# COMPUTING REAL ZEROS OF POLYNOMIALS WITH PARAMETRIC COEFFICIENTS

by

Attilio Colagrossi, Alfonso M. Miola
Istituto di Analisi dei Sistemi ed Informatica
Via Buonarroti 12, 00185 Roma, Italy

## ABSTRACT

The problem of localizing real zeros of polynomials with parametric coefficients is considered.

An informative solution to this problem is proposed and an algorithm based on a generalization of Sturm's method for univariate polynomials over the reals is presented. Thus, given a polynomial $P(x,y)$, where x is the variable and y a parameter, and a real interval $I_x$ for the variable x, the algorithm furnishes a list (eventually empty) of real intervals $I_{iy}$ for the parameter y, such that there exist i real simple zeros of $P(x,y)$ in the two dimension interval determined by $I_x$ and $I_{iy}$.

## 1. INTRODUCTION

It is well known that analytical solutions of univariate polynomial equations can be obtained only for those equations of degree less than five. It is also well known that numerical and algebraic methods can be applied for computing real zeros of univariate polynomials with numerical coefficients (i.e. integers, floating-points numbers, gaussian integers).

But, in the other hand, many real problems in scientific applications involve operations on polynomials with parametric coefficients. For these kind of polynomials where the degree is higher than four there is not a possibility of finding their roots as a formal analytical expression of the coefficients.

In this case the numerical approach is available only if numerical values are assigned to the parameters, even if several attempts for several different values can be performed.

Therefore a complete formal solution of those kind of problems is generally not allowed. However methods to get many useful and powerful informations on the possible numerical solutions are definible without specifying numerical values for the parameters.

In the following a method of this type, based on a natural generalization of the Sturm's method, will be presented in order to achieve as many informations as possible on the given problems.

In fact, if a closed form solution for a given problem is not possible, some informations from a qualitative analysis on, for instance, the continuity or the differentiability of a function, together with the knowledge of the existence of points of singularity, of zeros, flexes, etc. are very helpful and sufficient to proceed. These informations altogether furnish a sort of an "informative solution" to the given problem. Let now

$$P(x,y) = \sum_{i=0}^{n} q_i(y)x^i \tag{1}$$

be a polynomial over $R[y]$, the univariate polynomials over the reals.

Let us consider the following problem: given a polynomial $P(x,y)$ as in (1), a real interval $I_x = (a,b]$ and an integer m with $0 < m \leq n$, determine a (eventually empty) real interval $I_y$ such that there exist m values $x_1,\ldots,x_m$ of the interval $I_x$ for each of which it is $P(x_i,y) = 0,\ \forall y \in I_y$.

Actually more than one of such intervals $I_y$ could exist for a fixed value of m. Furthermore this problem can also be posed for several values of m, with $0 < m \leq n$. Therefore a more general problem can be formulated as follows:

PROBLEM: given $P(x,y)$ as in (1), let $I_x = (a,b]$ be a real interval for x.
Determine $F_y = \{I_{oy}, I_{1y}, \dots, I_{my}\}$, $m \leq n$, a family of sets $I_{ky}$ of real intervals $I_y$ for the y variable, such that $\forall I_y \in I_{ky}$, $0 \leq k \leq m$, $\forall \bar{y} \in I_y$ there exist k values $x_1, x_2, \dots, x_k$ belonging to $I_x$ such that:

$$P(x_j, \bar{y}) = 0, \quad \forall j, \ 1 \leq j \leq k$$

The root finding problem for univariate polynomials has been well investigated for both numerical and algebraic methods by Heindel [3], that proposed Sturm's approach and Collins [1], that gave the theory of algebraic cylindrical decomposition. In this paper some results of both those works are used.

## 2. A GENERALIZATION OF THE STURM'S METHOD

In order to solve this more general problem let:

$$S = \{P_o(x,y), P_1(x,y), P_2(x,y), \dots, P_k(x,y)\} \qquad (2)$$

be a bivariate polynomial sequence constructed in the following way:

$$P_o(x,y) = P(x,y)$$

$$P_1(x,y) = \frac{\partial}{\partial x} P(x,y)$$

$$P_i(x,y) = -\mathrm{prem}(P_{i-2}, P_{i-1}), \quad i = 2, \dots, k$$

where $\mathrm{prem}(P_{i-2}, P_{i-1})$ is the pseudo-remainder of $P_{i-2}(x,y)$ and $P_{i-1}(x,y)$, polynomials as in (1).
Given the real interval $I_x = (a,b]$, we obtain the following sequences:

$$\begin{aligned} S_a &= \{P_o^{(a)}(y), P_1^{(a)}(y), P_2^{(a)}(y), \dots, P_k^{(a)}(y)\} \\ S_b &= \{P_o^{(b)}(y), P_1^{(b)}(y), P_2^{(b)}(y), \dots, P_k^{(b)}(y)\} \end{aligned} \qquad (3)$$

by evaluating the sequence S in (2) for x = a and x = b respectively.

Now, given $\bar{y} \in \mathbb{R}$ and the following real numerical sequences:

$$\begin{aligned} \bar{S}_a &= \{P_o^{(a)}(\bar{y}), P_1^{(a)}(\bar{y}), P_2^{(a)}(\bar{y}), \dots, P_k^{(a)}(\bar{y})\} \\ \bar{S}_b &= \{P_o^{(b)}(\bar{y}), P_1^{(b)}(\bar{y}), P_2^{(b)}(\bar{y}), \dots, P_k^{(b)}(\bar{y})\} \end{aligned} \qquad (4)$$

let $v_a$ and $v_b$ be the variation in sign of $\bar{S}_a$ and $\bar{S}_b$ respectively.

Then , by Sturm's theorem, the univariate polynomial in x, $P(x,\bar{y})$, has exactly $v_a - v_b$ real zeros when $x \in I_x$.

To solve the problem previously stated in the introduction, we will use this result together with the following definition:

DEFINITION: A "uniformity interval" (in sign) for a univariate polynomial $Q(y)$ over the reals, is a real interval $(c,d)$ such that:

$$\mathrm{sign}(Q(h)) = \mathrm{sign}(Q(t)), \ \forall h,t \in (c,d), \ h \neq t$$

and

$$Q(y) \neq 0, \quad \forall y \in (c,d)$$

Using this definition, let us consider a real interval $I_y$ which is a uniformity interval for all the polynomials $P_i^{(a)}(y)$ and $P_i^{(b)}(y)$, for $i=0,\dots,k$, of the sequences $S_a$ and $S_b$ in (3).

Given a value $\bar{y} \in I_y$, let $v_a$ and $v_b$ be the variations in sign of $\bar{S}_a$ and $\bar{S}_b$ as in (4); whatever we choose the value $\bar{y} \in I_y$, the sequences $\bar{S}_a$ and $\bar{S}_b$, by definition of uniformity interval always present $v_a$ and $v_b$ variations in sign. Then the given polynomial $P(x,\bar{y})$ has $v_a - v_b$ real zeros, $\forall x \in I_x$ and $\forall \bar{y} \in I_y$.

Then Sturm's theorem is also applicable to bivariate polynomials when a real interval for x and at least one uniformity interval for all the polynomials $P_i^{(a)}(y)$ and $P_i^{(b)}(y)$ are given.

## 3. AN OUTLINE OF THE ALGORITHM FOR COMPUTING REAL ZEROS OF POLYNOMIALS WITH PARAMETRIC COEFFICIENTS

Let us now present a description of the computational steps to be performed to solve the problem stated in the first paragraph, according to the above results.

Given a bivariate polynomial $P(x,y)$ as in (1) and a real interval $I_x = (a,b]$, the following steps must be performed to solve the problem posed at beginning:

Step 1): construct the bivariate polynomial sequence S as in (2) for $P(x,y)$;

Step 2): evaluate S for $x = a$ and $x = b$, obtaining respectively $S_a$ and $S_b$ as in (3);

Step 3): determine the uniformity intervals for each polynomials of $S_a$ and $S_b$;

Step 3.1): determine two sets of all the real zeros of the polynomials belonging to $S_a$ and $S_b$:

$$M_i^{(a)} = \{\mu_{i,1}^{(a)}, \mu_{i,2}^{(a)}, \ldots, \mu_{i,s}^{(a)}\}$$

$$M_i^{(b)} = \{\mu_{i,1}^{(b)}, \mu_{i,2}^{(b)}, \ldots, \mu_{i,r}^{(b)}\}$$

$\forall i, \quad 0 \leq i \leq k, \quad$ with:

$$s \leq \sum_{j=0}^{k} \deg(P_j^{(a)})$$

$$r \leq \sum_{j=0}^{k} \deg(P_j^{(b)}) \ ;$$

Step 3.2): arrange in increasing order the elements of:

$$M = (\bigcup_{i=0}^{k} M_i^{(a)}) \cup (\bigcup_{i=0}^{k} M_i^{(b)})$$

obtaining:

$$M = \{\mu_1, \mu_2, \ldots, \mu_z\}$$

with $z \leq r+s$ and $\mu_i \neq \mu_j$ for $i \neq j$, $i = 1,\ldots,z; \quad j = 1,\ldots,z$ ;

Step 3.3): the uniformity intervals for all the polynomials of $S_a$ and $S_b$ are the following:

$$I_1 = (-\infty, \mu_1)$$
$$I_2 = (\mu_1, \mu_2)$$
$$\vdots$$
$$I_z = (\mu_{z-1}, \mu_z)$$
$$I_{z+1} = (\mu_z, +\infty);$$

Step 4): Compute the number of real zeros of $P(x,y)$:

Step 4.1): define the operator $s(Q,I_y)$ in the following way:

$$s(Q,I_y) = \begin{cases} +1 \text{ if } Q(y) > 0 & \forall y \in I_y \\ -1 \text{ if } Q(y) < 0 & \forall y \in I_y \ ; \end{cases}$$

Step 4.2): compute the following $2(z+1)$ sequences:

$$V_1^{(a)} = \{s(P_o^{(a)},I_1), s(P_1^{(a)},I_1), \ldots, s(P_k^{(a)},I_1)\}$$

$$\vdots$$

$$V_{z+1}^{(a)} = \{s(P_o^{(a)},I_{z+1}, s(P_1^{(a)},I_{z+1}), \ldots, s(P_k^{(a)},I_{z+1})\}$$

$$V_1^{(b)} = \{s(P_o^{(b)},I_1), s(P_1^{(b)},I_1), \ldots, s(P_k^{(b)},I_1)\}$$

$$\vdots$$

$$V_{z+1}^{(b)} = \{s(P_o^{(b)},I_{z+1}), s(P_1^{(b)},I_{z+1}), \ldots, s(P_k^{(b)},I_{z+1})\};$$

Step 4.3): compute $v_j^{(a)}$, the number of variations in sign of $V_j^{(a)}$, and $v_j^{(b)}$, the number of variations in sign of $V_j^{(b)}$, $\forall j$, $1 \leq j \leq z+1$, to get the number of real zeros of $P(x,y)$, $\forall y \in I_y$, $x \in (a,b]$.

## 4. FURTHER DEVELOPMENTS

The proposed algorithm can be applied in several practical situations. In fact many applications require the knowledge of the behaviour of the real zeros of polynomials including parameters in their coefficients.

Given a polynomial $P(x,y)$ in one variable x and in one parameter y, and a real interval $I_x$ for the variable x, the algorithm furnish a list$\{I_{yi}\}$ of real intervals for the parameter y such that there exist i real zeros of $P(x,y)$, $\forall x \in I_x$, $\forall y \in I_{yi}$, $i = 1,\ldots,\ell$, where $\ell \leq \deg(P)$.

However, the following cases can also be given:

a) given a polynomial $P(x,y)$ and a real interval $I_y$ for the parameter y, find the existence of real zeros for $y \in I_y$;

b) given a polynomial $P(x,y)$, a real interval $I_x$ for the variable x and a real interval $I_y$ for the parameter y, find the existence of real zeros for $x \in I_x$ and $y \in I_y$.

The cases a) and b) occur often in real practice, and the algorithm given in the previous section can be easily arranged to answer such questions.

However the proposed approach is one of the possible ways to look at the zerofinding problem for parametric polynomial. It would be actually interesting to investigate other research directions.

For instance, a probabilistic approach using a generalization of Rabin's algorithm [5] to the multivariate case may be attempted following the classical schema:

- map the problem in a finite field
- apply Rabin's algorithm
- lift to the true solutions

The crucial point of the proposed algorithm is the computation of the uniformity intervals, that is, the computation of real zeroes of some univariate polynomials. Since we approximate a real zero by a pair of rational numbers, we also may use a p-adic arithmetic to cope the numerical errors of the computation.

The mapping between rationals and p-adic numbers together with suitable p-adic algorithms as described in [4] are actually available.

REFERENCES


[1] G.E. COLLINS: Quantifier elimination for real closed fields by cylindrical algebraic decomposition, in Second GI Conference on Automata Theory and Formal Languages, vol. 33 of Lecture Notes in Computer Science. Springer-Verlag, Berlin 1975, pp. 134-183.

[2] R.T. GREGORY: Error-free computation with rational numbers, BIT 21(1981), 194-202.

[3] L.E. HEINDEL: Integer arithmetic algorithms for polynomial real zero determination, J. ACM, vol. 18, n. 4, oct. 71.

[4] A.M. MIOLA: The conversion of Hensel codes to their rational equivalents, SIGSAM Bulletin, this issue.

[5] M.O. RABIN: Probabilistic algorithm in finite field, SIAM J. Comp., vol. 9, n. 2, May '80.

[6] J.V. USPENSKY: Theory of equations; Mc Graw-Hill, New York, 1948.