THE EVALUATION, MANIPULATION AND IDENTIFICATION OF NONDIMENSIONAL NUMBERS

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ABSTRACT

This paper describes a computer program which evaluates, manipulates and identifies nondimensional numbers (pi terms, dimensionless numbers or factors) generated using the Buckingham Pi Theorem. Computations are performed using SYMBOLANG, an algebraic (symbol) manipulation package (solutions are kept in symbolic form). Manipulation occurs in two ways. First, many allowable combinations of pi terms (up to 720) are generated by changing the ordering of the dimensional equations which are solved. Second, a small group of operators (squaring, square-rooting, cube-rooting, reciprocating, and division of both the minimum and maximum exponent) are applied to each solution (these too are solutions and also kept in symbolic form). Numerical substitution into the symbolic forms produce scalars (evaluated solutions) for subsequent use (plotting, regression, factor analysis, etc.). Finally, every symbolic solution is compared with well-known nondminsional numbers (e.g., Reynold's, Weber's, etc.). If a match is not found, this information is displayed.

"It is out of the question to formulate and carry out experiments nowadays without making use of similarity and dimensionality concepts".[1]

I. INTRODUCTION

One measure of the importance of a scientific area is the number of papers written in that area. Dimensional analysis has generated over 600 publications[2], however, only a handful of these papers deal with computerized dimensionality[3,4,5,6,7].

There are several reasons why so little computer work has been done in dimensional analysis. First, much work has been devoted to methodologies and specific solutions in highly specialized areas. Second, and more important, is that computer codes have been inflexible. Only one solution is usually generated for a given ordering of dimensional equations. Therefore, fruitful and/or well recognized solutions might not be generated. This, just because of reordering. Next, identifiable and well recognized solutions are not identified. The user does needless computational work when this occurs. Finally, with the exception of the work of Cohen and Ferrari, (Reference 6), solutions are not usually numerically evaluated. Therefore, a great deal of computational work must be performed before suitable data is available for plotting, regression, clustering, or factor analysis.

The program described in this paper overcomes most of the named deficiencies. That is, those of inflexibility. Computer solutions are generated using the Buckingham Pi Theorem (Section II). Computations are performed using SYMBOLANG (Section III), an algebraic (symbol) manipulation package. Solutions to dimensional equations are kept in symbolic form. In addition, solutions are manipulated in two ways. First, many allowable combinations of pi terms (up to 720) can be generated by the program automatically changing the order of the dimensional equations. Second, a small group of operators (squaring, cubing, square-rooting, cube-rooting, reciprocating, and division by both the minimum and maximum exponent) are applied to each generated solution. Various transformations applied to solutions produce new solutions. Products of solutions also produce new solutions. Provision has been made for numerical substitution into the symbolic forms producing scalars (evaluated solutions) for subsequent use. In addition, every symbolic solution is compared with well-known nondimensional numbers (e.g., Reynold's, Weber's, etc.). Well-known numbers are selected from the Land Table[8]. When a match between a symbolic solution and a Land number is found, the name of the Land number is printed for all such numbers which match (i.e., there may be more than one name printed for a particular solution).

II. THE BUCKINGHAM PI THEOREM

The Buckingham Pi Theorem[9] summarizes the entire theory of dimensional analysis[10]. The result of a dimensional analysis is the reduction of the number of variables in a problem. Application of the Pi theorem itself provides the method of solution of a set of dimensional equations. Simply stated, the Pi theorem asserts:

If there are n variables involving N fundamental units, these may be combined to forn n-N dimensionless *parameters each involving N + 1 variables.*

The usual method of applying the Pi theorem is for one to write equations describing the physical system one is interested in in terms of a set of fundamental units (force, length, time, etc.). The equations are then systematically exponentiated and multiplied together (hence "Pi" theorem for the mathematical symbol for multiplication (π). The resulting exponentials form an N by N set of linear equations whose solution is applied back to the variables of the problem (see the example below). One can see that it is tedious to work a problem involving many variables by hand. In fact, there is no good indicator of just how many variables one should include in a problem (that is true even for a computerized solution).

The following example (taken from Housner and Hudson[11]) used throughout the remainder of the paper illustrates the mechanisms of the Buck-

Consider a drag force (F) acting upon a body moving through a fluid. Assume a constant velocity (V) through the fluid of density (ρ) and viscosity (μ). If the analysis is applied to bodies of a specific shape, the cross-sectional area (A) may be used as a measure of the body's size.

The following variables and fundamental units enter into the problem:

Variable		Fundamental Units
F	=	F
μ	=	$FL^{-2}T$
A	=	L^2
ρ	=	$FL^{-4}T^2$
V	=	LT^{-1}

where F = Force, L = Length and T = Time,

According to the Pi theorem, two terms can be formed from the five equations (each equation is expressed in terms of the three fundamental units). The two solutions will each contain four of the variables. The pi terms formed with this ordering are:

$$\pi_1 = FA^{\alpha}\rho^{\beta}V^{\gamma} = F^{1+\beta}L^{2\alpha-4\beta+\gamma}T^{2\beta-\gamma}$$

and

$$\pi_2 = \mu A^{\alpha}\rho^{\beta}V^{\gamma} = F^{1+\beta}L^{-2+2\alpha-4\beta+\gamma}T^{1+2\beta-\gamma}$$

Solving these equations we find: π_1 has the solution $\alpha = -1$, $\beta = -1$, $\gamma = -2$; π_2 has the solution $\alpha = -1/2$, $\beta = -1$, $\gamma = -1$; so the resulting dimensionless pi terms are:

$$\pi_1 = \frac{F}{A\rho V^2} \quad \text{and} \quad \pi_2 = \frac{\mu}{A^{1/2}\rho V}$$

π_1 is a pressure coefficient and π_2 is the reciprocal of the Reynold's number[10]. One can see how the equations are systematically selected and visualize the ease with which such an algorithm can be programmed.

The above represents one set of solutions. In this particular case, eight other solutions are possible depending upon the reordering of the five basic equations. The full set of solutions is listed below:

(1)	$\frac{F}{A\rho V^2}$	(6)	$\frac{A\rho V^2}{F}$
(2)	$\frac{\mu}{A^{1/2}\rho V}$	(7)	$\frac{VA^{1/2}\rho^{1/2}}{F^{1/2}}$
(3)	$\frac{A\mu^2 V^2}{F^2}$	(8)	$\frac{\mu}{F^{1/2}\rho^{1/2}}(A^{o})$
(4)	$\frac{\rho F(V^{o})}{\mu^2}$	(9)	$\frac{V\mu A^{1/2}}{F}$
(5)	$\frac{\mu}{\rho^{1/2}F^{1/2}}(V^{o})$	(10)	$\frac{\rho F}{\mu^2}(A^{o})$

In this example, only solutions (1) and (2) are fruitful (contain information). The other eight solutions can all be derived from solutions (1) and (2). We must note, however, that solutions (4) and (10) are minimal in the sense that they (a) contain the fewest possible number of variables and (b) the sum of their (integer) exponents is a minimum (see References 4 and 5 for a further discussion of minimal solutions).

III. SYMBOLANG

SYMBOLANG[12,13], a high-level FORTRAN language for algebraic (symbol) manipulation, was used to form the symbolic products (pi terms) of the fundamental units. This application was well suited for SYMBOLANG. Not only were solutions generated, but, one was able to see the development of the pi terms as the products were being formed. In addition, the final equation was also displayed. SYMBOLANG is of pedagogical value in demonstrating the mechanisms of the Buckingham Pi Theorem.

IV. THE PROGRAM

The name of the main program is BUCKY. BUCKY initializes the SLIP-SYMBOLANG working storage area, then reads and displays the inputs (sample inputs appear in Figure 1). BUCKY next calculates the number of combinations (orderings) possible based upon the number of equations input, however; only 720 permutations are allowed. BUCKY next breaks the equations into two pieces;

Figure 1. Sample Inputs

VARIABLE	ENGLISH UNITS	METRIC UNITS
Force (F)	10^4 Lbs	444820×10^4 Dynes
Velocity (V)	10^3 Ft/Sec	$30480 \frac{cm}{sec}$
Area (A)	$10^2 Ft^2$	$92903.04\ cm^2$
Density (δ)	$.00237 \frac{Lb\ Sec}{Ft^4}$	$.0012144 \frac{Dyne\ Sec}{cm^4}$
Viscosity (μ)	$4 \times 10^{-7} \frac{Lb\ Sec}{Ft^2}$	$1.915201 \frac{Dyne\ Sec}{cm^2}$

PROGRAM INPUTS

```
    8
FORCED             444820.        + 04-9999.9
VISCOSITY          1.915201       - 04-9999.9
AREA                92903.04           -9999.9
DENSITY            .00122144           -9999.9
VELOCITY            30480.             -9999.9
FORCE              -9999.9
LENGTH             -9999.9
TIME               -9999.9
    5
    3
      FORCED=FORCE$$
      VISCOSITY=FORCE*TIME/LENGTH**2$$
      AREA=LENGTH**2$$
      DENSITY-FORCE*TIME**2/LENGTH**4$$
      VELOCITY=LENGTH/TIME$$
NOMOR
$$
```

Figure 2. Sample Program Outputs

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                                                              PI THEOREM SOLVER
NUMBER OF PRIMITIVES          8

LAND CANDIDATES REFERS TO
A COMPILATION OF DIMENSIONLESS NUMBERS BY NORMAN S. LAND
NASA SP-274, USGPO STOCK NUMBER 3300-0408, 1972

PRIMITIVES USED IN EVALUATION
PRIMITIVES                      VALUES
FORCED           0.44482000E 10     -0.99999000E 04       0.00000000E 00
VISCOSITY        0.19152010E-03     -0.99999000E 04       0.00000000E 00
AREA             0.92903040E 05     -0.99999000E 04       0.00000000E 00
DENSITY          0.12214400E-02     -0.99999000E 04       0.00000000E 00
VELOCITY         0.30480000E 05     -0.99999000E 04       0.00000000E 00
FORCE           -0.99999000E 04      0.00000000E 00       0.00000000E 00
LENGTH          -0.99999000E 04      0.00000000E 00       0.00000000E 00
TIME            -0.99999000E 04      0.00000000E 00       0.00000000E 00

THERE ARE       5 FORMULAS INVOLVING     3 VARIABLES

THERE ARE       5 POSSIBLE COMBINATIONS
       FORCED=FORCE$$
       VISCOSITY=FORCE*TIME/LENGTH**2$$
       AREA=LENGTH**2$$
       DENSITY=FORCE*TIME**2/LENGTH**4$$
       VELOCITY=LENGTH/TIME$$

THERE ARE       2 PI TERMS
      LTV = FORCE**(1     +    B)*LENGTH**(2*A     -    4*B     + C)
         *TIME**(2*B     -    C)
$
$END OF EXPRESSION

SOLUTION OR MANIPULATED SOLUTION FOR PI TERM
FORCED       **  .10000000E 01
AREA         ** -.10000000E 01
DENSITY      ** -.10000000E 01
VELOCITY     ** -.20000000E 01

VALUE(S) OF PI TERM FOLLOW

VALUES USED IN EVALUATING PI TERM
      0.44482000E 10      0.92903040E 05      0.12214400E-02      0.30480000E 05

EVALUATED TERM =       0.42194157E-01  RECIPROCAL =       0.23699964E 02

LAND CANDIDATES
  BOND         WEBER         EOTVOS       MAG INTER
```

the one to the left of the equal sign are variables and those to the right of the equal sign are fundamental units. Now, the Pi theorem algorithm is invoked. This is where the exponentiation and multiplication of fundamental units occurs. The result of this step is the system of equations which is solved using a matrix inverse routine. The solution of the linear equations is performed in subroutine BUCKSV. Next, the appropriate numerical operator (squaring, etc.) is established and a numerical substitution made. A scalar is produced.

Evaluation is performed in subroutine EVALP. EVALP is an inelegant routine whose main virtue lies in producing the correct scalar value. Up to five values may be input for each of 24 variables (a considerable number for a dimensional analysis). Reciprocating is also performed in EVALP.

After evaluation, the exponents of the symbolic solution are passed to TABLUK, which compares them with well-known dimensionless numbers taken from the Land Table. When a match is found, the name(s) of all numbers in the table fitting the match is printed.

Following the table lookup, a new numerical operator is selected (subroutine PRMTE) and the evaluation and lookup process repeated. When all numerical operators have been processed, a new ordering of the equations is generated (this is a random process performed in ONEMR which also keeps track of which permutations have already been made) and the entire solution process is repeated. Up to 720 solutions are permitted, so all solutions will be generated even for relatively large problems. Sample program outputs appear in Figure 2.

V. DISCUSSION

In an earlier section, we have seen what program BUCKY does. Namely, it forms solutions to the Buckingham Pi Theorem, evaluates these solutions, and finally, identifies them when possible. In addition, reordering the dimensional equations allows for different solution sets to be formed. When there are few equations to be solved, every possible solution is formed; therefore, an optimal solution is always generated[3,4,5]. Furthermore, numeric substitution into the symbolic form generates scalar values (evaluated solutions) which may be kept in a data base for further use. When an investigator finds a solution particularly suited to his needs, selective retrieval of evaluated solutions allow data to be plotted as well as used in regression, clustering, or factor analysis. As a last step, an attempt is made to identify each solution (primary and algebraically manipulated) by comparing solutions with well-known nondimensional numbers. It should also be noted that by keeping solutions in symbolic form one does not need to refer to other documents to determine which variables are involved in the solution.

There are some shortcomings to program BUCKY. First, no attempt is made to verify that there is a consistent set of equations (none of the other computer programs does this either). Failure to provide a consistent set of equations result in a singular matrix which cannot be inverted. Second, the program does not provide for an automatic change of units[6]. This is correctable by providing conversion factors and having a separate conversion calculation phase before evaluation occurs. Finally, while no other program attempts to identify solutions, not all well-known solutions are tabled. This can be easily corrected by adding solutions to the table. In addition, some well-known solutions are not recognized because of dimensional substitutions which can be but are not made. For instance,

$$\frac{A^{1/2}\rho V}{\mu}$$

is a Reynold's number; however, the program does not realize that $A^{1/2} = L$ (A is area, L is length). This fault can also be corrected by substituting fundamental units for variables, however, the expense of making all such substitutions does not seem to warrant the gain (there are always numerous little things one can do!)

No further extension of this work is contemplated at the present time.

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