LOGICS OF TRUTH AND DISPOSITION

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1. Logics Without Logical Operators

It is well known that truth functions of any number of arguments can be defined in terms of a single logical operator. Viz. either D or X, where D is read "not-both" and X is read "neither-nor."[1]

It is less obvious that propositional calculi can be designed so as to make *all* logical operators unnecessary.

This can be done by asserting that propositions are not only true or false, but are also, at the same time, *disposed* in one of a number of ways.

Portions of the first section of this paper were presented in slightly different form at the Eighth Annual Conference on Numerical Mathematics and Computing, University of Manitoba, 1978.

[1] Let p and q be propositional variables, and let t be any true proposition and f be any false proposition. Then Dtt=f, Dtf=Dft=t, and Dff=f. Similarly Xtt=Xtf=Xft=f, and Xff=t. So Np ("not p") can be defined as either Xpp or Dpp; and Kpq ("p and q") can be defined as XNpNq ("neither not p nor not q") or as NDpq ("not not-both p and q"); and Apq ("p or q") can be defined as NXpq ("not neither p nor q") or as DNpNq ("not-both not p and not q"); etc. Cf. Peirce (1) and Sheffer(2).

2

The analogy is with chemical combination. A false atomic proposition may be disposed to form molecular propositions of only a conjunctive nature. Or a true atomic proposition may be disposed to form molecular propositions of only a disjunctive nature. And so on.[2] Some propositions may be dispositionless--and therefore *necessarily* susceptible to the combinatorial dispositions of other propositions--, and some propositions with dispositions may be truth-valueless. What results is a kind of propositional valency.

Consider, as an example, propositions which may be true or false, and which may be either undisposed or disposed to conjunction, disjunction, or negation.[3]

K, A, and N are not used as logical operators, but are re-introduced as dispositions for conjunction, disjunction, and negation, respectively.

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[2] Dispositions which are pre-existing, essential, or innate will be called predispositions. But the factual nature of disposition is not an especial prerogative of a logic of dispositions, even as the factual nature of truth is not an especial prerogative of a logic of truth. It suffices that there exists some symbolic mechanism for inflecting disposition, to the extent this is somehow necessary to a logic at issue. At most, it can be said that what is taken as logically constant and what is taken as logically variable--truth or disposition--must be in some manner flexible, when truth and disposition are assumed to be different logical dimensions of a single logical object. Cf. below.

[3] Other dispositional schemes are clearly possible; and dispositions are easily inter-defined.

0 is introduced for propositions which are dispositionless, or truth-valueless. a, b, c, ... are introduced as truth variables; and x, y, z, ... are introduced as disposition variables.[4]

In this case, propositional valuations are two-dimensional[5] quantities of the following kind: t_0, t_K, t_A, t_N, f_0, f_K, f_A, and f_N.[6]

These quantities can be taken to obey the rules of combination:

$$t_K t_K = t_K t_0 = t_0 t_K = t_0$$

$$t_K f_K = t_K f_0 = t_0 f_K = f_0$$

$$f_K t_K = f_K t_0 = f_0 t_K = f_0$$

$$f_K f_K = f_K f_0 = f_0 f_K = f_0$$

$$t_A t_A = t_A t_0 = t_0 t_A = t_0$$

$$t_A f_A = t_A f_0 = t_0 f_A = t_0$$

$$f_A t_A = f_A t_0 = f_0 t_A = t_0$$

$$f_A f_A = f_A f_0 = f_0 f_A = f_0$$

$$t_N = f_0 \qquad f_N = t_0$$

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4 These variables are used as constants in appropriate contexts.

5 The idea of two-dimensional logics and two-dimensional languages goes back to Jaskowski (3) and Post (4), and has been recently resuscitated by Herzberger (5). But these writers are interpreting logical operators as truth-functional in one dimension of a language and non-truth-functional in another; and this is quite another matter from interpreting propositions in a two-dimensional way, and thereby eliminating the need for logical operators altogether.

4

Quantities of the form $a_x b_y$ do not combine, for $x \neq y$, unless x or else[7] y is either 0 or N.

Propositions of higher molecular weight can be punctuated in a normal way. E.g.

$$(a_K b_K)_A (c_K d_K)_A$$

as distinguished from:

$$a_K (b_A (c_K d_K)_A)_K$$

From the above, logical operators can be introduced as abbreviations:[8]

$$Kab = a_K b_K = a_K b_0 = a_0 b_K$$

$$Aab = a_A b_A = a_A b_0 = a_0 b_A$$

$$Na = a_N$$

$$a = a_0 = (a_N)_N$$

$$Cab = a_N b_A$$

$$Eab = (a_N b_A)_K (b_N a_A)_K$$

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[6] It is sometimes convenient to re-write these expressions in this way: $N = 0_N$, $P = t_A$, $Q = t_K$, $R = f_A$ $S = f_K$, $T = t_0 = f_N$, $F = f_0 = t_N$. In this form, the rules of combination would be: QQ = QT = TQ = T; QS = QF = TS = F; SQ = ST = FQ = F; SS= FS = F. PP = PT = TP = T; PR = PF = TR = T; RP = RT = FP = T; RR = RF = FR = F. NT = F; NF = T.

[7] This locution is meant to exclude the expressions $a_0 b_N$ and $a_N b_0$.

[8] C is read "if-then" and E is read "if and only if". Modal operators can also be replaced by modal dispositions in a two-dimensional context. E.g. L for a disposition to necessity and M for a disposition to possibility. All modal and non-modal operators can be defined in terms of a single triadic disposition, based on the operator of Hallden (6). Truth and disposition can be fuzzified in a two-dimensional setting, in the manner of Gaines (7).

5

What is wanted now is a general mechanism for solving two-dimensional logical equations in several unknowns of several kinds, and this will be provided in a sequel.[9] It is to be expected that well-developed logics of truth and disposition will find application in several areas. For example, versions of these logics which are somehow _actively recombinative_ may make the problem of assigning probabilities to logical formulae of high molecular weight interesting and difficult, and may raise questions or essential predisposition and likely modality in an unusually natural way.

2. Higher-Order Dispositions

A variegated mass of predisposed atomic propositions can combine and recombine itself into molecular propositions of greater and greater weight, given the right sort of conditions. And already, at this stage, certain issues suggest themselves.[10]

But before developing this idea it will be desirable to generalize the combination rules presented before, so as to allow for reticular patterns of propositional growth.[11]

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[9] This talk of equation theory is deliberate. The interesting question is no longer what it was in Boole's day, but rather how to make logical algebras as _numerical_ as possible.

[10] For example, that of locally active centers of punctuation. (PP)(PP) $\neq$ ((PP)P)P, etc., so it obviously matters how centers for propositional growth are distributed. But cf. footnote 13 below.

[11] The object here is to blur the distinction between the more or less straightforward combinatorics of logic and the rather occult combinatorics of the world. One thinks of Wheeler's suggestion that the physical world is built up of objects of essentially "logical" character. Cf. (8), pp. 1211-2.

6

One way to do this is to notice that rules of the form[12] can be

$$a_x b_x = c_0$$

can be modified to read

$$a_x b_x = c_y$$

with $y \neq 0$. I.e. the rules can be re-written to assert the results of combinations are not (always) dispositionless.

E.g. one might have $a_A b_A = c_A$ which says that combinations resulting from dispositions to disjunction are, themselves, disposed to disjunction.[13] One might equally have $a_A b_A = c_K$ or even some mix of such possibilities, suitably expressed, existing for different propositions.

But it will be useful to derive this full range of possibilites from the idea of <u>second-order dispositions</u>.

A second-order disposition, y, will be written in the form

$$a_{xy}$$

where this says that an atomic proposition of truth-value a is disposed to form molecular propositions of kind x, and is further disposed to dispose the resulting molecular propositions to form molecular propositions of kind y.

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[12] The weaker forms $a_x b_0$, etc., are suppressed here, and in what follows.

[13] Note the effect this has on the punctuation problem stated in footnote 10. In this case (PP)(PP) = ((PP)P)P.

A second-order disposition is realized when[14]

$$a_{wx}b_{yz}$$

obtains, such that w=y (or else w (or else y) = 0 or N) and x=z (or else x (or else z) = 0 or N).[15] So that, for example, $a_{0A}b_{A0}=c_A$.

In this sense, the rules $a_x b_x=c_0$ can be re-written in the form

$$a_{x0}b_{x0}=c_0$$

Care must be taken not to confuse a_{xy} with $(a_x)_y$. The former y is second-order; the latter is not.[16]

n-order disposition can be introduced in the form of (n+1)-dimensional propositions:

$$a_{xy\ldots z}$$

so that, for example:

$$(a_{xyz}b_{xyz})c_{yz}=d_{yz}c_{yz}=e_z$$

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[14]This is only one possible scheme, informally articulated.

[15]A characteristic expression is $a_{xy}b_{xy}=(a_x b_x)_y$.

[16]A characteristic expression is $(a_{0x})_y=a_{yx}$. The case of disposition to negation is special. Here $a_{NN}=(a_N)_N$.

3. Appendix: An Operatorless Propositional Calculus

This axiom system treats dispositions as logical constants, and is therefore completely classical in character:[17]

Axioms:

1. $((a_A a_A)_N)_A a_A$
2. $(a_N)_A(a_A b_A)_A$
3. $(((a_N)_A b_A)_N)_A((((b_N)_A c_A)_N)_A((a_N)_A c_A)_A)_A$

Rule:

$$\alpha_0,\ (\alpha_N)_A\beta_A \longrightarrow \beta_0$$

8. References

(1) C.S. Peirce, Collected Papers 4.12-20, 264-5, Harvard U.P., 1931-5.

(2) H.M. Sheffer, "A Set of Five Independent Postulates for Boolean Algebras, with Applications to Logical Constants," Transactions of the American Mathematical Society, 14, 481-488.

(3) S. Jaskowski, "Recherches sur le systeme de la logique intuitioniste," in S. McCall, Polish Logic 1920-1939, 259-63, Oxford 1967.

(4) E. Post, "Introduction to a General Theory of Elementary Propositions," American Journal of Mathematics, 1921.

(5) H. Herzberger, "Supervaluations in Two Dimensions," Proceedings of the 1975 International Symposium on Multiple-Valued Logic, 429-35.

(6) S. Hallden, "A Reduction of the Primitive Symbols of the Lewis Calculi," Portugaliae Mathematica, 8, 85-88, 1949.

(7) B.Gaines, "Fuzzy Reasoning and the Logic of Uncertainty," Proceedings of the Sixth Int. Symposium on Multiple-Valued Logic, 179-88.

(8) C.W. Misner,K.S. Thorne, J.A. Wheeler, Gravitation, Freeman, 1973.

(9) R. Joseph, "What Does Mr. Johnson Mean by a Proposition?," Mind, v. 36.

(10) J. Lukasiewicz, Elementy logiki mathematycznej, Warszawa, 1929.

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[17] Cf. footnote 2. The axioms here are the AN-forms of the C-N system of Lukasiewicz (10). Viz. of CCNppp, CpCNpq, and CCpqCCqrCpr. The rule is the AN-form of detachment. K and C are introduced as definitions. I.e., $a_K b_K = ((a_N)_A(b_N)_A)_N$, and $a_C b_C = a_N b_A$.