DIGITAL TO ANALOG CONVERSION (A SPECULATION)

Peter Kugel

Boston College

Chestnut Hill, MA 02167

I used to think that anything that could be done by an analog computer could be done by a digital one - at least from a theoretical point of view in which one ignores questions of efficiency. There are some vague, rather intuitive, ideas that are beginning to convert me from this view and although they hardly provide anything resembling proofs that analog computers have theoretical capabilities that digital machines do not, they do suggest that a theory of analog computation might be worth developing even though it might be hard to imagine what the machines that it is about would "look like".

Consider the following three examples:

(1) The infinite "shortest path" problem*

Consider the problem of trying to find the shortest road trip from City A to City B. In a digital computer, one might convert a road map to a graph representation and then search the graph for a shortest path. Although this can take time, it is not hard. But there is a simpler analog method. Make a string model of the road map (or the corresponding graph). Grab the point representing A with one hand and the point representing B with the other. Pull, and the shortest path is the one represented by the taut string.

This does not prove that an analog procedure can do something that a digital one cannot do. It only shows that it can do the job "faster". But now change the problem so that there are infinitely many paths from A to B. It is easy to see that no digital procedure can compute a shortest path (although such a path can be found by a limiting computation of the type discussed by Gold (1) and Putnam (2).) But, if we assume that we can lift the infinite string model (which, alas, has infinite weight) then we can solve this problem by a one step "computation". Pull!

If the infinite bundle of strings puts you off, let the map be made of rubber and consider the problem of finding the shortest path from A to B without being forced to stick to a road. (Let the rubber map be one of those topographical maps that has bumps representing hills.) Now the infinite string bundle has become a finite rubber sheet and you can lift it. And there is a sense in which you can "compute" a solution to the problem by just pulling. (There remains a problem that I want to ignore but which can be summarized by the question "How do you define 'stretch'?")

You might object that one can still come arbitrarily close by a digital approximation and that may be true. But you can come arbitrarily close to the square root of two by a rational approximation and that does not prove that irrational numbers do not exist. (Or does it? It rather depends on what you mean by "exist".)

(2) The Halting Problem

Recall that the general halting problem - to determine whether an arbitrary machine M, with input i, will or will not halt - cannot be solved by a digital computer. Now imagine the following (physically impossible) situation. We have built a computer that can replicate itself (with input and program) and when it does, its offspring is half its size. This however, has a merit because, being half the size, the offspring has smaller (easier to switch) components and shorter wires which allow it to operate at twice the speed of its parent. To make things better, reproduction takes 0 time and as soon as an offspring is produced, it reproduces itself at once in the same way.

To solve the halting problem, we get such a machine, set it up with the problem for simulating M and give it the input i. Then we instruct it to "reproduce and go (i.e. run)". Imagine that this computer (and each of its offspring) is endowed with a little red light that it turns on when it halts. To solve the M-i case of the halting problem we simply look down this infinite sequence of replications for a glimmer of red. If it appears (and it appears in what we might call an "epsilon" of time if it appears at all) then we conclude that the machine M halts with input i. Otherwise it does not.

Again we have solved a problem that requires a trial and error (or limiting computable) procedure on a digital machine in finite (and computably long) time which, I think, allows us to think of the process involved as a normal (non-limiting) computation. But again, we have a problem with actual realizability of the theoretical procedure. As our machines get smaller and smaller, they are going to run, not only into quantum effects (which might make them unreliable), but into limits of smallest size that will make them unrealizable. The red light will get so small that we won't be able to see it, and so forth. But similar considerations give the diagonal of the unit square a rational length and that does not seem to stop geometers from saying that it does not have a rational length.

(3) The British Are Coming (A Bicentennial Problem)**

The American troops in Concord and surrounding Middlesex County need to know when the British are coming. We had thought of putting a lookout in the steeple of the Old North Church and having him signal when he sees the arrival of the troops on the beach. But this will not do. It will not do because our troops are (anachronistically) equipped with missiles and they need advance warning so that they can ready their missiles for firing. The more time they have for this purpose, the greater the odds of their hitting the troops and saving our boys. Luckily, we have a two pronged alternative developed by our scientists. We can try to use a digital simulation of the British ships (which we will assume to be sailing up from New York one month from now) or we can run an analog simulation. Both attempt to predict arrival time. The latter is a little model of the ocean with little model ships on it. It runs in "real time". The British are not sailing for a month so this model allows us to predict (with perfect accuracy and when the first month ends) when the British will arrive. Since they cannot arrive before the prediction is made, and the accuracy is perfect, this gives us a (theoretical) probability of hitting these ships of 1.0.

Things are not so good with the digital simulation. The trouble with it is that if we want accuracy from it, we have to shorten the time-increments of the simulation. Halve the time increments and you double the accuracy of the arrival-time prediction - but you also double the running time. The trouble with that is that if you double the running time, you run the risk that the British will arrive before the simulation tells you that they will and that you will therefore miss them altogether. (The British, being obtuse, do not view this possibility as negatively as the Americans.)

For specifics, imagine that we know (for sure) that the British will sail in one month and arrive within two months. But we don't know when in that interval they will arrive. If we set our digital simulation to time intervals of 1 day, then our simulation will stay ahead of the British through the end of the second month and the odds of the digital simulation predicting the arrival in time will be 1. But the accuracy of the prediction will be such that the odds of hitting the British will be only .5 for a total probability of success of .5. Suppose that, if we set the interval to a 1/2 day, we increase the probability of hitting the British (if we get a timely warning) to .75. But, if the British arrive after the middle of the (second) month, our simulation will predict the arrival too late and we have no chance of a hit. Assuming a nice flat probability distribution, our odds of knowing in time are now .5 so that our total probability of hitting them is .5x.75=.375. And things get worse as our simulation gets "better" if we assume that halving the interval adds 50% to the accuracy and cuts the time coverage in half.

Our digital simulation generates predictions that become, in some sense, "worse and worse" as it slices up the analog universe into increasingly smaller pieces to increase accuracy. Here then is an example of a digital process that does not "approach" a corresponding analog one, even in the limit. (Of course, this does not prove that in this case, no digital simulation could approach it, but that is another matter.)

So what are we to make of all this? In a sense, all I have done is to relate three anecdotes. (One could hardly call them "theorems".) I would like to sum up by relating a fourth anecdote. (One could hardly call it a "conclusion".)

(4) <u>The Day That Mathematics Was Not Invented</u>

Once upon a time, in the town of Croton on the island of Sicily, there lived a group of mystics who thought that there were numbers in all things. They were not the only mystics on the shores of the Mediterranean. But most of the others despaired of the possibility of scientific knowledge. Some of these others thought that everything was always changing so that no scientific knowledge was possible and others thought that nothing ever changed so that no scientific knowledge was necessary. But our mystics thought that scientific knowledge was possible and that it was to be obtained by trying to find the numbers in all things.

Now what these people called "numbers" were what we call the "rational numbers" since, knowing only the usual arithmetic operations of addition, subtraction, multiplication and division, they could not see how any other kinds of numbers could arise. One day, one of these mystics by the name of Sarogothyp, put together two theorems that he had figured out on the beach. According to one, the length of the diagonal of the unit square was the square root of two and according to the other square root of two was not a number. (Today we would say it was an irrational number but for people who knew only rational numbers, the square root of two was simply not a number.) Here was a thing - the diagonal of a square whose side was length one - that did not (gasp!) have a number in it.

Sarogothyp decided to keep this secret, for he felt that the other non-scientific mystics on the Mediterranean might use this fact against the scientific views of his followers. He succeeded, and mathematics, as we know it, was never invented.

It seems to me that limiting computations, of the kind that arise in my first three anecdotes, extend the idea of a computation in very much the same way that the idea of passing to a limit can extend the idea of a rational number to the idea of irrational ones. And it seems to me that the same arguments that can be used against calling limiting computations "computations" can be used against calling irrational numbers "numbers". The square root of two is not "really" irrational because no "real" line segment can be of exactly that length. A rational approximation of some degree is always exact enough. And so is a digital approximation of an analog process.

So perhaps we should keep the kinds of non-Churchian computations that my first three anecdotes illustrate secret too.

NOTES

* I think I have gotten the basic idea for this example from conversations with either H. Putnam or H. Dreyfus, but I am not sure. In any case, they are not to be held responsible for what I say about it here.

** I got the basic idea for this example from conversations with H. Margolis but he too should be granted the customary absolution.

REFERENCES

(1) E.M. Gold, "Limiting Recursion", Journal of Symbolic Logic, Vol. XXX, 1965, pp. 28-48.

(2) H. Putnam, "Trial and Error Predicates and the Solution to a Problem of Mostowski", ibid, pp. 49-57.