Symbolic Summation with Single-Nested Sum Extensions

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ABSTRACT

We present a streamlined and refined version of Karr's summation algorithm. Karr's original approach constructively decides the telescoping problem in $\Pi\Sigma$-fields, a very general class of difference fields that can describe rational terms of arbitrarily nested indefinite sums and products. More generally, our new algorithm can decide constructively if there exists a so called single-nested $\Pi\Sigma$-extension over a given $\Pi\Sigma$-field in which the telescoping problem for f can be solved in terms that are not more nested than f itself. This allows to eliminate an indefinite sum over f by expressing it in terms of additional sums that are not more nested than f. Moreover, our refined algorithm contributes to definite summation: it can decide constructively if the creative telescoping problem for a fixed order can be solved in single-nested Σ^*-extensions that are less nested than the definite sum itself.

Categories and Subject Descriptors

I.1.1 [**Expressions and Their Simplification**]: Simplification of Expressions; I.1.2 [**Symbolic and Algebraic Manipulation**]: Algebraic algorithms

General Terms

Algorithms

Keywords

Difference field extensions, telescoping, creative telescoping

1. INTRODUCTION

Let $(\mathbb{F},\sigma)$ be a difference field, i.e., a field[1] $\mathbb{F}$ together with a field automorphism $\sigma:\mathbb{F}\to\mathbb{F}$, and let $\mathbb{K}$ be its constant field, i.e., $\mathbb{K}=\text{const}_\sigma\mathbb{F}:=\{k\in\mathbb{F}\,|\,\sigma(k)=k\}$. Then Problem *PFLDE* plays an important role in symbolic summation.

Problem PFLDE: Solving **P**arameterized **F**irst Order **L**inear **D**ifference **E**quations.
Given $a_1,a_2\in\mathbb{F}^*$ and $(f_1,\dots,f_n)\in\mathbb{F}^n$. *Find all* $g\in\mathbb{F}$ and $(c_1,\dots,c_n)\in\mathbb{K}^n$ with $a_1\,\sigma(g)+a_2\,g=\sum_{i=1}^n c_i\,f_i$.

For instance, if one takes the field of rational functions $\mathbb{F}=\mathbb{K}(k)$ with the shift $\sigma(k)=k+1$ and specializes to $n=1$, $a_1=1$ and $a_2=-1$, one considers the telescoping problem for a rational function $f_1=f'(k)\in\mathbb{K}(k)$. Moreover, if $\mathbb{K}=\mathbb{K}'(m)$ and $f_i=f'(m+i-1,k)\in\mathbb{K}'(m)(k)$ for $1\le i\le n$, one formulates the creative telescoping problem [15] of order $n-1$ for definite rational sums.
More generally, $\Pi\Sigma$-fields, introduced in [6, 7], are difference fields $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ where $\mathbb{F}:=\mathbb{K}(t_1)\dots,(t_e)$ is a rational function field and the application of σ on the t_i's is recursively defined over $1\le i\le e$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ for $\alpha_i,\beta_i\in\mathbb{K}(t_1)\dots(t_{i-1})$; we omitted some technical conditions given in Section 2. Note that $\Pi\Sigma$-fields enable one to describe a huge class of sequences, like hypergeometric terms, as shown in [13], or most d'Alembertian solutions [1, 9], a subclass of Liouvillian solutions [5] of linear recurrences. More generally, $\Pi\Sigma$-fields allow to describe rational terms consisting of arbitrarily nested indefinite sums and products. We want to emphasize that the nested depth of these sums and products gives a measure of the complexity of expressions. This can be carried over to $\Pi\Sigma$-fields by introducing the *depth* of t_i as the number of recursive definition steps that are needed to describe the application of σ on t_i; for more details see Section 2. Moreover, the depth of $f\in\mathbb{F}$ is the maximum depth of the t_i's that occur in f, and the depth of $(\mathbb{F},\sigma)$ is the maximum depth of all the t_i.
The main result in [6] is an algorithm that solves Problem *PFLDE* and therefore the telescoping and creative telescoping problem for a given $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ where the constant field $\mathbb{K}$ is *σ-computable*. This means that **(1)** for any $k\in\mathbb{K}$ one can decide if $k\in\mathbb{Z}$, **(2)** polynomials in $\mathbb{K}[t_1,\dots,t_n]$ can be factored over $\mathbb{K}$, and **(3)** one knows how to compute a basis of $\{(n_1,\dots,n_k)\in\mathbb{Z}^k\,|\,c_1^{n_1}\dots c_k^{n_k}=1\}$ for $(c_1,\dots,c_k)\in\mathbb{K}^k$ which is a submodule of $\mathbb{Z}^k$ over $\mathbb{Z}$. For instance, any rational function field $\mathbb{K}=\mathbb{A}(x_1,\dots,x_r)$ over an algebraic number field $\mathbb{A}$ is σ-computable; see [13].
In this paper we will present a streamlined and simplified version of Karr's original algorithm [6] for Problem *PFLDE* using Bronstein's denominator bound [2] and results from [6, 12, 10, 11]. Afterwards we will extend this approach to an algorithm that can solve

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[1]Throughout this paper all fields will have characteristic 0.

Problem RS: **R**efined **S**ummation. *Given* a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with depth d, constant field $\mathbb{K}$ and $(f_1,\dots,f_n)\in\mathbb{F}^n$. *Decide* constructively if there are $(0,\dots,0)\neq(c_1,\dots,c_n)\in\mathbb{K}^n$ and $g\in\mathbb{F}(x_1)\dots(x_e)$ for $\sigma(g)-g=\sum_{i=1}^n c_i\,f_i$ in an extended $\Pi\Sigma$-field $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ with depth d and $\sigma(x_i)=\alpha_i\,x_i+\beta_i$ where $\alpha_i,\beta_i\in\mathbb{F}$.

Suppose we fail to find a solution g with $\sigma(g)-g=f$ in a given $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with depth d and $f\in\mathbb{F}^*$ with depth d, but there exists such an extended $\Pi\Sigma$-field $(\mathbb{F}(x_1)\dots(x_e),\sigma)$ and a solution g with depth d for $\sigma(g)-g=f$. Then our new algorithm can compute such an extension with such a solution g. As a side result we will show that it suffices to restrict to the sum case, i.e., $\sigma(x_i)-x_i\in\mathbb{F}$. In some sense our results shed new constructive light on Karr's Fundamental Theorem [6].

For instance, in Karr's approach [6] one can find the right hand side in (1) only by setting up manually the corresponding $\Pi\Sigma$-field in terms of the harmonic numbers $H_n:=\sum_{i=1}^n\frac{1}{i}$ and the generalized versions $H_n^{(r)}:=\sum_{i=1}^n\frac{1}{i^r}$, $r>1$, whereas with our new algorithm the underlying $\Pi\Sigma$-field is constructed completely automatically. Additional examples are

$$\sum_{k=1}^n\frac{1}{k}\sum_{j=1}^k\frac{1}{j}\sum_{i=1}^j\frac{1}{i}=\frac{1}{6}\big[H_n^3+3H_n\,H_n^{(2)}+2\,H_n^{(3)}\big],\qquad(1)$$

$$\sum_{k=1}^n\frac{1}{k}\sum_{i=1}^k\frac{1}{H_i}=-n+H_n\sum_{i=1}^n\frac{1}{H_i}+\sum_{i=1}^n\frac{1}{iH_i},$$

$$\sum_{k=0}^a\left(\sum_{i=0}^k\binom{n}{i}\right)^2=(n-a)\binom{n}{a}\sum_{i=0}^a\binom{n}{i}-\frac{n-2a-2}{2}\left(\sum_{i=0}^a\binom{n}{i}\right)^2-\frac{n}{2}\sum_{i=1}^n\binom{n}{i}^2.$$

Our new approach also refines creative telescoping: we might find a recurrence of smaller order by introducing additional sums with depths smaller than the definite sum.

All these algorithms have been implemented in form of the summation package *Sigma* in the computer algebra system Mathematica. The wide applicability of this new approach is illustrated for instance in [9, 8, 4].

2. REFINED SUMMATION IN $\Pi\Sigma$-FIELDS

First we introduce some notations and definitions. Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, $\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}^2$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{F}^n$. For any $\boldsymbol{h}=(h_1,\dots,h_n)\in\mathbb{F}^n$ and $p\in\mathbb{F}$ we write $\boldsymbol{f}\,\boldsymbol{h}:=\sum_{i=1}^n f_i\,h_i$, $\sigma(\boldsymbol{h}):=(\sigma(h_1),\dots,\sigma(h_n))$, and $\boldsymbol{h}\,p:=(h_1\,p,\dots,h_n\,p)$. We call $\boldsymbol{a}$ *homogeneous over* $\mathbb{F}$ if $a_1\,a_2\neq0$ and $a_1\,\sigma(g)+a_2\,g=0$ for some $g\in\mathbb{F}^*$.

Now let $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$ and suppose that $\boldsymbol{a}\neq\boldsymbol{0}$. Then we define the *solution space* $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{V})$ as the subspace $\big\{(c_1,\dots,c_n,g)\in\mathbb{K}^n\times\mathbb{V}\,|\,a_1\sigma(g)+a_2g=\sum_{i=1}^n c_if_i\big\}$ of the vector space $\mathbb{K}^n\times\mathbb{F}$ over $\mathbb{K}$. By difference field theory [3], the dimension is at most $n+1$; see also [9, 10]. Therefore Problem *PFLDE* is equivalent to find a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.

A difference field $(\mathbb{E},\sigma')$ is a *difference field extension* of $(\mathbb{F},\sigma)$ if $\mathbb{F}$ is a subfield of $\mathbb{E}$ and $\sigma'(g)=\sigma(g)$ for $g\in\mathbb{F}$; note that from now σ and σ' are not distinguished anymore.

A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a Π- *(resp. Σ^*-) extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t)=a\,t$ ($\sigma(t)=t+a$ resp.) for some $a\in\mathbb{F}^*$ and $\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$. A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a Σ-*extension* if $\mathbb{F}(t)$ is a rational function field, $\sigma(t)=\alpha\,t+\beta$ for some $\alpha,\beta\in\mathbb{F}^*$, $\text{const}_\sigma\mathbb{F}(t)=\text{const}_\sigma\mathbb{F}$, and the following two properties hold: (1) there does not exist a $g\in\mathbb{F}(t)\setminus\mathbb{F}$ with $\frac{\sigma(g)}{g}\in\mathbb{F}$, and (2) if there is a $g\in\mathbb{F}^*$ and $n\neq0$ with $\sigma(g)/g=\alpha^n$ then there is also a $g\in\mathbb{F}^*$ with $\sigma(g)/g=\alpha$. Note that any Σ^*-extension is also a Σ-extension; for more details see [6, 7, 2, 9, 13]. A $\Pi\Sigma$-*extension* is either a Π- or a Σ-extension.

A difference field extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is a *(nested) $\Sigma^*/\Pi\Sigma$-extension* if $(\mathbb{F}(t_1)\dots(t_i),\sigma)$ is a $\Sigma^*/\Pi\Sigma$-extension of $(\mathbb{F}(t_1)\dots(t_{i-1}),\sigma)$ for all $1\leq i\leq e$; for $i=0$ we define $\mathbb{F}(t_1)\dots(t_{i-1})=\mathbb{F}$. Note that $e=0$ gives the trivial extension.

For $\mathbb{H}\subseteq\mathbb{F}$, a $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is *single-nested over* $\mathbb{H}$, or in short *over* $\mathbb{H}$, if $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ with $\alpha_i,\beta_i\in\mathbb{H}$ for all $1\leq i\leq e$. A $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ is called single-nested, if it is single-nested over $\mathbb{F}$.

Finally, a $\Pi\Sigma$-*field* $(\mathbb{F},\sigma)$ *over* $\mathbb{K}$ is a $\Pi\Sigma$-extension of $(\mathbb{K},\sigma)$ with $\text{const}_\sigma\mathbb{K}=\mathbb{K}$, i.e., $\text{const}_\sigma\mathbb{F}=\mathbb{K}$.

In [6] alternative definitions of $\Pi\Sigma$-extensions are introduced that allow to decide constructively if an extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-extension under the assumption that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$. For instance, for Σ^*-extensions there is the following result given in [7, Theorem 2.3] or [9, Corollary 2.2.4].

THEOREM 1. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$. Then this is a Σ^*-extension iff $\sigma(t)=t+\beta$, $t\notin\mathbb{F}$, $\beta\in\mathbb{F}$, and there is no $g\in\mathbb{F}$ with $\sigma(g)-g=\beta$.*

In particular, this result states that indefinite summation and building up Σ^*-extensions are closely related. Namely, if one fails to find a $g\in\mathbb{F}$ with $\sigma(g)-g=\beta\in\mathbb{F}$, i.e., one cannot solve the telescoping problem in $\mathbb{F}$, one can adjoin the solution t with $\sigma(t)+t=\beta$ to $\mathbb{F}$ in form of the Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$.

Our refined simplification strategy for a given sum is as follows: If we fail to solve the telescoping problem, we do not adjoin immediately the sum in form of a Σ^*-extension, but we first try to find an appropriate $\Pi\Sigma$-extension in which the sum can be formulated less nested. These ideas can be clarified further with the depth-function. Let $\mathbb{F}=\mathbb{K}(t_1,\dots,t_e)$ be a function field over $\mathbb{K}$. Then for $g=\frac{g_1}{g_2}\in\mathbb{F}^*$ with $g_i\in\mathbb{K}[t_1,\dots,t_e]$ and $\gcd_{\mathbb{K}[t_1,\dots,t_e]}(g_1,g_2)=1$ we define the support of g, in short $\text{supp}_\mathbb{F}(g)$, as those t_i that occur in g_1 or g_2. Then for a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ over $\mathbb{K}$ with $\mathbb{F}:=\mathbb{K}(t_1)\dots(t_e)$ and $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ for $\alpha_i,\beta_i\in\mathbb{K}(t_1)\dots(t_{i-1})$, the depth-function $\text{depth}:\mathbb{F}\to\mathbb{N}_0$ is defined recursively as follows. For any $g\in\mathbb{K}$ set $\text{depth}(g)=0$. If the depth-function is defined for $(\mathbb{K}(t_1)\dots(t_{i-1}),\sigma)$ with $i>1$, we first define $\text{depth}(t_i)=\max(\text{depth}(\alpha_i),\text{depth}(\beta_i))+1$ and then define $\text{depth}(g)=\max(\{\text{depth}(x)\,|\,x\in\text{supp}_{\mathbb{K}(t_1,\dots,t_i)}(g)\}\cup\{0\})$ for any $g\in\mathbb{K}(t_1)\dots(t_i)$. The depth of $(\mathbb{F},\sigma)$, in short $\text{depth}(\mathbb{F})$, is the maximal depth of all elements in $\mathbb{F}$, i.e., $\text{depth}(\mathbb{F})$ is equal to $\max(0,\text{depth}(t_1),\dots,\text{depth}(t_e))$. We say that a $\Pi\Sigma/\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ has maximal depth d if $\text{depth}(t_i)\leq d$ for all $1\leq i\leq e$.

Now we can reformulate Problem *RS* as follows. *Given* a $\Pi\Sigma$-field $(\mathbb{F},\sigma)$ with depth d and $\boldsymbol{f}\in\mathbb{F}^n$. *Decide* constructively if there is a single-nested $\Pi\Sigma$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$

with maximal depth d, $g \in \mathbb{E}$ and $\mathbf{0} \neq \boldsymbol{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f}$.

Example 1. Denote the left hand side in (1) with $S_n^{(3)}$ and define $S_n^{(1)} := \sum_{i=1}^n \frac{1}{i}$ and $S_n^{(2)} := \sum_{j=1}^n S_j^{(1)}/j$. In the straightforward summation approach one applies usual telescoping which results in the $\Pi\Sigma$-field $(\mathbb{Q}(t_1)(t_2)(t_3)(t_4), \sigma)$ over $\mathbb{Q}$ with $\sigma(t_1) = t_1 + 1$, $\sigma(t_2) = t_2 + \frac{1}{t_1+1}$, $\sigma(t_3) = t_3 + \sigma(\frac{t_2}{t_1})$ and $\sigma(t_4) = t_4 + \sigma(\frac{t_3}{t_1})$, i.e., there is no $g \in \mathbb{Q}(t_1)$ with $\sigma(g) - g = \frac{1}{t_1+1}$ and no $g \in \mathbb{Q}(t_1)\dots(t_r)$ with $\sigma(g) - g = \sigma(\frac{t_r}{t_1})$ for $r = 2, 3$. Then t_r represents $S_n^{(r-1)}$ with $\mathrm{depth}(t_r) = r$ for $r = 2, 3, 4$, and $\mathrm{depth}(\mathbb{Q}(t_1)\dots(t_4)) = 4$. But with our refined summation approach we obtain the following improvement starting from the $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ with $\mathbb{F} := \mathbb{Q}(t_1)(t_2)$. We find the Σ^*-extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s) = s + \frac{1}{(t_1+1)^2}$ with the solution $g := \frac{t_2^2+s}{2}$ for $\sigma(g) - g = \sigma(\frac{t_2}{t_1})$ that represents the sum $S_n^{(2)}$. Moreover, we find the Σ^*-extension $(\mathbb{F}(s)(s'), \sigma)$ of $(\mathbb{F}(s), \sigma)$ with $\sigma(s') = s' + \frac{1}{(t_1+1)^3}$ and the solution $g' = \frac{1}{6}(t_2^3 + 3t_2\,s + 2s')$ for $\sigma(g') - g' = \sigma(\frac{g}{t_1})$. Then $S_n^{(3)}$ is represented by g' with $\mathrm{depth}(g') = 2$ which gives the right hand side of identity (1).

Besides refined indefinite summation, we obtain a generalized version of creative telescoping in $\Pi\Sigma$-fields. Suppose that the sequences $f'(m + i - 1, k)$ can be represented with $f_i \in \mathbb{F}$ for $i \geq 1$ in a $\Pi\Sigma$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}(m)$ with $\mathrm{depth}(f_i) = d$. Moreover assume that we do not find a $g \in \mathbb{F}$ and $\mathbf{0} \neq \boldsymbol{c} \in \mathbb{K}(m)^n$ with $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f}$ for $\boldsymbol{f} = (f_1, \dots, f_n)$. Then the usual strategy is to increase n, i.e., the order of the possibly resulting creative telescoping recurrence. But if we find a solution for Problem *RS*, we derive a recurrence of order $n - 1$ in terms of sum extensions with maximal depth d. Summarizing, for telescoping and creative telescoping we are interested in finding a single-nested $\Pi\Sigma$-extension in which a nontrivial linear combination of $(f_1, \dots, f_n)$ in the solution space exists. More generally, we will ask for those extensions that will give us additional linear combinations. To make this more precise, we define for any $\mathbb{A} \subseteq \mathbb{F}^{n+1}$ the set $\Pi_n(\mathbb{A}) := \{(a_1, \dots, a_n) \mid (a_1, \dots, a_n, a_{n+1}) \in \mathbb{A}\}$.

Definition 1. Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma$-field over $\mathbb{K}$ with depth d, $1 \leq \delta \leq d+1$, and $\boldsymbol{f} \in \mathbb{E}^n$. We call a $\Pi\Sigma$-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ *single-nested δ-complete* for $\boldsymbol{f}$ if for all single-nested $\Pi\Sigma$-extensions $(\mathbb{H}, \sigma)$ of $(\mathbb{E}, \sigma)$ with maximal depth δ we have

$$\Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{H})) \subseteq \Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{G})). \qquad (2)$$

In this paper we solve the following problem. *Given* a $\Pi\Sigma$-field $(\mathbb{E}, \sigma)$ over a σ-computable $\mathbb{K}$ with depth d, $\boldsymbol{f} \in \mathbb{E}^n$ and $\delta \in \mathbb{N}$ with $1 \leq \delta \leq d+1$; *compute* a single-nested Σ^*-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ with maximal depth δ which is single-nested δ-complete for $\boldsymbol{f}$, and compute a basis of $\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{G})$. Note that Problem *RS* for single-nested $\Pi\Sigma$-extension is contained in this problem by setting $\delta := d$.

3. A MORE GENERAL PROBLEM

In order to treat the problem stated in the previous paragraph, we solve the more general problem to find an $\mathbb{F}$-complete extension of $(\mathbb{E}, \sigma)$ for $\boldsymbol{f}$ defined in

Definition 2. Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $\boldsymbol{f} \in \mathbb{E}^n$. We call a $\Pi\Sigma$-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ *single-nested $\mathbb{F}$-complete for $\boldsymbol{f}$*, or in short *$\mathbb{F}$-complete for $\boldsymbol{f}$*, if (2) holds for all $\Pi\Sigma$-extensions $(\mathbb{H}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$.

The following lemma is crucial in order to show in Theorem 2 that there exists a Σ^*-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ which is $\mathbb{F}$-complete for $\boldsymbol{f}$. This means that it suffices to restrict to Σ^*-extensions. Moreover this lemma is needed to prove Theorem 6 which gives us the essential idea how one can compute such $\mathbb{F}$-complete extensions.

LEMMA 1. *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $f \in \mathbb{E}^*$. If there exists a single-nested $\Pi\Sigma$-extension $(\mathbb{G}, \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ with a $g \in \mathbb{G} \setminus \mathbb{E}$ such that $\sigma(g) - g = f$ then there exists a Σ^*-extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ with a $w \in \mathbb{E}$ such that $\sigma(s + w) - (s + w) = f$.*

PROOF. Let $(\mathbb{G}, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$, i.e., $\mathbb{G} = \mathbb{E}(t_1)\dots(t_e)$ with $\sigma(t_i) = \alpha_i\,t_i + \beta_i$ and $\alpha_i, \beta_i \in \mathbb{F}$, and suppose that there is a $g \in \mathbb{G} \setminus \mathbb{E}$ with $\sigma(g) - g = f$. Then by Karr's Fundamental Theorem [6, Theorem 24], see also [7, Section 4], it follows that $g = \sum_{i=0}^e c_i\,t_i + w$ for some $w \in \mathbb{E}$ and $c_i \in \mathbb{K}$, where $c_i = 0$ if $\sigma(t_i) - t_i \notin \mathbb{F}$. In particular, $\mathbf{0} \neq (c_1, \dots, c_e)$, since $g \notin \mathbb{E}$. Now let $\mathbb{E}(s)$ be a rational function field and suppose that the difference field extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(s) - s = \sum_{i=1}^e c_i\,(\sigma(t_i) - t_i) =: \beta \in \mathbb{F}$ is not a Σ^*-extension. Then by Theorem 1 we can take a $g' \in \mathbb{E}$ with $\sigma(g') - g' = \beta$. Let j be maximal such that $c_j \neq 0$. Then we have $\sigma(v) - v = \sigma(t_j) - t_j \in \mathbb{F}$ for $v := \frac{1}{c_j}\left(g' - \sum_{i=1}^{j-1} c_i\,t_i\right) \in \mathbb{E}(t_1)\dots(t_{j-1})$, and thus $(\mathbb{E}(t_1)\dots(t_{j-1})(t_j), \sigma)$ is not a Σ^*-extension of $(\mathbb{E}(t_1)\dots(t_{j-1}), \sigma)$ by Theorem 1, a contradiction. Hence $(\mathbb{E}(s), \sigma)$ is a Σ^*-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$, and $\sigma(s + w) - (s + w) = \sum_{i=1}^e c_i(\sigma(t_i) - t_i) + \sigma(w) - w = \sigma(g) - g = f$. $\square$

Observe that Lemma 1 follows immediately by Theorem 1 if one restricts to the special case $\mathbb{E} = \mathbb{F}$. For the case $\mathbb{F} \subsetneq \mathbb{E}$, in which we are actually interested, we needed Karr's Fundamental Theorem [6].

THEOREM 2. *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F}, \sigma)$ and $\boldsymbol{f} \in \mathbb{E}^n$. Then there is a Σ^*-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ which is $\mathbb{F}$-complete for $\boldsymbol{f}$.*

PROOF. Let $(\mathbb{G}, \sigma)$ be a Σ^*-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ which is not $\mathbb{F}$-complete for $\boldsymbol{f}$. Then we can take a $\boldsymbol{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f} \in \mathbb{E}$ has a solution in some $\Pi\Sigma$-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$, but no solution in $\mathbb{E}$. Then by Lemma 1 it follows that there is a Σ^*-extension $(\mathbb{E}(s), \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ with $\sigma(s + w) - (s + w) = f$ for some $w \in \mathbb{E}$. Observe that there also does not exist an $h \in \mathbb{G}$ with $\sigma(h) - h = \beta \in \mathbb{F}$. Otherwise we would have $\sigma(h + w) - (h + w) = \boldsymbol{c}\,\boldsymbol{f}$ with $h + w \in \mathbb{G}$, a contradiction. Consequently, by Theorem 1 also $(\mathbb{G}(s), \sigma)$ is a Σ^*-extension of $(\mathbb{G}, \sigma)$ with $\sigma(s) = s + \beta$ and therefore a Σ^*-extension of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$. Since $\Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{G}))$ is a proper subspace of $\Pi_n(\mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{G}(s)))$ and those spaces have dimension at most n, this argument can be repeated at most n times before an $\mathbb{F}$-complete Σ^*-extension is reached. $\square$

In the following we will represent the $\Pi\Sigma$-field $(\mathbb{E}, \sigma)$ in such a way that one can find a single-nested δ-complete extension of $(\mathbb{E}, \sigma)$ for $\boldsymbol{f}$ by finding an $\mathbb{F}$-complete extension over a certain subfield $\mathbb{F} \subseteq \mathbb{E}$.

The crucial idea is that the generators of a $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ can be reordered by increasing depth without changing the $\Pi\Sigma$ nature of the extension; this is trivial since no term can depend on terms of higher depth. For further details we refer to [14].

Hence one can reorder a $\Pi\Sigma$-field $(\mathbb{E},\sigma)$ over $\mathbb{K}$ with depth d and $1\leq\delta\leq d+1$ to a $\Pi\Sigma$-field $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ with $\text{depth}(\mathbb{F})=\delta-1$ and $\text{depth}(t_i)\geq\delta$ for all $1\leq i\leq e$. With this reordered $\Pi\Sigma$-field one obtains

LEMMA 2. *Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma$-field with $\delta:=\text{depth}(\mathbb{F})+1$ and $\text{depth}(t_i)\geq\delta$ for $1\leq i\leq e$, and let $(\mathbb{H},\sigma)$ be a single-nested $\Pi\Sigma$-extension of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$. Then this extension has maximal depth δ iff it is over $\mathbb{F}$.*

PROOF. Write $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_e)(s_1)\dots(s_u)$. First assume that the extension is over $\mathbb{F}$, i.e., $\sigma(s_i)=\alpha_i s_i+\beta_i$ with $\alpha_i,\beta_i\in\mathbb{F}$. Then, because of $\text{depth}(\mathbb{F})=\delta-1$ it follows that $\text{depth}(\beta_i)\leq\delta-1$ and $\text{depth}(\alpha_i)\leq\delta-1$, thus $\text{depth}(s_i)=\max(\text{depth}(\alpha_i),\text{depth}(\beta_i))+1\leq\delta$, and therefore the extension has maximal depth δ. Conversely, suppose that this extension has maximal depth δ, i.e. $\text{depth}(s_i)\leq\delta$. Then $\text{depth}(\alpha_i)\leq\delta-1$ and $\text{depth}(\beta_i)\leq\delta-1$, and consequently $\alpha_i,\beta_i\in\mathbb{F}$. $\square$

THEOREM 3. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-field where $\delta:=\text{depth}(\mathbb{F})+1$ and $\text{depth}(t_i)\geq\delta$ for $1\leq i\leq e$, and $\boldsymbol{f}\in\mathbb{E}^n$. Then a $\Pi\Sigma$-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ which is $\mathbb{F}$-complete for $\boldsymbol{f}$ has maximal depth δ and is single-nested δ-complete for $\boldsymbol{f}$.*

PROOF. Assume such an extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ is not single-nested δ-complete for $\boldsymbol{f}$. Then take a single-nested $\Pi\Sigma$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{E},\sigma)$ with maximal depth δ and $\boldsymbol{c}\in\Pi_n(\text{V}((1,-1),\boldsymbol{f},\mathbb{H}))\setminus\Pi_n(\text{V}((1,-1),\boldsymbol{f},\mathbb{G}))$. Since $\delta=\text{depth}(\mathbb{F})+1$ and $\text{depth}(t_i)\geq\delta$, $(\mathbb{H},\sigma)$ is an extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ by Lemma 2, and thus the extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ is not $\mathbb{F}$-complete for $\boldsymbol{f}$, a contradiction. Moreover, the extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ is single-nested with maximal depth δ by Lemma 2. $\square$

In Section 5 we will develop an algorithm that computes an $\mathbb{F}$-complete Σ^*-extension of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over $\mathbb{F}$ for $\boldsymbol{f}$. Then by Theorem 3 this extension will be also single-nested δ-complete for $\boldsymbol{f}$ with maximal depth δ.

4. A REDUCTION STRATEGY

We develop a streamlined version of Karr's summation algorithm [6] based on results of [2] and [9, 12, 10, 11] that solves Problem *PFLDE*. In particular, this approach will assist in finding $\mathbb{F}$-complete extensions over $\mathbb{F}$.

More precisely, let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$, $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, $\boldsymbol{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}(t)^2$ and $\boldsymbol{f}\in\mathbb{F}(t)^n$. We will introduce a simplified version of Karr's reduction strategy [6] that helps in finding a basis of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ over $\mathbb{K}$. If $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field, this reduction turns into a complete algorithm. Moreover, this reduction technique will deliver all the information to compute an $\mathbb{F}$-complete extension.

A special case. If $a_1\,a_2=0$, we have $\boldsymbol{g}=\boldsymbol{c}\,\sigma^{-1}(\frac{\boldsymbol{f}}{a_1})$ with $a_1\neq0$ or $\boldsymbol{g}=\boldsymbol{c}\frac{\boldsymbol{f}}{a_2}$ with $a_2\neq0$. Then it follows with $\boldsymbol{g}=(g_1,\dots,g_n)$ and the i-th unit vector $(0\dots,1,\dots,0)\in\mathbb{K}^n$ that $\{(0\dots,1,\dots,0,g_i)\}_{1\leq i\leq n}\subseteq\mathbb{K}^n\times\mathbb{F}(t)$ is a basis of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. Hence from now on we suppose $\boldsymbol{a}\in(\mathbb{F}(t)^*)^2$.

Clearing denominators/cancelling common factors. Compute $\boldsymbol{a'}=(a_1',a_2')\in(\mathbb{F}[t]^*)^2$, $\boldsymbol{f'}=(f_1',\dots,f_n')\in\mathbb{F}[t]^n$ such that $\gcd_{\mathbb{F}[t]}(f_1',\dots,f_n',a_1',a_2')=1$ and $\boldsymbol{a'}=\boldsymbol{a}\,q$, $\boldsymbol{f'}=\boldsymbol{f}\,q$ for some $q\in\mathbb{F}(t)^*$. Then we have $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))=\text{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(t))$. Therefore we may suppose that $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$ where the entries have no common factors.

In Karr's original approach [6] the solutions $g=p+q\in\mathbb{F}(t)$ in $(c_1,\dots,c_n,g)\in\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ are computed by deriving first the polynomial part $p\in\mathbb{F}[t]$ and afterwards the fractional part $q\in\mathbb{F}(t)$, i.e., the degree of the numerator is smaller than the degree of the denominator. We simplify this approach substantially by first computing a common denominator of all the possible solutions in $\mathbb{F}(t)$ and afterwards computing the numerator of the solutions over this common denominator.

Denominator bounding. In the first important reduction we bound the possible denominators of the solution space $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. More precisely, we look for a *denominator bound* d of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$, i.e., a polynomial $d\in\mathbb{F}[t]^*$ that satisfies

$$\forall(c_1,\dots,c_n,g)\in\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t)):\ d\,g\in\mathbb{F}[t].$$

Since $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ is a finite dimensional vector space over $\mathbb{K}$, a denominator bound must exist. Now suppose that we have given such a d and define $\boldsymbol{a'}:=(\frac{a_1}{\sigma(d)},\frac{a_2}{d})$. Then it follows that $\{(c_{i1},\dots,c_{in},g_i)\}_{1\leq i\leq r}$ is a basis of $\text{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}[t])$ if and only if $\{(c_{i1},\dots,c_{in},\frac{g_i}{d})\}_{1\leq i\leq r}$ is a basis of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$. For a proof we refer to [9, 12]. Hence, given a denominator bound d of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$, we can reduce the problem to search for a basis of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ to look for a basis of $\text{V}(\boldsymbol{a'},\boldsymbol{f},\mathbb{F}[t])$. By clearing denominators and cancelling common factors in $\boldsymbol{a}$ and $\boldsymbol{f}$, as above, we may also suppose that $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$.

Polynomial degree bounding. The next step consists of bounding the polynomial degrees in $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$. For convenience we introduce $\mathbb{F}[t]_b:=\{f\in\mathbb{F}[t]\mid\deg(f)\leq b\}$ for $b\in\mathbb{N}_0$ and $\mathbb{F}[t]_{-1}:=\{0\}$. Moreover, we define $\|b\|:=\deg b$ for $b\in\mathbb{F}[t]^*$, $\|0\|:=-1$, and $\|\boldsymbol{b}\|:=\max_{1\leq i\leq l}\|b_i\|$ for $\boldsymbol{b}=(b_1,\dots,b_l)\in\mathbb{F}[t]^l$. Then we look for a *polynomial degree bound* b of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$, i.e., a $b\in\mathbb{N}_0\cup\{-1\}$ such that

$$\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_b)=\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]),\ b\geq\max(-1,\|\boldsymbol{f}\|-\|\boldsymbol{a}\|).\quad(3)$$

Again, since $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$ is finite dimensional over $\mathbb{K}$, a degree bound must exist. Note that by the second condition in (3) it follows that $\boldsymbol{f}\in\mathbb{F}[t]_{\|\boldsymbol{a}\|+b}$ which is needed to proceed with the degree elimination technique below.
Due to [6, 7, 2] the problem to determine a denominator bound or degree bound is completely constructive if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$. The proofs and subalgorithms of these results can be found in [2, 10, 11].

THEOREM 4. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ and $\boldsymbol{f}\in\mathbb{F}[t]^n$. Then there are algorithms that compute a denominator bound of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t))$ or a degree bound of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t])$.*

Polynomial degree reduction. Finally we have to deal with the problem to compute a basis of $\text{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_\delta)$ for some $\delta\in\mathbb{N}_0\cup\{-1\}$ where $\boldsymbol{f}\in\mathbb{F}[t]^n_{\delta+l}$ with $l:=\|\boldsymbol{a}\|$. Here we follow exactly the idea in [6]. Namely, we first find the candidates of the leading coefficients $g_\delta\in\mathbb{F}$ for the solutions

$(c_1,\dots,c_n,g)\in \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_\delta)$ with $g=\sum_{i=0}^{\delta} g_i\,t^i$, plugging back its solution space and go on recursively to derive the candidates of the missing coefficients $g_i\in\mathbb{F}$. More precisely, define

$$\begin{aligned}\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}=(\tilde{a}_1,\tilde{a}_2)&:=\left(\operatorname{coeff}(a_1,l)\,\alpha^\delta,\operatorname{coeff}(a_2,l)\right),\\ \tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}&:=(\operatorname{coeff}(f_1,\delta+l),\dots,\operatorname{coeff}(f_n,\delta+l)).\end{aligned}$$

where $\mathbf{0}\neq\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}}\in\mathbb{F}^2$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}}\in\mathbb{F}^n$; $\operatorname{coeff}(p,l)$ gives the l-th coefficient of $p\in\mathbb{F}[t]$. Then the right linear combinations of a basis of $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}},\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}},\mathbb{F})$ enable one to construct partially the solutions $(c_1,\dots,c_n,g)\in\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_\delta)$, namely $(c_1,\dots,c_n)\in\mathbb{K}^n$ with the δ-th coefficient g_δ in $g\in\mathbb{F}[t]_\delta$. Given this basis of $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}},\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}},\mathbb{F})$, one plugs in the possible solutions for the leading coefficients, and ends up to find a basis of a solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f}_{\boldsymbol{\delta-1}},\mathbb{F}[t]_{\delta-1})$ which contains all the information for the remaining coefficients. Details how this vector $\boldsymbol{f}_{\boldsymbol{\delta-1}}$ with entries in $\mathbb{F}[t]_{\delta+l-1}$ can be computed are given in [12, 14]. Combining the bases of these two solution spaces, i.e., the possible leading coefficients and its remaining coefficients, one can reconstruct a basis of the original solution space $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_\delta)$. Applying this reduction technique recursively one has to deal with the problem to find bases of solution spaces $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_i},\mathbb{F}[t]_i)$ for $-1\leq i\leq\delta-1$ where the $\boldsymbol{f_i}$ have entries in $\mathbb{F}[t]_{i+l}$. This can be achieved by computing bases of the solution spaces $\mathrm{V}(\tilde{\boldsymbol{a}}_{\boldsymbol{i}},\tilde{\boldsymbol{f}}_{\boldsymbol{i}},\mathbb{F})$ for $0\leq i\leq\delta$ with $\tilde{\boldsymbol{a}}_{\boldsymbol{i}}\in\mathbb{F}^2$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{i}}\in\mathbb{F}^n$, and by dealing with the following

Base case I. In the incremental reduction we finally reach the problem to find a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_{-1})$ with $\mathbb{F}[t]_{-1}=\{0\}$. Then we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\{0\})=\operatorname{Nullspace}_{\mathbb{K}}(\boldsymbol{f})\times\{0\}$ where $\operatorname{Nullspace}_{\mathbb{K}}(\boldsymbol{f})=\{\boldsymbol{k}\in\mathbb{K}^n\mid\boldsymbol{f}\,\boldsymbol{k}=0\}$. Note that a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\{0\})$ can be computed by linear algebra if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$; for more details see [12].

Summarizing, let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $\boldsymbol{a}\in(\mathbb{F}[t]^*)^2$ with $l:=\|\boldsymbol{a}\|$ and $\boldsymbol{f}\in\mathbb{F}[t]^n_{\delta+l}$ for some $\delta\in\mathbb{N}_0\cup\{-1\}$. Then we can apply this reduction technique step by step and obtain an *incremental reduction* of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t]_\delta)$. We call $\{(\boldsymbol{a},\boldsymbol{f_\delta},\mathbb{F}[t]_\delta),\dots,(\boldsymbol{a},\boldsymbol{f_{-1}},\mathbb{F}[t]_{-1})\}$ the *incremental tuples* and $\{(\tilde{\boldsymbol{a}}_{\boldsymbol{\delta}},\tilde{\boldsymbol{f}}_{\boldsymbol{\delta}},\mathbb{F}),\dots,(\tilde{\boldsymbol{a}}_{\boldsymbol{0}},\tilde{\boldsymbol{f}}_{\boldsymbol{0}},\mathbb{F})\}$ the *coefficient tuples* of such an incremental reduction.

Example 2. Take the $\Pi\Sigma$-field $(\mathbb{Q}(t_1)(t_2),\sigma)$ over $\mathbb{Q}$ from Example 1, i.e., $\sigma(t_1)=t_1+1$ and $\sigma(t_2)=t_2+\frac{1}{t_1+1}$, and write $\mathbb{F}:=\mathbb{Q}(t_1)$. With our reduction strategy we will find a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_2))$ for $\boldsymbol{a}=(1,-1)\in\mathbb{F}(t_2)^2$ and $\boldsymbol{f}=(\sigma(t_2/t_1))=(\frac{1+(t_1+1)\,t_2}{(t_1+1)^2})\in\mathbb{F}(t_2)^1$. Clearing denominators gives $\boldsymbol{a}=((t_1+1)^2,-(t_1+1)^2)\in\mathbb{F}[t_2]^2$, $\boldsymbol{f}=(1+(t_1+1)\,t_2)\in\mathbb{F}[t_2]^1$. A den. bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_2))$ is 1, and a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t_2])$ is 2. Now we start the incremental reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}[t_2]_2)$. For the incremental tuple $(\boldsymbol{a},\boldsymbol{f_2},\mathbb{F}[t_2]_2)$ with $\boldsymbol{f_2}:=\boldsymbol{f}\in\mathbb{F}[t_2]^1_2$ we obtain the coefficient tuple $(\boldsymbol{a},(0),\mathbb{F})$. The basis $\{(1,0),(0,1)\}$ of $\mathrm{V}(\boldsymbol{a},(0),\mathbb{F})$ gives us the incremental tuple $(\boldsymbol{a},\boldsymbol{f_1},\mathbb{F}[t_2]_1)$ with $\boldsymbol{f_1}=(1+(t_1+1)\,t_2,-1-2(t_1+1)\,t_2)\in\mathbb{F}[t_2]^2_1$ and the coefficient tuple $(\boldsymbol{a},(t_1+1,-2(t_1+1)),\mathbb{F})$. Then taking the basis $\{(2,1,0),(0,0,1)\}$ of $\mathrm{V}(\boldsymbol{a},(1,-2),\mathbb{F})$, one obtains $\boldsymbol{f_0}=(1,-t_1-1)\in\mathbb{F}$, the incremental tuple $(\boldsymbol{a},\boldsymbol{f_0},\mathbb{F}[t_2]_0)$ and the coefficient tuple $(\boldsymbol{a},\boldsymbol{f_0},\mathbb{F})$. A basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_0},\mathbb{F})$ is $\{(0,0,1)\}$ which defines $\boldsymbol{f_{-1}}=(0)$. Finally, we end up in the base case $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_{-1}},\{0\})$ which immediately allows us to compute the basis $\{(1,0)\}$. Finally we can reconstruct the bases $\{(0,0,1)\}$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_i},\mathbb{F}[t_2]_i)$ for $i\in\{0,1\}$ and therefore the basis $\{(0,1)\}$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_2},\mathbb{F}[t_2]_2)$ and $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_2))$. For further details we refer to [12, 14].

A reduction to $\mathbb{F}$. Suppose that we have given not only a single but a nested $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ where we write $\mathbb{F}_i:=\mathbb{F}(t_1)\dots(t_i)$ for $0\leq i\leq e$, i.e., $\mathbb{F}_0=\mathbb{F}$. Let $\mathbf{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{F}_e$ and $\boldsymbol{f}\in\mathbb{F}_e^n$. Then we understand by a *reduction of* $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ *to* $\mathbb{F}$ a recursive application of the above reductions. More precisely, if $e=0$, we do nothing. Otherwise, suppose that $e>0$. If $a_1\,a_2=0$, we just apply the special case from above. Otherwise, within our reduction there is a denominator bound $d\in\mathbb{F}_{e-1}[t_e]^*$ which reduces the problem to find a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ to find one for $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}_{e-1}[t_e])$ for some $\boldsymbol{a'}\in(\mathbb{F}_{e-1}[t_e]^*)^2$ and $\boldsymbol{f'}\in\mathbb{F}_{e-1}[t_e]^n$; those are given by setting $\boldsymbol{a'}:=(a_1/\sigma(d),a_2/d)$, $\boldsymbol{f'}:=\boldsymbol{f}$, and clearing denominators and cancelling common factors. Next, with a degree bound b of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}_{e-1}[t_e])$ the incremental reduction of $(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}_{e-1}[t_e]_b)$ is applied. Within this reduction the coefficient tuples $(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}_{e-1})$ for $0\leq i\leq b$ give the subreductions of $(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}_{e-1})$ to $\mathbb{F}$ for $0\leq i\leq b$ that define recursively the whole reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ to $\mathbb{F}$.
We call T the *tuple set* of a reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ to $\mathbb{F}$ if besides $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)\in T$ the set T contains exactly all those coefficient tuples that occur in the recursively applied incremental reductions. Moreover, for $\boldsymbol{a_e}:=\boldsymbol{a}$ and $\boldsymbol{f_e}:=\boldsymbol{f}$ we call $\{(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}_i)\}_{r\leq i\leq e}\subseteq T$ *path-tuples* of $(\boldsymbol{a_r},\boldsymbol{f_r},\mathbb{F}_r)\in T$ if in the subreduction of $(\boldsymbol{a_{i+1}},\boldsymbol{f_{i+1}},\mathbb{F}_{i+1})$ to $\mathbb{F}$ the coefficient tuple $(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}_i)$ occurs for each $r\leq i<e$ in the incremental reduction. Finally, we introduce the $\mathbb{F}_r$*-critical tuple set* S in a reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ to $\mathbb{F}$ as that subset of the tuple set T of the reduction to $\mathbb{F}$ that contains all $(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}_r)\in T$ with the following property: for its path-tuples $\{(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}_i)\}_{r\leq i\leq e}$ we have that $\boldsymbol{a_i}$ is homogeneous for all $r\leq i\leq e$.

An algorithm. If the denominator bound problem and polynomial degree bound problem can be solved in the $\Pi\Sigma$-extensions $(\mathbb{F}_i,\sigma)$ of $(\mathbb{F}_{i-1},\sigma)$ for $1\leq i\leq e$ and one can compute a basis of any solution space in $(\mathbb{F},\sigma)$, the above reduction technique immediately turns into an algorithm to compute a basis of the solution space $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$. In particular our algorithm gives a reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_e)$ to $\mathbb{F}$ where we easily can collect the reduction tuple set of this reduction. Furthermore, if one stops collecting tuples in the subreductions of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}_i)$ to $\mathbb{F}$ when $\boldsymbol{a}$ is inhomogeneous, one can extract the $\mathbb{F}_r$-critical tuples in this reduction. Our algorithm can be found explicitly in [14].

Now assume that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, i.e., $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma$-field over $\mathbb{K}$. Then by Theorem 4 there are algorithms to solve the denominator and polynomial degree bound problem. Moreover, for the special case $\mathbb{F}=\mathbb{K}$ there is the following

Base case II. If $\mathrm{const}_\sigma\mathbb{K}=\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}=(a_1,a_2)\in\mathbb{K}^2$ and $\boldsymbol{f}=(f_1,\dots,f_n)\in\mathbb{K}^n$ then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{K})=\operatorname{Nullspace}_{\mathbb{K}}(\boldsymbol{f'})$ for $\boldsymbol{f'}=(f_1,\dots,f_n,-(a_1+a_2))$. A basis can be computed by linear algebra; see [10].

Hence, with our algorithm one can compute a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$ in a $\Pi\Sigma$-field $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over a σ-computable $\mathbb{K}$ and can extract the $\mathbb{F}$-critical tuples of the corresponding reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$ to $\mathbb{F}$.

Finally, we introduce reductions to $\mathbb{F}$ that are extension-stable. Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $\boldsymbol{a}\in(\mathbb{H}[t_e]^*)^2$ and $\boldsymbol{f}\in\mathbb{H}[t_e]^n$ for $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_{e-1})$. We call a denominator bound $d\in\mathbb{H}[t_e]^*$ of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_e))$ or a degree bound b of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}[t_e])$ *extension-stable over* $\mathbb{F}$ if $\boldsymbol{a}$ is inhomogeneous over $\mathbb{H}(t_e)$ or the following holds. Take any Σ^*-extension $(\mathbb{F}(t_1)\dots(t_e)(s),\sigma)$ of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over $\mathbb{F}$, and embed $\boldsymbol{a},\boldsymbol{f}$ in the reordered $\Pi\Sigma$-ext. $(\mathbb{F}(s)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$. Then also d embedded in $\mathbb{F}(s)(t_1)\dots(t_e)$ must be a denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s)(t_1)\dots(t_e))$. Similarly, b must be a degree bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s)(t_1)\dots(t_{e-1})[t_e])$. We call a reduction of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$ to $\mathbb{F}$ *extension-stable* if all denominator and degree bounds within the reduction to $\mathbb{F}$ are extension-stable over $\mathbb{F}$.
It has been shown in [10, Theorem 8.2] and [11, Theorem 7.3] that the algorithms proposed in [6] already compute extension-stable denominator and degree bounds in a $\Pi\Sigma$-field. Summarizing, we obtain

THEOREM 5. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{E}^2$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then there is an algorithm that computes a basis of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ with an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$. Moreover, during this computation, one can extract the $\mathbb{F}$-critical tuples.*

Example 3. In Example 2 the denominator and degree bounds are extension-stable. Consequently, this reduction of $((1,-1),(\sigma(t_2/t_1)),\mathbb{F}(t_2))$ to $\mathbb{F}$ is extension-stable. The $\mathbb{F}$-critical tuples are $(((t_1+1)^2,-(t_1+1)^2),\boldsymbol{f},\mathbb{F})$ for $\boldsymbol{f}\in\{(0),(t_1+1,-2(t_1+1)),(1,-(t_1+1))\}$.

5. REFINED SUMMATION ALGORITHMS

In the sequel let $(\mathbb{E},\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$ and $\boldsymbol{f}\in\mathbb{E}^n$. Then in Theorem 6 we will develop a constructive criterium which tells us if a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ is $\mathbb{F}$-complete for $\boldsymbol{f}$ and how such an extension can be constructed. For this task we first compute a basis of $\mathbb{V}:=\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ together with an extension-stable reduction of $((1,-1),\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$; see Theorem 5. If the dimension of $\mathbb{V}$ is $n+1$, the trivial extension $(\mathbb{E},\sigma)$ of $(\mathbb{E},\sigma)$ is clearly $\mathbb{F}$-complete for $\boldsymbol{f}$. Otherwise, we extract the $\mathbb{F}$-critical tuple set in our extension-stable reduction; see Theorem 5. Then the crucial observation is stated in Proposition 1 that depends on Lemma 3. This lemma is a special case of Karr's Fundamental Theorem [6, 7]; for a proof see [9, Proposition 4.1.2].

LEMMA 3. *If $(\mathbb{E},\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$, $\mathbf{0}\neq\boldsymbol{a}\in\mathbb{F}^2$ inhomogeneous over $\mathbb{F}$ and $\boldsymbol{f}\in\mathbb{F}^n$ then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$.*

PROPOSITION 1. *Let $(\mathbb{E}(s),\sigma)$ with $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(s)-s\in\mathbb{F}$ and consider the reordered $\Pi\Sigma$-extension $(\mathbb{F}(s)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$. Let $\boldsymbol{a}\in\mathbb{E}^2$ be homogeneous over $\mathbb{E}$, $\boldsymbol{f}\in\mathbb{E}^n$, and let S be an $\mathbb{F}$-critical tuple set of an extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$. If for all $(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})\in S$ we have $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(s))$ then $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(s))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s)(t_1)\dots(t_e))$.*

PROOF. The proof will be done by induction on the number e of extensions $\mathbb{F}(t_1)\dots(t_e)$. First consider the case $e=0$. Since $\boldsymbol{a}$ is homogeneous, $(\boldsymbol{a},\boldsymbol{f},\mathbb{F})\in S$ and therefore $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$. Now assume that the proposition holds for $e\geq 0$. Let $(\mathbb{F}(t_1)\dots(t_e)(t_{e+1})(s),\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ with $\sigma(s)-s\in\mathbb{F}$ and consider the reordered $\Pi\Sigma$-extension $(\mathbb{F}(s)(t_1)\dots(t_e)(t_{e+1}),\sigma)$ of $(\mathbb{F},\sigma)$. As shortcut write $\mathbb{E}:=\mathbb{F}(t_1)\dots(t_e)$, $\mathbb{H}:=\mathbb{F}(s)(t_1)\dots(t_e)$. Let $\boldsymbol{f}\in\mathbb{E}(t_{e+1})^n$, assume that $\boldsymbol{a}\in\mathbb{E}(t_{e+1})^2$ is homogeneous over $\mathbb{E}(t_{e+1})$, and take any extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))$ to $\mathbb{F}$ with the $\mathbb{F}$-critical tuple set S. Now suppose that $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(s))$ for all $(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})\in S$. Then we will show that

$$\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_{e+1})). \tag{4}$$

In the extension-stable reduction let $d\in\mathbb{E}[t_{e+1}]^*$ be the denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))$. Since $\boldsymbol{a}$ is homogeneous over $\mathbb{E}(t_{e+1})$, $d\in\mathbb{H}[t_{e+1}]$ is also a denominator bound of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{H}(t_{e+1}))$. After clearing denominators and cancelling common factors, we get $\boldsymbol{a'}:=(a_1/\sigma(d),a_2/d)\,q\in\mathbb{E}[t_{e+1}]^2$ and $\boldsymbol{f'}:=\boldsymbol{f}\,q\in\mathbb{E}[t_{e+1}]^n$ for some $q\in\mathbb{E}(t_{e+1})^*$ in our reduction. Note that $\boldsymbol{a'}$ is still homogeneous over $\mathbb{E}(t_{e+1})$. This follows from the fact that if for $h\in\mathbb{E}(t_{e+1})$ we have $a_1\,\sigma(h)+a_2\,h=0$ then $a'_1\,\sigma(h\,d)+a'_2\,h\,d=0$. Now it suffices to show that $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t_{e+1}])=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{E}[t_{e+1}])$, in order to show (4). In the given reduction let b be the degree bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{E}[t_{e+1}])$. Since $\boldsymbol{a'}$ is homogeneous over $\mathbb{E}(t_{e+1})$, b is a degree bound of $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t_{e+1}])$ too. Hence, if $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{E}[t_{e+1}]_b)=\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{H}[t_{e+1}]_b)$, also (4) is proven. Let $((\boldsymbol{a},\boldsymbol{f_b},\mathbb{E}[t_{e+1}]_b),\dots,(\boldsymbol{a},\boldsymbol{f_{-1}},\mathbb{E}[t_{e+1}]_{-1}))$ be the incremental tuples and $((\boldsymbol{\tilde{a}_b},\boldsymbol{\tilde{f}_b},\mathbb{E}),\dots,(\boldsymbol{\tilde{a}_0},\boldsymbol{\tilde{f}_0},\mathbb{E}))$ be the coefficient-tuples in the incr. reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E}[t_{e+1}]_b)$. We show that $\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E})=\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{H})$ for all $0\leq i\leq b$. By reordering of $(\mathbb{F}(t_1)\dots(t_{e+1})(s),\sigma)$ we get the $\Pi\Sigma$-extension $(\mathbb{F}(t_1)\dots(t_e)(s)(t_{e+1}),\sigma)$ of $(\mathbb{F},\sigma)$. First suppose that $\boldsymbol{\tilde{a}_i}$ is inhomogeneous over $\mathbb{E}$. Hence, $\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E})=\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E}(s))$ by Lemma 3, and therefore $\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E})=\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{H})$ by $(\mathbb{F}(t_1)\dots(t_e)(s),\sigma)\simeq(\mathbb{F}(s)(t_1)\dots(t_e),\sigma)$.
Otherwise, assume that $\boldsymbol{\tilde{a}_i}$ is homogeneous over $\mathbb{E}$. Then the extension-stable reduction of $(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))$ to $\mathbb{F}$ contains an extension-stable reduction of $(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E})$ to $\mathbb{F}$ and the $\mathbb{F}$-critical tuple set of the reduction of $(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E})$ is a subset of S. Hence with the induction assumption it follows that $\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{E})=\mathrm{V}(\boldsymbol{\tilde{a}_i},\boldsymbol{\tilde{f}_i},\mathbb{H})$. Since $\mathbb{E}[t_{e+1}]_{-1}=\mathbb{H}[t_{e+1}]_{-1}=\{0\}$, we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_{-1}},\mathbb{E}[t_{e+1}]_{-1})=\mathrm{V}(\boldsymbol{a},\boldsymbol{f_{-1}},\mathbb{H}[t_{e+1}]_{-1})$. Consequently by the construction of the incremental reduction, see [14], we can conclude that $\mathrm{V}(\boldsymbol{a},\boldsymbol{f_i},\mathbb{E}[t_{e+1}]_i)=\mathrm{V}(\boldsymbol{a},\boldsymbol{f_i},\mathbb{H}[t_{e+1}]_i)$ for all $-1\leq i\leq b$ and therefore we have proven (4). Hence $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1}))=\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E}(t_{e+1})(s))$ by reordering of $\mathbb{F}(s)(t_1)\dots(t_{e+1})$ to $\mathbb{F}(t_1)\dots(t_{e+1})(s)$. $\square$

Hence we have $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})\subsetneq\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}(s))$ for a Σ^*-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ if $\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})\subsetneq\mathrm{V}(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F}(s))$ in one of its $\mathbb{F}$-critical tuples $(\boldsymbol{a'},\boldsymbol{f'},\mathbb{F})$ in an extension-stable reduction to $\mathbb{F}$.

Example 4. Consider the $\Pi\Sigma$-fields from Example 1, 2 and 3. By Example 1 it follows that $\mathrm{V}((1,-1),(\frac{\sigma(t_2)}{t_1}),\mathbb{F}(t_2))$ is a proper subset of $\mathrm{V}((1,-1),(\frac{\sigma(t_2)}{t_1}),\mathbb{F}(t_2)(s))$. Hence looking at the $\mathbb{F}$-critical tuples of our extension stable reduction in Example 3, we know by Proposition 1 that there is an $\boldsymbol{f}\in\{(0),(t_1+1,-2(t_1+1)),(1,-(t_1+1))\}$ such that $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ with $\boldsymbol{a}=((t_1+1)^2,-(t_1+1)^2)$ is a proper subset of $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}(s))$. Indeed, we can choose $\boldsymbol{f}=(1,-(t_1+1))$ since there does not exist a $g\in\mathbb{F}$ with $\sigma(g)-g=\frac{1}{(t_1+1)^2}$, but there is the solution $g=s\in\mathbb{F}(s)$.

Next we provide a sufficient condition in Proposition 2 which tells us if a Σ^*-extension cannot contribute further to a given solution space.

PROPOSITION 2. *Let $(\mathbb{F},\sigma)$ be a difference field with $\boldsymbol{a} = (a_1, a_2) \in \mathbb{F}^2$ homogeneous over $\mathbb{F}$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$. If for all $1 \leq i \leq n$ there is a $g \in \mathbb{F}^*$ with $a_1\,\sigma(g) + a_2\,g = f_i$ then for any difference field (ring) extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathrm{const}_\sigma\mathbb{E} = \mathrm{const}_\sigma\mathbb{F}$ we have $\mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F}) = \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$.*

PROOF. Let $g_i \in \mathbb{F}$ with $a_1\,\sigma(g_0)+a_2\,g_0 = 0$ and $a_1\,\sigma(g_i)+a_2\,g_i = f_i$ for $1 \leq i \leq n$. Then observe that $(0,\dots,0,g_0)$, $(1,0,\dots,0,g_1),\dots,(0,\dots,0,1,g_n) \in \mathbb{K}^n \times \mathbb{F}$ forms a basis of $\mathbb{V} := \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{F})$ over $\mathbb{K} := \mathrm{const}_\sigma\mathbb{F}$. Since $\mathbb{V}$ is a subspace of $\mathbb{W} := \mathrm{V}(\boldsymbol{a},\boldsymbol{f},\mathbb{E})$ over $\mathbb{K}$ and the dimension of $\mathbb{W}$ is at most $n+1$, it follows that $\mathbb{V} = \mathbb{W}$. $\square$

This result allows us to specify a criterium in Theorem 6 if a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ is $\mathbb{F}$-complete for $\boldsymbol{f}$.

THEOREM 6. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$ and $\boldsymbol{f} \in \mathbb{E}^n$. Let $\{(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F})\}_{1\leq i\leq k}$ with $\boldsymbol{a_i} = (a_{i1}, a_{i2})$ and $\boldsymbol{f_i} = (f_{i1},\dots,f_{ir_i}) \in \mathbb{F}^{r_i}$ be the $\mathbb{F}$-critical tuple set of an extension-stable reduction of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ to $\mathbb{F}$. If $(\mathbb{G},\sigma)$ is a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ where for any $1 \leq i \leq k$ and $1 \leq j \leq r_i$ there is a $g \in \mathbb{G}^*$ with $a_{i1}\,\sigma(g) - a_{i2}\,g = f_{ij}$ then the extension is $\mathbb{F}$-complete for $\boldsymbol{f}$.*

PROOF. Suppose such an extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ is not $\mathbb{F}$-complete for $\boldsymbol{f}$. Then we can take a $\boldsymbol{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f}$ has a solution in some $\Pi\Sigma$-extension of $(\mathbb{E},\sigma)$, but no solution in $(\mathbb{G},\sigma)$ and therefore no solution in $(\mathbb{E},\sigma)$. Hence, by Lemma 1 there is a Σ^*-extension $(\mathbb{E}(s),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ and a $g \in \mathbb{E}(s)$ with $\sigma(g) - g = \boldsymbol{c}\,\boldsymbol{f}$. Consequently, by Proposition 1 there exists an i with $1 \leq i \leq k$ such that $\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}) \subsetneq \mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}(s))$ holds for the Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$. But by Proposition 2 we have $\mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}) = \mathrm{V}(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F}(s))$, a contradiction. $\square$

Example 5. Consider Examples 2 and 3. Since for any $f \in \{0, t_1+1, -2(t_1+1), 1, -(t_1+1)\}$ there is a $g \in \mathbb{F}(t_2)(s)$ with $\sigma(g) - g = f$, it follows that the Σ^*-extension $(\mathbb{F}(t_2)(s),\sigma)$ of $(\mathbb{F}(t_2),\sigma)$ is $\mathbb{F}$-complete for $(\sigma(t_2)/t_1)$.

Finally, in Proposition 3 we show that such an extension can be constructed that fulfills our sufficient criterium.

PROPOSITION 3. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma$-extension of $(\mathbb{F},\sigma)$, $(a_{i1}, a_{i2}) \in \mathbb{F}^2$ be homogeneous over $\mathbb{F}$ and $f_i \in \mathbb{F}$ for $1 \leq i \leq n$. Then there is a Σ^*-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ such that there is a $g \in \mathbb{G}^*$ with $a_{i1}\,\sigma(g) + a_{i2}\,g = f_i$ for all $1 \leq i \leq n$. If $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, such a $\Pi\Sigma$-field $(\mathbb{G},\sigma)$ can be computed.*

PROOF. Suppose that we have shown the existence for such a Σ^*-extension $(\mathbb{G},\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ for $1 \leq i \leq n$. Now let $(a_1, a_2) \in \mathbb{F}^2$ be homogeneous over $\mathbb{F}$ and $f \in \mathbb{F}$. If there is a $g \in \mathbb{G}$ with $a_1\,\sigma(g) + a_2\,g = f$, we have shown the induction step. Otherwise, construct the extension $(\mathbb{G}(s),\sigma)$ of $(\mathbb{G},\sigma)$ with s transcendental over $\mathbb{F}$ and $\sigma(s) = s - \frac{f}{h\,a_2} \in \mathbb{F}$ where $h \in \mathbb{F}^*$ with $a_1\,\sigma(h) + a_2\,h = 0$. Now suppose there is a $g' \in \mathbb{G}^*$ with $\sigma(g') - g' = -\frac{f}{h\,a_2}$. Then for $w := h\,g' \in \mathbb{G}^*$ we have $f = -a_2\,h(\sigma(g') - g') = a_1\,\sigma(h)\,\sigma(g') + a_2\,h\,g' = a_1\,\sigma(w) + a_2\,w$, a contradiction. Hence by Theorem 1 $(\mathbb{G}(s),\sigma)$ is a Σ^*-extension of $(\mathbb{G},\sigma)$ over $\mathbb{F}$. Furthermore, for $v := h\,s \in \mathbb{G}(s)$ we have that $a_1\,\sigma(v) + a_2\,v = f$, which follows by similar arguments as above for w. This closes the induction step.
Now suppose that $(\mathbb{F},\sigma)$ is a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$. Then by Theorem 5 one can decide if there exists a $g \in \mathbb{G}^*$ with $a_1\,\sigma(g) + a_2\,g = f$ and can compute an $h \in \mathbb{F}^*$ with $a_1\,\sigma(h) + a_2\,h = 0$. This shows, that the proof above becomes completely constructive. $\square$

Summarizing, we first compute a basis of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E})$ with an extension-stable reduction and extract the $\mathbb{F}$-critical tuples; this is possible by Theorem 5. Next we construct with Proposition 3 a Σ^*-extension of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ that fulfills the criterium in Theorem 6.

Example 6. Looking at Example 3 we obtain immediately the Σ^*-extension $(\mathbb{F}(t_2)(s),\sigma)$ of $(\mathbb{F}(t_2),\sigma)$ with $\sigma(s) = s + \frac{1}{(t_1+1)^2}$ which is $\mathbb{F}$-complete for $(\sigma(t_2/t_1)) \in \mathbb{F}(t_2)^1$ by following this strategy. Finally we restart our computation in this extension and obtain for $\mathrm{V}((1,-1),(\sigma(t_2/t_1)),\mathbb{F}(t_2)(s))$ the basis $\{(0,1),(2,t_2+s)\}$ which gives the result $g = \frac{t_2+s}{2}$ in Example 1.
Now we proceed, and try to find a $g' \in \mathbb{F}(t_2)(s)$ such that $\sigma(g') - g' = \sigma(g/t_1)$, but we fail. Therefore, we extract the $\mathbb{F}$-critical tuples $(((t_1+1)^3, -(t_1+1)^3), \boldsymbol{f}, \mathbb{F})$ with

$$\begin{aligned}\boldsymbol{f} \in \{&(-(t_1+1)^2, \frac{t_1+1}{2}, -2(t_1+1)), (2(t_1+1)^2, 0, 0),\\ &(0,0), (-3(t_1+1)^2, (t_1+1)^2, 0), ((t_1+1), 2, (t_1+1)^2)\}\end{aligned} \tag{5}$$

from our extension stable reduction to $\mathbb{F}$. Following Theorem 6 we construct a Σ^*-extension $(\mathbb{G},\sigma)$ of $(\mathbb{F}(t_2)(s),\sigma)$ over $\mathbb{F}$ such that there are $h \in \mathbb{G}$ with $\sigma(h) - h = \frac{f}{(t_1+1)^2}$ for all $f \in \mathbb{F}$ from (5). Following the algorithm given in the proof of Proposition 3 we obtain the Σ^*-extension $(\mathbb{F}(t_2)(s)(s'),\sigma)$ of $(\mathbb{F}(t_2)(s),\sigma)$ with $\sigma(s') = s' + \frac{2}{(t_1+1)^3}$; afterwards we cancel the constant factor 2. By Theorem 6 this extension is $\mathbb{F}$-complete for $(\sigma(g/t_1)) \in \mathbb{F}(t_2)(s)^1$. To this end we compute for the solution space $\mathrm{V}((1,-1),(\sigma(g/t_1)),\mathbb{F}(t_2)(s)(s'))$ the basis $\{(0,1),(6,(t_2^3+3t_2\,s+2s'))\}$ which gives the final result in Example 1.

Let $I \subseteq \{0,\dots,e\}$. Restricting Algorithm 1 to $I = \{0\}$ gives just the above strategy.
In addition, $\mathbb{F}_i := \mathbb{F}(t_1)\dots(t_i)$-complete extensions can be searched for all $i \in I$. This can be motivated as follows. $\mathbb{F}_i$-complete extensions $(\mathbb{E}_i,\sigma)$ of $(\mathbb{E},\sigma)$ with bigger i can give more solutions $\mathbb{W}_i := \Pi_n(\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_i))$; but they might be also more complicated, since they depend on more t_j (which are usually more nested). Hence, one should look for extensions with smallest possible i that give still interesting solutions in $\mathbb{W}_i$. Algorithm 1 enables one to search in one stroke for all those $\mathbb{F}_i$-complete extensions with $i \in I$.

Algorithm 1. `SingleNestedCompleteExt`$((\mathbb{E}_0,\sigma),\boldsymbol{f})$
Input: A $\Pi\Sigma$-field $(\mathbb{E}_0,\sigma)$ with $\mathbb{E}_0 := \mathbb{F}(t_1)\dots(t_e)$ over a σ-computable $\mathbb{K}$, $I = \{j_1 < \dots < j_\lambda\} \subseteq \{0,\dots,e\}$ and $\boldsymbol{f} \in \mathbb{E}_0^n$.
Output: Σ^*-extensions $(\mathbb{E}_i,\sigma)$ of $(\mathbb{E}_{i-1},\sigma)$ over $\mathbb{F}(t_1)\dots(t_{j_i})$ which are single-nested $\mathbb{F}(t_1)\dots(t_{j_i})$-complete for $\boldsymbol{f}$ for $1 \leq i \leq \lambda$; a basis of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_\lambda)$.

(1) Compute a basis B of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_0)$ by an extension-stable reduction to $\mathbb{F}$. Let $d:=\dim\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_0)$.
(2) IF $d=n+1$ RETURN $((\mathbb{E}_0,\sigma),B)$ FI
(3) FOR $i=1$ TO λ DO
Extract the $\mathbb{F}(t_1)\dots(t_{j_\lambda})$-critical tuples from our reduction, say $\{(\boldsymbol{a_i},\boldsymbol{f_i},\mathbb{F})\}_{1\leq i\leq k}$, where $\boldsymbol{a_i}=(a_{i1},a_{i2})$ and $\boldsymbol{f_i}=(f_{i1},\dots,f_{ir_i})\in\mathbb{F}^{r_i}$ with $r_i>0$. Construct a single-nested Σ^*-ext. $(\mathbb{E}_i,\sigma)$ of $(\mathbb{E}_{i-1},\sigma)$ over $\mathbb{F}(t_1)\dots(t_{j_\lambda})$ s.t. for any $1\leq i\leq k$ and $1\leq j\leq r_i$ there is a $g\in\mathbb{E}_i^*$ with $a_{i1}\,\sigma(g)-a_{i2}\,g=f_{ij}$.
OD
(4) IF $(\mathbb{E}_\lambda,\sigma)=(\mathbb{E}_0,\sigma)$ RETURN $((\mathbb{E}_0,\sigma),B)$ FI
(5) Compute a basis B' of $\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_\lambda)$ with dim. d'.
(6) IF $d=d'$ THEN
RETURN $((\mathbb{E}_0,\sigma),B)$ ELSE RETURN$((\mathbb{E}_\lambda,\sigma),B')$.
FI

THEOREM 7. *Let $(\mathbb{E}_0,\sigma)$ with $\mathbb{E}_0:=\mathbb{F}(t_1)\dots(t_e)$ be a $\Pi\Sigma$-field over a σ-computable $\mathbb{K}$, $I=\{j_1<\dots<j_\lambda\}\subseteq\{0,\dots,e\}$ and $\boldsymbol{f}\in\mathbb{E}_0^n$. Then with Algorithm 1 Σ^*-extensions $(\mathbb{E}_i,\sigma)$ of $(\mathbb{E}_{i-1},\sigma)$ over $\mathbb{F}(t_1)\dots(t_{j_i})$ can be computed which are $\mathbb{F}(t_1)\dots(t_{j_i})$-complete for $\boldsymbol{f}$ for $1\leq i\leq\lambda$.*

The Σ^*-extension $(\mathbb{E}_\lambda,\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ produced by Algorithm 1 can be reduced to a more compact extension that delivers the same solutions $\Pi_n(\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_\lambda))$. Namely, if $\mathbb{E}_\lambda:=\mathbb{E}(s_1)\dots(s_\epsilon)$, remove those s_i that do not occur in $\mathbb{W}_\lambda=\mathrm{V}((1,-1),\boldsymbol{f},\mathbb{E}_\lambda)$. Moreover, join all those s_i's to one single Σ^*-extension which occur in a basis element of $\mathbb{W}_i$; see Lemma 1. Furthermore, cancel constants from $\mathbb{K}$ that may occur in the summand $\sigma(s_i)-s_i$; see Example 6.

Observe that recursively applied indefinite summation can be treated more efficiently, if one reduces these extensions after each application of Algorithm 1.

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