PLANAR 3-COLORABILITY IS POLYNOMIAL COMPLETE†

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March 1973

The general problem of recognizing the set of pairs (G,k), where k is a positive integer and G is a graph which is k-colorable, is polynomial complete as defined by Karp [1]. It is shown here that this problem is still complete even for pairs (G,k) where k = 3 and G is a planar graph. We assume that the reader is familiar with the definitions and notation of [1].

The problems to be considered are the following.

3-COLORABILITY

INPUT: Graph G with nodes N and arcs A.

PROPERTY: There is a function f: $N \rightarrow \{1,2,3\}$ such that if u, v are adjacent then $f(u) \neq f(v)$.

PLANAR 3-COLORABILITY

INPUT: Planar graph G.

PROPERTY: Same as above.

†Work reported here in was supported in part by Project MAC, an M.I.T. research program sponsored by the Advanced Research Projects Agency, Department of Defense, under Office of Naval Research Contract Number N00014-70-A-0362-0006 and theNational Science Foundation under contract number GJ00-4327. Reproduction in whole or in part is permitted for any purpose the the United States Government.

2a) v_1, v_2, v_1', v_2' are nodes of some face of G (for some embedding of G in the plane) and they appear in that order as the cycle of edges bounding the face is traversed in some direction.

2b) v_1 and v_1' are bound and v_2 and v_2' are bound.

A cross-over graph G_C is shown in Fig. 2, although it may not be the simplest example. G_C is planar and satisfies condition (2a) by inspection. To verify that G_C is 3-colorable and satisfies (2b), consider the subgraph G_F of G_C shown in Fig. 3. Clearly G_F is 3-colorable and u and v are bound. This implies that v_2 and v_2' are bound in G_C. We leave it to the reader to convince himself that if v_1 and v_1' are also colored the same then a 3-coloration is possible, and if v_1 and v_1' are colored differently then a 3-coloration is impossible.

Now let G be a given graph with nodes $\{u_1,\ldots,u_n\}$. A planar graph G' is constructed such that G' is 3-colorable iff G is 3-colorable. The nodes of G' include a p(n) by n array of nodes $\{v_{ij} \mid i = 1,\ldots,p(n),$ $j = 1,\ldots,n\}$, for some polynomial $p(n) \le O(n^2)$.

G' has the property that for each row $i = 2,3,\ldots,p(n)$, there is a permutation $\sigma_i\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$ such that v_{1j} and $v_{i,\sigma_i(j)}$ are bound for all $j = 1,\ldots,n$. Each row $\{v_{i1},\ldots,v_{in}\}$ of nodes is "connected" to the next row $\{v_{i+1,1},\ldots,v_{i+1,n}\}$ by copies of G_C and G_F. The rows are connected in such a way that for each arc $\{u_k,u_\ell\}$ in G, there is some row i and some j, Now G is 3-colorable iff $D_1,\ldots,D_r$ is satisfiable. Suppose G is 3-colorable and let $f\colon N \to \{1,2,3\}$ be a coloring. The arc $\{t_1,t_2\}$ ensures that $f(t_1) \neq f(t_2)$ so we may assume that $f(t_1) = 1$ and $f(t_2) = 2$. Now $f(\sigma) \in \{2,3\}$ for all literals $\sigma \in L$ because of the arcs $\{t_1,\sigma\}$, and $f(u_i) \neq f(\bar{u}_i)$ for all i because of the arcs $\{u_i,\bar{u}_i\}$. The graphs G_i^* ensure that no clause contains literals all colored 2. Therefore S = $\{\sigma \in L \mid f(\sigma) = 3\}$ is a consistent truth assignment which satisfies $D_1,\ldots,D_r$. The converse is similar. This completes the proof of 1).

The proof above can easily be extended to show that k-colorability is complete for any fixed $k \geq 3$. In particular, let $G = (N_1,A_1)$ be any graph and $K_m = (N_2,A_2)$ be the complete graph on m nodes. If $G' = (N',A')$ is defined as $N' = N_1 \cup N_2$ and $A' = A_1 \cup A_2 \cup \{\{u,v\} \mid u \in N_1,\ v \in N_2\}$, then G' is (k+m)-colorable iff G is k-colorable.

2). 3-COLORABILITY $\propto$ PLANAR 3-COLORABILITY

The proof follows from the existence of a "cross-over" graph which enables one to eliminate cross-overs, thereby converting an arbitrary graph into a planar graph, while preserving 3-colorability.

We say that two nodes u,v of a given graph G are <u>3-color bound</u> (or simply <u>bound</u>) if u and v must be assigned the same color in any 3-coloration of G.

A cross-over graph is defined to be a finite graph G with the properties that

1) G is planar and 3-colorable.

2) There are four distinct nodes v_1,v_1',v_2,v_2' of G such that

Theorem. 3-COLORABILITY and PLANAR 3-COLORABILITY are polynomial complete.

Proof. 1) SATISFIABILITY WITH AT MOST 3 LITERALS PER CLAUSE $\propto$ 3-COLORABILITY.

Let $D_1,D_2,\dots,D_r$ be the clauses and $L = \{u_1,\dots,u_m,\bar{u}_1,\dots,\bar{u}_m\}$ the literals of a given satisfiability problem. Let $D_i = \{\sigma_{i1},\sigma_{i2},\sigma_{i3}\} \subset L$, $i = 1,\dots,r$. Consider the graph G^* with nodes $N^* = \{v_1,v_2,\dots,v_6\}$ and arcs A^* shown as solid lines in Fig. 1. Suppose G^* is connected to nodes s_1,s_2,s_3 as shown. G^* has the following properties:

i) If s_1,s_2,s_3 are constrained to have the same color c, then v_6 must be colored c in any 3-coloring of G^*.

ii) If s_1,s_2,s_3 are constrained to be colors at least two of which are different, then for all $c \in \{1,2,3\}$ there is a 3-coloring of G^* in which v_6 is colored c.

Let $G_i^* = (N_i^*,A_i^*)$, $i = 1,\dots,r$, be r copies of G^*. Let $N_i^* = \{v_{i1},\dots,v_{i6}\}$ as in Fig. 1. $G = (N,A)$ is the following.

$$N = \{t_1,t_2\} \cup \{u_1,\dots,u_m,\bar{u}_1,\dots,\bar{u}_m\} \cup \bigcup_{i=1}^{r} N_i^* .$$

$$A = \{\{t_1,t_2\}\} \cup \{\{t_1,\sigma\} \mid \sigma \in L\} \cup \{\{u_i,\bar{u}_i\} \mid i = 1,\dots,m\} \cup \{\{\sigma_{ij},v_{ij}\} \mid i = 1,\dots,r;\ j = 1,2,3\} \cup \{\{t_2,v_{i6}\} \mid i = 1,\dots,r\} \cup \bigcup_{i=1}^{r} A_i^* .$$

$1 \leq j \leq n-1$, such that v_{1k} and v_{ij} are bound and $v_{1\ell}$ and $v_{i,j+1}$ are bound. The arc $\{u_k, u_\ell\}$ in G can be added as the arc $\{v_{ij}, v_{i,j+1}\}$ in G' without destroying the planarity of G'.

A careful description of G' is somewhat tedious. Fig. 4 illustrates how G' is constructed for $G = K_5$. Copies of G_C and G_F have been abbreviated as in Fig. 2 and 3. Numbers written next to nodes indicate the bindings. All nodes with the same number are bound. The reader should have no trouble generalizing this construction to an arbitrary non-planar graph G.

G_F has been used in the above construction only to simplify the description of G'. Nodes in the array which are bound by a chain of copies of G_F can be merged into a single node while keeping G' planar. This completes the proof of 2).

It is known that 2-colorability can be checked in polynomial time for any graph [2] and that k-colorability, $k \geq 5$, is trivial in the planar case. The only open question, planar 4-colorability, hinges on the 4 color conjecture. However, it might be possible to show that any algorithm, A, which actually produces a 4-coloring of a planar graph input (or states that none exists if that is the case) is polynomial complete in the sense that some complete problem becomes deterministic polynomial time recognizable in the presence of an A subroutine.

References

1. Karp, R.M. Reducibility Among Combinatorial Problems, in Complexity of Computer Computations, R.E. Miller and J.W. Thatcher, ed., Plenum Press, N.Y., 85-104.

2. Berge, C. The Theory of Graphs and Its Applications, John Wiley and Sons, N.Y., 1962.

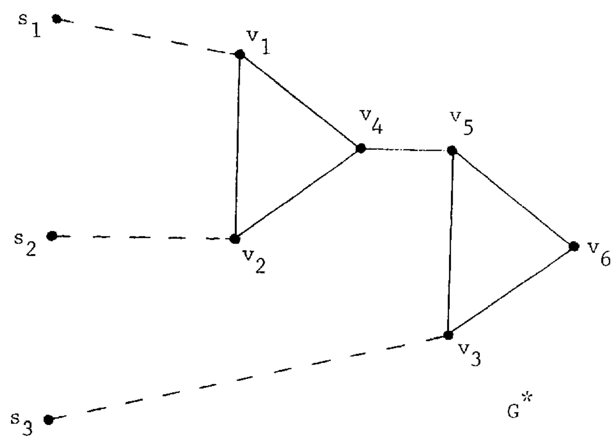


Figure 1.

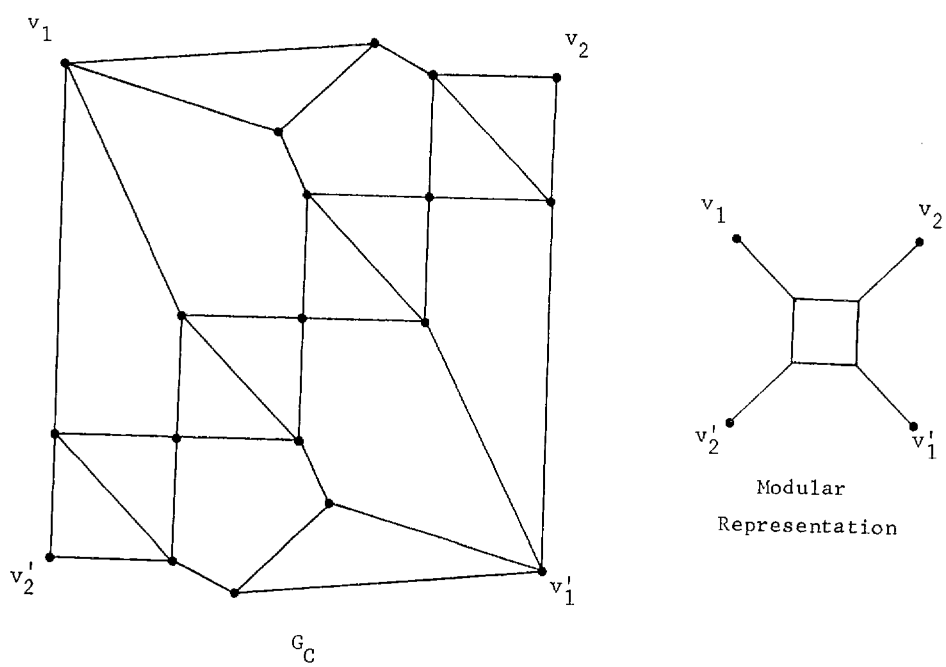
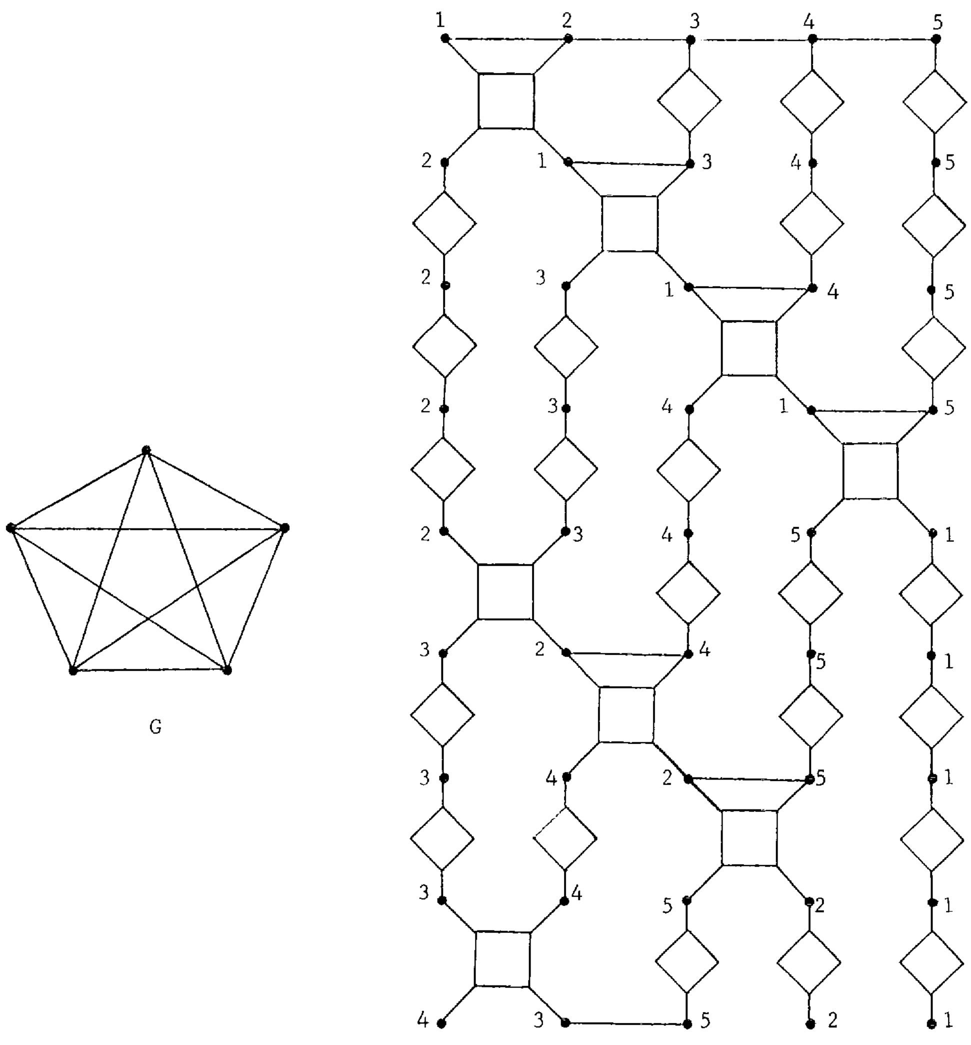


Figure 2.

Modular Representation

Figure 3.



G'

Figure 4.