THE POLYTOPE OF NON-CROSSING GRAPHS ON A PLANAR POINT SET

DAVID ORDEN AND FRANCISCO SANTOS

Abstract. For any finite set $\mathcal{A}$ of n points in $\mathbb{R}^2$, we define a $(3n-3)$-dimensional simple polyhedron whose face poset is isomorphic to the poset of "non-crossing marked graphs" with vertex set $\mathcal{A}$, where a marked graph is defined as a geometric graph together with a subset of its vertices. The poset of non-crossing graphs on $\mathcal{A}$ appears as the complement of the star of a face in that polyhedron.

The polyhedron has a unique maximal bounded face, of dimension $2n_i + n - 3$ where n_i is the number of points of $\mathcal{A}$ in the interior of $\mathrm{conv}(\mathcal{A})$. The vertices of this polytope are all the pseudo-triangulations of $\mathcal{A}$, and the edges are flips of two types: the traditional diagonal flips (in pseudo-triangulations) and the removal or insertion of a single edge.

As a by-product of our construction we prove that all pseudo-triangulations are infinitesimally rigid graphs.

1. Introduction

The set of (straight-line, or geometric) non-crossing graphs with a given set of vertices in the plane is of interest in Computational Geometry, Geometric Combinatorics, and related areas. In particular, much effort has been directed towards enumeration, counting and optimization on the set of maximal such graphs, that is to say, triangulations. A lower bound of $2^{n_i+2n-3} = \Omega(4^n)$ for any point set $\mathcal{A}$ is trivial (consider the subgraphs of any given triangulation) and an upper bound of type $O(c^n)$ for some constant c was first shown in [3]. The best current value for c is $59 \cdot 8 = 472$ [17]. In this paragraph and in the rest of the paper n is the total cardinality of $\mathcal{A}$ and n_i denotes the number of points in the interior of $\mathrm{conv}(\mathcal{A})$, respectively. For point sets in non-general position we will need to distinguish between vertices of $\mathrm{conv}(\mathcal{A})$ (*extremal* points) and other points in the boundary of $\mathrm{conv}(\mathcal{A})$, which we call *semi-interior* points. We denote n_v and n_s their respective numbers, so that $n = n_i + n_v + n_s$.

The poset structure of non-crossing graphs is only well understood if the points are in convex position. In this case the non-crossing graphs containing all the hull edges are the same as the polygonal subdivisions of the convex n-gon and, as is well-known, they form the face poset of the $(n-3)$-*associahedron*. The paper [7] contains several enumerative results about geometric graphs with vertices in convex position. In particular, it shows that there are $\Theta((6+4\sqrt{2})^n n^{-3/2})$ non-crossing graphs in total and gives explicit formulas for each fixed cardinality.

Date: February 13, 2003.

Research partially supported by project BMF2001-1153 of Spanish Dirección General de Investigación Científica.

In this paper we generalize the associahedron and construct from $\mathcal{A}$ a polytope whose face poset contains the poset of non-crossing graphs on $\mathcal{A}$ embedded in a very nice way:

Theorem 1.1. *Let $\mathcal{A}$ be a finite set of n points in the plane, not all contained in a line. Let n_i, n_s and n_v be the number of interior, semi-interior and extremal points of $\mathcal{A}$, respectively.*

There is a simple polytope $Y_f(\mathcal{A})$ of dimension $2n_i + n - 3$, and a face F of $Y_f(\mathcal{A})$ (of dimension $2n_i + n_v - 3$) such that the complement of the star of F in the face-poset of $Y_f(\mathcal{A})$ equals the poset of non-crossing graphs on $\mathcal{A}$ that use all the convex hull edges.

This statement deserves some words of explanation:

– Since convex hull edges are irrelevant to crossingness, the poset of *all* non-crossing graphs on $\mathcal{A}$ is the direct product of the poset in the statement and a Boolean poset of rank n_v.
– The equality of posets in Theorem 1.1 reverses inclusions. Maximal non-crossing graphs (triangulations of $\mathcal{A}$) correspond to minimal faces (vertices of $Y_f(\mathcal{A})$).
– We remind the reader that the *star* of a face F is the set of faces contained in the union of all the facets (maximal proper faces) containing F. In the complement of the star of F we must include the polytope $Y_f(\mathcal{A})$ itself, which corresponds to the graph with no interior edges.
– If our point set is in convex position then $Y_f(\mathcal{A})$ is the associahedron and the face F is the whole polytope, whose star we must interpret as being empty.
– We give a fully explicit facet description of $Y_f(\mathcal{A})$. It lives in $\mathbb{R}^{3n}$ and is defined by the 3 linear equalities (1) and the $\binom{n}{2} + n$ linear inequalities (4) and (5) of Section 3, with some of them turned into equalities. (With one exception: for technical reasons, if $\mathcal{A}$ contains three collinear boundary points we need to add extra points to its exterior and obtain $Y_f(\mathcal{A})$ as a face of the polytope $Y_f(\mathcal{A}')$ of the extended point set $\mathcal{A}'$).
– The f_{ij}'s in equations (4) and (5) and in the notation $Y_f(\mathcal{A})$ denote a vector in $\mathbb{R}^{\binom{n+1}{2}}$. Our construction starts with a linear cone $\overline{Y_0}(\mathcal{A})$ (Definition 3.1) whose facets are then translated using the entries of f to produce a polyhedron $\overline{Y_f}(\mathcal{A})$, of which the polytope $Y_f(\mathcal{A})$ is the unique maximal bounded face. Our proof goes by analyzing the necessary and sufficient conditions for f to produce a polytope with the desired properties and then proving the existence of valid choices of f. In particular, Theorem 3.7 shows one valid choice. This is essentially the same approach used in [16] for the polytope of pointed non-crossing graphs constructed there. That polytope is actually the face F of the statement of Theorem 1.1.
– Our results are valid for point sets in non-general position, an aspect which was left open in [16]. Our definition of non-crossing in non-general position is that if q is between p and r, then the edge pr cannot appear in a non-crossing graph, regardless of whether q is incident to any edge in the graph. This definition has the slight drawback that a graph which is non-crossing in a point set $\mathcal{A}$ may become crossing in $\mathcal{A} \cup \{p\}$, but it has the advantage that it makes maximal crossing-free graphs coincide with

the triangulations of $\mathcal{A}$ (with the convention, standard in Computational Geometry, that triangulations of $\mathcal{A}$ are required to use all the points of $\mathcal{A}$ as vertices).

It is worth relating our construction to other two constructions of polytopes whose vertices are triangulations of a point set:

(1) The *secondary polytope* of $\mathcal{A}$, of dimension $n-3$ (see [4]), which specializes also to the associahedron for points in convex position. Its face poset is that of *regular subdivisions* with vertex set *contained* in $\mathcal{A}$. That is to say, only regular (or "generalized Delaunay") triangulations of $\mathcal{A}$ appear as vertices, and the definition of triangulation allows for interior points to be used as vertices or not.
(2) The *universal polytope*, which is a 0/1 polytope living in $\mathbb{R}^{\binom{n}{3}}$ (one coordinate for each possible triangle in the configuration) and has dimension $\binom{n-1}{3}$ for point sets in general position. In the definition of [6], triangulations that do not use all the interior points as vertices are allowed. But those using all the interior points form a face, obtained setting to zero all the variables of non-empty triangles.

Both the secondary polytope and the universal polytope have been used for optimization or enumeration purposes in triangulations. But observe that there are no explicit facet descriptions of them: In the secondary polytope, facets correspond to the coarse polygonal subdivisions of $\mathcal{A}$, which have no easy characterization. In the universal polytope, the facet description in [6] gives only a linear programming relaxation of the polytope, which implies that integer programming is needed in order to optimize linear functionals in it.

The fact that the poset we are interested in is the complement of the star of a face and not just a subposet of the face lattice of $Y_f(\mathcal{A})$ has theoretical and practical implications. On the one hand, it implies that the poset is homeomorphic to a ball of dimension $2n_i+n-4$, since there is a shelling order ending precisely in the facets that contain F. It also says that the part of the boundary of $Y_f(\mathcal{A})$ that we are interested in becomes the (strict) lower envelope of a convex polyhedron via any projective transformation that sends a supporting hyperplane of the face F to the infinity.

Observe also that the dimension $2n_i + n - 3$ of the polytope we construct is the minimum possible one because it equals the number of interior edges in every triangulation.

Although this paper is (mostly) self-contained, the construction is greatly based on [16]. There, a polyhedron $\overline{X_f}(\mathcal{A})$ of dimension $2n-3$ is constructed whose face poset is (opposite to) that of *pointed* non-crossing graphs on $\mathcal{A}$. A straight-line graph embedded in the plane is called pointed if the edges incident to every vertex span an angle of at most 180 degrees (but there is a subtlety in our understanding of pointedness at semi-interior points. See Definition 5.2). The polyhedron $\overline{X_f}(\mathcal{A})$ has a unique maximal bounded face $X_f(\mathcal{A})$, of dimension $2n_i+n_v-3$, the *polytope of pointed pseudo-triangulations* of $\mathcal{A}$.

Our main new ingredient is that we consider "marked" non-crossing graphs, meaning non-crossing graphs together with the specification of a subset of their pointed vertices. With ideas similar to those of [16] but with n extra coordinates for the n possible marks, we get a polyhedron $\overline{Y_f}(\mathcal{A})$ of dimension $3n-3$. Facets of $\overline{Y_f}(\mathcal{A})$ correspond to the edges or marks available in $\mathcal{A}$, which are $\binom{n}{2} + n$ in general position and less than that in special position, because edges whose relative interior meets $\mathcal{A}$ do not produce facets. $\overline{Y_f}(\mathcal{A})$ has a unique maximal bounded face, which is the polytope $Y_f(\mathcal{A})$ in the statement of Theorem 1.1. In the absence of semi-interior points, this face is just the intersection of the $2n_v$ facets corresponding to marks in boundary vertices or edges of $\mathrm{conv}(\mathcal{A})$. The face F of the statement of Theorem 1.1 is the intersection with the facets of the remaining $n - n_v = n_i$ marks. If semi-interior points exist then our construction is slightly indirect, as mentioned above.

The technical tools both in our construction and in [16] are *pseudo-triangulations* of planar point sets and their relation to structural rigidity of non-crossing graphs. Pseudo-triangulations, first introduced by Pocchiola and Vegter around 1995 (see [14]), have by now been used in many Computational Geometry applications, among them visibility [13, 15, 14, 18], ray shooting [9], and kinetic data structures [1, 12]. Streinu [19] introduced the *minimum* or *pointed pseudo-triangulations*, and used them to prove the Carpenter's Rule Theorem (the first proof of which was given shortly before by Connelly et al [5]). Pointed pseudo-triangulations turn out to coincide with the maximal non-crossing and pointed graphs; that is to say, with the vertices of the polyhedron $X_f(\mathcal{A})$ of [16] (the face F of Theorem 1.1). Our method extends that construction to cover all pseudo-triangulations, with a suitable definition of pseudo-triangulation for point sets in non-general position (Definition 5.1). In particular:

Theorem 1.2. *The vertex set of the polytope $Y_f(\mathcal{A})$ of Theorem 1.1 is in bijection to the set of all pseudo-triangulations of $\mathcal{A}$. The 1-skeleton of $Y_f(\mathcal{A})$ is the graph of flips between them.*

The flips between pseudo-triangulations that we consider are introduced in Section 2 (see Definition 2.5) for point sets in general position and in Section 5 (see Definition 5.5 and Figures 8 and 9) for point sets with collinearities. The definition is new (to the best of our knowledge) but in the case of general position it has independently been considered in [2], where flips between pseudo-triangulations are related to geometric flips between polyhedral terrains.

Our flips restrict to the ones in [19] and [16] when the two pseudo-triangulations involved are pointed, and they are also related to the flips of Pocchiola and Vegter [14] as follows: Pocchiola and Vegter were interested in pseudo-triangulations of a set $\mathcal{O} := \{o_1, \ldots, o_n\}$ of convex bodies, and they defined a graph of flips between them. That graph is regular of degree $3n-3$. Pocchiola (personal communication) has shown that taking each o_i to be a sufficiently small convex body around each point, our graph is obtained from the one in [14] by contraction of certain edges. In particular, this shows that our graph has diameter $O(n^2)$ since that is the case for the graph in [14].

Our construction has also rigid-theoretic consequences. A generically rigid graph (for dimension 2) is a graph which becomes rigid in almost all its straight-line embeddings in the plane. Generically rigid graphs need at least $2n-3$ edges, because that is the number of degrees of freedom of n points in the plane (after neglecting rigid motions). Generically rigid graphs with exactly $2n-3$ edges are called *isostatic* and they admit the following characterization, due to Laman (see, for example, [10]): they are the graphs with $2n-3$ edges and with the property that any subset of $k \leq n-2$ vertices is incident to at least $2k$ edges. Using this characterization, Ileana Streinu [19] proved that every pointed pseudo-triangulations is an isostatic graph. We have the following generalization:

Theorem 1.3. *Let T be a pseudo-triangulation of a planar point set $\mathcal{A}$ in general position. Let G be its underlying graph. Then:*

(1) *G is infinitesimally rigid, hence rigid and generically rigid.*
(2) *There are at least $2k+3l$ edges of T incident to any subset of k pointed plus l non-pointed vertices of T (assuming $k+l \leq n-2$).*

This result is true for points in non-general position, as long as they do not have boundary collinearities. In the presence of boundary collinearities, non-rigid pseudo-triangulations (for our definition) exist. For example, only six of the fourteen pseudo-triangulations of the point set of Figure 9 are rigid.

If we recall that a pseudo-triangulation with k non-pointed vertices has exactly $2n-3+k$ edges (see Proposition 2.2), Theorem 1.3 has the consequence that the space of self-stresses on a pseudo-triangulation has exactly dimension k. This fact follows also from the results of [2].

The structure of the paper is as follows: We first develop our construction for point sets in general position, in three steps: the combinatorics of the polyhedron we are seeking for is studied in Section 2, where we introduce in particular the graph of flips between pseudo-triangulations. Then, the construction of the polytope is given in Section 3 but the proof that it has the required properties depends on some assumption which is proved in Section 4 using rigidity theoretic ideas. In Section 5 everything is adapted to point sets with collinearities.

In closing, we propose two open questions:

- Can every planar and generically rigid graph be embedded as a pseudo-triangulation? We believe this is true. In the maximal case (combinatorial triangulations can be drawn with convex faces) it holds by Tutte's theorem. In the minimal case (planar Laman graphs can be embedded as pointed pseudo-triangulations) the result has been proved in [11].
- Is the poset of non-crossing graphs on $\mathcal{A}$ the poset of a polyhedron? A naive answer would be that the polyhedron can be obtained by just deleting from the facet definition of the polytope $Y_f(\mathcal{A})$ of Theorem 1.1 the facets containing F. We have checked that this is false in the simplest example of a single point in general position in the interior of a quadrilateral. In this example F itself is a facet, but its removal gives a polyhedron with two extra vertices, not present in $Y_f(\mathcal{A})$, and corresponding to graphs with crossings.

2. The graph of all pseudo-triangulations of $\mathcal{A}$

All throughout this section, $\mathcal{A}$ denotes a set of n points in general position in the plane, n_i of them in the interior of $\mathrm{conv}(\mathcal{A})$ and n_v in the boundary.

Definition 2.1. A *pseudo-triangle* is a simple polygon with only three convex vertices (called *corners*) joined by three inward convex polygonal chains (called *pseudo-edges* of the pseudo-triangle).

A *pseudo-triangulation of $\mathcal{A}$* is a geometric non-crossing graph with vertex set $\mathcal{A}$ and which partitions $\mathrm{conv}(\mathcal{A})$ into pseudo-triangles.

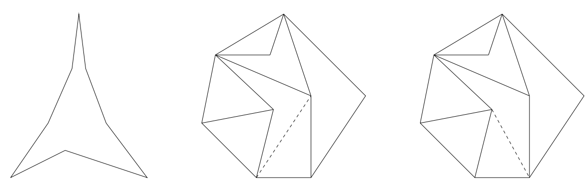
Part (a) of Figure 1 shows a pseudo-triangle. Parts (b) and (c) show two pseudo-triangulations.

Since the maximal non-crossing graphs on $\mathcal{A}$ (the *triangulations* of $\mathcal{A}$) are a particular case of pseudo-triangulations, they are the maximal pseudo-triangulations. As is well-known, they all have $2n_v + 3n_i - 3$ edges. It turns out that the pseudo-triangulations with the minimum possible number of edges are also very interesting from different points of view. We recall that a vertex of a geometric graph is called *pointed* if all its incident edges span an angle smaller than 180 degrees from that vertex. The graph itself is called pointed if all its vertices are pointed. The following statement comes originally from [19] and a proof can also be found in [16].

Proposition 2.2 (Streinu). *Let $\mathcal{A}$ be a planar point set as above. Then:*

1. *Every pseudo-triangulation of $\mathcal{A}$ with n_γ non-pointed vertices and n_ϵ pointed vertices has: $2n - 3 + n_\gamma = 3n - 3 - n_\epsilon$ edges.*
2. *Every pointed and planar graph on $\mathcal{A}$ has at most $2n - 3$ edges, and is contained in some pointed pseudo-triangulation of $\mathcal{A}$.*

Part 1 implies that, among pseudo-triangulations of $\mathcal{A}$, pointed ones have the minimum possible number of edges . For this reason they are sometimes called *minimum pseudo-triangulations*. Part 2 says that pointed pseudo-triangulations coincide with maximal non-crossing and pointed graphs.



(a) (b) (c)

FIGURE 1. (a) A pseudo-triangle. (b) A pointed pseudo-triangulation. (c) The dashed edge in (b) is flipped, giving another pointed pseudo-triangulation.

Another crucial property of pseudo-triangulations is the existence of a natural notion of flip. Let e be an interior edge in a pseudo-triangulation T of $\mathcal{A}$ and let σ be the union of the two pseudo-triangles incident to e. We regard σ as a graph, one of whose edges is e. We can consider $\sigma \setminus e$ to be a (perhaps degenerate) polygon, with a well-defined boundary cycle; in degenerate cases some edges and vertices may appear twice in the cycle. See an example of what we mean in Figure 2, in which the cycle of vertices is $pqrstsu$ and the cycle of edges is $pq, qr, rs, st, ts, su, up$. As in any polygon, each (appearance of a) vertex in the boundary cycle of $\sigma \setminus e$ is either concave or convex. In the figure, there are four convex vertices (corners), namely r, second appearance of s, u and q. Then:

Lemma 2.3. 1. *$\sigma \setminus e$ has either 3 or 4 corners.*

2. *It has 3 corners if and only if exactly one of the two end-points of e is pointed in σ. In this case $T \setminus e$ is still a pseudo-triangulation.*
3. *It has 4 corners if and only if both end-points of e are pointed in σ. In this case $T \setminus e$ is not a pseudo-triangulation and there is a unique way to insert an edge in $T \setminus e$ to obtain another pseudo-triangulation.*

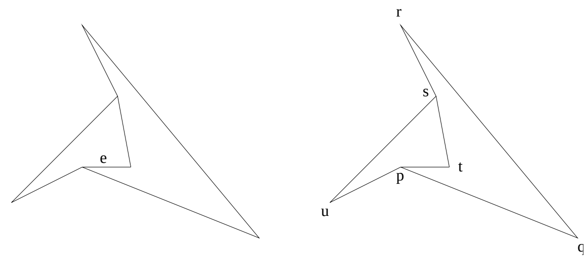


FIGURE 2. The region $\sigma \setminus e$ is a degenerate polygon with four corners

Proof. Let v_1 and v_2 be the two end-points of e. For each v_i, one of the following three things occur: (a) v_i is not-pointed in σ, in which case it is a corner of the two pseudo-triangles incident to e and is not a corner of $\sigma \setminus e$; (b) v_i is pointed in σ with the big angle exterior to σ, in which case it is a corner of both pseudo-triangles and of $\sigma \setminus e$ as well, or (c) v_i is pointed with its big angle interior, in which case it is a corner in only one of the two pseudo-triangles and not a corner in $\sigma \setminus e$.

In case (a), v_i contributes two more corners to the two pseudo-triangles than to $\sigma \setminus e$. In the other two cases, it contributes one more corner to the pseudo-triangles than to $\sigma \setminus e$. Since the two pseudo-triangles have six corners in total, $\sigma \setminus e$ has four, three or two corners depending on whether both, one or none of v_1 and v_2 are pointed in σ. The case of two corners is clearly impossible, which finishes the proof of 1. Part 2 only says that "degenerate pseudo-triangles" cannot appear.

Part 3 is equivalent to saying that a pseudo-quadrangle (even a degenerate one) can be divided into two pseudo-triangles in exactly two ways. Indeed, these two partitions are obtained drawing the geodesic arcs between two opposite corners. Such a geodesic path consists of a unique interior edge and (perhaps) some boundary edges. □

Cases (2) and (3) of the above lemma will define two different types of flips in a pseudo-triangulation. The inverse of the first one is the insertion of an edge, in case this keeps a pseudo-triangulation. The following statement states exactly when this happens:

Lemma 2.4. *Let T be a pseudo-triangle with k non-corners. Then, every interior edge dividing T into two pseudo-triangles makes non-pointed exactly one non-corner. Moreover, there are exactly k such interior edges, each making non-pointed a different non-corner.*

Proof. The first sentence follows from Lemma 2.3, which says that exactly one of the two end-points of the edge inserted is pointed (after the insertion). For each non-corner, pointedness at the other end of the edge implies that the edge is the one that arises in the geodesic arc that joins that non-corner to the opposite corner. This proves uniqueness and existence. □

Definition 2.5. (Flips in pseudo-triangulations) Let T be a pseudo-triangulation. We call *flips* in T the following three types of operations, all producing pseudo-triangulations. See examples in Figure 3:

- (*Deletion flip*). The removal of an edge $e \in T$, if $T \setminus e$ is a pseudo-triangulation.

- (*Insertion flip*). The insertion of an edge $e \notin T$, if $T \cup e$ is a pseudo-triangulation.
- (*Diagonal flip*). The exchange of an edge $e \in T$, if $T \setminus e$ is not a pseudo-triangulation, for the unique edge e' such that $(T \setminus e) \cup e'$ is a pseudo-triangulation.

The *graph of pseudo-triangulations* of $\mathcal{A}$ has as vertices all the pseudo-triangulations of $\mathcal{A}$ and as edges all flips of any of the types.

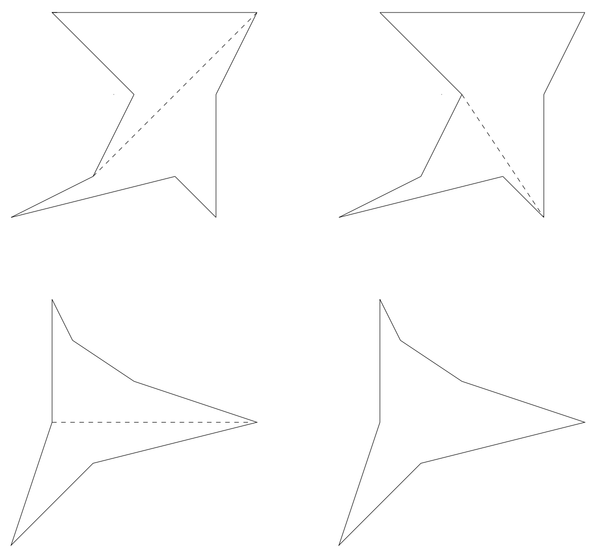


FIGURE 3. Above, a diagonal-flip. Below, an insertion-deletion flip.

Proposition 2.6. *The graph of pseudo-triangulations of $\mathcal{A}$ is connected and regular of degree $3n_i + n_v - 3 = 3n - 2n_v - 3$.*

Proof. There is one diagonal or deletion flip for each interior edge, giving a total of $3n - 3 - n_\epsilon - n_v$ by Proposition 2.2. There are as many insertion flips as pointed interior vertices by Lemma 2.4, giving $n_\epsilon - n_v$.

To establish connectivity, let p be a point on the convex hull of $\mathcal{A}$. The pseudo-triangulations of $\mathcal{A}$ with degree 2 at p coincide with the pseudo-triangulations of $\mathcal{A} \setminus \{p\}$ (together with the two tangents from p to $\mathcal{A} \setminus \{p\}$). By induction, we assume all those pseudo-triangulations to be connected in the graph. On the other hand, in pseudo-triangulations with degree greater than 2 at p all interior edges incident to e can be flipped and produce pseudo-triangulations with smaller degree at e. (Remark: if p is an interior point, then a diagonal-flip on an edge incident to p may create another edge incident to p; but for a boundary point this cannot be the case since p is a corner in the pseudo-quadrilateral $\sigma \setminus e$ of Lemma 2.3). Decreasing one by one the number of edges incident to p will eventually lead to a pseudo-triangulation with degree 2 at p. $\square$

Remarks 2.7. It is an immediate consequence of Lemma 2.3 that every interior edge in a pointed pseudo-triangulation is flippable. This shows that the graph of diagonal-flips between pointed pseudo-triangulations of $\mathcal{A}$ is regular of degree $2n_i + n_v - 3$, a crucial fact in [16].

As another remark, one may be tempted to think that two pseudo-triangulations are connected by a diagonal flip if and only if one is obtained from the other by the removal and insertion of a single edge, but this is not the case: The two pseudo-triangulations of Figure 4 are not connected by a diagonal flip, according to our definition, because the intermediate graph $T \setminus e$ is a pseudo-triangulation.

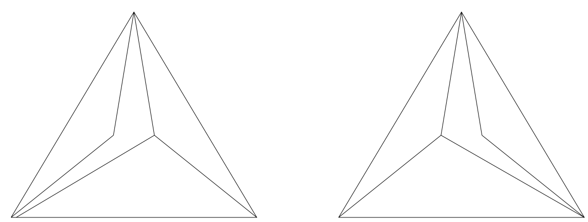


FIGURE 4. These two pseudo-triangulations are not connected by a flip.

Marked non-crossing graphs on $\mathcal{A}$. As happened with pointed pseudo-triangulations, Proposition 2.6 suggests that the graph of pseudo-triangulations of $\mathcal{A}$ may be the skeleton of a simple polytope of dimension $3n_i + n_v - 3$. As a step towards this result we first look at what the face poset of such a polytope should be. The polytope being simple means that we want to regard each pseudo-triangulation T as the upper bound element in a Boolean poset of order $3n - 3 - 2n_v$. This number equals, by Proposition 2.2, the number of interior edges plus interior pointed vertices in T:

Definition 2.8. A *marked graph* on $\mathcal{A}$ is a geometric graph with vertex set $\mathcal{A}$ together with a subset of its vertices, that we call "marked". We call a marked graph *non-crossing* if it is non-crossing as a graph and marks arise only in pointed vertices.

We call a non-crossing marked graph *fully-marked* if it is marked at all pointed vertices. If, in addition, it is a pseudo-triangulation, then we call it a *fully-marked pseudo-triangulation*, abbreviated as *f.m.p.t.*

Marked graphs form a poset by inclusion of both the sets of edges and of marked vertices. We say that a marked graph contains the boundary of $\mathcal{A}$ if it contains all the convex hull edges and convex hull marks. The following results follow easily from the corresponding statements for non-crossing graphs and pseudo-triangulations.

Proposition 2.9. *With the previous definitions:*

1. *Every marked pseudo-triangulation of $\mathcal{A}$ with n_γ non-pointed vertices, n_ϵ pointed vertices and n_m marked vertices, has $2n-3+n_\gamma+n_m = 3n-3-n_\epsilon+n_m$ edges plus marks. In particular, all fully-marked pseudo-triangulations have $3n-3$ edges plus marks, $3n-3-2n_v$ of them interior.*
2. *Fully-marked pseudo-triangulations of $\mathcal{A}$ are exactly the maximal non-crossing marked graphs on $\mathcal{A}$.*
3. **(Flips in marked pseudo-triangulations)** *In a fully-marked pseudo-triangulation of $\mathcal{A}$, every interior edge or interior mark can be* flipped*;*

once removed, there is a unique way to insert another edge or mark to obtain a different fully-marked pseudo-triangulation of $\mathcal{A}$. The graph of flips between fully-marked pseudo-triangulations of $\mathcal{A}$ equals the graph of pseudo-triangulations of $\mathcal{A}$ of Definition 2.5. □

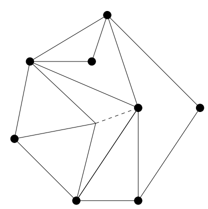


FIGURE 5. Two marked pseudo-triangulations (with marks represented by dots) related by a flip. An edge from the left is switched to a mark on the right.

These properties imply that, if the graph of pseudo-triangulations of $\mathcal{A}$ is to be the skeleton of a simple polytope, then the face poset of that polytope must be (opposite to) the inclusion poset of *non-crossing marked graphs containing the boundary of* $\mathcal{A}$. Indeed, this poset has the right "1-skeleton" and the right lower ideal below every fully-marked pseudo-triangulation (a Boolean lattice of order $3n-3-2n_v$).

3. The polyhedron of marked non-crossing graphs on $\mathcal{A}$

In the first part of this section we do not assume $\mathcal{A}$ to be in general position. Only after Definition 3.6 we need general position, among other things because we have not yet defined marked non-crossing graphs or pseudo-triangulations for point sets in special position. That will be done in Section 5.

The setting for our construction is close to the rigid-theoretic one used in [16]. There, the polytope to be constructed is embedded in the space $\mathbb{R}^{2n-3}$ of all infinitesimal motions of the n points $p_1, \ldots, p_n$. The space has dimension $2n-3$ because the infinitesimal motion of each point produces two coordinates (an infinitesimal velocity $v_i \in \mathbb{R}^2$) but global translations and rotations produce a 3-dimensional subspace of trivial motions which are neglected. Formally, this can be done by a quotient $\mathbb{R}^{2n}/M_0$, where M_0 is the 3-dimensional subspace of trivial motions, or it can be done by fixing three of the $2n$ coordinates to be zero. For example, if the points p_1 and p_2 do not lie in the same horizontal line, one can take

$$v_1^1 = v_1^2 = v_2^1 = 0.$$

In our approach, we will consider a third coordinate t_i for each point, related to the "marks" discussed in the previous paragraphs, or to pointedness of the vertices.

That is to say, given a set of n points $\mathcal{A} = \{p_1, \ldots, p_n\}$ in $\mathbb{R}^2$, we consider the following $(3n-3)$-dimensional space;

$$S := \{(v_1, \ldots, v_n, t_1, \ldots, t_n) \in \left(\mathbb{R}^2\right)^n \times \mathbb{R}^n : v_1^1 = v_1^2 = v_2^1 = 0\} \subset \mathbb{R}^{3n}. \quad (1)$$

In it we consider the following $\binom{n}{2} + n$ linear inequalities

$$H_{ij}^+ := \{(v,t) \in S : \langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) \geq 0\} \quad (2)$$

and

$$H_{0j}^+ := \{(v,t) \in S : t_j \geq 0\}. \tag{3}$$

We denote by $H_{i,j}$ and $H_{0,j}$ their boundary hyperplanes.

Definition 3.1. We call *expansion cone* of $\mathcal{A}$ and denote it $\overline{Y_0}(\mathcal{A})$ the positive region of the above hyperplane arrangement:

$$\overline{Y_0}(\mathcal{A}) := \bigcap_{i,j \in \{0,1,\ldots,n\}} H_{ij}^+$$

When clear from the context we will omit the point set $\mathcal{A}$ and use just $\overline{Y_0}$.

Observe that the equations defining $\overline{Y_0}$ imply that for every i, j:

$$\langle p_i - p_j, v_i - v_j \rangle \geq |p_i - p_j|(t_i + t_j) \geq 0.$$

In particular, the vector $(v_1, \ldots, v_n)$ is an expansive infinitesimal motion of the point set, in the standard sense.

Lemma 3.2. *The polyhedron $\overline{Y_0}(\mathcal{A})$ has full dimension $3n-3$ in $S \subset \mathbb{R}^{3n}$ and it is a pointed polyhedral cone. (Here, "pointed" means "having the origin as a vertex" or, equivalently, "containing no opposite non-zero vectors").*

Proof. The vector (v,t) with $v_i := p_i, t_i := \min_{k,l}\{|p_k - p_l|\}/4$ satisfies all the inequalities (2) and (3) strictly. In order to obtain a point in S we add to it a suitable infinitesimal trivial motion.

To prove that the cone is pointed, suppose that it contains two opposite vectors (v,t) and $-(v,t)$. Equivalently, that (v,t) lies in all the hyperplanes $H_{i,j}$ and $H_{0,i}$. That is to say, $t_i = 0$ for every i and

$$\langle v_j - v_i, p_j - p_i \rangle = 0$$

for all i, j. These last equations say that $(v_1, \ldots, v_n)$ is an infinitesimal flex of the complete graph on $\mathcal{A}$. Since the complete graph on every full-dimensional point set is infinitesimally rigid, $(v_1, \ldots, v_n)$ is a trivial motion and equations (1) imply that the motion is zero. $\square$

An edge p_ip_j or a point p_i are called *tight* for a certain vector $(v,t) \in \overline{Y_0}$ if (v,t) lies in the corresponding hyperplane $H_{i,j}$ or $H_{0,i}$. We call *supporting graph* of any (v,t) and denote it $T(v,t)$ the marked graph of tight edges for (v,t) with marks at tight points for (v,t).

Lemma 3.3. *Let $(v,t) \in \overline{Y_0}$. If $T(v,t)$ contains the boundary edges and vertices of a convex polygon, then $v_l = 0$ and $t_l = 0$ for every point p_l in the interior of the polygon. Therefore, $T(v,t)$ contains the complete marked graph on the set of vertices and interior points of the polygon.*

Observe that this statement says nothing about points in the relative interior of a boundary edge, if the polygon has collinear points in its boundary. Indeed, such points may have a non-zero v_l, namely the exterior normal to the boundary edge containing the point.

Proof. The hypotheses are equivalent to $t_i = 0$ and $\langle p_i - p_j, v_i - v_j \rangle = 0$ for all the boundary vertices p_i and boundary edges p_ip_j of the convex polygon. We first claim that the infinitesimal expansive motion $v = (v_1, \ldots, v_n)$ also preserves distances between non-consecutive polygon vertices. Since the sum of interior angles at vertices of an n-gon is independent of coordinates, a non-trivial motion fixing the lengths of boundary edges would decrease the interior angle at some polygon vertex p_i and then its adjacent boundary vertices get closer, what contradicts (2). Hence, v is a translation or rotation of the polygon boundary which, by equations (1), is zero. On the other hand, if $v_k \neq 0$ for any p_k interior to the polygon, then p_k gets closer to some boundary vertex, what using $t_k \geq 0$ contradicts (2) again.

Therefore, $v_l = 0$ for every point p_l enclosed in the polygon (what can be concluded from [16, Lemma 3.2(b)] as well). Then, the equation (2) corresponding to p_l and to any point p_i in the boundary of the polygon implies that $t_l \leq 0$. Together with the equation (3) corresponding to p_l this implies $t_l = 0$. □

Obviously, $\overline{Y_0}$ is not the polyhedron we are looking for, since its face poset does not have the desired combinatorial structure; it has a unique vertex while $\mathcal{A}$ may have more than only one fully-marked pseudo-triangulation. The right polyhedron for our purposes is going to be a convenient perturbation of $\overline{Y_0}$ obtained by translation of its facets.

Definition 3.4. For each $f \in \mathbb{R}^{\binom{n+1}{2}}$ (with entries indexed $f_{i,j}$, for $i, j \in \{0, \ldots, n\}$) we call *polyhedron of expansions constrained by* f, and denote it $\overline{Y_f}(\mathcal{A})$, the polyhedron defined by the $\binom{n}{2}$ equations

$$\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j) \geq f_{ij} \tag{4}$$

for every $p_i, p_j \in \mathcal{A}$ and the n equations

$$t_j \geq f_{0j}, \qquad \forall p_i \in \mathcal{A}. \tag{5}$$

From Lemma 3.2, we conclude that:

Corollary 3.5. *$\overline{Y_f}(\mathcal{A})$ is a $(3n-3)$-dimensional unbounded polyhedron with at least one vertex, for any f.* □

In the rest of this section and in Section 4 we assume $\mathcal{A}$ to be in general position. As before, to each feasible point $(v, t) \in \overline{Y_f}$ we associate the marked graph consisting of edges and vertices whose equations (4) and (5) are tight on (v, t). Similarly, to a face F of $\overline{Y_f}$ we associate the tight marked graph of any of its relative interior points. This gives an (order-reversing) embedding of the face poset of $\overline{Y_f}$ into the poset of all marked graphs of $\mathcal{A}$. Our goal is to show that for certain choices of the constraint parameters f, the face poset of $\overline{Y_f}$ coincides with that of non-crossing marked graphs on $\mathcal{A}$.

Definition 3.6. We define a choice of the constants f to be *valid* if the tight marked graph $T(F)$ of every face F of $\overline{Y_f}$ is non-crossing.

The proof that valid choices exist for any point set is postponed to Section 4, in order not to interrupt the current flow of ideas. In particular, Corollary 4.5 implies that the following explicit choice is valid:

Theorem 3.7. *The choice $f_{ij} := \det(O, p_i, p_j)^2$, $f_{0j} := 0$ is valid.*

The main statement in the paper is then:

Theorem 3.8. (The polyhedron of marked non-crossing graphs) *If f is a valid choice of parameters, then $\overline{Y_f}$ is a simple polyhedron of dimension $3n-3$ whose face poset equals (the opposite of) the poset of non-crossing marked graphs on $\mathcal{A}$. In particular:*

(a) *Vertices of the polyhedron are in 1-to-1 correspondence with fully-marked pseudo-triangulations of $\mathcal{A}$.*
(b) *Bounded edges correspond to flips of interior edges or marks in fully-marked pseudo-triangulations, i.e., to fully-marked pseudo-triangulations with one interior edge or mark removed.*
(c) *Extreme rays correspond to fully-marked pseudo-triangulations with one convex hull edge or mark removed.*

Proof. By Corollary 3.5, every vertex (v,t) of $\overline{Y_f}$ has at least $3n-3$ incident facets. By Proposition 2.9, if f is valid then the marked graph of any vertex of $\overline{Y_f}$ has exactly $3n-3$ edges plus marks and is a fully-marked pseudo-triangulation. This also implies that the polyhedron is simple. If we prove that all the fully-marked pseudo-triangulations appear as vertices of $\overline{Y_f}$ we finish the proof, because then the face poset of $\overline{Y_f}$ will have the right minimal elements and the right upper ideals of minimal elements (the Boolean lattices of subgraphs of fully-marked pseudo-triangulations) to coincide with the poset of non-crossing marked graphs on $\mathcal{A}$.

That all fully-marked pseudo-triangulations appear follows from connectedness of the graph of flips: Starting with any given vertex of $\overline{Y_f}$, corresponding to a certain f.m.p.t. T of $\mathcal{A}$, its $3n-3$ incident edges correspond to the removal of a single edge or mark in T. Moreover, if the edge or mark is not in the boundary, Lemma 3.3 implies that the edge (of $\overline{Y_f}$) corresponding to it is bounded because it collapses to the origin in $\overline{Y_0}$. Then, this edge connects the original vertex of $\overline{Y_f}$ to another one which can only be the f.m.p.t. given by the flip in the corresponding edge or mark of T. Since this happens for all vertices, and since all f.m.p.t.'s are reachable from any other one by flips, we conclude that they all appear as vertices. □

From Theorems 3.7 and 3.8 it is easy to conclude the statements in the introduction. The following is actually a more precise statement implying both:

Theorem 3.9. (The polytope of all pseudo-triangulations) *Let $Y_f(\mathcal{A})$ be the face of $\overline{Y_f}(\mathcal{A})$ defined turning into equalities the equations (4) and (5) which correspond to convex hull edges or convex hull points of $\mathcal{A}$, and assume f to be a valid choice. Then:*

(1) *$Y_f(\mathcal{A})$ is a simple polytope of dimension $2n_i + n - 3$ whose 1-skeleton is the graph of pseudo-triangulations of $\mathcal{A}$. (In particular, it is the unique maximal bounded face of $\overline{Y_f}(\mathcal{A})$).*
(2) *Let F be the face of $Y_f(\mathcal{A})$ defined by turning into equalities the remaining equations (5). Then, the complement of the star of F in the face-poset of $Y_f(\mathcal{A})$ equals the poset of non-crossing graphs on $\mathcal{A}$ that use all the convex hull edges.*

Proof. (1) That $Y_f(\mathcal{A})$ is a bounded face follows from Lemma 3.3 (it collapses to the zero face in $\overline{Y_0}(\mathcal{A})$). Since vertices of $\overline{Y_f}(\mathcal{A})$ are f.m.p.t.'s and since all f.m.p.t.'s contain all the boundary edges and vertices, $Y_f(\mathcal{A})$ contains all the vertices of $\overline{Y_f}(\mathcal{A})$. Hence, its vertices are in bijection with all f.m.p.t.'s which, in turn, are in bijection with pseudo-triangulations. Edges of $Y_f(\mathcal{A})$ correspond to f.m.p.t.'s minus one interior edge or mark, which are precisely the flips between f.m.p.t.'s, or between pseudo-triangulations.

(2) The facets containing F are those corresponding to marks in interior points. Then, the faces in the complement of the star of F are those in which none of

the inequalities (5) are tight; that is to say, they form the poset of "non-crossing marked graphs containing the boundary edges and marks but no interior marks", which is the same as the poset of non-crossing graphs containing the boundary. □

We now turn our attention to Theorem 1.3. Its proof is based in the use of the homogeneous cone $\overline{Y_0}(\mathcal{A})$ or, more preciesely, the set $\mathcal{H} := \{H_{ij} : i, j = 1, \dots, n\} \cup \{H_{0i} : i = 1, \dots, n\}$ of hyperplanes that define it.

Proof of Theorem 1.3. Observe now that the equations defining H_{ij}, specialized to $t_i = 0$ for every i, become the equations of the infinitesimal rigidity of the complete graph on $\mathcal{A}$. In particular, a graph G is rigid on $\mathcal{A}$ if and only if the intersection

$$(\cap_{ij \in G} H_{ij}) \cap (\cap_{i=1}^{n} H_{0i})$$

equals 0.

This happens for any pseudo-triangulation because Theorem 3.8 implies that the hyperplanes corresponding to the $3n-3$ edges and marks of any fully-marked pseudo-triangulation form a basis of the (dual of) the linear space S.

To prove part (2) we only need the fact that the $3n-3$ linear hyperplanes corresponding to a fully-marked pseudo-triangulation are independent. In particular, any subset of them is independent too. We consider the subset corresponding to the induced (marked) subgraph on the $n-k-l$ vertices other than the k pointed and l non-pointed ones we are interested in. They form an independent set involving only $3(n-k-l)$ coordinates, hence their number is at most $3(n-k-l)-3$ (we need to subtract 3 for the rigid motions of the $n-k-l$ points, and here is where we need $k+l \le n-2$). Since the fully-marked pseudo-triangulation has $3n-3$ edges plus marks, at least $3k+3l$ of them are incident to our subset of points. And exactly k marks are incident to our points, hence at least $2k+3l$ edges are. □

Actually, we can derive some consequences for general planar rigid graphs. Observe that every planar and generically rigid graph G must have between $2n-3$ and $3n-3$ edges (the extreme cases being an isostatic graph and a triangulation of the 2-sphere). Hence, we can say that the graph G has $2n-3+y$ edges, where and $0 \le y \le n-3$. If the graph can be embedded as a pseudo-triangulation then the embedding will have exactly y non-pointed vertices. In particular, the following statement is an indication that every planar and rigid graph can be embedded as a pseudo-triangulation:

Proposition 3.10. *Let G be a planar and generically rigid graph with n vertices and $2n-3+y$ edges. Then, there is a subset Y of cardinality y of the vertices of G such that every set of l vertices in Y plus k vertices not in Y is incident to at least $2k+3l$ edges, whenever $k+l \le n-2$.*

Proof. Consider G embedded planarly in a sufficiently generic straight-line manner. Since the embedding is planar, it can be completed to a pseudo-triangulation T. In particular, the set of edges of G represents an independent subset of $2n-3+y$ hyperplanes of $\mathcal{H}$. But since the graph is rigid, adding marks to all the vertices produces a spanning set of $3n-3+y$ hyperplanes. In between these two sets there must be a basis, consisting of the $2n-3+y$ edges of G plus $n-y$ marks. We call Y the vertices not marked in this basis, and the same argument as in the proof of Theorem 1.3 gives the statement. □

It has to be said however, that a planar graph G with a subset Y of its vertices satisfying Proposition 3.10 need not be generically rigid. Figure 6 shows an example (take as Y any three of the four six-valent vertices).

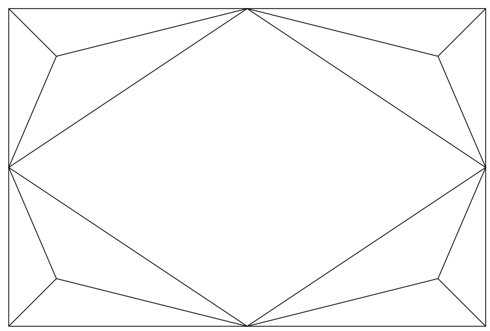


FIGURE 6. A planar graph satisfying the conclusion of Proposition 3.10 need not be rigid.

4. VALID CHOICES OF f

It remains to be proved that valid choices of parameters do exist. In particular, that the choice in Theorem 3.7 is valid. Our methods, again inspired on [16], give actually more: a full description of the set of valid choices via a set of $\binom{n}{4}$ linear inequalities, one for each 4-point subset of the n points.

Definition 4.1. Let G be a graph embedded on $\mathcal{A}$, with set of edges E and set of marked vertices V. In our context, a *stress* G is an assignment of scalars w_{ij} to edges and α_j to marked vertices of G, such that for every $(v,t) \in \mathbb{R}^{3n}$:

$$\sum_{ij\in E} w_{ij}(\langle p_i - p_j, v_i - v_j\rangle - |p_i - p_j|(t_i + t_j)) + \sum_{i\in V} \alpha_i t_i = 0 \tag{6}$$

Lemma 4.2. *Let $\sum_{i=1}^n \lambda_i p_i = 0$, $\sum \lambda_i = 0$, be an affine dependence on a point set $\mathcal{A} = \{p_1, \ldots, p_n\}$. Then,*

$$w_{ij} := \lambda_i \lambda_j \text{ for every } i, j$$

and

$$\alpha_i := \sum_{j:ij\in E} \lambda_i \lambda_j |p_i - p_j| \text{ for every } i$$

defines a stress of the complete graph G on $\mathcal{A}$.

Proof. The condition (6) on variables v gives

$$\sum_{ij\in E} w_{ij} \langle p_i - p_j, v_i - v_j\rangle = 0, \text{ for every } v \in \mathbb{R}^{2n} \tag{7}$$

what can be equivalently stated as saying that the w_{ij} form a stress on the underlying graph of G. This is fulfilled by the w_{ij}'s of the statement:

$$\sum_{j\neq i} \lambda_i \lambda_j (p_i - p_j) = \sum_{j=1}^n \lambda_i \lambda_j (p_i - p_j) = \lambda_i p_i \sum_{j=1}^n \lambda_j - \lambda_i \sum_{j=1}^n \lambda_j p_j = 0$$

where last equality comes from λ_i's being an affine dependence. Then, cancellation of the coefficient of t_i in equation (6) is equivalent to $\alpha_i = \sum_{j:ij\in E} w_{ij}|p_i - p_j|$. $\square$

Let us consider the case of four points in general position in $\mathbb{R}^2$, which have a unique (up to constants) affine dependence. The coefficients of this dependence are:

$$\lambda_i = (-1)^i \det([p_1, \ldots, p_4]\backslash\{p_i\})$$

Hence, if we divide the w_{ij}'s and α_j's of the previous lemma by the constant

$$-\det(p_1,p_2,p_3)\det(p_1,p_2,p_4)\det(p_1,p_3,p_4)\det(p_2,p_3,p_4)$$

we obtain the following expressions:

$$w_{ij} = \frac{1}{\det(p_i,p_j,p_k)\det(p_i,p_j,p_l)}, \ \alpha_i = \sum_{j:ij\in E} w_{ij}|p_i - p_j| \tag{8}$$

where, in that of $w_{i,j}$, k and l denote the two indices other than i and j. The reason why we perform the previous rescaling is that the expressions obtained in this way have a key property which will turn out to be fundamental later on; see Figure 7:

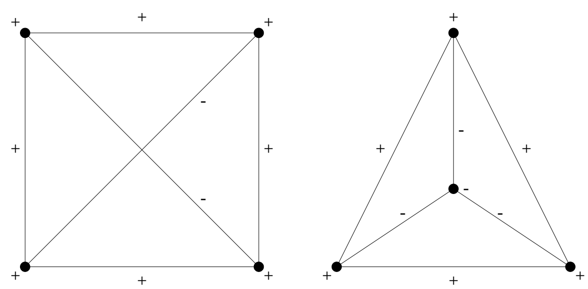


FIGURE 7. The negative parts of these two marked graphs are the excluded minors in non-crossing marked graphs of a point set in general position

Lemma 4.3. *For any four points in general position, the previous expressions give positive w_{ij} and α_j on boundary edges and points and negative w_{ij} and α_j on interior edges and points.*

Proof. In order to check the part concerning w_{ij}'s we use that $\det(q_1,q_2,q_3)$ is two times the signed area of the triangle spanned by q_1,q_2,q_3: For a boundary edge the two remaining points lie on the same side of the edge, so they have the same sign. For an interior edge, they lie on opposite sides and therefore they have different signs.

For the α_i's, if i is an interior point then all the $w_{i,j}$'s in the formula for α_i are negative and, hence, α_i is also negative. If i is a boundary point then two of the $w_{i,j}$ are positive and the third one is negative. But, since

$$\sum_{j\in\{1,2,3,4\}\backslash i} w_{i,j}(p_i - p_j) = 0,$$

the triangle inequality implies that the two positive summands $w_{ij}|p_i - p_j|$ in the expression of α_i add up to a greater absolute value than the negative one. Hence α_i is positive. $\square$

The previous statement is crucial to us, because no matter whether the four points are in convex position or one of them is inside the convex hull of the other three, the fully-marked pseudo-triangulations on the four points can be characterized as the marked graphs with nine edges plus marks and in which the missing edge or mark is interior (two f.m.p.t.'s for points in convex position, four of them for a triangle plus an interior point).

We conclude that:

Theorem 4.4. *An $f \in \mathbb{R}^{\binom{n+1}{2}}$ is valid if and only if for every four points $\{p_1, p_2, p_3, p_4\}$ of $\mathcal{A}$ the following inequality holds,*

$$\sum_{1 \le i < j \le 4} w_{ij} f_{ij} + \sum_{j=1}^{4} \alpha_j f_{0j} > 0 \tag{9}$$

where the w_{ij}'s and α_j's are those of (8).

Proof. Suppose first that $\mathcal{A}$ has only four points. The polyhedron $\overline{Y_f}(\mathcal{A})$ is nine-dimensional, what implies that for every vertex (v,t) of the polyhedron, the set $T(v,t)$ contains at least nine edges plus marks on those four points. Therefore, $T(v,t)$ is the complete marked graph with an edge or mark removed.

Let us denote by G_k and G_{kl} the complete marked graph with a non-marked vertex k or a missing edge kl, respectively. Recall that by Lemma 4.3 the choice of stress on four points has the property that G_k and G_{kl} are fully-marked pseudo-triangulations if and only if α_k and w_{kl} (corresponding respectively to the removed mark or edge) are negative. Let us see that this is equivalent to f being valid:

By the definition of stress,

$$\sum_{1 \le i < j \le 4} w_{ij} (\langle p_i - p_j, v_i - v_j \rangle - |p_i - p_j|(t_i + t_j)) + \sum_{j=1}^{4} \alpha_j t_j$$

equals zero. In the case of G_k, in which every edge and vertex except k are tight, that expression equals

$$\sum_{1 \le i < j \le 4} w_{ij} f_{ij} + \sum_{j=1}^{4} \alpha_j f_{0j} + \alpha_k (t_k - f_{0k}).$$

In the case of G_{kl}, where every vertex and edge except kl are tight, it equals

$$\sum_{1 \le i < j \le 4} w_{ij} f_{ij} + \sum_{j=1}^{4} \alpha_j f_{0j} + w_{kl} (\langle p_k - p_l, v_k - v_l \rangle - |p_k - p_l|(t_k + t_l) - f_{kl}).$$

Since $\langle p_k - p_l, v_k - v_l \rangle - |p_k - p_l|(t_k + t_l) - f_{kl} \ge 0$ and $t_k - f_{0k} \ge 0$, by (4) and (5), we conclude that in the first and second cases above, α_k and w_{kl} respectively are negative if, and only if, f is valid.

Now we turn to the case of a general $\mathcal{A}$ and our task is to prove that a choice of parameters f is valid if and only if it is valid when restricted to any four points. Observe that if $\mathcal{A}' \subset \mathcal{A}$ then $Y_f(\mathcal{A}')$ equals the intersection of $Y_f(\mathcal{A})$ with the subspace where $v_i = 0$ and $t_i = 0$ for all $p_i \in \mathcal{A} \setminus \mathcal{A}'$. In particular, the marked graphs on $\mathcal{A}'$ corresponding to faces of $Y_f(\mathcal{A}')$ are subgraphs of marked graphs of faces of $Y_f(\mathcal{A})$. Moreover, non-crossingness of a marked graph on $\mathcal{A}$ is equivalent to non-crossingness of every induced marked graph on four vertices: indeed, a crossing of two edges appears in the marked graph induced by the four end-points of the two edges, and a non-pointed marked vertex appears in the marked graph induced on the four end-points involved in any three edges forming a non-pointed "letter Y" at the non-pointed vertex.

Hence: if f is valid for every four points, then none of the 4-point minors forbidden by non-crossingness appear in faces of $Y_f(\mathcal{A})$ and f is valid for $\mathcal{A}$. Conversely, if f is not valid on some four point subset $\mathcal{A}'$, then the marked graph on $\mathcal{A}'$ corresponding to any vertex of $Y_f(\mathcal{A}')$ would be the complete graph minus one boundary edge or vertex, that is to say, it would not be non-crossing. Hence f would not be valid on $\mathcal{A}$ either. □

Corollary 4.5. *For any $a, b \in \mathbb{R}^2$, the choice $f_{ij} := \det(a, p_i, p_j)\det(b, p_i, p_j)$, $f_{0j} := 0$ is valid.*

Proof. Consider the four points p_i as fixed and regard $R := \sum w_{ij} f_{ij} + \sum \alpha_j f_{0j} = \sum w_{ij} f_{ij}$ as a function of a and b:

$$R(a,b) = \sum_{1 \le i < j \le 4} \det(a, p_i, p_j)\det(b, p_i, p_j) w_{ij}.$$

We have to show that $R(a, b)$ is always positive. We actually claim it to be always 1. Observe first that $R(p_i, p_j)$ is trivially 1 for $i \neq j$. Since any three of our points are an affine basis and since $R(a, b)$ is an affine function of b for fixed a, we conclude that $R(p_i, b)$ is one for every $i \in \{1, 2, 3, 4\}$ and for every b. The same argument shows that $R(a, b)$ is constantly 1: for fixed b it is an affine function of a and is equal to 1 on an affine basis. □

5. Points in special position

In this section we show that almost everything we said so far applies equally to point sets with collinear points. We will essentially follow the same steps as in Sections 2, 3 and 4. Two subtleties are that our definitions of pointedness or pseudo-triangulations can only be fully justified a posteriori, and that the construction of the polyhedron for point sets with boundary collinearities is slightly indirect: it relies in the choice of some extra exterior points to make colliniarities go to the interior.

The graph of all pseudo-triangulations of $\mathcal{A}$.

Definition 5.1. Let $\mathcal{A}$ be a finite point set in the plane, possibly with collinear points.

(1) A graph G with vertex set $\mathcal{A}$ is called *non-crossing* if no edge intersects another edge of G or point of $\mathcal{A}$ except at its end-points. In particular, if p_1, p_2 and p_3 are three collinear points, in this order, then the edge $p_1 p_3$ cannot appear in a non-crossing graph, independently of whether there is an edge incident to p_2 or not.
(2) A *pseudo-triangle* is a simple polygon with only three interior angles smaller than 180 degrees. A *pseudo-triangulation* of $\mathcal{A}$ is a non-crossing graph with vertex set $\mathcal{A}$, which partitions $\operatorname{conv}(\mathcal{A})$ into pseudo-triangles and such that no point in the interior of $\operatorname{conv}(\mathcal{A})$ is incident to more than one angle of 180 degrees.

Figure 8 shows the eight pseudo-triangulations of a certain point set. We have drawn them connected by certain flips, to be defined later, and with certain points marked. The graph on the right of the figure is not a pseudo-triangulation because it fails to satisfy the last condition in our definition. Intuitively, the reason why we do not allow it as a pseudo-triangulation is that we are considering angles of exactly 180 degrees as being reflex, and we do not want a vertex to be incident to two reflex angles.

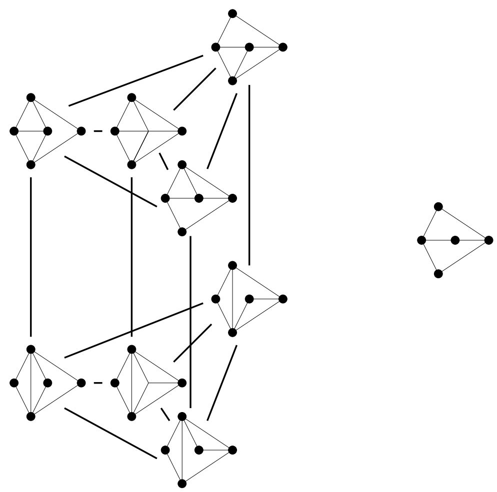


FIGURE 8. The eight pseudo-triangulations of a point set with interior collinearities (left) plus a non-crossing graph with pseudo-triangular faces but which we do not consider a pseudo-triangulation (right)

But if collinearities happen in the boundary of $\mathrm{conv}(\mathcal{A})$, as in Figure 9, we treat things differently. The exterior angle of 180 degrees is not counted as reflex, and hence the middle point in a boundary collinearity is allowed to be incident to an interior angle of 180 degrees. The following definition can be restated as "a vertex is pointed if and only if it is incident to a reflex angle", where reflex is meant as in these last remarks.

Definition 5.2. A vertex p in a non-crossing graph on $\mathcal{A}$ is considered *pointed* if either (1) it is a vertex of $\mathrm{conv}(\mathcal{A})$, (2) it is semi-interior and not incident to any edge going through the interior of $\mathrm{conv}(\mathcal{A})$ or (3) it is interior and its incident edges span at most 180 degrees.

A *non-crossing marked graph* is a non-crossing graph with marks at some of its pointed vertices. If all pointed vertices are marked we say the non-crossing graph is *fully-marked*. Marks at interior and semi-interior points will be called *interior marks*.

For example, all the graphs of Figures 8 and 9 are fully-marked. That is to say, big dots correspond exactly to pointed vertices. Of course, fully-marked pseudo-triangulations are just pseudo-triangulations with marks at all their pointed vertices. Observe that we are calling interior marks and edges exactly those which

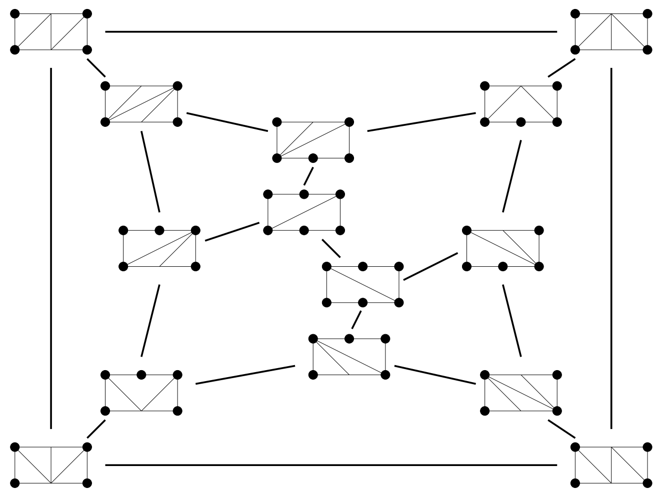


FIGURE 9. The graph of pseudo-triangulations of a point set with boundary collinearities

do not appear in all pseudo-triangulations. From now on, we denote by n_i, n_s and n_v the number of interior, semi-interior and extremal points of $\mathcal{A}$. Finally, $n = n_v + n_s + n_i$ denotes the total number of points in $\mathcal{A}$. The following two statements essentially say that Proposition 2.9 is valid for non-generic configurations.

Lemma 5.3. *Fully-marked pseudo-triangulations are exactly the maximal marked non-crossing graphs on $\mathcal{A}$. They all have $3n-n_s-3$ edges plus marks and $2n_i+n-3$ interior edges plus interior marks.*

Proof. The first sentence is equivalent to saying that every non-crossing graph G can be completed to a pseudo-triangulation without making any pointed vertex non-pointed. The proof of this is that if G is not a pseudo-triangulation then either it has a face with more than three corners, in which case we insert the diagonal coming from the geodesic between any two non-adjacent corners, or there is an interior vertex with two angles of 180 degrees, in which case we choose to consider one of them as reflex and the other as convex, and insert the diagonal joining the convex angle to the opposite corner of the pseudo-triangle containing it.

To prove the cardinality of pseudo-triangulations, let n_ϵ denote the number of marks. Let us think of boundary collinearities as if they were concave boundary chains in our graph, and triangulate the polygons formed by these chains by adding (combinatorially, or topologically) n_s edges in total. If, in addition, we consider interior angles of 180 degrees or more as reflex and the others as convex, we get a graph with all the combinatorial properties of pseudo-triangulations and, in particular, a graph for which Proposition 2.2 can be applied, since its proof is purely combinatorial (a double counting of convex angles, combined with Euler's relation). In particular, the extended graph has $3n - 3$ edges plus marks, and the original graph has $3n - 3 - n_s$ of them. Since there are exactly $n_s + n_v$ exterior edges and n_v exterior marks in every pseudo-triangulation, the last sentence follows. $\square$

Lemma 5.4. *If an interior edge or mark is removed from a fully-marked pseudo-triangulation then there is a unique way to insert another edge or mark to obtain a different fully-marked pseudo-triangulation.*

Proof. If an edge is removed then there are three possibilities: (1) the removal does not create any new reflex angle, in which case the region obtained by the removal is a pseudo-quadrangle (that is, it has four non-reflex angles), because the two regions incident to it had six corners in total and the number of them decreases by two. We insert the opposite diagonal of it. (2) the removal creates a new reflex angle at a vertex which was not pointed. Then the region obtained is a pseudo-triangle and we just add a mark at the new pointed vertex. (3) the removal creates a new reflex angle at a vertex that was already pointed. This means that after the removal the vertex has two reflex angles, that is to say two angles of exactly 180 degrees each. We insert the edge joining this vertex to the opposite corner of the pseudo-triangle containing the original reflex angle.

If a mark is removed, then the only possibility is: (4) the pointed vertex holding the mark is incident to a unique reflex angle (remember that we consider interior angles of 180 degrees as reflex). We insert the edge joining the vertex to the opposite corner of the corresponding pseudo-triangle. □

Definition 5.5. Two fully-marked pseudo-triangulations are said to differ by a *flip* if they differ by just one edge or mark. Cases (1), (2), (3) and (4) in the previous proof are called, respectively, *diagonal flip*, *deletion flip*, *mirror flip* and *insertion flip*.

Of course, our definition of flips specializes to the one for points in general position, except that mirror flips can only appear in the presence of collinearities. An example of a mirror flip can be seen towards the upper right corner of Figure 8.

Corollary 5.6. *The graph of flips between fully-marked pseudo-triangulations of a planar point set is connected and regular of degree* $2n_i + n - 3$.

The reader will have noticed that the graphs of Figures 8 and 9 are more than regular of degrees 4 and 3 respectively. They are the graphs of certain simple polytopes of dimensions 4 and 3. (Figure 8 is a prism over a simplex).

The case with only interior collinearities. Now we assume that our point set $\mathcal{A}$ has only interior collinearities.

For each $f \in \mathbb{R}^{n+1}$ let $\overline{Y_f}(\mathcal{A})$ be the polyhedron defined in Section 3. Recall that everything we said in that section, up to Corollary 3.5, is valid for points in special position. Our main result here is that Theorems 3.7 and 3.8 hold word by word in the case with no boundary collinearities, except that a precision needs to be made regarding the concept of validity.

Recall that for a given choice of $f \in \mathbb{R}^{\binom{n+1}{2}}$, an edge p_ip_j or a point p_i are called *tight* for a certain $(v,t) \in \mathbb{R}^{3n-3}$ or for a face F of $\overline{Y_f}(\mathcal{A})$ if the corresponding equation (4) or (5) is satisfied with equality.

Definition 5.7. We call *strict supporting graph* of a $(v,t) \in \mathbb{R}^{3n-3}$ (or face F of $\overline{Y_f}(\mathcal{A})$) the marked graph of all its tight edges and points, and denote it $T(v,t)$. We call *weak supporting graph* of a (v,t) or face the marked subgraph consisting of edges and points of $T(v,t)$ which define facets of $\overline{Y_f}(\mathcal{A})$.

A choice of f is called *weakly valid* (resp., *strictly valid*) if the weak (resp., strict) supporting graphs of all the faces of $\overline{Y_f}(\mathcal{A})$ are non-crossing marked graphs.

Observe that from any weakly valid choice f one can obtain strictly valid ones: just decrease by arbitrary positive amounts the coordinates of f corresponding to equations which do not define facets of $\overline{Y_f}(\mathcal{A})$. Hence, we could do what follows only in terms of strict validity and would obtain the same polyhedron. But weak validity is needed, as we will see in Remark 5.13, if we want our construction to depend continuously on the coordinates of the point set $\mathcal{A}$.

To obtain the equations that valid choices must satisfy we proceed as in Section 4. The crucial point there was that a marked graph is non-crossing if and only if it does not contain the negative parts of the unique stress in certain subgraphs.

Lemma 5.8. *Let $\mathcal{A}$ be a point set with no three collinear boundary points. Then, a marked graph on $\mathcal{A}$ is non-crossing if and only if it does not contain any of the following four marked subgraphs: the negative parts of the marked graphs displayed in Figure 7 and the negative parts of the marked graphs displayed in Figure 10.*

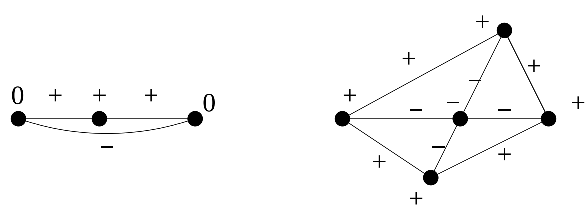


FIGURE 10. The two additional excluded minors for non-crossing marked graphs of a point set with interior collinearities

Proof. Exclusion of the negative parts of the left graphs in both figures are our definition of crossingness for an unmarked graph. An interior vertex is pointed if and only if none of the negative parts of the right graphs appear. □

Lemma 5.9. *The two graphs in Figure 10 have a stress with signs as in the figure.*

Proof. For the left part it is easy to show that the following is a stress:

$$w_{12} = \frac{1}{|p_2 - p_1|}, \quad w_{13} = -\frac{1}{|p_3 - p_1|}, \quad w_{23} = \frac{1}{|p_3 - p_2|}, \quad \alpha_1 = \alpha_3 = 0, \quad \alpha_2 = 2.$$

For the right part, observe that, by definition, stresses on a marked graph form a linear space. Let the four exterior points be p_1, p_2, p_3 and p_4, in cyclic order, and let the interior point be p_5. We know three different stresses of the complete graph on these five points: the one we used in Section 4 for the four exterior points and the two that we have just introduced for the two collinear triplets. From these three we can eliminate the coordinates of edges p_1p_3 and p_2p_4 and we get a stress with the stated signs. □

Theorem 5.10. *Let $\mathcal{A}$ be a point set with no three collinear boundary points. Then, a choice of f is weakly valid if it satisfies equations (9) for all quadruples of points in general position plus the following sets of equations:*

- *For any three points p_1, p_2 and p_3 collinear in this order:*

$$\frac{f_{12}}{|p_2 - p_1|} - \frac{f_{13}}{|p_3 - p_1|} + \frac{f_{23}}{|p_3 - p_1|} + 2f_{02} \geq 0 \tag{10}$$

- *For any five points as in the right part of Figure 10, the following equation where the w_{ij}'s and the α_i's form a stress with signs as indicated in the figure (by convention, w_{ij} equals zero for the two missing edges in the graph):*

$$\sum_{1 \leq i < j \leq 5} w_{ij} f_{ij} + \sum_{j=1}^{5} \alpha_j f_{0j} > 0 \tag{11}$$

The choice is strictly valid if and only if, moreover, the equations (10) of all collinear triplets are satisfied strictly.

Proof. Equations (9) guarantee that no weak or strict tight graph contains the two excluded marked graphs of negative edges and points of Figure 7. Equations (10) and (11) with strict inequality, do the same for the graphs of Figure 10. That these equations are equivalent to strict validity is proved exactly as in Section 4. The reason why we allow equality in equations (10) if we only want a weakly valid choice is that the negative part of the stress consists of a single edge. If the equation is satisfied with equality then the hyperplane corresponding to this edge is a supporting hyperplane of the face of $\overline{Y_f}$ given by the intersection of the three hyperplanes of the positive part of the stress. □

Corollary 5.11. *Let $\mathcal{A}$ be a point set with no collinear boundary points. Any choice of f satisfying equations (9) for every four points in general position plus the following ones for every collinear triplet is weakly valid:*

$$\frac{f_{12}}{|p_2 - p_1|} - \frac{f_{13}}{|p_3 - p_1|} + \frac{f_{23}}{|p_3 - p_1|} + 2f_{02} = 0 \tag{12}$$

In particular, the choices of Corollary 4.5 and Theorem 3.7 are weakly valid.

Proof. For the first assertion, we need to show that equations (11) follow from equations (9) and (12). But this is straightforward: from our proof of Lemma 5.9 it follows that equation (11) is just the one obtained substituting in (9) the values for w_{13} and w_{24} obtained from the two equations (12).

For the last assertion, we already proved in Corollary 4.5 that the choices of f introduced there satisfy equations (9). It is easy, and left to the reader, to show that they also satisfy (12). □

Theorem 5.12. (Main theorem, case without boundary collinearities) *Let $\mathcal{A}$ be a point set with no three collinear points in the boundary of* $\operatorname{conv}(\mathcal{A})$*, and let f be a weakly valid choice of parameters. Then, $\overline{Y_f}$ is a simple polyhedron of dimension $3n - 3$ with all the properties stated in Theorems 3.8 and 3.9.*

Proof. Recall that if no three boundary points are collinear then every fully-marked pseudo-triangulation (i.e., maximal marked non-crossing graph) has $3n - 3$ edges plus marks, exactly as in the general position case (Lemma 5.3). In particular, it is still true, for the same reasons as in the general position case, that $\overline{Y_f}(\mathcal{A})$ is simple and all its vertices correspond to f.m.p.t.'s, for any valid choice of f. The rest of the arguments in the proof of Theorem 3.8 rely on the graph of flips being connected, a property that we still have. As for Theorem 3.9, the face $Y_f(\mathcal{A})$ is bounded because Lemma 3.3 still applies. The rest is straightforward. □

Remark 5.13. It is interesting to observe that taking the explicit valid choice of f of Theorem 3.7 the equations defining $\overline{Y_f}(\mathcal{A})$ depend continuously on the coordinates of the points in $\mathcal{A}$. When three points become collinear, the hyperplane corresponding to the (now) forbidden edge becomes, as we said in the proof of Theorem 5.10 a supporting hyperplane of a codimension 3 face of $\overline{Y_f}(\mathcal{A})$. The combinatorics of the polytope changes but maintaining its simplicity. This continuity of the defining hyperplanes would clearly be impossible if we required our choice to be strictly valid for point sets with collinearities.

Example 5.14. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be the two point sets with five points each whose pseudo-triangulations are depicted in Figures 11 and 12. The first one has three collinear points and the second is obtained by perturbation of the collinearity. These two examples were computed with the software `CDD+` of Komei Fukuda [8] before we had a clear idea of what the right definition of pseudo-triangulation for points in special position should be. To emphasize the meaning of weak validity, in Figure 11 we are showing the weak supporting graphs of the vertices of $\overline{Y_f}(\mathcal{A}_1)$, rather than the strict ones.

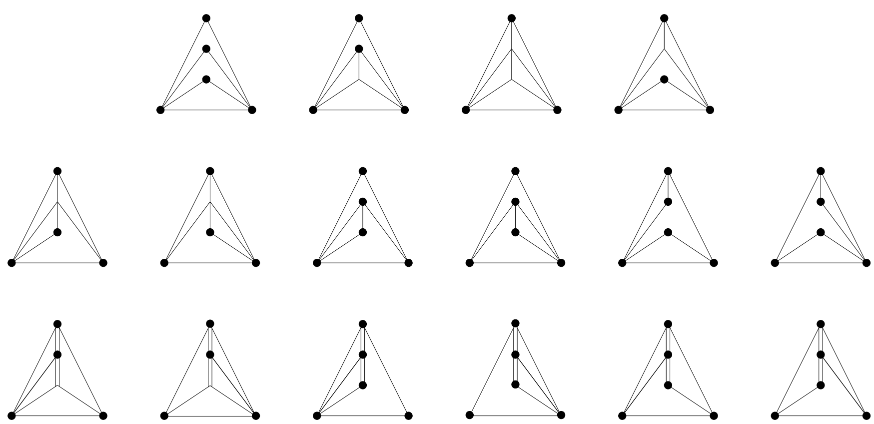


FIGURE 11. The 16 pseudo-triangulations of $\mathcal{A}_1$

The 10 first pseudo-triangulations are common to both figures (upper two rows). Only the pseudo-triangulations of $\mathcal{A}_1$ using the two collinear edges plus the mark at the central point of the collinearity are affected by the perturbation of the point set. This is no surprise, since these two edges plus this mark are the positive part of the stress involved in the collinearity. At each of these six pseudo-triangulations, the hyperplane of the big edge is tangent to the vertex of $\overline{Y_f}(\mathcal{A}_1)$ corresponding to the pseudo-triangulation. When the collinearity is perturbed, this hyperplane moves in one of the two possible ways: away from the polyhedron, in which case the combinatorics is not changed, or towards the interior of the polyhedron, in which case the old vertex disappears and some new vertices are cut by this hyperplane. In our case, these two behaviors appear each in three of the six "non-strict" pseudo-triangulations of $\mathcal{A}_1$. When the hyperplane moves towards the interior, four new vertices appear where there was one.

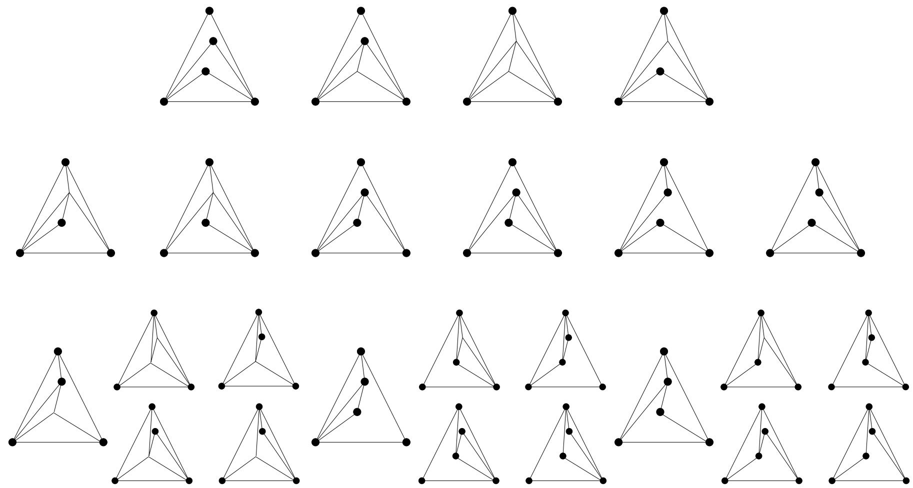


FIGURE 12. The 25 pseudo-triangulations of $\mathcal{A}_2$

Boundary collinearities. In the presence of boundary collinearities, Lemma 5.3 implies a significant difference: with the equations we have used so far, the face of $\overline{Y_0}(\mathcal{A})$ defined by tightness at boundary edges and vertices is not the origin, but an unbounded cone of dimension n_s. Indeed, for each semi-interior point p_i, the vector (v, t) with v_i an exterior normal to the boundary of $\mathrm{conv}(\mathcal{A})$ at p_i and every other coordinate equal to zero defines an extremal ray of that face. As a consequence, the corresponding face in $\overline{Y_f}(\mathcal{A})$ is unbounded.

We believe that it should be possible to obtain a polyhedron with the properties we want by just intersecting the polyhedron of our general definition with n_s hyperplanes. But instead of doing this we use the following simple trick to reduce this case to the previous one. From a point set $\mathcal{A}$ with boundary collinearities we construct another point set $\mathcal{A}'$ adding to $\mathcal{A}$ one point in the exterior of each edge of $\mathrm{conv}(\mathcal{A})$ that contains semi-interior points.

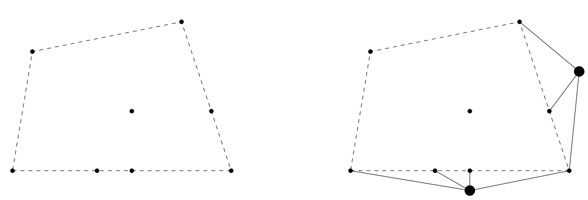


FIGURE 13. The extended point set $\mathcal{A}'$ and the relation between non-crossing marked graphs on $\mathcal{A}$ and $\mathcal{A}'$

Lemma 5.15. *A marked graph G on $\mathcal{A}$ is non-crossing if and only if it becomes a non-crossing graph on $\mathcal{A}'$ when we add to it the marks on all points of $\mathcal{A}' \setminus \mathcal{A}$ and the edges connecting each of these points to all the points of $\mathcal{A}$ lying in the corresponding edge of* $\mathrm{conv}(\mathcal{A})$.

Proof. Straightforward. □

In particular, we can construct the polyhedron $\overline{Y_f}(\mathcal{A}')$ for this extended point set $\mathcal{A}'$ (taking any f valid on $\mathcal{A}'$), and call $\overline{Y_f}(\mathcal{A})$ the face of $\overline{Y_f}(\mathcal{A}')$ corresponding to the edges and marks mentioned in the statement of Lemma 5.15. Then:

Corollary 5.16. (Main theorem, case with boundary collinearities) $\overline{Y_f}(\mathcal{A})$ *is a simple polyhedron of dimension* $3n-3-n_s$ *with all the properties stated in Theorem 3.8.*

Let $Y_f(\mathcal{A})$ *be the face of* $\overline{Y_f}(\mathcal{A})$ *corresponding to the* n_v+n_s *edges between consecutive boundary points and the* n_v *marks at vertices of* $\operatorname{conv}(\mathcal{A})$*. Let* F *be the face of* $Y_f(\mathcal{A})$ *corresponding to the remaining* $n-n_v$ *marks. Then,* $Y_f(\mathcal{A})$ *is a polytope of dimension* $2n_i+n-3$ *and* F *is a face of it of dimension* $2n_i+n_v-3$*. They satisfy all the properties stated in Theorem 3.9.*

Remark 5.17. The reader may wonder about the combinatorics of the polyhedron $\overline{Y_f}(\mathcal{A})$ that one would obtain with the equations of the generic case. Clearly, the tight graphs of its faces will not contain any of the four forbidden subgraphs of Lemma 5.8. It can be checked that the maximal marked graphs without those subgraphs all have $3n-3$ edges plus marks and have the following characterization: as graphs they are pseudo-triangulations in which all the semi-interior vertices are incident to interior edges, and they have marks at all the boundary points and at the pointed interior points. In other words, they would be the fully-marked pseudo-triangulations if we treated semi-interior points exactly as interior ones, hence forbidding them to be incident to two angles of 180 degrees and considering them always pointed since they are incident to one angle of 180 degrees.

For example, in the point set of Figure 9 there are 6 such graphs, namely the ones shown in Figure 14.

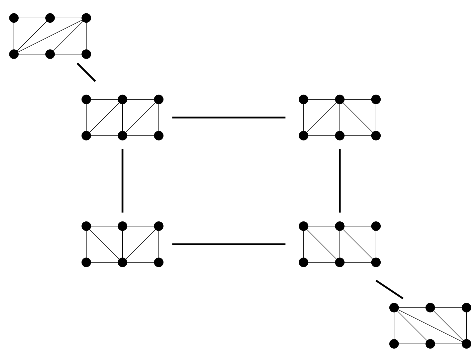


FIGURE 14. The bounded part of $\overline{Y_f}(\mathcal{A})$ for a point set with boundary collinearities

This implies that the polyhedron $\overline{Y_f}(\mathcal{A})$ is still simple. The reason why we prefer the definitions we have given is that the polyhedron no longer has a unique maximal bounded face (it has three in the example of Figure 14) and the graph of flips is no longer regular.

Observe finally that the proof of Theorem 1.3 given at the end of Section 3 is valid for points in special position, without much change: in all cases the hyperplanes corresponding to the edges of a pseudo-triangulation are independent in $\overline{Y_0}(\mathcal{A})$.

References

[1] P. Agarwal, J. Basch, L. Guibas, J. Hershberger, and L. Zhang. Deformable free space tilings for kinetic collision detection. In B. R. Donald, K. Lynch, and D. Rus, editors, *Algorithmic and Computational Robotics: New Directions,* 4*th Int. Workshop on Alg. Found. Robotics (WAFR)*, pages 83–96. A K Peters, 2000.

[2] O. Aicholzer, F. Aurenhammer and H. Krasser, Pseudo-triangulations from surfaces and a novel type of flip, manuscript 2002.

[3] M. Ajtai, V. Chvátal, M. M. Newborn, E. Szemerédi, crossing-free subgraphs. In "Theory and practice of combinatorics", (A. Rosa, G. Sabidussi and J. Turgeon, eds.), North-Holland Math. Stud., **60**, North-Holland, Amsterdam, 1982, pp. 9–12.

[4] L. J. Billera, P. Filliman and B. Sturmfels, Constructions and complexity of secondary polytopes, *Adv. Math.*, **83** (1990), 155-179.

[5] R. Connelly, E. Demaine, and G. Rote, Straightening polygonal arcs and convexifying polygonal cycles, *Discrete Comput. Geom.* to appear. Preliminary version in Proc. 41st Annual Symp. on Found. of Computer Science (FOCS), Redondo Beach, California, 2000, pp. 432-452.

[6] J. A. de Loera, S. Hoşten, F. Santos, B. Sturmfels, The polytope of all triangulations of a point configuration, *Doc. Math. J. DMV* **1** (1996), 103–119.

[7] P. Flajolet and M. Noy, Analytic combinatorics of non-crossing configurations, *Discrete Math.* **204** (1999) 203–229.

[8] K. Fukuda, software available at `http://www.cs.mcgill.ca/~fukuda/soft/cdd_home/cdd.html`

[9] M. Goodrich and R. Tamassia. Dynamic ray shooting and shortest paths in planar subdivisions via balanced geodesic triangulations. *J. Algorithms*, **23** (1997) 51–73.

[10] J. Graver, B. Servatius and H. Servatius. *Combinatorial Rigidity*, Amer. Math. Soc., Graduate Studies in Mathematics vol.2, 1993.

[11] R. Haas, D. Orden, G. Rote, F. Santos, B. Servatius, H. Servatius, D. Souvaine, I. Streinu and W. Whiteley, Planar Minimally Rigid Graphs and Pseudo-Triangulations, preprint 2002.

[12] D. Kirkpatrick, J. Snoeyink, and B. Speckmann. Kinetic collision detection for simple polygons. *International Journal of Computational Geometry and Applications*, 12:3–27, 2002.

[13] M. Pocchiola and G. Vegter. Pseudo-triangulations: Theory and applications. In *Proc. 12th Ann. ACM Sympos. Comput. Geom.*, pages 291–300. ACM Press, may 1996.

[14] M. Pocchiola and G. Vegter. Topologically sweeping visibility complexes via pseudo-triangulations. *Discrete and Computational Geometry*, 16:419–453, 1996.

[15] M. Pocchiola and G. Vegter. The visibility complex. *Int. J. Comput. Geom. Appl.*, 6:279–308, 1996.

[16] G. Rote, F. Santos, and I. Streinu. Expansive motions and the polytope of pointed pseudo-triangulations, preprint 2002, `http://arXiv.org/abs/math.CO/0206027`. To appear in *Discrete and Computational Geometry – The Goodman-Pollack Festschrift*, (B. Aronov, S. Basu, J. Pach, M. Sharir, eds), Algorithms and Combinatorics, Springer Verlag, Berlin.

[17] F. Santos and R. Seidel, A better upper bound on the number of triangulations of a planar point set, preprint `http://arXiv.org/abs/math.CO/0204045`, 6 pages. To appear in *J. Combin. Theory Ser. A* .

[18] B. Speckmann and C. D. Tóth. Allocating vertex π-guards in simple polygons via pseudo-triangulations. In *Proc. 14th ACM-SIAM Symposium on Discrete Algorithms*, 2003 (To appear).

[19] I. Streinu. A combinatorial approach to planar non-colliding robot arm motion planning. In *Proc. 41st Ann. Symp. on Found. of Computer Science (FOCS 2000), Redondo Beach, California*, pages 443–453, 2000.

Dpto. de Matemáticas, Estadística y Computación. Universidad de Cantabria. 39005 Santander, SPAIN

E-mail address: {`ordend,santos`}`@matesco.unican.es`