# Constant-Depth Frege Systems with Counting Axioms Polynomially Simulate Nullstellensatz Refutations 

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#### Abstract

We show that constant-depth Frege systems with counting axioms modulo $m$ polynomially simulate Nullstellensatz refutations modulo $m$. Central to this is a new definition of reducibility from formulas to systems of polynomials with the property that, for most previously studied translations of formulas to systems of polynomials, a formula reduces to its translation. When combined with a previous result of the authors, this establishes the first size separation between Nullstellensatz and polynomial calculus refutations. We also obtain new, small refutations for certain CNFs by constant-depth Frege systems with counting axioms.


## 1 Introduction

This paper studies proof sizes in propositional systems that utilize modular counting in limited ways. The complexity of propositional proofs has received much attention in recent years because of its connections to computational and circuit complexity 17202517 . In particular, $N P$ equals coNP if and only if there exists a propositional proof system that proves every tautology in size polynomial in the size of the tautology [17. But before we can prove lower bounds for all proof systems, it seems necessary that we be able to prove lower bounds for specific proof systems. There was much initial success showing lower bounds for constant-depth proof systems 12224. While these proof systems can simulate many powerful theorem proving techniques, such as resolution, they cannot perform reasoning that involves modular counting. For this reason, there has been much interest in recent years regarding proof systems that incorporate modular counting in different ways. Three such systems are: constant-depth Frege systems augmented with counting axioms [23651427819] (counting axioms state that a set of size $N$ cannot be partitioned into sets of size $m$ when

[^0]$N$ is indivisible by $m$ ), the Nullstellensatz system 51481512, which captures static polynomial reasoning, and the polynomial calculus [1626181013, which captures iterative polynomial reasoning.

We show that constant-depth Frege systems with counting axioms modulo $m$ polynomially simulate Nullstellensatz refutations modulo $m$. This allows us to transform Nullstellensatz refutations into constant-depth Frege with counting axioms proofs with a small increase in size, and to infer size lower bounds for Nullstellensatz refutations from size lower bounds for constant-depth Frege with counting axioms proofs. In particular, this method establishes the first superpolynomial size separation between Nullstellensatz and polynomial calculus refutations.

Our simulation also shows that previously used proof techniques were not only sufficient but necessary. Papers such as 514819 prove size lower bounds for constant-depth Frege systems with counting axioms by converting small proofs into low degree Nullstellensatz refutations. The existence of such low degree Nullstellensatz refutations is then disproved by algebraic and combinatorial means. Low degree Nullstellensatz refutations are small (because there are few monomials), so our simulation shows that if there were such a low degree Nullstellensatz refutations, there would be a small constant-depth Frege with counting axioms proof. Therefore, Nullstellensatz degree lower bounds are necessary for size lower bounds for constant-depth Frege systems with counting axioms.

It is not immediately clear how to compare constant-depth Frege systems with Nullstellensatz refutations because Frege systems prove propositional formulas in connectives such as $\bigwedge, \bigvee$ and $\neg$, and the Nullstellensatz system shows that systems of polynomials have no common roots. We propose a new definition of reducibility from propositional formulas to systems of polynomials: a formula $F$ reduces to a system of polynomials over $\mathbb{Z}_{m}$ if we can use $F$ to define an $m$-partition (a partition in which every class consists of exactly $m$ elements) on the satisfied monomials of the polynomials. The simulation shows that if a formula has a small reduction to a set of polynomials with a small Nullstellensatz refutation, then the formula has a small refutation in constant-depth Frege with counting axioms. This notion of reduction seems natural in that for previously studied translations of formulas into systems of polynomials, a formula reduces to its translation.

### 1.1 Outline of the Paper

In section 2 we give some definitions that we will we use in the rest of the paper.
The simulation of Nullstellensatz refutations modulo $m$ by constant-depth Frege systems with counting axioms modulo $m$ works by defining two different $m$-partitions on the satisfied monomials in the expansion of the Nullstellensatz refutation. One covers the satisfied monomials perfectly, and the other leaves out exactly one satisfied monomial. In section 3 we show that Frege systems with counting axioms can prove in constant depth and polynomial size that such a partition can not exist.

Section 4 formalizes our definition of reducibility from propositional formulas to systems of polynomials and proves the main simulation theorem.

In section 5 we show that, for several methods of translating propositional formulas into systems of polynomials, a formula efficiently reduces to its translation.

We explore some applications of the simulation in section 6 First, we obtain small constant-depth Frege with counting axioms refutations for unsolvable systems of linear equations in which each equation contains a small number of variables. This class of tautologies includes the Tseitin tautologies and the " $\tau$ formulas" for Nisan-Wigderson pseudorandom generators built from the parity function 421. The Tseitin tautologies on a constant degree expander can be expressed as an unsatisfiable set of constant-width clauses, and are known to require exponential size to refute in constant-depth Frege systems [9. Therefore, as a corollary, we obtain an exponential separation of constant-depth Frege systems with counting axioms and constant-depth Frege systems with respect to constant-width CNFs.

## 2 Definitions, Notation and Conventions

In this paper, we perform many manipulations on partitions of sets into pieces of a fixed size. We make use of the following definitions:

Definition 1. Let $S$ be a set. The set $[S]^{m}$ is the collection of m element subsets of $S ;[S]^{m}=\left\{e|e \subseteq S,|e|=m\}\right.$. For $e, f \in[S]^{m}$, we say that $e$ conflicts with $f, e \perp f$, if $e \neq f$ and $e \cap f \neq \emptyset$.

When $N$ is a positive integer, we write $[N]$ for the set of integers $\{i \mid 1 \leq$ $i \leq N\}$. The collection of $m$ element subsets of $[N]$ are denoted by $[N]^{m}$, not by $[[N]]^{m}$.

Throughout this paper, we use the word polynomial to mean "multivariate polynomial."

Definition 2. A monomial is a product of variables. A term is scalar multiple of a monomial.

Definition 3. For a monomial $t=\prod_{i \in I} x_{i}^{\alpha_{i}}$, its multilinearization, $\bar{t}$, is defined as $\bar{t}=\prod_{i \in I} x_{i}$. Let $f=\sum_{t} c_{t} t$ be a polynomial. The multilinearization of $f, \bar{f}$, is defined as $\bar{f}=\sum_{t} c_{t} \bar{t}$. We say that a polynomial $f$ is multilinear if $f=\bar{f}$.

Definition 4. Let $n>0$ be given, and let $x_{1}, \ldots, x_{n}$ be variables. Let $I \subseteq[n]$ be given. The monomial $x_{I}$ is defined to be $\prod_{i \in I} x_{i}$.

Notice that a multilinear polynomial $f$ in the variables $x_{1}, \ldots, x_{n}$ can be written as $\sum_{I \subseteq[n]} a_{I} x_{I}$.

### 2.1 Proof Systems

Propositional proof systems are usually viewed as deriving tautologies by applying inference rules to a set of axioms. However, it can be useful to take the dual view that such proof systems establish that a set of hypotheses is unsatisfiable by deriving FALSE from the hypotheses and axioms. Such systems are called refutation systems. The Nullstellensatz and polynomial calculus systems demonstrate that sets of polynomials have no common solution, and are inherently refutation systems. Frege systems are traditionally viewed as deriving tautologies, but for ease of comparison, we treat them as refutation systems.

Furthermore, we will be discussing propositional formulas and polynomials in the same set of variables. This is justified by identifying the logical constant FALSE with the field element 0 and the logical constant TRUE with the field element 1.

Constant-Depth Frege Systems A Frege system is a sound, implicationally complete propositional proof system over a finite set of connectives with a finite number of axiom schema and inference rules. By the methods of Cook and Reckhow [17], any two Frege systems simulate one another up to a polynomial factor in size and a linear factor in depth. For concreteness, the reader can keep in mind the following Frege system whose connectives are NOT gates, $\neg$, and unbounded fan-in OR gates, $\bigvee$, and whose inference rules are: (1) Axioms $\overline{A \vee \neg A}$, (2) Weakening $\frac{A}{A \vee B}$ (3) Cut $\frac{A \vee B(\neg A) \vee C}{B \vee C}$ (4) Merging $\frac{\vee X \vee \vee Y}{V(X \cup Y)}$ (5) Unmerging $\frac{\vee(X \cup Y)}{V X \vee \vee Y}$.

Let $\mathcal{H}$ be a set of formulas. A derivation from $\mathcal{H}$ is a sequence of formulas $f_{1}, \ldots, f_{m}$ so that for each $i \in[m]$, either $f_{i}$ is a substitution instance of an axiom, $f_{i}$ is an element of $\mathcal{H}$, or there exist $j, k<i$ so that $f_{i}$ follows from $f_{j}$ and $f_{k}$ by the application of an inference rule to $f_{j}$ and $f_{k}$.

For a given formula $F$, a proof of $F$ is a derivation from the empty set of hypotheses whose final formula is $F$.

For fixed set of hypotheses $\mathcal{H}$, a refutation of $\mathcal{H}$ is a derivation from $\mathcal{H}$ whose final formula is FALSE.

The size of a derivation is the total number of symbols appearing in it.
We say that a family of tautologies $\tau_{n}$, each of size $s(n)$, has polynomial size constant-depth Frege proofs (refutations) if there are constants $c$ and $d$ so that for all $n$, there is a proof (refutation) of $\tau_{n}$ so that each formula in the proof has depth at most $d$, and the proof (refutation) has size $O\left(s^{c}(n)\right)$.

Counting Axioms Modulo $\boldsymbol{m}$ Constant-depth Frege with counting axioms modulo $m$ is the extension of constant-depth Frege systems that has axioms that state for integers $m, N, m \geq 2$ and $N \not \equiv_{m} 0$, it is impossible to partition a set of $N$ elements into pieces of size $m$.

Definition 5. Let $m>1$ and $N \not \equiv_{m} 0$ be given. Let $V$ be a set of $N$ elements. For each $e \in[V]^{m}$, let there be a variable $x_{e}$.

$$
\text { Count } t_{m}^{V}=\bigvee_{v \in V}\left(\bigwedge_{\substack{e \in[V]^{m} \\ e \ni v}} \neg x_{e}\right) \vee \bigvee_{\substack{e, f \in[V]^{m} \\ e \perp f}}\left(x_{e} \wedge x_{f}\right)
$$

Frege with counting modulo $m$ derivations are Frege derivations that allow the use of substitution instances of $\operatorname{Count}_{m}^{[N]}\left(\right.$ with $\left.N \not \equiv_{m} 0\right)$ as axioms.

Nullstellensatz Refutations One way to prove that a system of polynomials $f_{1}, \ldots, f_{k}$ has no common roots is to give a list of polynomials $p_{1}, \ldots, p_{k}$ so that $\sum_{i=1}^{k} p_{i} f_{i}=1$. Because we are interested in translations of propositional formulas, we add the polynomials $x^{2}-x$ as hypotheses to guarantee all roots are zero-one roots.

Definition 6. For a system of polynomials $f_{1}, \ldots, f_{k}$ in variables $x_{1}, \ldots, x_{n}$ over a field $F$, a Nullstellensatz refutation of $f_{1}, \ldots, f_{k}$ is a list of polynomials $p_{1}, \ldots, p_{k}, r_{1}, \ldots, r_{n}$ satisfying the following equation:

$$
\sum_{i=1}^{k} p_{i} f_{i}+\sum_{j=1}^{n} r_{j}\left(x_{j}^{2}-x_{j}\right)=1
$$

For a polynomial $q$, a Nullstellensatz derivation of $q$ from $f_{1}, \ldots, f_{k}$ is a list of polynomials $p_{1}, \ldots, p_{k}, r_{1}, \ldots, r_{n}$ satisfying the following equation:

$$
\sum_{i=1}^{k} p_{i} f_{i}+\sum_{j=1}^{n} r_{j}\left(x_{j}^{2}-x_{j}\right)=q
$$

The degree of the refutation (derivation) is the maximum degree of the polynomials $p_{i} f_{i}, r_{j}\left(x_{j}^{2}-x_{j}\right)$.

We define the size of a Nullstellensatz refutation (derivation) to be the number of monomials appearing in $p_{1}, \ldots, p_{k}$ and $f_{1}, \ldots, f_{k}$.

Hilbert's weak Nullstellensatz guarantees that over a field, all unsatisfiable systems of polynomials have Nullstellensatz refutations [23. We can define Nullstellensatz refutations over any ring, but such systems are no longer complete. In this paper, we work with Nullstellensatz refutations of polynomials over $\mathbb{Z}_{m}$, and for the sake of generality, we make no assumptions on $m$ unless otherwise stated.

## Polynomial Calculus

Definition 7. Let $f_{1}, \ldots, f_{k}$ be polynomials over a field $F$. A polynomial calculus refutation of $f_{1}, \ldots, f_{k}$ over $F$ is a sequence of polynomials $g_{1}, \ldots, g_{m}$ so that, $g_{m}=1$, and for each $i \in[m]$, either $g_{i}$ is $f_{l}$ for some $l \in[k], g_{i}$ is $x_{l}{ }^{2}-x_{l}$ for some $l \in[n]$, $g_{i}$ is $a g_{j}+b g_{l}$ for some $j, l<i, a, b \in F$, or $g_{i}$ is $x_{l} g_{j}$ for some $j<i, l \in[n]$.

The size of a polynomial calculus refutation is the total number of monomials appearing in the polynomials of the refutation. The degree of a polynomial calculus refutation is the maximum degree of a polynomial that appears in the refutation.

## 3 Contradictory Partitions of Satisfied Variables

To simulate Nullstellensatz refutations in constant-depth Frege systems with counting axioms, we construct two partitions on the satisfied monomials of the refutation: one which covers the satisfied monomials exactly, and another which covers the satisfied monomials with $m-1$ new points. This is impossible, and in this section, we show that constant-depth Frege systems with counting axioms can prove that this is impossible with polynomial size proofs.

Definition 8. Let positive integers $n$ and $k$ be given. Let $u_{1}, \ldots, u_{n}$ be a set of Boolean variables. For each $e \in[n]^{m}$, let $y_{e}$ be a variable, and for each $e \in$ $[n+k]^{m}$, let $z_{e}$ be a variable. $\mathrm{CP}_{m}^{n, k}(\boldsymbol{u}, \boldsymbol{y}, \boldsymbol{z})$ is the negation of the conjunction of the following formulas:

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"every variable covered by the first partition is satisfied"
    for each e }\in[n\mp@subsup{]}{}{m},\mp@subsup{y}{e}{}->\mp@subsup{\bigwedge}{i\ine}{}\mp@subsup{u}{i}{
"every satisfied variable is covered by the first partition"
    for each i }\in[n],\mp@subsup{u}{i}{}->\mp@subsup{\bigvee}{e\nii}{}\mp@subsup{y}{e}{
"no two overlapping edges are used by the first partition"
    for each e, f\in[n] m}\mathrm{ with e }\perpf,\neg\mp@subsup{y}{e}{}\vee\neg\mp@subsup{y}{f}{
"every variable covered by the second partition is satisfied"
    for each e }\in[n+k\mp@subsup{]}{}{m},\mp@subsup{z}{e}{}->\mp@subsup{\bigwedge}{\begin{subarray}{c}{i\ine}\\{i\leqn}\end{subarray}}{\begin{subarray}{c}{i}\end{subarray}
"every satisfied variable is covered by the second partition"
    for each i\in[n],ui}->\mp@subsup{\}{e\nii}{}\mp@subsup{z}{e}{
"every extra point is covered by the second partition"
    for each i, n+1\leqi\leqn+k, \bigvee e\nii}\mp@subsup{z}{e}{
"no two overlapping edges are used by the second partition"
    for each e, f\in[n+k]}\mp@subsup{]}{}{m}\mathrm{ with e }\perpf,\neg\mp@subsup{z}{e}{}\vee\neg\mp@subsup{z}{f}{
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Lemma 1. Fix $m$ and $k$ so that $m$ is not divisible by $k$. For all $n$, the tautology $C P_{m}^{n, k}$ has a constant depth, size $O\left(n^{m}\right)$ proof in constant-depth Frege with counting modulo $m$ axioms.

Proof. Fix $m, n$ and $k$. The proof of $\mathrm{CP}_{m}^{n, k}$ is by contradiction. We define a set $U$ of size $m n+k$ and formulas $\phi_{e}$ for each $e \in[U]^{m}$ so that we can derive $\left(\neg\right.$ Count $\left._{m}^{U}\right)\left[x_{e} \leftarrow \phi_{e}\right]$ in size $O\left(n^{m}\right)$ from the hypothesis $\neg \mathrm{CP}_{m}^{n, k}$.

Let $U$ be the set consisting of the following points: $p_{r, i}, r \in[m], i \in[n]$ (the $r$ 'th copy of the row of variables) and $p_{m, i}, n+1 \leq i \leq k$ (the extra points.)
"when $u_{i}$ is unset, we group together its copies"
for each $i \in[n], \phi_{\left\{p_{1, i}, \ldots, p_{m, i}\right\}}=\neg u_{i}$
"in the first $m-1$ rows, use the partition given by the $y_{e}$ 's"
for each $r \in[m-1]$, each $i_{1}, \ldots, i_{m} \in[n], \phi_{\left\{p_{r, i_{1}}, \ldots, p_{r, i_{m}}\right\}}=y_{\left\{i_{1}, \ldots, i_{m}\right\}}$
"in the last row, use the the partition given by the $z_{e}$ 's"
for each $i_{1}, \ldots, i_{m} \in[n+k], \phi_{\left\{p_{m, i_{1}}, \ldots, p_{m, i_{m}}\right\}}=z_{\left\{i_{1}, \ldots, i_{m}\right\}}$
other edges are not used
for all other $e \in[U]^{m}, \phi_{e}=0$
Now we sketch the derivation of $\left(\neg\right.$ Count $\left._{m}^{U}\right)\left[x_{e} \leftarrow \phi_{e}\right]$ from $\neg \mathrm{CP}_{m}^{n, k}$. It is easily verified that the derivation has constant depth and size $O\left((m n+k)^{m}\right)=$ $O\left(n^{m}\right)$.
"Every point of $U$ is covered by the partition."
Let $p_{r, i} \in U$ with $i \in[n], r \in[m-1]$ be given. From $\neg \mathrm{CP}_{m}^{n, k}$ derive $u_{i} \rightarrow$ $\bigvee_{\substack{f \in[n]^{m} \\ f \rightrightarrows i}} y_{f}$. Because $\bigvee_{\substack{f \in[n]^{m} \\ f \ni i}} y_{f}$ is a sub-disjunction of $\bigvee_{\substack{e \in[U]^{m} \\ e \ni r_{r, i}}} \phi_{e}$, we may derive $u_{i} \xrightarrow{f \ni i} \bigvee_{\substack{e \in[U]^{m} \\ e \ni p_{r, i}}} \phi_{e}$ with a weakening inference. Because ${ }_{\substack{\left.f \ni p_{r, i} \\ \phi_{1, i}, \ldots, p_{m, i}\right\}}}=\neg u_{i}$, we may derive $\neg u_{i} \rightarrow \bigvee_{\substack{\left.e \in\lceil U]^{m} \\ e \exists\right]_{r, i}}} \phi_{e}$. Combining these two formulas yields $\bigvee_{\substack{e \in[U]^{m} \\ e \ni p_{r, i}}} \phi_{e}$. The case for $p_{m, i}, i \in[n]$ is similar.

For a point $p_{m, i}, n+1 \leq i \leq n+k$, from $\neg \mathrm{CP}_{m}^{n, k}$ derive $\bigvee_{\substack{f \in[n+k]^{m} \\ f \ni i}} z_{f}$. A weakening inference applied to this derives $\bigvee_{e \ni p_{m, i}} \phi_{e}$.
"No overlapping edges are used."
Let $e_{1}, e_{2} \in[U]^{m}$ be given so that $e_{1} \perp e_{2}$, and neither $\phi_{e_{1}}$ nor $\phi_{e_{2}}$ is identically 0 .

If $\phi_{e_{1}}=\neg u_{i}$ and $\phi_{e_{2}}=y_{f}$, then $e_{1}$ is $\left\{p_{r, i} \mid r \in[m]\right\}$ and $e_{2}$ is $\left\{p_{r, j} \mid j \in f\right\}$ for some $r \in[m]$ and $f \in[n]^{m}$ so that $i \in f$. From $\neg \mathrm{CP}_{m}^{n, k}$ derive $y_{f} \rightarrow u_{i}$. From this, derive $\neg \neg u_{i} \vee \neg y_{f}=\neg \phi_{e_{1}} \vee \neg \phi_{e_{2}}$.

If $\phi_{e_{1}}=y_{f_{1}}$ and $\phi_{e_{2}}=y_{f_{2}}$, then $e_{1}$ is $\left\{p_{r_{1}, i} \mid i \in f_{1}\right\}$ and $e_{2}$ is $\left\{p_{r_{2}, i} \mid i \in f_{2}\right\}$ with $r_{1}=r_{2}$ and $f_{1} \perp f_{2}$. From $\neg \mathrm{CP}_{m}^{n, k}$ derive $\neg y_{f_{1}} \vee \neg y_{f_{2}}=\neg \phi_{e_{1}} \vee \neg \phi_{e_{2}}$.

The only other cases are when $\phi_{e_{1}}=\neg u_{i}$ and $\phi_{e_{2}}=z_{f}$ or $\phi_{e_{1}}=z_{f_{1}}$ and $\phi_{e_{2}}=z_{f_{2}}$, and these are handled similarly.

## 4 The Simulation

Because we work over $\mathbb{Z}_{m}$, a polynomial vanishes on a given assignment if and only if there is an $m$-partition on its satisfied monomials (recall that we treat a monomial with coefficient $a$ as having $a$ distinct copies.) The definability of this partition is the connection between refuting a propositional formula and refuting a system of polynomials.

### 4.1 Reducing Formulas to Systems of Equations

The method we use to reduce a formula to a system of polynomials is to define a partition on the satisfied monomials of the polynomials with small, constantdepth formulas and prove that these formulas define a partition using the formula as a hypothesis.

Because of the central role played by the sets of monomials appearing in each polynomial, we take a moment to define this notion precisely. First of all, because we are concerned only with $0 / 1$ assignments, a polynomial vanishes if and only if its multilinearization vanishes. For this reason, we restrict our attention to multilinear polynomials. We treat a term $a x_{I}$ as $a$ distinct copies of the monomial $x_{I}$. For this reason, when we talk about the "set of monomials" of a polynomial, we do not mean the set of monomials that appear in the polynomial, but a set which includes $a$ copies of each monomial with coefficient $a$. We will generally identify $a x_{I}$ with $a$ objects $m_{1, I}, \ldots, m_{a, I}$. Think of $m_{c, I}$ as the $c^{\prime}$ 'th copy of the monomial $x_{I}$. There should be little confusion of the dual use of the symbol " $m$ " because when the symbol appears without a subscript it denotes the modulus, and when it appear with a subscript it denotes a monomial.

Definition 9. Let $f=\sum_{I \subseteq[n]} a_{I} x_{I}$ be a multilinear polynomial over $\mathbb{Z}_{m}$. The set of monomials of $f$ is the following set:

$$
M_{f}=\left\{m_{c, I} \mid I \subseteq[n], c \in\left[a_{I}\right]\right\}
$$

Definition 10. Let $x_{1}, \ldots, x_{n}$ be Boolean variables. Let $f$ be a multilinear polynomial in the variables $x_{1}, \ldots, x_{n}$. For each $E \in\left[M_{f}\right]^{m}$, let $\theta_{E}$ be a formula in $\boldsymbol{x}$. We say that the $\theta$ 's form an m-partition the satisfied monomials of $f$ if the following formula holds:

$$
\begin{aligned}
& \bigwedge_{E \in\left[M_{f}\right]^{m}}\left(\theta_{E} \rightarrow \bigwedge_{m_{c, I} \in E} \bigwedge_{k \in I} x_{k}\right) \wedge\left(\bigwedge_{\substack{E, F \in\left[M_{f}\right]^{m} \\
E \perp F}} \neg \theta_{E} \vee \neg \theta_{F}\right) \\
& \wedge \bigwedge_{m_{c, I} \in M_{f}}\left(\left(\bigwedge_{k \in I} x_{k}\right) \rightarrow \bigvee_{\substack{E \in\left[M_{f}\right]^{m} \\
E \ni m_{c, I}}} \theta_{E}\right)
\end{aligned}
$$

Definition 11. Let $x_{1}, \ldots, x_{n}$ be Boolean variables. Let $\Gamma(\boldsymbol{x})$ be a propositional formula. Let $F=\left\{f_{1}, \ldots, f_{k}\right\}$ be a system of polynomials over $\mathbb{Z}_{m}$ with a Nullstellensatz refutation $p_{1}, \ldots, p_{k}, r_{1}, \ldots, r_{n}$. If, for each $i \in[k]$, there are formulas $\beta_{E}^{i}(\boldsymbol{x}), E \in\left[M_{\bar{f}_{i}}\right]^{m}$, so that there is a size $T$, depth $d$ Frege derivation from $\Gamma(\boldsymbol{x})$ that, for each $i$, the $\beta^{i}$ 's form an m-partition on the satisfied monomials of $\bar{f}_{i}$, then we say that $\Gamma$ reduces to $F$ in depth $d$ and size $T$.

### 4.2 The Simulation

Theorem 1. Let $m>1$ be an integer. Let $x_{1}, \ldots, x_{n}$ be Boolean variables. Let $\Gamma(\boldsymbol{x})$ be a propositional formula, and let $F$ be a system of polynomials over $\mathbb{Z}_{m}$ so that $\Gamma$ reduces to $F$ in depth $d$ and size $T$. If there is a Nullstellensatz refutation of $F$ with size $S$, then there is a depth $O(d)$ Frege with counting axioms modulo $m$ refutation of $\Gamma(\boldsymbol{x})$ with size $O\left(S^{2 m} T\right)$.

Proof. Let $p_{1}, \ldots, p_{k}, r_{1}, \ldots, r_{n}$ be a size $S$ Nullstellensatz refutation of $F$. Let $\beta_{E}^{i}(\boldsymbol{x})$, for $i \in[k], E \in\left[M_{\bar{f}_{i}}\right]^{m}$, be formulas so that from $\Gamma$ there is a size $T$, depth $d$ proof that for each $i$ the $\beta_{E}^{i}(\boldsymbol{x})$ 's form an $m$-partition on the satisfied monomials of $\bar{f}_{i}$.

We obtain contradictory partitions of the the monomials that appear in the expansion of $\sum_{i=1}^{k} \bar{p}_{i} \bar{f}_{i}$ in which polynomials are multiplied and multilinearized, but no terms are collected. In other words, the set is the collection, over $i \in[k]$, of all pairs of monomials from $\bar{p}_{i}$ and $\bar{f}_{i}$.

$$
V=\bigcup_{i=1}^{k}\left\{\left(m_{c, I}, m_{d, J}, i\right) \mid m_{c, I} \in M_{\overline{p_{i}}}, m_{d, J} \in M_{\bar{f}_{i}}\right\}
$$

Notice that $|V|=O\left(S^{2}\right)$.
For each $v \in V, v=\left(m_{c, I}, m_{d, J}, i\right)$, let $\gamma_{v}=\bigwedge_{k \in I \cup J} x_{k}$. Think of these as the monomials. We will give formulas $\theta_{E}$, that define a partition on the satisfied monomials with $m-1$ many extra points, and $\eta_{E}$, that define a partition on the satisfied monomials with no extra points. We will give a $O\left(|V|^{m}+T\right)=$ $O\left(S^{2 m}+T\right)$ derivation from $\Gamma$ of the following:

$$
\neg \mathrm{CP}_{m}^{|V|, m-1}\left[u_{v} \leftarrow \gamma_{v}, y_{E} \leftarrow \theta_{E}, z_{E} \leftarrow \eta_{E}\right]
$$

On the other hand, by lemma $1 \mathrm{CP}_{m}^{|V|, m-1}$ has constant depth Frege proofs of size $O\left(|V|^{m}\right)$, so $\mathrm{CP}_{m}^{|V|, m-1}\left[u_{v} \leftarrow \gamma_{v}, y_{E} \leftarrow \theta_{E}, z_{E} \leftarrow \eta_{E}\right]$ has a constant depth Frege proof of size $O\left(|V|^{m} T\right)$. Therefore, $\Gamma$ has a depth $O(d)$ Frege refutation of size $O\left(S^{2 m} T\right)$.

The Partition with $m-1$ Extra Points
Notice that we have the following equation:

$$
\overline{\sum_{i=1}^{k} \bar{p}_{i} \bar{f}_{i}}=\overline{\sum_{i=1}^{k} p_{i} f_{i}+\sum_{j=1}^{n} r_{j}\left(x_{j}^{2}-x_{j}\right)}=1
$$

So when we collect terms after expanding $\sum_{i=1}^{k} \bar{p}_{i} \bar{f}_{i}$ and multilinearizing, the coefficient of every nonconstant term is 0 modulo $m$, and the constant term is 1 modulo $m$.

For each $S \subseteq[n]$, let $\left.V_{S}=\left\{\left(m_{c, I}, m_{d, J}, i\right)\right) \in V \mid I \cup J=S\right\}$. Think of these as the occurrences of $x_{S}$ in the multilinearized expansion.

For each $S \subseteq[n], S \neq \emptyset$, there is an $m$-partition on $V_{S}$, call it $\mathcal{P}_{S}$. Likewise, there is an $m$-partition on $V_{\emptyset} \cup[m-1]$, call it $\mathcal{P}_{\emptyset}$.

Define the formulas $\theta_{E}$ as follows: for each $E \in([V] \cup[m-1])^{m}$, if $E \in \mathcal{P}_{S}$ for some $S \subseteq[n]$ then $\theta_{E}=\bigwedge_{k \in S} x_{k}$, otherwise $\theta_{E}=0$.

Constant-depth Frege can prove that this is a m-partition of the satisfied monomials of $\sum_{i=1}^{k} \overline{\bar{p}_{i} \bar{f}_{i}}$ with $m-1$ extra points. The proof has size $O\left(|V|^{m}\right)$ and depth $O(1)$. It is trivial from the definition of $\theta_{E}$ that the edges cover only satisfied monomials. That every satisfied monomial $\bigwedge_{k \in S} x_{k}$ is covered is also trivial: the edge from $\mathcal{P}_{S}$ is used if and only if the term $x_{S}$ is satisfied. Finally, it
easily shown that the formulas for two overlapping edges are never both satisfied: only edges from $\mathcal{P}_{S}$ are used (regardless of the values of the $x$ 's), so for any pair of overlapping edges, $E \perp F$, one of the two formulas $\theta_{E}$ or $\theta_{F}$ is identically 0 .

The Partition with No Extra Points
The idea is that an $m$-partition on the satisfied monomials on $\bar{f}_{i}$ can be used to build an $m$-partition on the satisfied monomials of $t \bar{f}_{i}$, for any monomial $t$.

For each $E \in[V]^{m}$, define $\eta_{E}$ as follows: if $E=\left\{\left(m_{c, I}, m_{d_{l}, J_{l}}, i\right) \mid l \in[m]\right\}$ for some $i \in[k], m_{c, I} \in M_{f_{i}}$, then $\eta_{E}=\bigwedge_{k \in I} x_{k} \wedge \beta_{\left\{m_{d_{l}, J_{l}} \mid l \in[m]\right\}}$, otherwise, $\eta_{E}=0$.

There is a size $O\left(S+|V|^{m}\right)$, depth $O(d)$ Frege derivation from $\Gamma$ that the $\eta_{E}$ 's form an $m$-partition on the satisfied monomials of $\sum_{i=1}^{k} \overline{\bar{p}_{i} \bar{f}_{i}}$. We briefly sketch how to construct the proof. Begin by deriving from $\Gamma$, for each $i$, that the $\beta_{E}^{i}$ 's form an $m$-partition on the satisfied monomials of $\bar{f}_{i}$.
"Every satisfied monomial is covered." Let $\left(m_{c, I}, m_{d, J}, i\right) \in V$ be given. If $\bigwedge_{k \in I \cup J} x_{k}$ holds, then so do $\bigwedge_{k \in I} x_{k}$ and $\bigwedge_{k \in J} x_{k}$. Because the $\beta^{i}$ s form an $m$ partition on the satisfied monomials of $\bar{f}_{i}$, we may derive $\bigvee_{F \in\left[M_{f_{i}}\right]^{m}} \beta_{F}^{i}$. From this derive $\bigvee_{F \in\left[M_{f_{i}}\right]^{m}} \bigwedge_{k \in I} x_{k} \wedge \beta_{F}^{i}$. A weakening inference applied to this yields $\bigvee_{E \in[V]^{m}} \eta_{E}$.
"Every monomial covered is satisfied." Let $v=\left(m_{c, I}, m_{d, J}, i\right) \in V$ be given so that $v \in E$ and $\eta_{E}$ holds. For this to happen, $E=\left\{\left(m_{c, I}, m_{d_{l}, J_{l}}, i\right) \mid l \in[m]\right\}$. By definition,,$\eta_{E}=\bigwedge_{k \in I} x_{k} \wedge \beta_{\left\{m_{d_{l}, J_{l}} l l \in[m]\right\}}^{i}$, and therefore $\bigwedge_{k \in I} x_{k}$ holds. Because the $\beta^{i}$ 's form an $m$-partition on the satisfied monomials of $\bar{f}_{i}$, we have that $\bigwedge_{k \in J} x_{k}$ holds. Therefore $\bigwedge_{k \in I \cup J} x_{k}$ holds.
"No two conflicting edges $E$ and $F$ can have $\eta_{E}$ and $\eta_{F}$ simultaneously satisfied." If $E \perp F$, and neither $\theta_{E}$ nor $\theta_{F}$ is identically 0 , then they share the same $\bar{p}_{i}$ component. That is, there exists $i, m_{c, I} \in M_{\overline{p_{i}}}$ so that $E=\left\{\left(m_{c, I}, m_{d_{l}, J_{l}}, i\right) \mid l \in\right.$ $[m]\}$, and $F=\left\{\left(m_{c, I}, m_{d^{\prime}}, J^{\prime} J_{l}, i\right) \mid l \in[m]\right\}$. Because $E \perp F$, we have $\left\{m_{d_{l}, J_{l}} \mid\right.$ $l \in[m]\} \perp\left\{m_{d^{\prime},{ }_{l}, J^{\prime}{ }_{l}} \mid l \in[m]\right\}$. Because the $\beta^{i}$ 's form an $m$-partition on the satisfied monomials of $\bar{f}_{i}$, we can derive $\left.\neg \beta_{\left\{m_{d_{l}, J_{l}} \mid l \in[m]\right\}}^{i} \vee \neg \beta_{\left\{m_{d^{\prime} l}, J^{\prime} l\right.}^{i} \mid l \in[m]\right\}$. We weaken this formula to obtain $\left.\neg \beta_{\left\{m_{d_{l}, J_{l}} \mid l \in[m]\right\}}^{i} \vee \neg \beta_{\left\{m_{d^{\prime} l}, J^{\prime} l\right.}^{i} \mid l \in[m]\right\}, ~ \vee \bigvee_{k \in I} \neg x_{k}$, and from that derive $\neg\left(\bigwedge_{k \in I} x_{k} \wedge \beta_{\left\{m_{d_{l}, J_{l}} \mid l \in[m]\right\}}^{i}\right) \vee \neg\left(\bigwedge_{k \in I} x_{k} \wedge \beta_{\left\{m_{d^{\prime}, J^{\prime}{ }_{l}} \mid l \in[m]\right\}}^{i}\right)=$ $\neg \eta_{E} \vee \neg \eta_{F}$.

## 5 Translations of Formulas into Polynomials

### 5.1 Direct Translation of Clauses

For sets of narrow clauses, a common way to translate the clauses into polynomials is to map $x$ to $1-x, \neg x$ to $x$ and replace "OR" by multiplication. This is most commonly used for constant-width CNFs, and in this case, we show that clauses efficiently reduce to their translations.

Definition 12. [11] For a clause $C$ in variables $\boldsymbol{x}$, the direct translation of $C$, $\operatorname{tr}(C)$, is defined recursively as follows: (i) $\operatorname{tr}(\emptyset)=1$ (ii) $\operatorname{tr}(A \vee x)=\operatorname{tr}(A)(1-x)$ (iii) $\operatorname{tr}(A \vee \neg x)=\operatorname{tr}(A) x$

For a CNF $F$, the direct translation of $F, \operatorname{tr}(F)$, is the set $\{\operatorname{tr}(C) \mid C \in F\}$.
It is easily verified by induction that an for any clause $C$, a Boolean assignment satisfies $C$ if and only if it is a root of $\operatorname{tr}(C)$.

Whenever $C$ is satisfied, there exists an $m$-partition on the satisfied monomials of $\operatorname{tr}(C)$. Moreover, if $C$ contains at most $w$ variables, then the $m$-partition can be defined by depth two formulas of size $O\left(2^{w}\right)$, and by the completeness of constant-depth Frege systems, there is a constant depth derivation from $C$ of size $2^{O(w)}$ that these formulas define an $m$-partition on the satisfied monomials of $\operatorname{tr}(C)$. Therefore, $C$ reduces to $\operatorname{tr}(C)$ in constant depth and size $O\left(2^{w}\right)$.

Lemma 2. If $F$ is an unsatisfiable $C N F$ of $m$ clauses of width $w$, then $F$ is reducible to $\operatorname{tr}(F)$ in size $m 2^{O(w)}$ and depth $O(1)$.

### 5.2 Translations That Use Extension Variables

More involved translations of formulas into sets of polynomials use extension variables that represent sub-formulas. The simplest way of doing this would be to reduce an unbounded fan-in formula $\Gamma$ to a bounded fan-in formula, and then introduce one new variable $y_{g}$ per gate $g$, with the polynomial that says $y_{g}$ is computed correctly from its inputs. It is easy to give a reduction from $\Gamma$ to this translation, of depth $\operatorname{depth}(\Gamma)$ and size poly $(|\Gamma|)$. (We can define $y_{g}$ by the subformula rooted at $g$ and every polynomial would have constant size, so defining the partition is trivial.) However, this translation reveals little for our purposes because there is usually no small degree Nullstellensatz refutation of the resulting system of polynomials, even for trivial $\Gamma$. For example, say that we translated the formula $x_{1}, \neg\left(\left(\left(\left(x_{1} \vee x_{2}\right) \vee \cdots \vee x_{n}\right)\right.\right.$ this way. The resulting system of polynomials is weaker than the induction principles (see the end of this section) which require $\Omega(\log n)$ degree NS refutations [15].

We give an alternative translation of formulas into sets of polynomials so that the formula is unsatisfiable if the set of polynomials has no common root. A formula $f$ reduces to the set of polynomials with depth $O(\operatorname{depth}(f))$ and size $O(|f|)$. Moreover, for many previously studied unsatisfiable CNFs (such as the negated counting principles), this translation is the same as the previously studied translations (up to constant-degree Nullstellensatz derivations).

Definition 13. Let $f$ be a formula in the variables $x_{1}, \ldots, x_{n}$ and the connectives $\{\bigvee, \neg\}$. For each pair of subformulas $g_{1}$ and $g_{2}$ of $f$, we write $g_{1} \rightarrow g_{2}$ if $g_{1}$ is an input to $g_{2}$. Canonically order the subformulas of $f$, and write $g_{1}<g_{2}$ if $g_{1}$ precedes $g_{2}$ in this ordering. For each subformula $g$ of $f$, let there be a variable $y_{g}$ - the value of $g$. For each pair of subformulas of $f, g_{1}$ and $g_{2}$, so that the top connective of $g_{2}$ is $\bigvee$ and $g_{1} \rightarrow g_{2}$, let there be a variable $z_{g_{1}, g_{2}}$-" $g_{1}$ is the first satisfied input of $g_{2}$ ". The polynomial translation of $f, \operatorname{POLY}(f)$, is the following set of polynomials:

For each variable $x_{i}$ :
"The value of subformula $x_{i}$ is equal to $x_{i} "$

$$
y_{x_{i}}-x_{i}
$$

For each subformula $g$ whose top connective is $\bigvee$ :
"if $g_{1}<g_{2}, g_{1} \rightarrow g, g_{2} \rightarrow g$, and $g_{1}$ is satisfied,
then $g_{2}$ is not the first satisfied input of $g$ "
$y_{g_{1}} z_{g_{2}, g}$
"if $g_{1}$ is the first satisfied input of $g$,
then $g_{1}$ is satisfied"
$z_{g_{1}, g} y_{g_{1}}-z_{g_{1}, g}$
" $g$ is satisfied if and only if the some input to $g$ is the first satisfied input of $g$ "
$y_{g}-\sum_{g_{1} \rightarrow g} z_{g_{1}, g}$
For each subformula $g$ whose top connective is $\neg$ :
Let $g_{1}$ the unique input of $g$,
"if $g_{1}$ is satisfied if and only if $g$ is not satisfied" $y_{g_{1}}+y_{g}-1$
The formula $f$ is satisfied:
$y_{f}-1$
One can show by induction that if $f$ is satisfiable then $\operatorname{POLY}(f)$ has a common root. By the contrapositive, if $\operatorname{POLY}(f)$ has no common roots, then $f$ is unsatisfiable.

Lemma 3. Let $f$ be a Boolean formula in the variables $x_{1}, \ldots, x_{n}$. If $f$ is satisfiable, then $P O L Y(f)$ has a common 0/1 root.

Proof. Let $\alpha$ be a $0 / 1$ assignment to $x_{1}, \ldots, x_{n}$. For any propositional formula $g$, let $\alpha(g)$ denote the value of $g$ under the assignment $\alpha$.

Suppose that $\alpha(f)=1$. We extend $\alpha$ to the variables of $\operatorname{POLY}(f)$ as follows: For each subformula $g$ of $f$, let $\alpha\left(y_{g}\right)=\alpha(g)$. When $g=\bigvee g_{i}$ and $\alpha(g)=1$, let $i_{0}$ be the first input to $g$ so that $\alpha\left(g_{i}\right)=1$. Set $\alpha\left(z_{g_{i_{0}}}\right)=1$ and for $i \neq i_{0}$, set $\alpha\left(z_{g_{i}}\right)=0$. When $g=\bigvee g_{i}$ and $\alpha(g)=0, \alpha\left(z_{g_{i}, g}\right)=0$ for all $i$.

We now show by induction that $\alpha$ is a root of $\operatorname{POLY}(f)$. Clearly, for each variable $x_{i}, \alpha$ is a root of $y_{x_{i}}-x_{i}$. Consider a subformula $\neg g$. Because $\alpha\left(y_{\neg g}\right)=$ $\alpha(\neg g)$ and $\alpha\left(y_{g}\right)=\alpha(g)=1-\alpha(\neg g), \alpha$ is a root of $y_{\neg g}+y_{g}-1$. Consider a subformula $g=\bigvee_{i} g_{i}$. If $\alpha(g)=0$, then for all $i, \alpha\left(z_{g_{i}, g}\right)=0, \alpha\left(y_{g_{i}}\right)=0$ and $\alpha\left(y_{g}\right)=0$. In this case, $\alpha$ is clearly a root to $z_{g_{i}, g} y_{g_{i}}-z_{g_{i}, g}, y_{g_{i}} z_{g_{j}, g}$ and $y_{g}-\sum_{i} z_{g_{i}, g}$. In the case when $\alpha(g)=1$, there exists $i_{0}$ so that $\alpha\left(z_{g_{i_{0}}, g}\right)=1$ and for all $j \neq i_{0}, \alpha\left(z_{g_{j}, g}\right)=0$. Moreover, $\alpha\left(y_{g_{i_{0}}}\right)=1, \alpha\left(y_{g}\right)=1$ and for all $j<i_{0}, \alpha\left(y_{g_{j}}\right)=0$. Therefore, $\alpha$ is a root to $y_{g_{j}} z_{g_{i}, g}$ for all $i<j, z_{g_{i}, g} y_{g_{i}}-z_{g_{i}, g}$ for all $i$, and $y_{g}-\sum_{i} z_{g_{i}, g}$. Finally, $\alpha$ is a root of $y_{f}-1$ because $\alpha(f)=1$ by assumption.

The argument of lemma 3 can be carried in Frege systems with depth $O(\operatorname{depth}(f))$ and size $O(|f|)$.

Theorem 2. If $f$ is a formula in the variables $x_{1}, \ldots, x_{n}$ and the connectives $\{\bigvee, \neg\}$, then $f$ is reducible to $P O L Y(f)$ in depth $O(\operatorname{depth}(f))$ and size polynomial in $|f|$.

Proof. We proceed in two stages. First, we give a set of formulas, $\operatorname{EXT}(f)$, that is in the variables $x_{i}, y_{g}$ and $z_{g_{1}, g_{2}}$ and is analogous to the translation of $f$ into polynomials. We show that this translation has a constant depth, polynomial size reduction to $\operatorname{POLY}(f)$ and then show that $f$ has a depth $O(\operatorname{depth}(f))$ reduction to $\operatorname{EXT}(f)$ of size polynomial in $|f|$.

Let $\operatorname{EXT}(f)$ be the following set of formulas:

For each variable $x_{i}$ :

$$
y_{x_{i}} \leftrightarrow x_{i}
$$

For each subformula $g$ whose top connective is $\bigvee$ :
"if $g_{1}<g_{2}, g_{1} \rightarrow g, g_{2} \rightarrow g$, and $g_{1}$ is satisfied, then $g_{2}$ is not the first satisfied input of $g "$ $\neg y_{g_{1}} \vee \neg z_{g_{2}, g}$
"if $g_{1}$ is the first satisfied input of $g$, then $g_{1}$ is satisfied" $z_{g_{1}, g} \rightarrow y_{g_{1}}$
" $g$ is satisfied if and only if some input to $g$ is the first satisfied input of $g$ $y_{g} \leftrightarrow \bigvee_{g_{1} \rightarrow g} z_{g_{1}, g}$
For each subformula $g$ whose top connective is $\neg$ :
Let $g_{1}$ the unique input of $g$,
"if $g_{1}$ is satisfied then $g$ is not satisfied" $y_{g_{1}} \leftrightarrow \neg y_{g}$
The formula $f$ is satisfied:
$y_{f}$

There is a straightforward constant-depth, polynomial-size reduction of $\operatorname{EXT}(f)$ to $\operatorname{POLY}(f)$. For each polynomial of $\operatorname{POLY}(f)$, there is a formula of $\operatorname{EXT}(f)$ that reduces to the polynomial; the formula associated with each polynomial is given in table V.1. For the constant-size polynomials of $\mathrm{POLY}(f)$, the corresponding formula of $\operatorname{EXT}(f)$ implies that there is an $m$-partition on the satisfied variables of the polynomial. Because the polynomial involves a constant number of variables, the partition may be defined and proved correct in constant size, depth two.

Table 1. Polynomials and their Associated Formulas

| polynomial | associated formula |
| :---: | :---: |
| $y_{x_{i}}-x_{i}$ | $y_{x_{i}} \leftrightarrow x_{i}$ |
| $y_{g_{1}} z_{g_{2}, g}$ | $\neg y_{g_{1}} \vee \neg z_{g_{2}, g}$ |
| $z_{g_{1}, g} y_{g_{1}}-z_{g_{1}, g}$ | $z_{g_{1}, g} \rightarrow y_{g_{1}}$ |
| $y_{g_{1}}+y_{g}-1$ | $y_{g_{1}} \leftrightarrow \neg y_{g}$ |
| $y_{f}-1$ | $y_{f}$ |

The only polynomials of $\operatorname{POLY}(f)$ that involve a non-constant number of variables are those of the form $y_{g}-\sum_{g_{1} \rightarrow g} z_{g_{1}, g}$, and from the hypotheses of $\operatorname{EXT}(f)$ it can be shown that $y_{g}$ is satisfied if only if exactly one of the $z_{g_{1}, g}$ 's is satisfied. Because there are $(m-1)$ copies of each $z_{g_{1}, g}$ in such a polynomial, we can group $y_{g}$ with these copies of $z_{g_{1}, g}$ whenever $z_{g_{1}, g}$ is satisfied.

To reduce $f$ to $\operatorname{EXT}(f)$, it is easy to check to that there is a polynomial size, depth $O(\operatorname{depth}(f))$ derivation of the following substitution instance of $\operatorname{EXT}(f)$ from the hypothesis $f$. (The substitution instances of each formula are given in table V.2.)

$$
\operatorname{EXT}(f)\left[y_{g} \leftarrow g, z_{g_{1}, g} \leftarrow\left(g_{1} \wedge \bigwedge_{\substack{g_{2}<g_{1} \\ g_{2} \rightarrow g}} \neg g_{2}\right)\right]
$$

Table 2. Formulas and their Substitution Instances

| formula | substitution instance | comment |
| :---: | :---: | :---: |
| $y_{x_{i}} \leftrightarrow x_{i}$ | $x_{i} \leftrightarrow x_{i}$ |  |
| $\neg y_{g_{1}} \vee \neg z_{g_{2}, g}$ | $\neg g_{1} \vee \neg\left(g_{2} \wedge \bigwedge_{\substack{g_{3}<g_{2} \\ g_{3} \rightarrow g}} \neg g_{3}\right)$ | $g_{1}<g_{2}$ |
| $z_{g_{1}, g} \rightarrow y_{g_{1}}$ | $\left(g_{1} \wedge \bigwedge_{\substack{g_{2}<g_{1} \\ g_{2} \rightarrow g}} \neg g_{2}\right) \rightarrow g_{1}$ |  |
| $y_{g} \leftrightarrow \vee_{g_{1} \rightarrow g} z_{g_{1}, g}$ | $g \leftrightarrow \vee_{g_{1} \rightarrow g}\left(g_{1} \wedge \bigwedge_{\substack{g_{2}<g_{1} \\ g_{2} \rightarrow g}} \neg g_{2}\right)$ | $g=\vee_{g_{1} \rightarrow g} g_{1}$ |
| $y_{g_{1}} \leftrightarrow \neg y_{g}$ | $g_{1} \leftrightarrow \neg g$ | $g=\neg g_{1}$ |
| $y_{f}$ | $f$ |  |

Example: We illustrate our translation with a the clauses of the negated counting principles. The translation of this set of clauses turns out to be same (up to constant degree Nullstellensatz derivations) as the polynomial formulation of the counting principles previously studied.

Let $V$ be a set of cardinality indivisible by $m$. The clauses are $F_{v}=\bigvee_{e \ni v} x_{e}$ for $v \in V$ and $G_{e, f}=\neg x_{e} \vee \neg x_{f}$ for $e, f \in[V]^{m}$ with $e \perp f$. The standard translation of these systems has the polynomials $\sum_{e \ni v} x_{e}$, for $v \in V$, and $x_{e} x_{f}$, for $e \perp f$.

The polynomials introduced by the translation of $G_{e, f}$ are: $y_{x_{e}}-x_{e}, y_{x_{f}}-$ $x_{f}, y_{\neg x_{e}}+y_{x_{e}}-1, y_{\neg x_{f}}+y_{x_{f}}-1, y_{\neg x_{e}} z_{\neg x_{f}, G_{e, f}}, z_{\neg x_{e}, G_{e, f}} y_{\neg x_{e}}-z_{\neg x_{e}, G_{e, f}}$, $z_{\neg x_{f}, G_{e, f}} y_{\neg x_{f}}-z_{\neg x_{f}, G_{e, f}}, y_{G_{e, f}}-z_{\neg x_{e}, G_{e, f}}-z_{\neg x_{f}, G_{e, f}}$ and $y_{G_{e, f}}-1$. It is easy to check that thee is a constant degree derivation of $x_{e} x_{f}$ from these polynomials (in particular, a non-optimal but constant-degree derivation is given by the completeness of the Nullstellensatz system).

The polynomials introduced by the translation of $F_{v}$ are: $y_{x_{e}}-x_{e}, z_{y_{x_{e}}, F_{v}} y_{f}$ (for $e, f \ni v$ and $e<f$ ), $z_{y_{x_{e}}, F_{v}} y_{x_{e}}-z_{y_{x_{e}}, F_{v}}\left(\right.$ for $e \ni v$ ), $y_{F_{v}}-\sum_{e \ni v} z_{y_{x_{e}}, F_{v}}$ and $y_{F_{v}}-1$. With a degree two Nullstellensatz derivation we may derive $\sum_{e \ni v} z_{e, F_{v}} x_{e}-$ 1. Multiplying this by $\sum_{e \ni v} x_{e}$, and reducing using the previously derived polynomials $x_{e} x_{f}$ and the axioms $x_{e}^{2}-x_{e}$, yields $\sum_{e \ni v} z_{e, F_{v}} x_{e}-\sum_{e \ni v} x_{e}$. Subtracting this from $\sum_{e \ni v} z_{e, F_{v}} x_{e}-1$ yields $\sum_{e \ni v} x_{e}$.

## A Note on Translations of Formulas to Polynomials Using Extension Variables

Definition 14. The induction principle of length $M, \operatorname{IND}(M)$, is the following system of polynomials: $y_{1}, y_{r+1} y_{r}-y_{r+1}($ for $r<M)$ and $y_{M}-1$.

Theorem 3. [1512] The $I N D(M)$ system has Nullstellensatz refutations of degree $O(\log M)$ over any field. Moreover, over any field the system requires degree $\Omega(\log M)$ Nullstellensatz refutations.

The "standard" translation of $x_{n}, \neg\left(\left(\left(\left(\left(x_{n} \vee x_{n-1}\right) \vee \cdots \vee x_{1}\right)\right)\right)\right)$ into polynomials using extension variables introduces new variables $z_{1}, \ldots, z_{n-1}$, with polynomials $x_{n}-1,1-\left(1-x_{n}\right)\left(1-x_{n-1}\right)-z_{n-1}, 1-\left(1-z_{n-1}\right)\left(1-x_{n-2}\right)-z_{n-2}$, $\ldots, 1-\left(1-z_{2}\right)\left(1-x_{1}\right)-z_{1}$, and $z_{1}$. (The indices have been reversed from those of subsection 5.1 to ease the reduction.)

We may define this set of polynomials from $\operatorname{IND}(n)$ using the following definitions: $x_{i}:=y_{i}$ for $i, 1 \leq i \leq n$, and $z_{i}:=y_{i}$, for $i \leq n-1$. The polynomials $z_{1}=y_{1}$ and $x_{n}-1=y_{n}-1$ are belong to $\operatorname{IND}(n)$, and for each $r, 1 \leq r \leq n-2$,

$$
\begin{array}{ll}
1-\left(1-z_{r+1}\right)\left(1-x_{r}\right)-z_{r} & =1-\left(1-y_{r+1}\right)\left(1-y_{r}\right)-y_{r} \\
=1-\left(1+y_{r+1} y_{r}-y_{r}-y_{r+1}\right)-y_{r} & =-\left(y_{r+1} y_{r}-y_{r+1}\right)
\end{array}
$$

Similarly, $1-\left(1-x_{n}\right)\left(1-x_{n-1}\right)-z_{n-1}=-\left(y_{n} y_{n-1}-y_{n}\right)$.
Because there is a constant degree reduction from $\operatorname{IND}(n)$ to the standard translation of $x_{n}, \neg\left(\left(\left(\left(\left(x_{n} \vee x_{n-1}\right) \vee \ldots x_{1}\right)\right)\right)\right)$ into polynomials, this translation requires super-constant degree to refute in the Nullstellensatz system.

## 6 An Application to Unsatisfiable Systems of Constant-Width Linear Equations

Many tautologies studied in propositional proof complexity, such as Tseitin's tautologies [9] and the $\tau$ formulas of Nisan-Wigderson generators built from parity functions, can be expressed as inconsistent systems of linear equations over a field $\mathbb{Z}_{q}$ in which each equation involves only a small number of variables. We show that in such situations, constant-depth Frege with counting axioms modulo $q$ can prove these principles with polynomial size proofs.

Fix a prime number $q$. Let $A$ be an $m \times n$ matrix over $\mathbb{Z}_{q}$, let $x_{1}, \ldots, x_{n}$ be variables and let $\boldsymbol{b} \in \mathbb{Z}_{q}^{m}$ be so that $A \boldsymbol{x}=\boldsymbol{b}$ has no solutions. Let $w$ be the maximum number of non-zero entries in any row of $A$.

For each $i \in[m]$, let $A_{i}$ be the $i$ 'th row of $A$, and let $p_{i}$ be the polynomial $A_{i} \boldsymbol{x}-b_{i}$. Let $C_{i}$ the CNF that is satisfied if and only $p_{i}(\boldsymbol{x})=0$. Notice that $C_{i}$ has size at most $2^{w}$. The explicit encoding of $A \boldsymbol{x}=\boldsymbol{b}$ is the $\operatorname{CNF} \bigwedge_{i=1}^{m} C_{i}$.

The methods of subsection 5.1 show that $\bigwedge_{i=1}^{m} C_{i}$ is reducible to the system of polynomials $\left\{p_{1}, \ldots, p_{m}\right\}$ via a constant depth reduction of size $m 2^{O(w)}$. Moreover, the system of polynomials $\left\{p_{1}, \ldots, p_{m}\right\}$ has a degree one Nullstellensatz refutation given by Gaussian elimination. Moreover, degree one refutations are of size $O(m n)$. Thus we have the following theorem:

Theorem 4. Fix a prime number $q$. Let $A$ be an $m \times n$ matrix, let $x_{1}, \ldots, x_{n}$ be variables and let $\boldsymbol{b} \in \mathbb{Z}_{q}^{m}$ be so that $A \boldsymbol{x}=\boldsymbol{b}$ has no solutions. Let $w$ be the maximum number of non-zero entries in any row of $A$.

There is a constant depth Frege with counting axioms modulo $q$ refutation of the explicit encoding of $A \boldsymbol{x}=\boldsymbol{b}$ of size polynomial in $m, n$ and $2^{w}$.

The Tseitin graph tautologies on an expander graph are known to require exponential size constant-depth Frege proofs [9. Because these principles can be represented as an unsatisfiable system of linear equations, they have polynomial size constant-depth Frege with counting axioms proofs.
Corollary 1. There exists a family of unsatisfiable sets of constant width clauses that require exponential size constant-depth Frege refutations, but have polynomial size constant-depth Frege with counting axioms refutations.

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