PROPOSITIONS WITH MULTIPLE DISPOSITIONS AND MULTIPLE TRUTH VALUES

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Propositional calculi can be designed so as to make all logical operators unnecessary. This is done by interpreting propositions in a two-dimensional way. Atomic propositions are not only true or false but are also disposed to form specific kinds of molecular propositions.[1]

1. Multivalent Propositions

One way to generalize the combination rules presented in "Logics of Truth and Disposition"[2] is to introduce propositions with multiple dispositions. Such propositions will be termed multivalent.

For example, an atomic proposition may be disposed to form conjunctive and disjunctive molecular propositions. This case will be written[3]

$$a_{A,K}$$

and the general case will be written

$$a_{x,y,\ldots,z}$$

Multivalent propositions are generally more susceptible to combination than univalent propositions, but are less susceptible to combination than dispositionless propositions.

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[1]Cf. Derus, K. H. and Hansen, J. C., "Logics of Truth and Disposition," SIGACT News; summer 1979, vol. 11, no. 1, pp. 36-43. Dispositions which are themselves disposed give rise to actively recombinative logic. (pp. 40-42.)

[2]p. 38.

[3]A bivalent second-order disposition could be written $a_{x(y,z)}$ and it is easy to adapt this symbolism to systematically and sporadically m-valent and n-order propositions.

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(Both $a_{A,K}b_A$ and $a_{A,K}b_K$ combine when neither $a_K b_A$ nor $a_A b_K$ does. But $a_{A,K}b_C$ does not combine,[4] even though $a_0 b_C$ does.)

The number of truth-functionally definable dispositions involving n propositions is:[5]

$$2^{(2^n)}$$

Included among these are the interesting dispositions a_T and a_F: the dispositions to form true and false propositions, respectively.

$a_T b_T = t_0$ and $a_F b_F = f_0$, for all valuations of a and b. Using A, K, and N as primary dispositions[6], $a_T b_T$ can be defined as $(a_A a_N)_K (b_A b_N)_K$ and $a_F b_F$ can be defined as $(a_K a_N)_A (b_K b_N)_A$.

The number of multivalent propositions definable in terms of truth-functional dispositions which are themselves definable in terms of n propositions is:

$$2^{(2^{(2^n)})}$$

Included among these are the interesting multivalent propositions a_0, already introduced, and a_Z: the proposition disposed in <u>all</u> truth-functionally definable ways.[7]

Let $\underline{x}$ and $\underline{y}$ be the sets of dispositions associated with the truth-variables a and b, respectively. Then $a_{\underline{x}}$ and $b_{\underline{y}}$ combine as

$$a_{\underline{x}\cap\underline{y}}\, b_{\underline{x}\cap\underline{y}}$$

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[4]Here $a_C b_C = a_N b_A$, without being substitutable *qua* disposition. The notations C and C' can be used for dispositions to antecedency and consequency, respectively.

[5]These can all be defined in terms of a single truth-functional disposition. Viz. D or X, where D is read "not-both" and X is read "neither-nor." Naturally there are many important dispositions that are not truth-functional, e.g. modal dispositions. A useful non-truth-functional disposition is $\emptyset$: the disposition to never form molecular propositions. $a_\emptyset$ must remain atomic.

[6]One must distinguish between "real" dispositions and those introduced as abbreviations. Cf. footnote 4.

[7]$a_0 b_x = a_Z b_x$, for $x \neq 0$, and in general $a_0 b_{\underline{x}} = a_Z b_{\underline{x}}$, when $\underline{x}$ is a non-empty set of dispositions.

If it is stipulated that multivalent propositions can combine only with univalent[8] propositions, the resulting valuations are straightforward, and yet compatible with recombinative schemes of some richness and variety.

For the case where multivalent propositions are allowed to combine with other multivalent propositions, there is no wholly compelling valuation theory.

One possible valuation scheme will be sketched briefly. Let x,y, ...,z be the dispositions contained in $\underline{x}\cap\underline{y}$: then[9]

$$a_{x,y,\ldots,z}b_{x,y,\ldots,z} = \left\{a_x b_x,\ a_y b_y,\ \ldots,\ a_z b_z\right\} = \left\{c_{(x)},\ d_{(y)},\ \ldots,\ e_{(z)}\right\}$$

Under this interpretation, atomic propositions which are disposed to combine in a number of ways <u>do</u> combine--in all of these ways--, with each disposition determing a truth-value for the resulting molecular proposition. A single proposition can have multiple truth-values, even as it can have multiple dispositions.

(For example, $t_{A,K}f_{A,K}$ is evaluated as $\left\{t_A f_A,\ t_K f_K\right\}$. I.e. as $\left\{t_{(A)},\ f_{(K)}\right\}$.)

As an alternative to indexing, the values $c_{(x)}, d_{(y)}, \ldots, e_{(z)}$ can be ordered in some fashion. For example, if A, C, and K are taken as the components of multivalent propositions, the $\left\{t_{(A)},\ f_{(K)}\right\}$ might be written as $\langle t,0,f\rangle$ and $\left\{f_{(A)},\ t_{(C)}\right\}$ might be written as $\langle f,t,0\rangle$. These expressions are <u>3-dimensional</u> truth-values.

In a universe admitting n kinds of dispositions, $|\underline{x}\cap\underline{y}|$-valent propositions combine to yield n-dimensional truth-variables involving $|\underline{x}\cap\underline{y}|$ non-zero values. This idea is easy to generalize, and easy to formalize.

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[8] Dispositionless propositions are not univalent.

[9] It follows that $a_{x,y} = a_{y,x}$, which is to be expected. The "indexing" of truth-values is at variance with ordinary practice, which need not concern itself with the "ancestry" of truth; but this indexing should not suggest that there is no longer one "kind" of truth. Only one kind of truth figures in expressions like $t_{(x)}$ and $t_{(y)}$, even when $x \neq y$.

2. n-Dimensional Truth Variables

n-Dimensional truth-variables can be disposed in various ways, as for example via higher-order dispositions associated with component variables.[10]

These can be taken to combine according to the rule:

$$\langle a,b,\ \ldots,\ c\rangle_x \langle a'b',\ \ldots,\ c'\rangle_x = \langle a_x a'_x,\ b_x b'_x,\ \ldots,\ c_x c'_x\rangle$$

(In particular, for the 2-dimensional case $\langle a,b\rangle_A \langle c,d\rangle_A = \langle a_A c_A,\ b_A d_A\rangle$.)

The rationale for this is fairly clear. If $\langle a,b\rangle = \{t_{(A)},\ f_{(K)}\}$ and $\langle c,d\rangle = \{t_{(A)},\ t_{(K)}\}$, then these sets of indexed truth-values are manipulated in an "orderly" way: the (A)-indexed values are disposed to combine in a particular way, and similarly with the (K)-indexed values. Each $a_{(x)}$ participates in its own "world" of combinations.[11] What results is logic with broadly linear properties.[12]

Expressions like

$$\langle a,b,\ \ldots,\ c\rangle_{x,y,\ldots z} \langle a',b',\ \ldots,\ c'\rangle_{x,y,\ldots,z}$$

can be managed as

$$\{\langle a_x a'_x,\ b_x b'_x,\ \ldots,\ c_x c'_x\rangle, \langle a_y a'_y,\ b_y b'_y,\ \ldots,\ c_y c'_y\rangle, \ldots$$
$$\langle a_z a'_z,\ b_z b'_z,\ \ldots,\ c_z c'_z\rangle\} = \{\langle e,f,\ \ldots,\ g\rangle_{(x)},\ \langle e',f',\ \ldots,$$
$$g'\rangle_{(y)},\ \ldots, \langle e'',f'',\ \ldots,\ g''\rangle_{(z)}\} = \langle \ldots \langle e,f,\ \ldots,\ g\rangle,\ \ldots,$$
$$\langle e',f',\ \ldots,\ g'\rangle\ \ldots, \langle e'',f'',\ \ldots,\ g''\rangle,\ \ldots\rangle$$

The zeros in n-dimensional truth-variables behave as truth-value gaps, so that for example

$$\langle a',b',0,0\rangle_x \langle a,b,c,d\rangle_x = \langle a'_x a_x,\ b'_x b_x,\ 0_x c_x,\ 0_x d_x\rangle = \langle e,f,c,d\rangle$$

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[10] Higher-order dispositions can be handled in a variety of ways. E.g. $a_{(x,y,\ldots,z)x'}$ can be distributed as $a_{xx',\ yx',\ \ldots,\ zx'}$; or quantities like $a_{xx',yy',\ldots,zz'}$ can be somehow operative.

[11] One thinks instinctively of "possible" worlds here, but the analogy is not particularly apt.

[12] Here and indeed throughout the authors have contrived to be technical without being precise, as Lord Balfour might say.

And a kind of "scalar multiplication" of truth-variables should also prove of some interest:

$$a'_x\langle a,b,\ \ldots,\ c\rangle_x = \langle a'_x a_x,\ a'_x b_x,\ \ldots,\ a'_x c_x\rangle$$

3. Other Dispositions Having Classes as Objects

Ordered and unordered sets of truth-variables can be introduced in ways having nothing to do with valuations of multivalent propositions; as for example via a non-truth-functional disposition Δ , which obeys the combination rule:

$$a_\Delta\, b_\Delta = \langle a,b\rangle_0$$

so that $(a_\Delta\, b_\Delta)_0 c_\Delta = \langle a,b,c\rangle_0$, etc.[13]

The recombinative behavior of a mass of propositions involving Δ can be studied as a kind of population game. E.g. one might start with $\underline{a}_{\Delta\Delta,KK}$, $\underline{b}_\Delta$, and $\underline{c}_K$ of various cardinalities and distributions[14], and then consider the likely dimension of the resulting molecular propositions.

Dispositions of the form mx, $m \gtrsim 2$, can be introduced, subject to the rule that $\underline{a}_x$ and b_{mx} combine if and only if $|\underline{a}| = m$ for unordered sets, or $\dim(\underline{a}) = m$ for ordered sets.[15]

(For example, $\langle a,b,c,d\rangle_K$ and e_{4K} combine, when $\langle a,b,c,d\rangle_K$ and e_{5K} do not.)

Here too, there is a rather simple-minded analogy with chemical combination, and it would seem the possibilities are endless.

4. Truth Variables with Numerical Properties

For some purposes one may want $0_x a_x = 0$, and one can easily imagine a truth-value gap with the "numerical" property of being susceptible to the additive nature of disjunction and the multiplicative nature of conjunction, as for example in this completely ad hoc definition of truth-variables with properties of complex numbers:

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[13] Alternatively, $a_\Sigma b_\Sigma = \{a,b\}_0$, etc.

[14] CF. "Logics of Truth and Disposition", footnote 10.

[15] The generalized case of $\underline{b}_{mx}$, with $|\underline{b}| \neq m$, is problematic.

$$\langle a,b\rangle_A \langle c,d\rangle_A = \langle a_A c_A,\ b_A d_A \rangle$$
$$\langle a,b\rangle_K \langle c,d\rangle_K = \langle (a_K c_K)_A (b_K d_K)_N)_A,\quad (a_K d_K)_A (b_K c_K)_A \rangle$$

Here $a_A 0_A = a$ but $a_K 0_K = 0$.

Truth-variables with variously nonamalgamative, noncommutative, nonassociative, and nondistributive properties can be introduced into the present scheme in much the way one might provide an n-dimensional linear space with a complex or hypercomplex field.[16] Or such variables can be introduced, in other contexts, as dispositions of special kinds.

Consider, as an example a "conditional" disposition to second-order negation defined as follows

$$(a_{\pi})_K = (a_{0N})_K = a_{KN}$$
$$(a_{\pi})_A = (a_{0\pi})_A = a_{A\pi}$$

And write $t_N = F$, $f_N = T$, $t_\pi = T'$, $f_\pi = F'$, $P' = (t_\pi)_A$, $Q' = (t_\pi)_K$, $R' = (f_\pi)_A$, $S' = (f_\pi)_K$.

Then this results in the following: $Q'Q' = (t_\pi)_K(t_\pi)_K = (t_{0N})_K(t_{0N})_K = t_{KN}t_{KN} = (t_K t_K)_N = (t_0)_N = f_0 = F$, etc.[17]

So that T' and F' behave as "purely" imaginary truth values.[18]

The rationale for introducing quasi-complex quantities is presumably equation-theoretic. E.g. to provide "solutions" to equations of the form:

$$(t_x)_K(f_x)_K = t_0 \quad \text{when} \quad (t_x)_K(t_x)_K = f_0$$

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[16] Obviously there is no reason to mimic complex numbers too closely! A logic involving truth-values with properties of complex numbers could be expected to give rise to a curious proof theory, because some but not all real-valued tautologies can be re-written as imaginary-valued contradictions, etc. (Complex-valued equivalences can involve propositions having differing truth-values.)

[17] Q'Q' = Q'T' = T'Q' = F; Q'S' = Q'F' = T'S' = T; S'Q' = S'T' = F'Q' = T; S'S' = S'F' = F'S' = T. P'P' = P'T' = T'P' = T'; P'R' = P'F' = T'R' = T'; R'P' = R'T' = F' P' = T'; R'R' = R'F' = F'R' = F'. NT' = F'; NF' = T'. Cf. "Logics of Truth and Disposition," footnote 6.

[18] π isn't intended to give "proper" values for expressions like Q'T and QT', and is not intended to generalize to anything like a complex-valued case. (Cf. footnote 16.) It raises the interesting question whether falsehood is the negation of truth. Note the sense in which π is truth-functional.