: x \in X\right\rangle$. The permutation group $\operatorname{Mlt}(X)$ generated by all translations $L_{x}$ and $\mathbf{R}_{x}$ is called the multiplication group of a birack.

We will say that a birack $\left(X, \circ, \backslash_{\circ}, \bullet, / \bullet\right)$ is left distributive, if for every $x, y, z \in X$ :

$$
\begin{equation*}
x \circ(y \circ z)=(x \circ y) \circ(x \circ z) \quad \Leftrightarrow \quad L_{x} L_{y}=L_{x \circ y} L_{x}, \tag{2.6}
\end{equation*}
$$

and it is right distributive, if for every $x, y, z \in X$ :

$$
\begin{equation*}
(y \bullet z) \bullet x=(y \bullet x) \bullet(z \bullet x) \quad \Leftrightarrow \quad \mathbf{R}_{x} \mathbf{R}_{z}=\mathbf{R}_{z \bullet x} \mathbf{R}_{x} \tag{2.7}
\end{equation*}
$$

The birack is distributive if it is left and right distributive. Each permutational birack is distributive.
Remark 2.3. A right rack is a right distributive right quasigroup. These two conditions refer to second and third Reidemeister moves [9]. Biracks generalize racks in the following way. Let $\left(X, \circ, \_{\circ}, \bullet / \bullet\right)$ be a birack with all left translations being the identity permutation, i.e. for every $x, y, \in X$ one has

$$
x \circ y=y .
$$

Then $(X, \bullet, / \bullet)$ is a rack.

A birack is involutive if it additionally satisfies, for every $x, y \in X$ :

$$
\begin{align*}
& (x \circ y) \circ(x \bullet y)=x,  \tag{2.8}\\
& (x \circ y) \bullet(x \bullet y)=y . \tag{2.9}
\end{align*}
$$

By Condition (2.8), operations $\bullet$ and $/ \bullet$ in an involutive birack are uniquely determined by operations $\circ$ and $\backslash_{\circ}$ since $x \bullet y=(x \circ y) \backslash \circ x$. This allows to treat involutive biracks as left quasigroups satisfying additional conditions [21, Proposition 1]. In case of distributive involutive biracks one obtains the one-to-one correspondence to 2 -reductive racks [16, Corollary 5.5].

In involutive biracks, $\operatorname{LMlt}(X)=\operatorname{RMlt}(X)=\operatorname{Mlt}(X)$. Moreover, by [16, Corollary 5.8] an involutive birack is left distributive if and only if it is right distributive.

Example 2.4. Let ( $X, \circ, \_{\circ}$ ) be a left distributive left quasigroup (left rack). Define operations $\bullet, / \bullet: X \times X \rightarrow X$ as $x \bullet y=x=x / \bullet y$. Then the structure $\mathbf{B}_{L}\left(X, \circ, \_{\circ}\right)=\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is a left distributive birack. Symmetrically, starting from a right rack $(Y, \triangleright, / \triangleright)$ and defining operations $\triangleleft, / \triangleleft: Y \times Y \rightarrow Y$ as $x \triangleleft y=y=x / \triangleleft y$, one obtains a right distributive birack $\mathbf{B}_{R}(Y, \triangleright, / \triangleright)=$ $\left(Y, \triangleleft, \_{\triangleleft}, \triangleright, / \triangleright\right)$. We call such biracks left and right derived biracks, respectively. They are involutive only if they are projection ones.

Example 2.5. Let $\left(X, *, \_{*}\right)$ be a left rack and let $(Y, \Delta, / \Delta)$ be a right rack. Then the product $\mathbf{B}_{L}\left(X, *, \backslash_{*}\right) \times \mathbf{B}_{R}(Y, \Delta, / \Delta)$ is a distributive birack with $\operatorname{Mlt}(X \times Y) \cong \operatorname{LMlt}(X) \times \operatorname{RMlt}(Y)$.
Example 2.6. Let $(G, \cdot, e)$ be a group. Defining on the set $G$ binary operations as follows: $x \circ y=$ $x y^{-1} x^{-1}, x \backslash \circ y=x^{-1} y^{-1} x$, and $x \bullet y=x y^{2}, x / \bullet y=x y^{-2}$, for $x, y \in G$, we obtain the birack $\left(G, \circ, \_{\circ}, \bullet, / \bullet\right)$ known as the Wada switch or Wada biquandle (see [8, Subsection 2.1(3)]).

Let $x, y, z \in G$. Direct calculations show that

$$
\begin{aligned}
x \circ(y \circ z) & =x y z y^{-1} x^{-1}, \\
(x \circ y) \circ(x \circ z) & =x y^{-1} z y x^{-1},
\end{aligned}
$$

and the birack is left distributive if and only if $y^{2} z=z y^{2}$, for all $y, z \in G$, that means $y^{2} \in Z(G)$, for all $y \in G$. Furthermore,

$$
\begin{aligned}
(y \bullet z) \bullet x & =y z^{2} x^{2}, \quad \text { and } \\
(y \bullet x) \bullet(z \bullet x) & =y x^{2} z x^{2} z x^{2} .
\end{aligned}
$$

This implies that the birack is right distributive if and only if $x^{2} z=z x^{-2}$, for all $x, z \in G$, which is equivalent to $x^{4}=e$ and $x^{2} \in Z(G)$, for all $x \in G$. Thus, if the birack is right distributive then it is left distributive as well.

Moreover,

$$
\begin{aligned}
& (x \circ y) \circ(x \bullet y)=x\left(y^{-1} x^{-1}\right)^{2}, \quad \text { and } \\
& (x \circ y) \bullet(x \bullet y)=x y x y^{2} .
\end{aligned}
$$

Hence, by (2.8) and (2.9), the birack is involutive if and only if $(x y)^{2}=e$, for all $x, y \in G$, that means if ( $G, \cdot, e$ ) is an elementary abelian 2 -group.

For instance, there are five groups of order 8 . One of them is cyclic of exponent 8 and therefore its Wada switch is not distributive. One of them is elementary abelian and its Wada switch is a projection birack. The other three groups (namely $\mathbb{Z}_{4} \times \mathbb{Z}_{2}, D_{8}$ and $Q_{8}$ ) are of exponent 4 and all their square elements fall within the centers and therefore these groups yield non-involutive distributive biracks.

Example 2.7. Let $(G,+, 0)$ be an abelian group. Then, the birack operations defined in Example 2.6 look as follows: $x \circ y=x \backslash \circ y=-y$ and $x \bullet y=x+2 y, x / \bullet y=x-2 y$, for $x, y \in G$. Clearly,
such a birack is always left distributive. It is a non-involutive distributive birack if and only if $G$ is an abelian group of exponent exactly 4.

Lemma 2.8. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a birack. The following are equivalent:
(i) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is left distributive;
(ii) $\left(X, \circ, \bigwedge_{0}, \bullet, / \bullet\right)$ satisfies, for every $x, y \in X$,

$$
\begin{equation*}
L_{x}=L_{x} \cdot y=L_{\mathbf{R}_{y}(x)} ; \tag{2.10}
\end{equation*}
$$

(iii) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ satisfies, for every $x, y \in X$,

$$
\begin{equation*}
L_{x}=L_{x / \bullet y}=L_{\mathbf{R}_{y}^{-1}(x)}, \tag{2.11}
\end{equation*}
$$

(iv) Left translations by elements taken from the same orbit of the action of the group $\operatorname{RMlt}(X)$ on a set $X$ are equal permutations on $X$.

Proof. Indeed, by (2.3) and (2.1), we have for $x, y, z \in X$

$$
\begin{aligned}
& (x \circ y) \circ(x \circ z)=x \circ(y \circ z) \quad \Leftrightarrow \quad(x \circ y) \circ(x \circ z)=(x \circ y) \circ((x \bullet y) \circ z) \quad \Leftrightarrow \\
& x \circ z=(x \bullet y) \circ z \Leftrightarrow L_{x}=L_{x} \bullet y .
\end{aligned}
$$

Additionally, by (2.2), substituting of $x$ by $x / \bullet y$ in (2.10) we immediately obtain

$$
L_{x}=L_{x / \bullet y}
$$

Similarly, substituting of $x$ by $x \bullet y$ in (2.11) we have

$$
L_{x \bullet y}=L_{x} .
$$

Finally, (ii) $\Leftrightarrow$ (iv) follows by the fact that for any $x \in X$ its orbit of the action $\mathrm{RMlt}(X)$ on $X$ consists exactly of elements $\alpha(x)$ for $\alpha \in \operatorname{RMlt}(X)$.

Analogously, due to (2.5) and (2.2), a birack is right distributive if and only if

$$
\begin{equation*}
\mathbf{R}_{x}=\mathbf{R}_{y \circ x}=\mathbf{R}_{L_{y}(x)}, \tag{2.12}
\end{equation*}
$$

or equivalently, right translations by elements taken from the same orbit of the action of the group LMlt $(X)$ on $X$ are equal permutations on $X$.

By results of [16, Section 3] left (right) distributivity in involutive biracks is equivalent to commutativity of the left (right) multiplication group. For a non-involutive distributive birack it is not always true (see Example 2.12). But even then left and right multiplication groups commute.
Lemma 2.9. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a distributive birack. Then,

$$
\begin{equation*}
[\operatorname{LMlt}(X), \operatorname{RMlt}(X)]=\{\operatorname{id}\} . \tag{2.13}
\end{equation*}
$$

Proof. For $x, y, z \in X$ one has,

$$
\begin{aligned}
& L_{x} \mathbf{R}_{y}(z) \stackrel{\sqrt{2.10]}}{=} L_{x \bullet(z \circ y)} \mathbf{R}_{y}(z)=(x \bullet(z \circ y)) \circ(z \bullet y) \stackrel{(2.4]}{=} \\
& (x \circ z) \bullet((x \bullet z) \circ y)=\mathbf{R}_{(x \bullet z) \circ y} L_{x}(z) \stackrel{(2.12]}{=} \mathbf{R}_{y} L_{x}(z) .
\end{aligned}
$$

Lemmas 2.8 and 2.9 allow us to characterize distributive biracks in an alternative way.
Proposition 2.10. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a structure with four binary operations. Then ( $X, \circ, \_{\bullet}, \bullet, / \bullet$ ) is a distributive birack if and only if the following conditions are satisfied
(i) $\left(X, \circ, \_{\circ}\right)$ is a left rack and $(X, \bullet, / \bullet)$ is a right rack,
(ii) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ satisfies (2.10), (2.12) and (2.13).

Proof. If $\left(X, \circ, \_{\circ}\right)$ is a left rack then (2.3) and (2.10) are equivalent, as we showed in Lemma 2.8. Analogously, (2.5) and (2.12) are equivalent when $(X, \bullet, / \bullet)$ is a right rack. Finally, when supposing (2.12) and (2.10), Conditions (2.4) and (2.13) are equivalent, as we saw in the proof of Lemma 2.9.

Remark 2.11. Condition (ii) in Proposition 2.10 can be formulated in the equivalent way as follows: for all $x, y, z \in X$

$$
\begin{aligned}
& (x \bullet y) \circ z=x \circ z, \\
& x \bullet(y \circ z)=x \bullet z, \\
& x \circ(y \bullet z)=(x \circ y) \bullet z
\end{aligned}
$$

Obviously, the left distributivity of $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ means that all left translations are automorphisms of $\left(X, \circ, \backslash_{\circ}\right)$. Additionally, directly from (2.6) we obtain that the left distributivity implies, for every $x, y \in X$,

$$
\begin{equation*}
L_{x \circ y}=L_{x} L_{y} L_{x}^{-1} \quad \text { and } \quad L_{x \backslash \circ y}=L_{x}^{-1} L_{y} L_{x} \tag{2.14}
\end{equation*}
$$

Note also that, for an arbitrary automorphism $\alpha$ of $\left(X, \circ, \_{\circ}\right)$, we have

$$
\begin{equation*}
L_{\alpha(x)}(y)=\alpha(x) \circ y=\alpha\left(x \circ \alpha^{-1}(y)\right)=\alpha L_{x} \alpha^{-1}(y) \tag{2.15}
\end{equation*}
$$

Similarly, for a right distributive birack $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$, we have

$$
\begin{equation*}
\mathbf{R}_{x \bullet y}=\mathbf{R}_{y} \mathbf{R}_{x} \mathbf{R}_{y}^{-1} \quad \text { and } \quad \mathbf{R}_{x / \bullet y}=\mathbf{R}_{y}^{-1} \mathbf{R}_{x} \mathbf{R}_{y} \tag{2.16}
\end{equation*}
$$

Moreover, for an arbitrary automorphism $\beta$ of $(X, \bullet, / \bullet)$, we have

$$
\begin{equation*}
\mathbf{R}_{\beta(x)}(y)=y \bullet \beta(x)=\beta\left(\beta^{-1}(y) \bullet x\right)=\beta \mathbf{R}_{x} \beta^{-1}(y) \tag{2.17}
\end{equation*}
$$

Example 2.12. Let $\left(X, \circ, \backslash_{\circ}, \bullet, / \bullet\right)$ be the following structure: $X=\{1,2,3,4,5,6\}$ and

$$
\begin{aligned}
& L_{1}=(3546) \quad L_{2}=(6453) \\
& L_{3}=L_{4}=(12)(56) \\
& \mathbf{R}_{1}=\mathbf{R}_{2}=\mathrm{id} \\
& \mathbf{R}_{3}=\mathbf{R}_{4}=\mathbf{R}_{5}=\mathbf{R}_{6}=(34)(56)
\end{aligned}
$$

By hand, or using a GAP library named ' RiG ' [13], we can show that the automorphism group of $\left.(X, \circ,\rangle_{\circ}, \bullet, / \bullet\right)$, is the group generated by the permutations $L_{1}, L_{3}$ and $L_{5}$. We can now easily prove that this structure is a distributive birack. Indeed, all $L_{x}$, for $x \in X$, are automorphims of $\left(X, \circ, \_{\circ}\right)$, as well as all $\mathbf{R}_{x}$, for $x \in X$, are automorphisms of $(X, \bullet, / \bullet)$ and therefore $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is both left and right distributive birack. The group $\operatorname{LMlt}(X)$ is a non-abelian group of order 8 having two orbits, namely $\{1,2\}$ and $\{3,4,5,6\}$. Condition (2.10) is satisfied since $\mathbf{R}_{1}=\mathbf{R}_{2}$ and $\mathbf{R}_{3}=\mathbf{R}_{4}=\mathbf{R}_{5}=\mathbf{R}_{6}$. The group $\operatorname{RMlt}(X)$ is a two-element group with four orbits $\{1\}$, $\{2\}$, $\{3,4\}$ and $\{5,6\}$. Condition (2.12) is satisfied since $L_{3}=L_{4}$ and $L_{5}=L_{6}$. The group RMlt( $X$ ) is equal to the center of $\operatorname{LMlt}(X)$ and we have therefore Condition (2.13). We may also notice that this example is not a Wada biquandle (Example 2.6) since there exists no 6-element group of exponent 4.

Proposition 2.13. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a distributive birack. Then, for each $x \in X$, the bijections $L_{x}$ and $\mathbf{R}_{x}$ are automorphisms of $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$.

Proof. The property $L_{x}(y \circ z)=L_{x}(y) \circ L_{x}(z)$ is the definition of left distributivity. Substituting $z \mapsto y \backslash \circ u$ we obtain $L_{x}(u)=L_{x}(y) \circ L_{x}(y \backslash \circ u)$ from where we get

$$
\begin{equation*}
L_{x}(y) \backslash \circ L_{x}(u)=L_{x}(y \backslash \circ u) \tag{2.18}
\end{equation*}
$$

Now

$$
\begin{aligned}
& L_{x}(y \bullet z)=L_{x} \mathbf{R}_{z}(y) \stackrel{[2.13]}{=} \mathbf{R}_{z} L_{x}(y) \stackrel{\left[\frac{[2.12]}{=}\right.}{=} \mathbf{R}_{L_{x}(z)} L_{x}(y)=L_{x}(y) \bullet L_{x}(z), \\
& L_{x}(y / \bullet z)=L_{x} \mathbf{R}_{z}^{-1}(y) \stackrel{[2.13]}{=} \mathbf{R}_{z}^{-1} L_{x}(y) \stackrel{[2.12]}{=} \mathbf{R}_{L_{x}(z)}^{-1} L_{x}(y)=L_{x}(y) / \bullet L_{x}(z),
\end{aligned}
$$

and therefore $L_{x}$ is a homomorphism of $(X, \bullet, / \bullet)$. The claim for $\mathbf{R}_{x}$ is proved analogously.
By Proposition 2.13) and Conditions (2.15), (2.17) one immediately obtains that groups LMlt( $X$ ), $\operatorname{RMlt}(X), \operatorname{Mlt}(X)$ are normal subgroups of the automorphism group of a distributive birack $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$.

## 3. Multipermutational biracks

Gateva-Ivanova characterized involutive multipermutation solutions of the Yang-Baxter equation in the language of some identities satisfied by corresponding biracks, see e.g. [10, Proposition 4.7]. We will generalize her result for (non-involutive) left distributive case. Let us start with some auxiliary definitions.

Definition 3.1. Let $m \in \mathbb{N}$. A birack $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is called:
(i) idempotent, if for every $x \in X$ :

$$
\begin{equation*}
x \circ x=x=x \bullet x \tag{3.1}
\end{equation*}
$$

(ii) left $m$-reductive, if for every $x_{0}, x_{1}, x_{2}, \ldots, x_{m} \in X$ :

$$
\begin{equation*}
\left(\ldots\left(\left(x_{0} \circ x_{1}\right) \circ x_{2}\right) \ldots\right) \circ x_{m}=\left(\ldots\left(\left(x_{1} \circ x_{2}\right) \circ x_{3}\right) \ldots\right) \circ x_{m} \tag{3.2}
\end{equation*}
$$

(iii) right $m$-reductive, if for every $x_{0}, x_{1}, x_{2}, \ldots, x_{m} \in X$ :

$$
\begin{equation*}
x_{0} \bullet\left(\ldots\left(x_{m-2} \bullet\left(x_{m-1} \bullet x_{m}\right)\right) \ldots\right)=x_{0} \bullet\left(\ldots\left(x_{m-3} \bullet\left(x_{m-2} \bullet x_{m-1}\right)\right) \ldots\right) \tag{3.3}
\end{equation*}
$$

(iv) left m-permutational, if for every $x, y, x_{1}, x_{2}, \ldots, x_{m} \in X$ :

$$
\begin{equation*}
\left(\ldots\left(\left(x \circ x_{1}\right) \circ x_{2}\right) \ldots\right) \circ x_{m}=\left(\ldots\left(\left(y \circ x_{1}\right) \circ x_{2}\right) \ldots\right) \circ x_{m} \tag{3.4}
\end{equation*}
$$

(v) right m-permutational, if for every $x, y, x_{1}, x_{2}, \ldots, x_{m} \in X$ :

$$
\begin{equation*}
x_{0} \bullet\left(\ldots\left(x_{m-2} \bullet\left(x_{m-1} \bullet x\right)\right) \ldots\right)=x_{0} \bullet\left(\ldots\left(x_{m-2} \bullet\left(x_{m-1} \bullet y\right)\right) \ldots\right) . \tag{3.5}
\end{equation*}
$$

A birack is $m$-reductive ( $m$-permutational) if it is both left and right $m$-reductive ( $m$-permutational).
Example 3.2. A distributive birack from Example 2.6 is right 2-reductive because $x \bullet(y \bullet z)=$ $x\left(y z^{2}\right)^{2}=x y z^{2} y z^{2}=x y^{2}=x \bullet y$, since $z^{2} \in Z(G)$ and $z^{4}=e$.

Lemma 3.3. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a left m-permutational birack.
(i) If $\left(X, \circ, \_{0}, \bullet, / \bullet\right)$ is idempotent then it is left m-reductive.
(ii) If $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is left distributive and $m \geq 2$ then it is left $m$-reductive.

Proof. (i) is evident. For (ii) we have

$$
\begin{aligned}
& \left(\cdots\left(\left(x_{0} \circ x_{1}\right) \circ x_{2}\right) \cdots\right) \circ x_{m} \stackrel{\mid 3.44}{=}\left(\cdots\left(\left(x_{1} \circ x_{1}\right) \circ x_{2}\right) \cdots\right) \circ x_{m}= \\
& \left(\cdots ( ( L _ { x _ { 1 } \circ x _ { 1 } } ( x _ { 2 } ) ) \cdots ) \circ x _ { m } \stackrel { \sqrt { 2 . 1 4 4 } } { = } \left(\cdots\left(\left(L_{x_{1}}\left(x_{2}\right)\right) \cdots\right) \circ x_{m}=\left(\cdots\left(x_{1} \circ x_{2}\right) \cdots\right) \circ x_{m},\right.\right.
\end{aligned}
$$

for every $x_{0}, x_{1}, x_{2}, \ldots, x_{m} \in X$.

By Example 2.2 there exists left 1-permutational left distributive birack that is not left 1reductive. It is a permutational birack with $f \neq \mathrm{id}$.

Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a birack. Etingof, Schedler and Soloviev defined in [6, Section 3.2] the relation

$$
\begin{equation*}
a \sim b \quad \Leftrightarrow \quad L_{a}=L_{b} \quad \Leftrightarrow \quad \forall x \in X \quad a \circ x=b \circ x . \tag{3.6}
\end{equation*}
$$

By their results, the relation $\sim$ is a congruence of an involutive birack, i.e. an equivalence relation on the set $X$ compatible with all four operations in a birack $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$. For a detailed definition see [15, Definition 3.1].

In the case of non-involutive biracks, the equivalence $\sim$ need not be a congruence (see [15, Example 3.4]) but it is so if the birack is left distributive.

Theorem 3.4. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a left distributive birack. Then the relation (3.6) is a congruence of $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$.
Proof. By (2.1), (2.14) and (2.10) the proof is straightforward. Let $a \sim x$ and $b \sim y$. Then

$$
\begin{aligned}
& L_{a \circ b} \stackrel{\sqrt{2.14]}}{=} L_{a} L_{b} L_{a}^{-1}=L_{x} L_{y} L_{x}^{-1}=L_{x \circ y} \Rightarrow a \circ b \sim x \circ y, \\
& L_{a \backslash \circ b} \stackrel{(2.14)}{=} L_{a}^{-1} L_{b} L_{a}=L_{x}^{-1} L_{y} L_{x}=L_{x \backslash \circ y} \Rightarrow a \backslash \circ b \sim x \backslash \circ y, \\
& L_{a \bullet b} \stackrel{(2.10)}{=} L_{a}=L_{x}=L_{x} \bullet y \Rightarrow a \bullet b \sim x \bullet y, \\
& L_{a / \bullet b} \stackrel{\sqrt{2.10}}{=} L_{(a / \bullet b) \bullet b} \stackrel{(2.1)}{=} L_{a}=L_{x}=L_{(x / \bullet y) \bullet y}=L_{x / \bullet y} \Rightarrow a / \bullet b \sim x / \bullet y .
\end{aligned}
$$

Lemma 3.5. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a left distributive birack. Then the quotient birack $\left(X / \sim, \circ, \_{\circ}, \bullet, / \bullet\right)$ is idempotent and equal to the left derived birack $\mathbf{B}_{L}\left(X / \sim, \circ, \_{\circ}\right)$.

Proof. By the left distributivity and (2.10), for every $x, y \in X$,

$$
x \sim x \bullet y .
$$

Furthermore, by (2.14), $L_{x}=L_{x} L_{x} L_{x}^{-1}=L_{x \circ x}$, which shows that $x \sim x \circ x$ and $\left(X / \sim, \circ, \_{\circ}, \bullet, / \bullet\right)$ is idempotent.

Analogously to (3.6), we can define symmetrical relation

$$
\begin{equation*}
a \backsim b \quad \Leftrightarrow \quad \mathbf{R}_{a}=\mathbf{R}_{b} \quad \Leftrightarrow \quad \forall x \in X \quad x \bullet a=x \bullet b \tag{3.7}
\end{equation*}
$$

and this relation is a congruence of every right distributive birack. If a birack is involutive then $a \sim b$ if and only if $a \backsim b$ [6, Proposition 2.2].

Definition 3.6. Let ( $\left.X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a left distributive birack. The left derived birack $\mathbf{B}_{L}\left(X / \sim, \circ, \_{\circ}\right)$ is called left retract of $X$ and denoted by $\operatorname{LRet}(X)$. One defines iterated left retraction in the following way: $\operatorname{LRet}^{0}(X)=\left(X, \circ, \_{\circ}, \bullet / \bullet\right)$ and $\operatorname{LRet}^{k}(X)=\operatorname{LRet}\left(\operatorname{LRet}^{k-1}(X)\right)$, for any natural number $k>1$.

The right retract and iterated right retraction are defined analogously.
Remark 3.7. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a distributive birack and let $\Theta$ be the join of the congruences $\sim$ and $\backsim$ (the least congruence containing both of them). Then the quotient birack ( $X / \Theta, \circ, \_{\bullet}, \bullet, / \bullet$ ) is the projection one.

The intersection of the two relations here defined is the relation

$$
\begin{equation*}
a \approx b \quad \Leftrightarrow \quad a \sim b \wedge a \backsim b_{8} \Leftrightarrow L_{a}=L_{b} \wedge \mathbf{R}_{a}=\mathbf{R}_{b} . \tag{3.8}
\end{equation*}
$$

When a birack is distributive then this equivalence is an intersection of two congruences and therefore a congruence. Nevertheless, it is a congruence even in the case of general biracks, however the proof is rather complicated and technical [15, Theorem 3.3].

Let $\left(X, \circ, \_{0}, \bullet, / \bullet\right)$ be a birack. The retract of $X$, denoted by $\operatorname{Ret}(X)$, is the quotient birack $\left(X / \approx, \circ, \bigcirc_{\circ}, \bullet, / \bullet\right)$. Similarly, as for the congruences $\sim$ and $\backsim$, one defines iterated retraction as $\operatorname{Ret}^{0}(X)=\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ and $\operatorname{Ret}^{k}(X)=\operatorname{Ret}\left(\operatorname{Ret}^{k-1}(X)\right)$, for any natural number $k>1$.

In the case of involutive biracks all three notions of retracts coincide.
Corollary 3.8. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a distributive birack. Then $\operatorname{Ret}(X)$ is an idempotent birack.
Example 3.9. Let $\left(G, \circ, \_{\circ}, \bullet, / \bullet\right)$ be the distributive birack from Example 2.6, It is easy to see that, for any $x, y \in G$,

$$
x \sim y \quad \Leftrightarrow \quad x y^{-1} \in Z(G) .
$$

and

$$
x \backsim y \quad \Leftrightarrow \quad x^{2}=y^{2} .
$$

Note that the relation $\approx$ is different than equality relation if and only if $Z(G)$ contains at least one element of order 2. It is also easy to see that $\sim$ is a subrelation of $\sim$ if and only if $Z(G)$ is an elementary abelian 2-group.

For instance, in the quaternion group $Q_{8}, \sim$ and $\backsim$ are different and the relation $\approx$ is equal to the conjugation relation. Thus $\left(Q_{8} / \approx, \circ, \_{\circ}, \bullet, / \bullet\right)$ is a 4 -element projection birack.

For an abelian group $(G,+, 0)$ satisfying the condition $4 x=0$, clearly the relations $\approx$ and $\sim$ coincide. In this case, the birack $\left(G / \approx, \circ, \_{\circ}, \bullet, / \bullet\right)$ is a projection birack.

## 4. Nilpotent permutation group

A birack is of multipermutation level $k$, if $\left|\operatorname{Ret}^{k}(X)\right|=1$ and $\left|\operatorname{Ret}^{k-1}(X)\right|>1$.
This means that applying $k$ times the congruence $\approx$ to the subsequent quotient biracks, one obtains the one-element birack.

In [10, Proposition 4.7] Gateva-Ivanova proved that an involutive birack ( $X, \circ, \_{\circ}, \bullet, / \bullet$ ) is of multipermutation level $k$ if and only if it is left $k$-permutational. The very same proof works for (non-involutive) left distributive biracks. In the distributive case we have an additional equivalent condition (nilpotency of the left multiplication group). Therefore, we decided to give a different proof that uses this condition.

Let $G$ be a group. We recall the definition of lower central series of the group $G$, for $i \in \mathbb{N}$ :

$$
\begin{aligned}
& \gamma_{0}(G)=G \\
& \gamma_{i}(G)=\left[\gamma_{i-1}(G), G\right]=\left[G, \gamma_{i-1}(G)\right], \text { for } i \geq 1
\end{aligned}
$$

Then $G$ is nilpotent of class $k$, if $k$ is the smallest number for which $\gamma_{k}(G)=\{1\}$. In particular, $G$ is nilpotent of class 0 if and only if $G$ is a trivial group. If $k>0$, nilpotency of class $k$ is equivalent to the property that $G / Z(G)$ is nilpotent of class $k-1$.

Theorem 4.1. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a left distributive idempotent birack and let $k \geq 1$. Then the following conditions are equivalent:
(i) $\left|\operatorname{LRet}^{k}(X)\right|=1$,
(ii) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is left $k$-reductive,
(iii) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is left $k$-permutational,
(iv) $\operatorname{LMlt}(X)$ is nilpotent of class at most $k-1$.

Proof. (ii) $\Leftrightarrow$ (iii) is Lemma 3.3.
(ii) $\Leftrightarrow$ (iv)

Let $k>1$. We translate the notion of $k$-reductivity in the language of permutations. Let $\left(x_{0}, \ldots, x_{k}\right) \in X^{k+1}$ and we denote this sequence by $\chi$. We write, by induction, $y_{0, \chi}=x_{0}$ and $y_{i+1, \chi}=y_{i, \chi} \circ x_{i+1}$ as well as $z_{1, \chi}=x_{1}$ and $z_{i+1, \chi}=z_{i, \chi} \circ x_{i+1}$. The equation of left $k$-reductivity then is $y_{k, \chi}=z_{k, \chi}$.

Let us write $\alpha_{-1, \chi}=$ id and $\alpha_{i+1, \chi}=\alpha_{i, \chi} L_{x_{i+1}} \alpha_{i, \chi}^{-1}$. We prove by induction that $L_{y_{i, \chi}}=\alpha_{i, \chi}$, for $0 \leq i<k$. The claim is clear for $i=0$ and

$$
L_{y_{i+1, \chi}}=L_{y_{i, \chi} \circ x_{i+1}}=L_{L_{y_{i, \chi}}\left(x_{i+1}\right)}=L_{\alpha_{i, \chi}\left(x_{i+1}\right)}=\alpha_{i, \chi} L_{x_{i+1}} \alpha_{i, \chi}^{-1}=\alpha_{i+1, \chi}
$$

Analogously, we write $\beta_{0, \chi}=\mathrm{id}$ and $\beta_{i+1, \chi}=\beta_{i, \chi} L_{x_{i+1}} \beta_{i, \chi}^{-1}$ and we get $L_{z_{i, \chi}}=\beta_{i, \chi}$. The left $k$-reductivity is, whenever $k>1$,

$$
y_{k, \chi}=z_{k, \chi} \quad \Leftrightarrow \quad y_{k-1, \chi} \circ x_{k}=z_{k-1, \chi} \circ x_{k} \quad \Leftrightarrow \quad L_{y_{k-1, \chi}}=L_{z_{k-1, \chi}} \quad \Leftrightarrow \quad \alpha_{k-1, \chi}=\beta_{k-1, \chi}
$$

for all $\chi \in X^{k+1}$.
We now prove by induction that the group $\gamma_{j}(\operatorname{LMlt}(X))$ is generated by the set $\left\{\alpha_{j, \chi}^{-1} \beta_{j, \chi}: \chi \in\right.$ $\left.X^{k+1}\right\}$. The claim is clear for $j=0$ since $\left\{\alpha_{0, \chi}^{-1} \beta_{0, \chi}=L_{x_{0}}^{-1}: x_{0} \in X\right\}$.

Let $j>0$. Then, by the induction hypothesis,

$$
\begin{aligned}
& \alpha_{j+1, \chi}^{-1} \beta_{j+1, \chi}=\alpha_{j, \chi} L_{x_{j+1}}^{-1} \alpha_{j, \chi}^{-1} \beta_{j, \chi} L_{x_{j+1}} \beta_{j, \chi}^{-1}=\alpha_{j, \chi} L_{x_{j+1}}^{-1}\left(\beta_{j, \chi}^{-1} \alpha_{j, \chi}\right)^{-1} L_{x_{j+1}}\left(\beta_{j, \chi}^{-1} \alpha_{j, \chi}\right) \alpha_{j, \chi}^{-1}= \\
& {\left[L_{x_{j+1}}, \beta_{j, \chi}^{-1} \alpha_{j, \chi}\right]^{\alpha_{j, \chi}^{-1} \in\left[\operatorname{LMlt}(X), \gamma_{j}(\operatorname{LMlt}(X))\right]=\gamma_{j+1}(\operatorname{LMlt}(X))}}
\end{aligned}
$$

which shows that $\left\langle\alpha_{j+1, \chi}^{-1} \beta_{j+1, \chi}: \chi \in X^{k+1}\right\rangle \subseteq \gamma_{j+1}(\operatorname{LMlt}(X))$.
Take $y \in X$ and $\alpha_{j, \chi}, \beta_{j, \chi}$ for some $\chi=\left(x_{0}, \ldots, x_{j}, \ldots, x_{k}\right)$. Consider the new sequence $\zeta=$ $\left(x_{0}, \ldots, x_{j}, y, x_{j+2}, \ldots, x_{k}\right)$. Obviously, $\beta_{j, \chi}^{-1} \alpha_{j, \chi}=\beta_{j, \zeta}^{-1} \alpha_{j, \zeta}$. Hence,

$$
\left[L_{y}, \beta_{j, \zeta}^{-1} \alpha_{j, \zeta}\right]=\alpha_{j, \zeta} \alpha_{j+1, \zeta}^{-1} \beta_{j+1, \zeta} \alpha_{j, \zeta}^{-1}
$$

which shows the inverse inclusion.
We are nearing the final argument. A birack $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is left $k$-reductive, for $k \geq 2$, if and only if $\alpha_{k-1, \chi}=\beta_{k-1, \chi}$ for each choice of the sequence $\chi$. This is equivalent to the fact that $\alpha_{k-1, \chi}^{-1} \beta_{k-1, \chi}=\mathrm{id}$ for each $\chi \in X^{k+1}$ or to the fact $\gamma_{k-1}(\operatorname{LMlt}(X))=\{\operatorname{id}\}$ and this is equivalent to $\operatorname{LMlt}(X)$ being nilpotent of class at most $k-1$.

For $k=1$, the group $\operatorname{LMlt}(X)$ is nilpotent of class 0 if and only if it is trivial which is clearly equivalent to $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ being 1-reductive, which completes the proof.
$(\mathrm{i}) \Leftrightarrow(\mathrm{iv})$ It is clear that the structure of the left retract $\operatorname{LRet}(X)$ may be formally defined as the $\operatorname{birack}\left(\tilde{X}=\left\{L_{x}: x \in X\right\}, \tilde{o}, \backslash \tilde{o}, \tilde{\bullet}, / \tilde{\bullet}\right)$ such that

$$
\begin{aligned}
& L_{x} \tilde{\circ} L_{y}=L_{x \circ y} \stackrel{\text { def }}{=} \mathcal{L}_{L_{x}}\left(L_{y}\right) \\
& L_{x} \backslash \tilde{\circ} L_{y}=L_{x \backslash ॰ y} \\
& L_{x} \tilde{\bullet} L_{y}=L_{x}=L_{x} / \tilde{\bullet} L_{y}
\end{aligned}
$$

It is so since the mapping $\kappa: X / \sim \rightarrow \tilde{X}, x / \sim \mapsto L_{x}$ is a well-defined isomorphism of biracks.
We define the following mapping $\Phi: \operatorname{LMlt}(X) \rightarrow \operatorname{LMlt}(\tilde{X}):$

$$
\Phi(\alpha)\left(L_{x}\right)=\alpha L_{x} \alpha^{-1}=L_{\alpha(x)}
$$

The mapping $\Phi$ is onto since $\Phi\left(L_{y}\right)=\mathcal{L}_{L_{y}}$. And it is a homomorphism since

$$
\Phi(\alpha \beta)\left(L_{x}\right)=L_{\alpha \beta(x)} \stackrel{(2.15)}{=} \Phi(\alpha)\left(L_{\beta(x)}\right) \stackrel{(2.15)}{=} \Phi(\alpha) \Phi(\beta)\left(L_{x}\right)
$$

Now we compute the kernel of the homomorphism:

$$
\Phi(\alpha)=\operatorname{id} \Leftrightarrow \Phi(\alpha)\left(L_{x}\right)=L_{x} \Leftrightarrow \alpha L_{x} \alpha^{-1}=L_{x} \Leftrightarrow \alpha \in Z(\operatorname{LMlt}(X)) .
$$

Hence $\operatorname{LMlt}(X)$ is nilpotent of class $k$ if and only if $\operatorname{LMlt}(\operatorname{LRet}(X))$ is nilpotent of class $k-1$. We finish the proof by noticing that $|\operatorname{LRet}(X)|=1$ if and only if $\operatorname{LMlt}(X)$ is nilpotent of class 0 .

The same theorem is not true for non-idempotent biracks. Non-idempotency of permutational birack is equivalent to the fact that $f \neq \mathrm{id}$. This means that $|\operatorname{LRet}(X)|=1$ but the left multiplication group $\operatorname{LMlt}(X)$ is non-trivial. However, this is the only exception.
Theorem 4.2. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a left distributive birack and let $k \geq 2$. Then the following conditions are equivalent:
(i) $\left|\operatorname{LRet}^{k}(X)\right|=1$,
(ii) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is left $k$-reductive,
(iii) $\left(X, \circ, \_{0}, \bullet, / \bullet\right)$ is left $k$-permutational,
(iv) $\operatorname{LMlt}(X)$ is nilpotent of class at most $k-1$.

Proof. (ii) $\Leftrightarrow$ (iii) is Lemma 3.3. The equivalence (ii) $\Leftrightarrow$ (iv) was proved in the proof of Theorem 4.1 as the idempotency was not used in this part of the proof. The only place where the idempotency was used was actually the case of $k=1$ in $(\mathrm{i}) \Leftrightarrow(\mathrm{iv})$. Hence we can reason again that $\operatorname{LMlt}(X)$ is nilpotent of class $k$ if and only if $\operatorname{LMlt}(\operatorname{LRet}(X))$ is nilpotent of class $k-1$. According to Lemma 3.5, $\operatorname{LRet}(X)$ is idempotent and therefore, according to Theorem 4.1, LMlt $(\operatorname{LRet}(X))$ is nilpotent of class $k-1$ if and only if $\left|\operatorname{LRet}^{k+1}(X)\right|=1$ and $\left|\operatorname{LRet}^{k}(X)\right|>1$.

Since the operation - was never used in the proof, we immediately get:
Corollary 4.3. Let $\left(X, \circ, \_{\circ}\right)$ be a left rack. Then $\left(X, \circ, \_{\circ}\right)$ is $k$-reductive if and only if $\operatorname{LMlt}(X)$ is nilpotent of class at most $k-1$.

Corollary 4.3 generalizes the result obtained by the authors for medial quandles (a proper subclass of idempotent left racks) [14, Theorem 5.3]. The proof given there used different methods.
Remark 4.4. It is worth emphasizing that all results established for left properties (distributivity, $m$-reductivity, $m$-permutationality, retracts) are also true for right ones, when using their dual versions.

Lemma 4.5. Let $G$ be a group and $H_{1}, H_{2}$ be its subgroups such that $\left[H_{1}, H_{2}\right]=\{1\}$ and $G=H_{1} H_{2}$. If both $H_{1}$ and $H_{2}$ are nilpotent of class at most $k$, for some $k \in \mathbb{N}$, then $G$ is nilpotent of class at most $k$. On the other hand, if $G$ is nilpotent of class $k$ then $H_{1}$ or $H_{2}$ are nilpotent of class at most $k$.

Proof. We will prove by induction that, for each $i \in \mathbb{N}$,

$$
\gamma_{i}(G)=\gamma_{i}\left(H_{1}\right) \gamma_{i}\left(H_{2}\right)
$$

By assumption,

$$
\gamma_{0}(G)=G=H_{1} H_{2}=\gamma_{0}\left(H_{1}\right) \gamma_{0}\left(H_{2}\right),
$$

hence the statement is true for $i=0$.
Let $i>0$. Take $a \in \gamma_{i}(G)$ and $b \in G$. Then by the induction hypothesis, there are $a_{1} \in \gamma_{i}\left(H_{1}\right)$, $a_{2} \in \gamma_{i}\left(H_{2}\right), b_{1} \in H_{1}, b_{2} \in H_{2}$ such that $a=a_{1} a_{2}$ and $b=b_{1} b_{2}$. Therefore,

$$
\begin{aligned}
& {[a, b]=a^{-1} b^{-1} a b=a_{2}^{-1} a_{1}^{-1} b_{2}^{-1} b_{1}^{-1} a_{1} a_{2} b_{1} b_{2}=a_{1}^{-1} b_{1}^{-1} a_{1} b_{1} a_{2}^{-1} b_{2}^{-1} a_{2} b_{2}=} \\
& {\left[a_{1}, b_{1}\right]\left[a_{2}, b_{2}\right] \in\left[\gamma_{i}\left(H_{1}\right), H_{1}\right]\left[\gamma_{i}\left(H_{2}\right), H_{2}\right]=\gamma_{i+1}\left(H_{1}\right) \gamma_{i+1}\left(H_{2}\right) .}
\end{aligned}
$$

This means that $\gamma_{i+1}(G) \subseteq \gamma_{i+1}\left(H_{1}\right) \gamma_{i+1}\left(H_{2}\right)$. The other inclusion uses the same argument.
The rest of the proof is now evident.

Theorem 4.6. Let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be a distributive birack and let $k \geq 2$. Then the following conditions are equivalent:
(i) $\left|\operatorname{Ret}^{k}(X)\right|=1$,
(ii) $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ is $k$-reductive,
(iii) $\left(X, \circ, \_{\circ}, \bullet / \bullet\right)$ is $k$-permutational,
(iv) $\operatorname{Mlt}(X)$ is nilpotent of class at most $k-1$.

Proof. Most of the claim, namely (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv), follows from Theorem 4.2, Lemmas 2.9 and 4.5, We only have to prove (i) $\Leftrightarrow$ (iv). The proof will be given by the induction on $k$.

First, let $\left(X, \circ, \_{\circ}, \bullet, / \bullet\right)$ be an idempotent birack. For such biracks, (i) $\Leftrightarrow$ (iv) is true even for $k=1$. It is evident that $|\operatorname{Ret}(X)|=1$ if and only if $\operatorname{Mlt}(X)$ is nilpotent of class 0 .

Let $k>1$. Similarly as in the proof of Theorem 4.1, the structure of the retract $\operatorname{Ret}(X)$ is formally defined as the birack $\left(\tilde{X}=\left\{\left(L_{x}, \mathbf{R}_{x}\right): x \in X\right\}, \tilde{o}, \backslash_{\tilde{o}}, \tilde{\bullet}, / \tilde{\boldsymbol{\bullet}}\right)$ such that

$$
\begin{aligned}
& \left(L_{x}, \mathbf{R}_{x}\right) \tilde{\circ}\left(L_{y}, \mathbf{R}_{y}\right)=\left(L_{x \circ y}, \mathbf{R}_{x \circ y}\right)=\left(L_{x \circ y}, \mathbf{R}_{y}\right) \stackrel{\text { def }}{=} \mathcal{L}_{\left(L_{x}, \mathbf{R}_{x}\right)}\left(\left(L_{y}, \mathbf{R}_{y}\right)\right), \\
& \left(L_{x}, \mathbf{R}_{x}\right) \backslash \tilde{o}\left(L_{y}, \mathbf{R}_{y}\right)=\left(L_{x \backslash \circ y}, \mathbf{R}_{x \backslash \circ y}\right)=\left(L_{x \backslash \circ y}, \mathbf{R}_{y}\right), \\
& \left(L_{x}, \mathbf{R}_{x}\right) \tilde{\bullet}\left(L_{y}, \mathbf{R}_{y}\right)=\left(L_{x \bullet y}, \mathbf{R}_{x \bullet y}\right)=\left(L_{x}, \mathbf{R}_{x \bullet y}\right) \stackrel{\text { def }}{=} \mathcal{R}_{\left(L_{y}, \mathbf{R}_{y}\right)}\left(\left(L_{x}, \mathbf{R}_{x}\right)\right), \\
& \left(L_{x}, \mathbf{R}_{x}\right) / \tilde{\bullet}\left(L_{y}, \mathbf{R}_{y}\right)=\left(L_{x / \bullet}, \mathbf{R}_{x / \bullet y}\right)=\left(L_{x}, \mathbf{R}_{x / \bullet y}\right) .
\end{aligned}
$$

We define the following mapping $\Phi: \operatorname{Mlt}(X) \rightarrow \operatorname{Mlt}(\tilde{X})$ :

$$
\Phi(\alpha)\left(\left(L_{x}, \mathbf{R}_{x}\right)\right)=\left(\alpha L_{x} \alpha^{-1}, \alpha \mathbf{R}_{x} \alpha^{-1}\right)=\left(L_{\alpha(x)}, \mathbf{R}_{\alpha(y)}\right) .
$$

The mapping $\Phi$ is onto since $\Phi\left(\left(L_{y}, \mathbf{R}_{y}\right)\right)=\mathcal{L}_{\left(L_{y}, \mathbf{R}_{y}\right)}$. And it is a homomorphism since

$$
\Phi(\alpha \beta)\left(\left(L_{x}, \mathbf{R}_{x}\right)\right)=\left(L_{\alpha \beta(x)}, \mathbf{R}_{\alpha \beta(x)}\right)=\Phi(\alpha)\left(\left(L_{\beta(x)}, \mathbf{R}_{\beta(x)}\right)\right)=\Phi(\alpha) \Phi(\beta)\left(\left(L_{x}, \mathbf{R}_{x}\right)\right) .
$$

Now we compute the kernel of the homomorphism:

$$
\Phi(\alpha)=\operatorname{id} \Leftrightarrow \Phi(\alpha)\left(\left(L_{x}, \mathbf{R}_{x}\right)\right)=\left(L_{x}, \mathbf{R}_{x}\right) \Leftrightarrow\left(\alpha L_{x} \alpha^{-1}, \alpha \mathbf{R}_{x} \alpha^{-1}\right)=\left(L_{x}, \mathbf{R}_{x}\right) \Leftrightarrow \alpha \in Z(\operatorname{Mlt}(X)) .
$$

Hence $\operatorname{Mlt}(X)$ is nilpotent of class $k$ if and only if $\operatorname{Mlt}(\operatorname{Ret}(X))$ is nilpotent of class $k-1$. This proves the equivalence for all idempotent distributive biracks. If $\left(X, \circ, \_{0}, \bullet, / \bullet\right)$ is non-idempotent and $k \geq 2$ the proof goes the same way since by Corollary 3.8 the retract $\operatorname{Ret}(X)$ is idempotent and in the inductive step we do not use the idempotency.

For non-distributive biracks, there is no equivalence between multipermutation and nilpotency, as we have remarked already in the introduction. There is no equivalence even on a small level - in the proof of [1, Theorem 3.1], Cedó, Jespers and Okniński gave an example of an involutive 3 -reductive birack with an abelian permutation group. This birack is non-distributive since the only involutive and distributive biracks are 2-reductive as shown in [16, Corollary 5.5].

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(P.J.) Department of Mathematics, Faculty of Engineering, Czech University of Life Sciences, Kamýcká 129, 16521 Praha 6, Czech Republic
(A.P., A.Z.) Faculty of Mathematics and Information Science, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland

E-mail address: (P.J.) jedlickap@tf.czu.cz
E-mail address: (A.P.) A.Pilitowska@mini.pw.edu.pl
E-mail address: (A.Z.) A.Zamojska-Dzienio@mini.pw.edu.pl

