Constant-Depth Frege Systems with Counting Axioms Polynomially Simulate Nullstellensatz Refutations

Russell Impagliazzo^{*} and Nathan Segerlind^{**}

Department of Computer Science University of California, San Diego La Jolla, CA 92093 {russell,nsegerli}@cs.ucsd.edu

Abstract. We show that constant-depth Frege systems with counting axioms modulo m polynomially simulate Nullstellensatz refutations modulo m. Central to this is a new definition of reducibility from formulas to systems of polynomials with the property that, for most previously studied translations of formulas to systems of polynomials, a formula reduces to its translation. When combined with a previous result of the authors, this establishes the first size separation between Nullstellensatz and polynomial calculus refutations. We also obtain new, small refutations for certain CNFs by constant-depth Frege systems with counting axioms.

1 Introduction

This paper studies proof sizes in propositional systems that utilize modular counting in limited ways. The complexity of propositional proofs has received much attention in recent years because of its connections to computational and circuit complexity [17,20,25,7]. In particular, NP equals coNP if and only if there exists a propositional proof system that proves every tautology in size polynomial in the size of the tautology [17]. But before we can prove lower bounds for all proof systems, it seems necessary that we be able to prove lower bounds for specific proof systems. There was much initial success showing lower bounds for constant-depth proof systems [1,22,24]. While these proof systems can simulate many powerful theorem proving techniques, such as resolution, they cannot perform reasoning that involves modular counting. For this reason, there has been much interest in recent years regarding proof systems are: constant-depth Frege systems augmented with counting axioms [2,3,6,5,14,27,8,19] (counting axioms state that a set of size N cannot be partitioned into sets of size m when

^{*} Research Supported by NSF Award CCR-9734911, grant #93025 of the joint US-Czechoslovak Science and Technology Program, NSF Award CCR-0098197, and USA-Israel BSF Grant 97-00188

^{**} Partially supported by NSF grant DMS-9803515

N is indivisible by m), the Nullstellensatz system [5,14,8,15,12], which captures static polynomial reasoning, and the polynomial calculus [16,26,18,10,13], which captures iterative polynomial reasoning.

We show that constant-depth Frege systems with counting axioms modulo m polynomially simulate Nullstellensatz refutations modulo m. This allows us to transform Nullstellensatz refutations into constant-depth Frege with counting axioms proofs with a small increase in size, and to infer size lower bounds for Nullstellensatz refutations from size lower bounds for constant-depth Frege with counting axioms proofs. In particular, this method establishes the first superpolynomial size separation between Nullstellensatz and polynomial calculus refutations.

Our simulation also shows that previously used proof techniques were not only sufficient but necessary. Papers such as [5,14,8,19] prove size lower bounds for constant-depth Frege systems with counting axioms by converting small proofs into low degree Nullstellensatz refutations. The existence of such low degree Nullstellensatz refutations is then disproved by algebraic and combinatorial means. Low degree Nullstellensatz refutations are small (because there are few monomials), so our simulation shows that if there were such a low degree Nullstellensatz refutations, there would be a small constant-depth Frege with counting axioms proof. Therefore, Nullstellensatz degree lower bounds are necessary for size lower bounds for constant-depth Frege systems with counting axioms.

It is not immediately clear how to compare constant-depth Frege systems with Nullstellensatz refutations because Frege systems prove propositional formulas in connectives such as \bigwedge , \bigvee and \neg , and the Nullstellensatz system shows that systems of polynomials have no common roots. We propose a new definition of reducibility from propositional formulas to systems of polynomials: a formula F reduces to a system of polynomials over \mathbb{Z}_m if we can use F to define an m-partition (a partition in which every class consists of exactly m elements) on the satisfied monomials of the polynomials. The simulation shows that if a formula has a small reduction to a set of polynomials with a small Nullstellensatz refutation, then the formula has a small refutation in constant-depth Frege with counting axioms. This notion of reduction seems natural in that for previously studied translations of formulas into systems of polynomials, a formula reduces to its translation.

1.1 Outline of the Paper

In section 2, we give some definitions that we will we use in the rest of the paper.

The simulation of Nullstellensatz refutations modulo m by constant-depth Frege systems with counting axioms modulo m works by defining two different m-partitions on the satisfied monomials in the expansion of the Nullstellensatz refutation. One covers the satisfied monomials perfectly, and the other leaves out exactly one satisfied monomial. In section 3, we show that Frege systems with counting axioms can prove in constant depth and polynomial size that such a partition can not exist. Section 4 formalizes our definition of reducibility from propositional formulas to systems of polynomials and proves the main simulation theorem.

In section 5 we show that, for several methods of translating propositional formulas into systems of polynomials, a formula efficiently reduces to its translation.

We explore some applications of the simulation in section 6. First, we obtain small constant-depth Frege with counting axioms refutations for unsolvable systems of linear equations in which each equation contains a small number of variables. This class of tautologies includes the Tseitin tautologies and the " τ formulas" for Nisan-Wigderson pseudorandom generators built from the parity function [4,21]. The Tseitin tautologies on a constant degree expander can be expressed as an unsatisfiable set of constant-width clauses, and are known to require exponential size to refute in constant-depth Frege systems [9]. Therefore, as a corollary, we obtain an exponential separation of constant-depth Frege systems with counting axioms and constant-depth Frege systems with respect to constant-width CNFs.

2 Definitions, Notation and Conventions

In this paper, we perform many manipulations on partitions of sets into pieces of a fixed size. We make use of the following definitions:

Definition 1. Let S be a set. The set $[S]^m$ is the collection of m element subsets of S; $[S]^m = \{e \mid e \subseteq S, |e| = m\}$. For $e, f \in [S]^m$, we say that e conflicts with $f, e \perp f, if e \neq f$ and $e \cap f \neq \emptyset$.

When N is a positive integer, we write [N] for the set of integers $\{i \mid 1 \leq i \leq N\}$. The collection of m element subsets of [N] are denoted by $[N]^m$, not by $[[N]]^m$.

Throughout this paper, we use the word polynomial to mean "multivariate polynomial."

Definition 2. A monomial is a product of variables. A term is scalar multiple of a monomial.

Definition 3. For a monomial $t = \prod_{i \in I} x_i^{\alpha_i}$, its multilinearization, \overline{t} , is defined as $\overline{t} = \prod_{i \in I} x_i$. Let $f = \sum_t c_t t$ be a polynomial. The multilinearization of f, \overline{f} , is defined as $\overline{f} = \sum_t c_t \overline{t}$. We say that a polynomial f is multilinear if $f = \overline{f}$.

Definition 4. Let n > 0 be given, and let x_1, \ldots, x_n be variables. Let $I \subseteq [n]$ be given. The monomial x_I is defined to be $\prod_{i \in I} x_i$.

Notice that a multilinear polynomial f in the variables x_1, \ldots, x_n can be written as $\sum_{I \subset [n]} a_I x_I$.

2.1 Proof Systems

Propositional proof systems are usually viewed as deriving tautologies by applying inference rules to a set of axioms. However, it can be useful to take the dual view that such proof systems establish that a set of hypotheses is unsatisfiable by deriving FALSE from the hypotheses and axioms. Such systems are called refutation systems. The Nullstellensatz and polynomial calculus systems demonstrate that sets of polynomials have no common solution, and are inherently refutation systems. Frege systems are traditionally viewed as deriving tautologies, but for ease of comparison, we treat them as refutation systems.

Furthermore, we will be discussing propositional formulas and polynomials in the same set of variables. This is justified by identifying the logical constant FALSE with the field element 0 and the logical constant TRUE with the field element 1.

Constant-Depth Frege Systems A Frege system is a sound, implicationally complete propositional proof system over a finite set of connectives with a finite number of axiom schema and inference rules. By the methods of Cook and Reckhow [17], any two Frege systems simulate one another up to a polynomial factor in size and a linear factor in depth. For concreteness, the reader can keep in mind the following Frege system whose connectives are NOT gates, \neg , and unbounded fan-in OR gates, \bigvee , and whose inference rules are: (1) Axioms $\overline{A \vee \neg A}$, (2) Weakening $\frac{A}{A \vee B}$ (3) Cut $\frac{A \vee B}{B \vee C}$ (4) Merging $\frac{\bigvee X \vee \bigvee Y}{\bigvee (X \cup Y)}$ (5) Unmerging $\frac{\bigvee (X \cup Y)}{X \vee \bigvee Y}$.

Let \mathcal{H} be a set of formulas. A derivation from \mathcal{H} is a sequence of formulas f_1, \ldots, f_m so that for each $i \in [m]$, either f_i is a substitution instance of an axiom, f_i is an element of \mathcal{H} , or there exist j, k < i so that f_i follows from f_j and f_k by the application of an inference rule to f_j and f_k .

For a given formula F, a proof of F is a derivation from the empty set of hypotheses whose final formula is F.

For fixed set of hypotheses \mathcal{H} , a refutation of \mathcal{H} is a derivation from \mathcal{H} whose final formula is FALSE.

The size of a derivation is the total number of symbols appearing in it.

We say that a family of tautologies τ_n , each of size s(n), has polynomial size constant-depth Frege proofs (refutations) if there are constants c and d so that for all n, there is a proof (refutation) of τ_n so that each formula in the proof has depth at most d, and the proof (refutation) has size $O(s^c(n))$.

Counting Axioms Modulo m Constant-depth Frege with counting axioms modulo m is the extension of constant-depth Frege systems that has axioms that state for integers $m, N, m \ge 2$ and $N \not\equiv_m 0$, it is impossible to partition a set of N elements into pieces of size m.

Definition 5. Let m > 1 and $N \not\equiv_m 0$ be given. Let V be a set of N elements. For each $e \in [V]^m$, let there be a variable x_e .

$$Count_m^V = \bigvee_{v \in V} \left(\bigwedge_{\substack{e \in [V]^m \\ e \ni v}} \neg x_e \right) \quad \lor \bigvee_{e, f \in [V]^m \\ e \perp f} (x_e \land x_f)$$

Frege with counting modulo m derivations are Frege derivations that allow the use of substitution instances of $\operatorname{Count}_{m}^{[N]}$ (with $N \neq_{m} 0$) as axioms.

Nullstellensatz Refutations One way to prove that a system of polynomials f_1, \ldots, f_k has no common roots is to give a list of polynomials p_1, \ldots, p_k so that $\sum_{i=1}^{k} p_i f_i = 1$. Because we are interested in translations of propositional formulas, we add the polynomials $x^2 - x$ as hypotheses to guarantee all roots are zero-one roots.

Definition 6. For a system of polynomials f_1, \ldots, f_k in variables x_1, \ldots, x_n over a field F, a Nullstellensatz refutation of f_1, \ldots, f_k is a list of polynomials $p_1, \ldots, p_k, r_1, \ldots, r_n$ satisfying the following equation:

$$\sum_{i=1}^{k} p_i f_i + \sum_{j=1}^{n} r_j \left(x_j^2 - x_j \right) = 1$$

For a polynomial q, a Nullstellensatz derivation of q from f_1, \ldots, f_k is a list of polynomials $p_1, \ldots, p_k, r_1, \ldots, r_n$ satisfying the following equation:

$$\sum_{i=1}^{k} p_i f_i + \sum_{j=1}^{n} r_j \left(x_j^2 - x_j \right) = q$$

The degree of the refutation (derivation) is the maximum degree of the polynomials $p_i f_i$, $r_j (x_j^2 - x_j)$.

We define the size of a Nullstellensatz refutation (derivation) to be the number of monomials appearing in p_1, \ldots, p_k and f_1, \ldots, f_k .

Hilbert's weak Nullstellensatz guarantees that over a field, all unsatisfiable systems of polynomials have Nullstellensatz refutations [23]. We can define Nullstellensatz refutations over any ring, but such systems are no longer complete. In this paper, we work with Nullstellensatz refutations of polynomials over \mathbb{Z}_m , and for the sake of generality, we make no assumptions on m unless otherwise stated.

Polynomial Calculus

Definition 7. Let f_1, \ldots, f_k be polynomials over a field F. A polynomial calculus refutation of f_1, \ldots, f_k over F is a sequence of polynomials g_1, \ldots, g_m so that, $g_m = 1$, and for each $i \in [m]$, either g_i is f_l for some $l \in [k]$, g_i is $x_l^2 - x_l$ for some $l \in [n]$, g_i is $ag_j + bg_l$ for some $j, l < i, a, b \in F$, or g_i is x_lg_j for some $j < i, l \in [n]$.

The size of a polynomial calculus refutation is the total number of monomials appearing in the polynomials of the refutation. The degree of a polynomial calculus refutation is the maximum degree of a polynomial that appears in the refutation.

3 Contradictory Partitions of Satisfied Variables

To simulate Nullstellensatz refutations in constant-depth Frege systems with counting axioms, we construct two partitions on the satisfied monomials of the refutation: one which covers the satisfied monomials exactly, and another which covers the satisfied monomials with m-1 new points. This is impossible, and in this section, we show that constant-depth Frege systems with counting axioms can prove that this is impossible with polynomial size proofs.

Definition 8. Let positive integers n and k be given. Let u_1, \ldots, u_n be a set of Boolean variables. For each $e \in [n]^m$, let y_e be a variable, and for each $e \in [n+k]^m$, let z_e be a variable. $CP_m^{n,k}(\boldsymbol{u}, \boldsymbol{y}, \boldsymbol{z})$ is the negation of the conjunction of the following formulas:

"every variable covered by the first partition is satisfied" for each e ∈ [n]^m, y_e → Λ_{i∈e} u_i
"every satisfied variable is covered by the first partition" for each i ∈ [n], u_i → V_{e∋i} y_e
"no two overlapping edges are used by the first partition" for each e, f ∈ [n]^m with e ⊥ f, ¬y_e ∨ ¬y_f
"every variable covered by the second partition is satisfied" for each e ∈ [n + k]^m, z_e → Λ_{i∈e} u_i
"every satisfied variable is covered by the second partition" for each i ∈ [n], u_i → V_{e∋i} z_e
"every extra point is covered by the second partition"

for each $i, n+1 \le i \le n+k, \bigvee_{e \ge i} z_e$

"no two overlapping edges are used by the second partition" for each $e, f \in [n+k]^m$ with $e \perp f, \neg z_e \vee \neg z_f$

Lemma 1. Fix m and k so that m is not divisible by k. For all n, the tautology $CP_m^{n,k}$ has a constant depth, size $O(n^m)$ proof in constant-depth Frege with counting modulo m axioms.

Proof. Fix m, n and k. The proof of $\operatorname{CP}_m^{n,k}$ is by contradiction. We define a set U of size mn + k and formulas ϕ_e for each $e \in [U]^m$ so that we can derive $(\neg \operatorname{Count}_m^U) [x_e \leftarrow \phi_e]$ in size $O(n^m)$ from the hypothesis $\neg \operatorname{CP}_m^{n,k}$.

Let U be the set consisting of the following points: $p_{r,i}, r \in [m], i \in [n]$ (the r'th copy of the row of variables) and $p_{m,i}, n+1 \leq i \leq k$ (the extra points.)

"when u_i is unset, we group together its copies"

for each $i \in [n], \phi_{\{p_{1,i},...,p_{m,i}\}} = \neg u_i$

"in the first m-1 rows, use the partition given by the y_e 's"

for each $r \in [m-1]$, each $i_1, \ldots, i_m \in [n]$, $\phi_{\{p_{r,i_1}, \ldots, p_{r,i_m}\}} = y_{\{i_1, \ldots, i_m\}}$

"in the last row, use the the partition given by the $z_e \, {\rm 's'}$

for each $i_1, \ldots, i_m \in [n+k], \phi_{\{p_{m,i_1}, \ldots, p_{m,i_m}\}} = z_{\{i_1, \ldots, i_m\}}$

other edges are not used

for all other $e \in [U]^m$, $\phi_e = 0$

Now we sketch the derivation of $(\neg \operatorname{Count}_m^U) [x_e \leftarrow \phi_e]$ from $\neg \operatorname{CP}_m^{n,k}$. It is easily verified that the derivation has constant depth and size $O((mn+k)^m) = O(n^m)$.

"Every point of U is covered by the partition."

Let $p_{r,i} \in U$ with $i \in [n]$, $r \in [m-1]$ be given. From $\neg CP_m^{n,k}$ derive $u_i \rightarrow \bigvee_{\substack{f \in [n]^m \\ f \ni i}} y_f$. Because $\bigvee_{\substack{f \in [n]^m \\ f \ni i}} y_f$ is a sub-disjunction of $\bigvee_{\substack{e \in [U]^m \\ e \ni p_{r,i}}} \phi_e$, we may derive $u_i \rightarrow \bigvee_{\substack{e \in [U]^m \\ e \ni p_{r,i}}} \phi_e$ with a weakening inference. Because $\phi_{\{p_{1,i},\dots,p_{m,i}\}} = \neg u_i$, we may derive $\neg u_i \rightarrow \bigvee_{\substack{e \in [U]^m \\ e \ni p_{r,i}}} \phi_e$. Combining these two formulas yields $\bigvee_{\substack{e \in [U]^m \\ e \ni p_{r,i}}} \phi_e$. The case for $p_{m,i}, i \in [n]$ is similar.

For a point $p_{m,i}$, $n+1 \leq i \leq n+k$, from $\neg \operatorname{CP}_m^{n,k}$ derive $\bigvee_{\substack{f \in [n+k]^m \\ f \ni i}} z_f$. A weakening inference applied to this derives $\bigvee_{e \ni p_{m,i}} \phi_e$.

"No overlapping edges are used."

Let $e_1, e_2 \in [U]^m$ be given so that $e_1 \perp e_2$, and neither ϕ_{e_1} nor ϕ_{e_2} is identically 0.

If $\phi_{e_1} = \neg u_i$ and $\phi_{e_2} = y_f$, then e_1 is $\{p_{r,i} \mid r \in [m]\}$ and e_2 is $\{p_{r,j} \mid j \in f\}$ for some $r \in [m]$ and $f \in [n]^m$ so that $i \in f$. From $\neg \operatorname{CP}_m^{n,k}$ derive $y_f \to u_i$. From this, derive $\neg \neg u_i \lor \neg y_f = \neg \phi_{e_1} \lor \neg \phi_{e_2}$.

If $\phi_{e_1} = y_{f_1}$ and $\phi_{e_2} = y_{f_2}$, then e_1 is $\{p_{r_1,i} \mid i \in f_1\}$ and e_2 is $\{p_{r_2,i} \mid i \in f_2\}$ with $r_1 = r_2$ and $f_1 \perp f_2$. From $\neg \operatorname{CP}_m^{n,k}$ derive $\neg y_{f_1} \lor \neg y_{f_2} = \neg \phi_{e_1} \lor \neg \phi_{e_2}$.

The only other cases are when $\phi_{e_1} = \neg u_i$ and $\phi_{e_2} = z_f$ or $\phi_{e_1} = z_{f_1}$ and $\phi_{e_2} = z_{f_2}$, and these are handled similarly.

4 The Simulation

Because we work over \mathbb{Z}_m , a polynomial vanishes on a given assignment if and only if there is an *m*-partition on its satisfied monomials (recall that we treat a monomial with coefficient *a* as having *a* distinct copies.) The definability of this partition is the connection between refuting a propositional formula and refuting a system of polynomials.

4.1 Reducing Formulas to Systems of Equations

The method we use to reduce a formula to a system of polynomials is to define a partition on the satisfied monomials of the polynomials with small, constantdepth formulas and prove that these formulas define a partition using the formula as a hypothesis. Because of the central role played by the sets of monomials appearing in each polynomial, we take a moment to define this notion precisely. First of all, because we are concerned only with 0/1 assignments, a polynomial vanishes if and only if its multilinearization vanishes. For this reason, we restrict our attention to multilinear polynomials. We treat a term ax_I as a distinct copies of the monomial x_I . For this reason, when we talk about the "set of monomials" of a polynomial, we do not mean the set of monomials that appear in the polynomial, but a set which includes a copies of each monomial with coefficient a. We will generally identify ax_I with a objects $m_{1,I}, \ldots, m_{a,I}$. Think of $m_{c,I}$ as the c'th copy of the monomial x_I . There should be little confusion of the dual use of the symbol "m" because when the symbol appears without a subscript it denotes the modulus, and when it appear with a subscript it denotes a monomial.

Definition 9. Let $f = \sum_{I \subseteq [n]} a_I x_I$ be a multilinear polynomial over \mathbb{Z}_m . The set of monomials of f is the following set:

$$M_f = \{m_{c,I} \mid I \subseteq [n], \ c \in [a_I]\}$$

Definition 10. Let x_1, \ldots, x_n be Boolean variables. Let f be a multilinear polynomial in the variables x_1, \ldots, x_n . For each $E \in [M_f]^m$, let θ_E be a formula in \boldsymbol{x} . We say that the θ 's form an m-partition the satisfied monomials of f if the following formula holds:

$$\bigwedge_{E \in [M_f]^m} \left(\theta_E \to \bigwedge_{m_{c,I} \in E} \bigwedge_{k \in I} x_k \right) \land \left(\bigwedge_{\substack{E, F \in [M_f]^m \\ E \perp F}} \neg \theta_E \lor \neg \theta_F \right) \\
\land \bigwedge_{m_{c,I} \in M_f} \left(\left(\bigwedge_{k \in I} x_k \right) \to \bigvee_{\substack{E \in [M_f]^m \\ E \ni m_{c,I}}} \theta_E \right)$$

Definition 11. Let x_1, \ldots, x_n be Boolean variables. Let $\Gamma(\mathbf{x})$ be a propositional formula. Let $F = \{f_1, \ldots, f_k\}$ be a system of polynomials over \mathbb{Z}_m with a Null-stellensatz refutation $p_1, \ldots, p_k, r_1, \ldots, r_n$. If, for each $i \in [k]$, there are formulas $\beta_E^i(\mathbf{x}), E \in [M_{\bar{f}_i}]^m$, so that there is a size T, depth d Frege derivation from $\Gamma(\mathbf{x})$ that, for each i, the β^i 's form an m-partition on the satisfied monomials of \bar{f}_i , then we say that Γ reduces to F in depth d and size T.

4.2 The Simulation

Theorem 1. Let m > 1 be an integer. Let x_1, \ldots, x_n be Boolean variables. Let $\Gamma(\mathbf{x})$ be a propositional formula, and let F be a system of polynomials over \mathbb{Z}_m so that Γ reduces to F in depth d and size T. If there is a Nullstellensatz refutation of F with size S, then there is a depth O(d) Frege with counting axioms modulo m refutation of $\Gamma(\mathbf{x})$ with size $O(S^{2m}T)$.

Proof. Let $p_1, \ldots, p_k, r_1, \ldots, r_n$ be a size S Nullstellensatz refutation of F. Let $\beta_E^i(\boldsymbol{x})$, for $i \in [k], E \in [M_{\bar{f}_i}]^m$, be formulas so that from Γ there is a size T, depth d proof that for each i the $\beta_E^i(\boldsymbol{x})$'s form an m-partition on the satisfied monomials of \bar{f}_i .

We obtain contradictory partitions of the the monomials that appear in the expansion of $\sum_{i=1}^{k} \bar{p}_i \bar{f}_i$ in which polynomials are multiplied and multilinearized, but no terms are collected. In other words, the set is the collection, over $i \in [k]$, of all pairs of monomials from \bar{p}_i and \bar{f}_i .

$$V = \bigcup_{i=1}^{k} \{ (m_{c,I}, m_{d,J}, i) \mid m_{c,I} \in M_{\bar{p}_i}, \ m_{d,J} \in M_{\bar{f}_i} \}$$

Notice that $|V| = O(S^2)$.

For each $v \in V$, $v = (m_{c,I}, m_{d,J}, i)$, let $\gamma_v = \bigwedge_{k \in I \cup J} x_k$. Think of these as the monomials. We will give formulas θ_E , that define a partition on the satisfied monomials with m - 1 many extra points, and η_E , that define a partition on the satisfied monomials with no extra points. We will give a $O(|V|^m + T) = O(S^{2m} + T)$ derivation from Γ of the following:

$$\neg \mathrm{CP}_{m}^{|V|,m-1}\left[u_{v}\leftarrow\gamma_{v}, y_{E}\leftarrow\theta_{E}, z_{E}\leftarrow\eta_{E}\right]$$

On the other hand, by lemma 1, $\operatorname{CP}_m^{|V|,m-1}$ has constant depth Frege proofs of size $O(|V|^m)$, so $\operatorname{CP}_m^{|V|,m-1}[u_v \leftarrow \gamma_v, y_E \leftarrow \theta_E, z_E \leftarrow \eta_E]$ has a constant depth Frege proof of size $O(|V|^m T)$. Therefore, Γ has a depth O(d) Frege refutation of size $O(S^{2m}T)$.

The Partition with m-1 Extra Points

Notice that we have the following equation:

$$\sum_{i=1}^{k} \bar{p}_i \bar{f}_i = \sum_{i=1}^{k} p_i f_i + \sum_{j=1}^{n} r_j (x_j^2 - x_j) = 1$$

So when we collect terms after expanding $\sum_{i=1}^{k} \bar{p}_i \bar{f}_i$ and multilinearizing, the coefficient of every nonconstant term is 0 modulo m, and the constant term is 1 modulo m.

For each $S \subseteq [n]$, let $V_S = \{(m_{c,I}, m_{d,J}, i)) \in V \mid I \cup J = S\}$. Think of these as the occurrences of x_S in the multilinearized expansion.

For each $S \subseteq [n]$, $S \neq \emptyset$, there is an *m*-partition on V_S , call it \mathcal{P}_S . Likewise, there is an *m*-partition on $V_{\emptyset} \cup [m-1]$, call it \mathcal{P}_{\emptyset} .

Define the formulas θ_E as follows: for each $E \in ([V] \cup [m-1])^m$, if $E \in \mathcal{P}_S$ for some $S \subseteq [n]$ then $\theta_E = \bigwedge_{k \in S} x_k$, otherwise $\theta_E = 0$.

Constant-depth Frege can prove that this is a *m*-partition of the satisfied monomials of $\sum_{i=1}^{k} \overline{p_i} \overline{f_i}$ with m-1 extra points. The proof has size $O(|V|^m)$ and depth O(1). It is trivial from the definition of θ_E that the edges cover only satisfied monomials. That every satisfied monomial $\bigwedge_{k \in S} x_k$ is covered is also trivial: the edge from \mathcal{P}_S is used if and only if the term x_S is satisfied. Finally, it

easily shown that the formulas for two overlapping edges are never both satisfied: only edges from \mathcal{P}_S are used (regardless of the values of the x's), so for any pair of overlapping edges, $E \perp F$, one of the two formulas θ_E or θ_F is identically 0.

The Partition with No Extra Points

The idea is that an *m*-partition on the satisfied monomials on f_i can be used to build an *m*-partition on the satisfied monomials of $t\bar{f}_i$, for any monomial t.

For each $E \in [V]^m$, define η_E as follows: if $E = \{(m_{c,I}, m_{d_l,J_l}, i) \mid l \in [m]\}$ for some $i \in [k]$, $m_{c,I} \in M_{f_i}$, then $\eta_E = \bigwedge_{k \in I} x_k \wedge \beta_{\{m_{d_l,J_l} \mid l \in [m]\}}$, otherwise, $\eta_E = 0$.

There is a size $O(S + |V|^m)$, depth O(d) Frege derivation from Γ that the η_E 's form an *m*-partition on the satisfied monomials of $\sum_{i=1}^k \overline{p_i f_i}$. We briefly sketch how to construct the proof. Begin by deriving from Γ , for each *i*, that the β_E^i 's form an *m*-partition on the satisfied monomials of $\overline{f_i}$.

"Every satisfied monomial is covered." Let $(m_{c,I}, m_{d,J}, i) \in V$ be given. If $\bigwedge_{k \in I \cup J} x_k$ holds, then so do $\bigwedge_{k \in I} x_k$ and $\bigwedge_{k \in J} x_k$. Because the β^i 's form an *m*-partition on the satisfied monomials of \bar{f}_i , we may derive $\bigvee_{F \in [M_{f_i}]^m} \beta_F^i$. From this derive $\bigvee_{F \in [M_{f_i}]^m} \bigwedge_{k \in I} x_k \wedge \beta_F^i$. A weakening inference applied to this yields $\bigvee_{E \in [V]^m} \eta_E$.

"Every monomial covered is satisfied." Let $v = (m_{c,I}, m_{d,J}, i) \in V$ be given so that $v \in E$ and η_E holds. For this to happen, $E = \{(m_{c,I}, m_{d_l,J_l}, i) \mid l \in [m]\}$. By definition,, $\eta_E = \bigwedge_{k \in I} x_k \wedge \beta^i_{\{m_{d_l,J_l} \mid l \in [m]\}}$, and therefore $\bigwedge_{k \in I} x_k$ holds. Because the β^i 's form an *m*-partition on the satisfied monomials of \bar{f}_i , we have that $\bigwedge_{k \in J} x_k$ holds. Therefore $\bigwedge_{k \in I \cup J} x_k$ holds.

"No two conflicting edges E and F can have η_E and η_F simultaneously satisfied." If $E \perp F$, and neither θ_E nor θ_F is identically 0, then they share the same \bar{p}_i component. That is, there exists $i, m_{c,I} \in M_{\bar{p}_i}$ so that $E = \{(m_{c,I}, m_{d_l,J_l}, i) \mid l \in [m]\}$, and $F = \{(m_{c,I}, m_{d'_l,J'_l}, i) \mid l \in [m]\}$. Because $E \perp F$, we have $\{m_{d_l,J_l} \mid l \in [m]\} \perp \{m_{d'_l,J'_l} \mid l \in [m]\}$. Because the β^i 's form an *m*-partition on the satisfied monomials of \bar{f}_i , we can derive $\neg \beta^i_{\{m_{d_l,J_l} \mid l \in [m]\}} \lor \neg \beta^i_{\{m_{d'_l,J'_l} \mid l \in [m]\}}$. We weaken this formula to obtain $\neg \beta^i_{\{m_{d_l,J_l} \mid l \in [m]\}} \lor \neg \beta^i_{\{m_{d'_l,J'_l} \mid l \in [m]\}} \lor \forall \downarrow_{k \in I} \neg x_k$, and from that derive $\neg \left(\bigwedge_{k \in I} x_k \land \beta^i_{\{m_{d_l,J_l} \mid l \in [m]\}} \right) \lor \neg \left(\bigwedge_{k \in I} x_k \land \beta^i_{\{m_{d'_l,J'_l} \mid l \in [m]\}} \right) = \neg \eta_E \lor \neg \eta_F$.

5 Translations of Formulas into Polynomials

5.1 Direct Translation of Clauses

For sets of narrow clauses, a common way to translate the clauses into polynomials is to map x to 1 - x, $\neg x$ to x and replace "OR" by multiplication. This is most commonly used for constant-width CNFs, and in this case, we show that clauses efficiently reduce to their translations.

Definition 12. [11] For a clause C in variables x, the direct translation of C, tr(C), is defined recursively as follows: (i) $tr(\emptyset) = 1$ (ii) $tr(A \lor x) = tr(A)(1-x)$ (iii) $tr(A \lor \neg x) = tr(A)x$

For a CNF F, the direct translation of F, tr(F), is the set $\{tr(C) \mid C \in F\}$.

It is easily verified by induction that an for any clause C, a Boolean assignment satisfies C if and only if it is a root of tr(C).

Whenever C is satisfied, there exists an m-partition on the satisfied monomials of tr(C). Moreover, if C contains at most w variables, then the m-partition can be defined by depth two formulas of size $O(2^w)$, and by the completeness of constant-depth Frege systems, there is a constant depth derivation from C of size $2^{O(w)}$ that these formulas define an m-partition on the satisfied monomials of tr(C). Therefore, C reduces to tr(C) in constant depth and size $O(2^w)$.

Lemma 2. If F is an unsatisfiable CNF of m clauses of width w, then F is reducible to tr(F) in size $m2^{O(w)}$ and depth O(1).

5.2 Translations That Use Extension Variables

More involved translations of formulas into sets of polynomials use extension variables that represent sub-formulas. The simplest way of doing this would be to reduce an unbounded fan-in formula Γ to a bounded fan-in formula, and then introduce one new variable y_g per gate g, with the polynomial that says y_g is computed correctly from its inputs. It is easy to give a reduction from Γ to this translation, of depth $depth(\Gamma)$ and size $poly(|\Gamma|)$. (We can define y_g by the subformula rooted at g and every polynomial would have constant size, so defining the partition is trivial.) However, this translation reveals little for our purposes because there is usually no small degree Nullstellensatz refutation of the resulting system of polynomials, even for trivial Γ . For example, say that we translated the formula $x_1, \neg((((x_1 \vee x_2) \vee \cdots \vee x_n)$ this way. The resulting system of polynomials is weaker than the induction principles (see the end of this section) which require $\Omega(\log n)$ degree **NS** refutations [15].

We give an alternative translation of formulas into sets of polynomials so that the formula is unsatisfiable if the set of polynomials has no common root. A formula f reduces to the set of polynomials with depth O(depth(f)) and size O(|f|). Moreover, for many previously studied unsatisfiable CNFs (such as the negated counting principles), this translation is the same as the previously studied translations (up to constant-degree Nullstellensatz derivations).

Definition 13. Let f be a formula in the variables x_1, \ldots, x_n and the connectives $\{\bigvee, \neg\}$. For each pair of subformulas g_1 and g_2 of f, we write $g_1 \rightarrow g_2$ if g_1 is an input to g_2 . Canonically order the subformulas of f, and write $g_1 < g_2$ if g_1 precedes g_2 in this ordering. For each subformula g of f, let there be a variable y_g - the value of g. For each pair of subformulas of f, g_1 and g_2 , so that the top connective of g_2 is \bigvee and $g_1 \rightarrow g_2$, let there be a variable z_{g_1,g_2} - " g_1 is the first satisfied input of g_2 ". The polynomial translation of f, POLY(f), is the following set of polynomials:

For each variable x_i :

"The value of subformula x_i is equal to x_i "

 $y_{x_i} - x_i$ For each subformula g whose top connective is \bigvee : "if $g_1 < g_2, g_1 \rightarrow g, g_2 \rightarrow g$, and g_1 is satisfied, then g_2 is not the first satisfied input of g" $y_{g_1} z_{g_2,g}$ "if q_1 is the first satisfied input of q, then g_1 is satisfied" $z_{g_1,g}y_{g_1} - z_{g_1,g}$ "g is satisfied if and only if the some input to g is the first satisfied input of g" $y_g - \sum_{g_1 \to g} z_{g_1,g}$ For each subformula g whose top connective is \neg : Let q_1 the unique input of q_1 , "if g_1 is satisfied if and only if g is not satisfied" $y_{g_1} + y_g - 1$ The formula f is satisfied: $y_{f} - 1$

One can show by induction that if f is satisfiable then POLY(f) has a common root. By the contrapositive, if POLY(f) has no common roots, then f is unsatisfiable.

Lemma 3. Let f be a Boolean formula in the variables x_1, \ldots, x_n . If f is satisfiable, then POLY(f) has a common 0/1 root.

Proof. Let α be a 0/1 assignment to x_1, \ldots, x_n . For any propositional formula g, let $\alpha(g)$ denote the value of g under the assignment α .

Suppose that $\alpha(f) = 1$. We extend α to the variables of POLY(f) as follows: For each subformula g of f, let $\alpha(y_g) = \alpha(g)$. When $g = \bigvee g_i$ and $\alpha(g) = 1$, let i_0 be the first input to g so that $\alpha(g_i) = 1$. Set $\alpha(z_{g_{i_0}}) = 1$ and for $i \neq i_0$, set $\alpha(z_{g_i}) = 0$. When $g = \bigvee g_i$ and $\alpha(g) = 0$, $\alpha(z_{g_{i_g}}) = 0$ for all i.

We now show by induction that α is a root of POLY(f). Clearly, for each variable x_i , α is a root of $y_{x_i} - x_i$. Consider a subformula $\neg g$. Because $\alpha(y_{\neg g}) = \alpha(\neg g)$ and $\alpha(y_g) = \alpha(g) = 1 - \alpha(\neg g)$, α is a root of $y_{\neg g} + y_g - 1$. Consider a subformula $g = \bigvee_i g_i$. If $\alpha(g) = 0$, then for all i, $\alpha(z_{g_i,g}) = 0$, $\alpha(y_{g_i}) = 0$ and $\alpha(y_g) = 0$. In this case, α is clearly a root to $z_{g_i,g}y_{g_i} - z_{g_i,g}$, $y_{g_i}z_{g_j,g}$ and $y_g - \sum_i z_{g_i,g}$. In the case when $\alpha(g) = 1$, there exists i_0 so that $\alpha(z_{g_{i_0},g}) = 1$ and for all $j \neq i_0$, $\alpha(z_{g_j,g}) = 0$. Moreover, $\alpha(y_{g_{i_0}}) = 1$, $\alpha(y_g) = 1$ and for all $j < i_0$, $\alpha(y_{g_j}) = 0$. Therefore, α is a root to $y_{g_j}z_{g_i,g}$ for all i < j, $z_{g_i,g}y_{g_i} - z_{g_{i_g},g}$ for all i, and $y_g - \sum_i z_{g_i,g}$. Finally, α is a root of $y_f - 1$ because $\alpha(f) = 1$ by assumption.

The argument of lemma 3 can be carried in Frege systems with depth O(depth(f))and size O(|f|).

Theorem 2. If f is a formula in the variables x_1, \ldots, x_n and the connectives $\{\bigvee, \neg\}$, then f is reducible to POLY(f) in depth O(depth(f)) and size polynomial in |f|.

12

Proof. We proceed in two stages. First, we give a set of formulas, EXT(f), that is in the variables x_i , y_g and z_{g_1,g_2} and is analogous to the translation of f into polynomials. We show that this translation has a constant depth, polynomial size reduction to POLY(f) and then show that f has a depth O(depth(f)) reduction to EXT(f) of size polynomial in |f|.

Let EXT(f) be the following set of formulas:

For each variable x_i : $y_{x_i} \leftrightarrow x_i$ For each subformula g whose top connective is \bigvee : "if $g_1 < g_2, g_1 \rightarrow g, g_2 \rightarrow g$, and g_1 is satisfied. then g_2 is not the first satisfied input of g" $\neg y_{g_1} \lor \neg z_{g_2,g}$ "if g_1 is the first satisfied input of g, then g_1 is satisfied" $z_{g_1,g} \to y_{g_1}$ "g is satisfied if and only if some input to gis the first satisfied input of g $y_g \leftrightarrow \bigvee_{g_1 \to g} z_{g_1,g}$ For each subformula g whose top connective is \neg : Let g_1 the unique input of g, "if g_1 is satisfied then g is not satisfied" $y_{g_1} \leftrightarrow \neg y_g$ The formula f is satisfied: y_f

There is a straightforward constant-depth, polynomial-size reduction of EXT(f) to POLY(f). For each polynomial of POLY(f), there is a formula of EXT(f) that reduces to the polynomial; the formula associated with each polynomial is given in table V.1. For the constant-size polynomials of POLY(f), the corresponding formula of EXT(f) implies that there is an *m*-partition on the satisfied variables of the polynomial. Because the polynomial involves a constant number of variables, the partition may be defined and proved correct in constant size, depth two.

 Table 1. Polynomials and their Associated Formulas

polynomial	associated formula
$y_{x_i} - x_i$	$y_{x_i} \leftrightarrow x_i$
$y_{g_1} z_{g_2,g}$	$\neg y_{g_1} \vee \neg z_{g_2,g}$
$z_{g_1,g}y_{g_1} - z_{g_1,g}$	$z_{g_1,g} \to y_{g_1}$
$y_{g_1} + y_g - 1$	$y_{g_1} \leftrightarrow \neg y_g$
$y_f - 1$	y_f

The only polynomials of POLY(f) that involve a non-constant number of variables are those of the form $y_g - \sum_{g_1 \to g} z_{g_1,g}$, and from the hypotheses of EXT(f) it can be shown that y_g is satisfied if only if exactly one of the $z_{g_1,g}$'s is satisfied. Because there are (m-1) copies of each $z_{g_1,g}$ in such a polynomial, we can group y_g with these copies of $z_{g_1,g}$ whenever $z_{g_1,g}$ is satisfied.

To reduce f to EXT(f), it is easy to check to that there is a polynomial size, depth O(depth(f)) derivation of the following substitution instance of EXT(f)from the hypothesis f. (The substitution instances of each formula are given in table V.2.)

$$\mathrm{EXT}(f)[y_g \leftarrow g, \ z_{g_1,g} \leftarrow (g_1 \land \bigwedge_{g_2 < g_1 \atop g_2 \to g} \neg g_2)]$$

formula	substitution instance	comment
$y_{x_i} \leftrightarrow x_i$	$x_i \leftrightarrow x_i$	
$\neg y_{g_1} \vee \neg z_{g_2,g}$	$\neg g_1 \lor \neg (g_2 \land \bigwedge_{g_3 < g_2 \atop g_3 \rightarrow g} \neg g_3)$	$g_1 < g_2$
$z_{g_1,g} \to y_{g_1}$	$(g_1 \land \bigwedge_{\substack{g_2 < g_1 \\ g_2 \to g}} \neg g_2) \to g_1$	
$y_g \leftrightarrow \vee_{g_1 \to g} z_{g_1,g}$	$g \leftrightarrow \bigvee_{g_1 \to g} (g_1 \land \bigwedge_{g_2 < g_1 \atop g_2 \to g} \neg g_2)$	$g = \vee_{g_1 \to g} g_1$
$y_{g_1} \leftrightarrow \neg y_g$	$g_1 \leftrightarrow \neg g$	$g = \neg g_1$
y_f	f	

 Table 2. Formulas and their Substitution Instances

Example: We illustrate our translation with a the clauses of the negated counting principles. The translation of this set of clauses turns out to be same (up to constant degree Nullstellensatz derivations) as the polynomial formulation of the counting principles previously studied.

Let V be a set of cardinality indivisible by m. The clauses are $F_v = \bigvee_{e \ni v} x_e$ for $v \in V$ and $G_{e,f} = \neg x_e \lor \neg x_f$ for $e, f \in [V]^m$ with $e \perp f$. The standard translation of these systems has the polynomials $\sum_{e \ni v} x_e$, for $v \in V$, and $x_e x_f$, for $e \perp f$.

The polynomials introduced by the translation of $G_{e,f}$ are: $y_{x_e} - x_e$, $y_{x_f} - x_f$, $y_{\neg x_e} + y_{x_e} - 1$, $y_{\neg x_f} + y_{x_f} - 1$, $y_{\neg x_e} z_{\neg x_f,G_{e,f}}$, $z_{\neg x_e,G_{e,f}} y_{\neg x_e} - z_{\neg x_e,G_{e,f}}$, $z_{\neg x_f,G_{e,f}} y_{\neg x_f} - z_{\neg x_f,G_{e,f}}$, $y_{G_{e,f}} - z_{\neg x_e,G_{e,f}} - z_{\neg x_f,G_{e,f}}$ and $y_{G_{e,f}} - 1$. It is easy to check that thee is a constant degree derivation of $x_e x_f$ from these polynomials (in particular, a non-optimal but constant-degree derivation is given by the completeness of the Nullstellensatz system).

The polynomials introduced by the translation of F_v are: $y_{x_e} - x_e$, $z_{y_{x_e},F_v}y_f$ (for $e, f \ni v$ and e < f), $z_{y_{x_e},F_v}y_{x_e} - z_{y_{x_e},F_v}$ (for $e \ni v$), $y_{F_v} - \sum_{e \ni v} z_{y_{x_e},F_v}$ and $y_{F_v} - 1$. With a degree two Nullstellensatz derivation we may derive $\sum_{e \ni v} z_{e,F_v}x_e - 1$. Multiplying this by $\sum_{e \ni v} x_e$, and reducing using the previously derived polynomials $x_e x_f$ and the axioms $x_e^2 - x_e$, yields $\sum_{e \ni v} z_{e,F_v} x_e - \sum_{e \ni v} x_e$. Subtracting this from $\sum_{e \ni v} z_{e,F_v} x_e - 1$ yields $\sum_{e \ni v} x_e$.

A Note on Translations of Formulas to Polynomials Using Extension Variables

Definition 14. The induction principle of length M, IND(M), is the following system of polynomials: y_1 , $y_{r+1}y_r - y_{r+1}$ (for r < M) and $y_M - 1$.

Theorem 3. [15,12] The IND(M) system has Nullstellensatz refutations of degree $O(\log M)$ over any field. Moreover, over any field the system requires degree $\Omega(\log M)$ Nullstellensatz refutations.

The "standard" translation of x_n , $\neg(((((x_n \lor x_{n-1}) \lor \cdots \lor x_1))))$ into polynomials using extension variables introduces new variables z_1, \ldots, z_{n-1} , with polynomials $x_n - 1$, $1 - (1 - x_n)(1 - x_{n-1}) - z_{n-1}$, $1 - (1 - z_{n-1})(1 - x_{n-2}) - z_{n-2}$, \ldots , $1 - (1 - z_2)(1 - x_1) - z_1$, and z_1 . (The indices have been reversed from those of subsection 5.1 to ease the reduction.)

We may define this set of polynomials from IND(n) using the following definitions: $x_i := y_i$ for $i, 1 \le i \le n$, and $z_i := y_i$, for $i \le n - 1$. The polynomials $z_1 = y_1$ and $x_n - 1 = y_n - 1$ are belong to IND(n), and for each $r, 1 \le r \le n - 2$,

$$1 - (1 - z_{r+1})(1 - x_r) - z_r = 1 - (1 - y_{r+1})(1 - y_r) - y_r$$

= 1 - (1 + y_{r+1}y_r - y_r - y_{r+1}) - y_r = -(y_{r+1}y_r - y_{r+1})

Similarly, $1 - (1 - x_n)(1 - x_{n-1}) - z_{n-1} = -(y_n y_{n-1} - y_n).$

Because there is a constant degree reduction from IND(n) to the standard translation of $x_n, \neg(((((x_n \lor x_{n-1}) \lor \ldots x_1)))))$ into polynomials, this translation requires super-constant degree to refute in the Nullstellensatz system.

6 An Application to Unsatisfiable Systems of Constant-Width Linear Equations

Many tautologies studied in propositional proof complexity, such as Tseitin's tautologies [9] and the τ formulas of Nisan-Wigderson generators built from parity functions, can be expressed as inconsistent systems of linear equations over a field \mathbb{Z}_q in which each equation involves only a small number of variables. We show that in such situations, constant-depth Frege with counting axioms modulo q can prove these principles with polynomial size proofs.

Fix a prime number q. Let A be an $m \times n$ matrix over \mathbb{Z}_q , let x_1, \ldots, x_n be variables and let $\mathbf{b} \in \mathbb{Z}_q^m$ be so that $A\mathbf{x} = \mathbf{b}$ has no solutions. Let w be the maximum number of non-zero entries in any row of A.

For each $i \in [m]$, let A_i be the *i*'th row of A, and let p_i be the polynomial $A_i \boldsymbol{x} - b_i$. Let C_i the CNF that is satisfied if and only $p_i(\boldsymbol{x}) = 0$. Notice that C_i has size at most 2^w . The explicit encoding of $A\boldsymbol{x} = \boldsymbol{b}$ is the CNF $\bigwedge_{i=1}^m C_i$.

The methods of subsection 5.1 show that $\bigwedge_{i=1}^{m} C_i$ is reducible to the system of polynomials $\{p_1, \ldots, p_m\}$ via a constant depth reduction of size $m2^{O(w)}$. Moreover, the system of polynomials $\{p_1, \ldots, p_m\}$ has a degree one Nullstellensatz refutation given by Gaussian elimination. Moreover, degree one refutations are of size O(mn). Thus we have the following theorem: **Theorem 4.** Fix a prime number q. Let A be an $m \times n$ matrix, let x_1, \ldots, x_n be variables and let $\mathbf{b} \in \mathbb{Z}_q^m$ be so that $A\mathbf{x} = \mathbf{b}$ has no solutions. Let w be the maximum number of non-zero entries in any row of A.

There is a constant depth Frege with counting axioms modulo q refutation of the explicit encoding of $A\mathbf{x} = \mathbf{b}$ of size polynomial in m,n and 2^w .

The Tseitin graph tautologies on an expander graph are known to require exponential size constant-depth Frege proofs [9]. Because these principles can be represented as an unsatisfiable system of linear equations, they have polynomial size constant-depth Frege with counting axioms proofs.

Corollary 1. There exists a family of unsatisfiable sets of constant width clauses that require exponential size constant-depth Frege refutations, but have polynomial size constant-depth Frege with counting axioms refutations.

7 Acknowledgements

The authors would like to thank Sam Buss for useful conversations and insightful suggestions.

References

- M. Ajtai. The complexity of the pigeonhole principle. In Proceedings of the Twenty-Ninth Annual IEEE Symposium on the Foundations of Computer Science, pages 346–355, 1988.
- M. Ajtai. Parity and the pigeonhole principle. In FEASMATH: Feasible Mathematics: A Mathematical Sciences Institute Workshop. Birkhauser, 1990.
- 3. M. Ajtai. The independence of the modulo *p* counting principles. In *Proceedings* of the Twenty-Sixth Annual ACM Symposium on the Theory of Computing, pages 402–411, 1994.
- M. Alekhnovich, E. Ben-Sasson, A. A. Razborov, and A. Wigderson. Pseudorandom generators in propositional proof complexity. In *Proceedings of the Forty-first Annual IEEE Symposium on Foundations of Computer Science*, pages 43–53, 2000.
- P. Beame, R. Impagliazzo, J. Krajíček, T. Pitassi, and P. Pudlák. Lower bounds on Hilbert's Nullstellensatz and propositional proofs. In *Proceedings of the Thirty-fifth* Annual IEEE Symposium on Foundations of Computer Science, pages 794–806, 1994.
- P. Beame and T. Pitassi. An exponential separation between the parity principle and the pigeonhole principle. Annals of Pure and Applied Logic, 80(3):195–228, 26 August 1996.
- P. Beame and T. Pitassi. Propositional proof complexity: Past, present, and future. Bulletin of the European Association for Theoretical Computer Science, 65:66–89, 1998.
- P. Beame and S. Riis. More one the relative strength of counting principles. In P. Beame and S. Buss, editors, *Proof Complexity and Feasible Arithmetics*, pages 13–35. American Mathematical Society, 1998.
- E. Ben-Sasson. Hard examples for bounded depth frege. In Proceedings of the Thirty-fourth Annual ACM Symposium on Theory of Computing, pages 563–572, 2002.

- E. Ben-Sasson and R. Impagliazzo. Random CNFs are hard for the polynomial calculus. In Proceedings of the Fortieth Annual IEEE Symposium on Foundations of Computer Science, pages 415–421, 1999.
- M. L. Bonet and N. Galesi. A study of proof search algorithms for resolution and polynomial calculus. In Proceedings of the Fortieth Annual IEEE Symposium on Foundations of Computer Science, pages 422–431, 1999.
- J. Buresh-Oppenheim, M. Clegg, R. Impagliazzo, and T. Pitassi. Homogenization and the polynomial calculus. In Proceedings of the Twenty-seventh International Colloquium on Automata, Languages and Programming, pages 926–937, 2000.
- S. Buss, D. Grigoriev, R. Impagliazzo, and T. Pitassi. Linear gaps between degrees for the polynomial calculus modulo distinct primes. In *Proceedings of the Thirtyfirst Annual ACM Symposium on Theory of Computing*, pages 547–556, 1999.
- S. Buss, R. Impagliazzo, J. Krajíček, P. Pudlák, R. Razborov, and J. Sgall. Proof complexity in algebraic systems and bounded depth Frege systems with modular counting. *Computational Complexity*, 6(3):256–298, 1997.
- S. Buss and T. Pitassi. Good degree bounds on Nullstellensatz refutations of the induction principle. In Proceedings of the Eleventh Annual IEEE Conference on Computational Complexity, 1996.
- 16. Matthew Clegg, Jeffery Edmonds, and Russell Impagliazzo. Using the Groebner basis algorithm to find proofs of unsatisfiability. In *Proceedings of the Twenty-Eighth Annual ACM Symposium on the Theory of Computing*, pages 174–183, 1996.
- 17. S. Cook and R. Reckhow. The relative efficiency of propositional proof systems. *The Journal of Symbolic Logic*, 44(1):36 50, March 1979.
- 18. R. Impagliazzo, P. Pudlak, and J. Sgall. Lower bounds for the polynomial calculus and the Groebner basis algorithm. *Computational Complexity*, 8(2):127–144, 1999.
- R. Impagliazzo and N. Segerlind. Counting axioms do not polynomially simulate counting gates (extended abstract). In *Proceedings of the Forty-Second Annual IEEE Symposium on Foundations of Computer Science*, pages 200–209, 2001.
- J. Krajíček. Bounded Arithmetic, Propositional Logic and Complexity Theory. Cambridge University Press, 1995.
- J. Krajíček. Tautologies from pseudo-random generators. The Bulletin of Symbolic Logic, 7(2):197–212, 2001.
- J. Krajíček, P. Pudlák, and A. Woods. An exponential lower bound to the size of bounded depth Frege proofs of the pigeonhole principle. RSA: Random Structures and Algorithms, 7, 1995.
- 23. T. Pitassi. Algebraic propositional proof systems. In Neil Immerman and Phokion G. Kolaitis, editors, *Descriptive Complexity and Finite Models*, volume 31 of *DIMACS: Series in Discrete Mathematics and Theoretical Computer Science*. American Mathematical Society, 1997.
- T. Pitassi, P. Beame, and R. Impagliazzo. Exponential lower bounds for the pigeonhole principle. *Computational Complexity*, 3(2):97–140, 1993.
- P. Pudlák. The lengths of proofs. In S. R. Buss, editor, Handbook of Proof Theory, pages 547–637. Elsevier North-Holland, 1998.
- 26. A. Razborov. Lower bounds for the polynomial calculus. *Computational Complexity*, 7(4):291–324, 1998.
- S. Riis. Count(q) does not imply Count(p). Annals of Pure and Applied Logic, 90(1-3):1-56, 15 December 1997.