A Note on the Multiplication of 4x4 Matrices

R.K. Shyamasundar

National Centre for Software Development
and Computing Techniques
Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400 005

In this note, it is shown that the multiplication of two 4 x 4 matrices can be done using only 48 multiplications, using a result due to Winograd (1968).

Winograd's Result is stated below:

Winograd's Result (Winograd (1968)):

Let E be an m x n matrix, and F and n x p matrix. Performing the product is equivalent to giving N = m + p vectors and performing m.p inner products. Now the inner product of two n dimensional vectors x and y is given by

$$(x,y) = \begin{cases} \sum_{j=1}^{\lfloor n/2 \rfloor} (x_{2j-1} + y_{2j})(x_{2j} + y_{2j} - 1) - \xi - \eta & \text{if } n \text{ is even} \\ \sum_{j=1}^{\lfloor n/2 \rfloor} (x_{2j-1} + y_{2j})(x_{2j} + y_{2j-1}) - \xi - \eta + x_n y_n & \text{if } n \text{ is odd} \end{cases}$$

where $\xi = \sum_{j=1}^{\lfloor n/2 \rfloor} x_{2j-1} \; x_{2j}$,

$$\eta = \sum_{j=1}^{\lfloor n/2 \rfloor} y_{2j-1} \; y_{2j} \text{ , and}$$

$\lfloor t \rfloor$ denotes the integer part of t.

It can be easily observed that the number of multiplications necessary for multiplying E and F is given by

$$(m + p)\, n + (m.p - m - p) \lfloor (n+1)/2 \rfloor .$$

Applying this result to the computation of two (4 x 4) matrices we get the following result:

The number of multiplications required for multiplying two 4 x 4 matrices is equal to $\frac{1}{2}(4 \times 4 \times 4) + (16 - 4 - 4) \lfloor 5/2 \rfloor = 48$.

<u>Remarks:</u> It should be noted this cannot be used recursively unlike the Strassen's scheme (See Aho, Hopcroft and Ullman (1974)). This is due to the fact that Winograd's result uses commutativity. Thus this result improves the complexity only by a constant factor.

The point to be noticed is the fact that 4 x 4 matrices can be multiplied using only 7^2 multiplications (using Strassen's scheme). An asymptotic improvement can be obtained if 4 x 4 matrices can be multiplied using only 48 multiplications (see Aho, Hopcroft and Ullman (1974)).

References

A.V. Aho, J.E. Hopcroft and J.D. Ullman (1974) "The Design and Analysis of Computer Algorithms, Addison-Wesley, Reading.

S. Winograd (1968), " A new algorithm for inner product", IEEE Trans., Vol. C-17, pp. 693-694.

RKS/pk:31.12.77