# Limitations of Quantum Coset States for Graph Isomorphism 

Sean Hallgren, Martin Rötteler, and Pranab Sen<br>NEC Laboratories America, Inc.<br>4 Independence Way, Suite 200<br>Princeton, NJ 08540, U.S.A.<br>\{hallgren, mroetteler, pranab\}@nec-labs.com


#### Abstract

It has been known for some time that graph isomorphism reduces to the hidden subgroup problem (HSP). What is more, most exponential speedups in quantum computation are obtained by solving instances of the HSP. A common feature of the resulting algorithms is the use of quantum coset states, which encode the hidden subgroup. An open question has been how hard it is to use these states to solve graph isomorphism. It was recently shown by Moore, Russell, and Schulman MRS05 that only an exponentially small amount of information is available from one, or a pair of coset states. A potential source of power to exploit are entangled quantum measurements that act jointly on many states at once. We show that entangled quantum measurements on at least $\Omega(n \log n)$ coset states are necessary to get useful information for the case of graph isomorphism, matching an information theoretic upper bound. This may be viewed as a negative result because highly entangled measurements seem hard to implement in general. Our main theorem is very general and also rules out using joint measurements on few coset states for some other groups, such as $\operatorname{GL}\left(n, \mathbb{F}_{p^{m}}\right)$ and $G^{n}$ where $G$ is finite and satisfies a suitable property.


## 1 Introduction

Almost all exponential speedups that have been achieved in quantum computing are obtained by solving some instances of the Hidden Subgroup Problem (HSP). In particular, the problems underlying Shor's algorithms for factoring and discrete logarithm [Sho97], as well as Simon's problem [Sim94], can be naturally generalized to the HSP: given a function $f: G \rightarrow S$ from a group $G$ to a set $S$ that is constant on left cosets of some subgroup $H \leq G$ and distinct on different cosets, find a set of generators for $H$. Ideally, we would like to find $H$ in time polynomial in the input size, i. e. $\log |G|$. The abelian HSP Kit95, BH97 ME98], i. e., when $G$ is an abelian group, lies at the heart of efficient quantum algorithms for important number-theoretic problems like factoring, discrete logarithm, Pell's equation, unit group of a number field etc. [Sho97 Hal02, Hal05 SV05].

It has been known for some time that graph isomorphism reduces to the HSP over the symmetric group [Bea97, EHK99a], a non-abelian group. While the non-abelian HSP has received much attention as a result, efficient algorithms are known only for some special classes of groups [IMS03, FIM ${ }^{+}$03] MRRS04, BCD05]. On the other hand, the HSP presents a systematic way to try and approach the graph isomorphism problem, and this approach is rooted in developing a deeper understanding of how far techniques and tools that have worked in the abelian case can be applied. To the best of our knowledge, the only other approach to solve graph isomorphism on a quantum computer is by creating a uniform superposition of all graphs
isomorphic to a given graph. It has been proposed to create this superposition via quantum sampling of Markov chains [AT03], however, very little is known about this.

One of the key features of a quantum computer is that it can compute functions in superposition. This fact alone does not lend itself to exponential speedups, for instance for unstructured search problems it merely leads to a polynomial speedup [Gro96, BBBV97]. On the other hand, the quantum states resulting from HSP instances have far more structure since they capture some periodicity aspects of the function $f$. Evaluating the function $f$ in superposition and ignoring the function value results in a random coset state. Coset states are quantum states of the form $|g H\rangle=\frac{1}{\sqrt{|H|}} \sum_{h \in H}|g h\rangle$, in other words, a coset state is a uniform superposition over the elements of the left coset $g H$. The challenge in using coset states lies in the fact that $g$ is a random element of the group, beyond our control, that is, we only have the mixed state $\sigma_{H}^{G}=\frac{1}{|G|} \sum_{g \in G}|g H\rangle\langle g H|$ and we have to determine $H$ from it. Though it is conceivable that some advantage can be had by making use of the function values, currently there are no proposals for using function values in any meaningful way.

How much information can be extracted from coset states? The most general way to extract classical information from quantum states are POVMs [NC00]. A fixed POVM operates on a fixed number $k$ of coset states at once. This induces a probability distribution over the set of classical outcomes associated with the POVM. A potential source of power with no classical analog is that the distribution induced by a POVM on $k$ coset states may have significantly more information than a POVM that acts on just one coset state at a time. In other words, the resulting distribution when the POVM is applied to $k$ coset states can be far from a product distribution. In this case we say that the POVM is an entangled measurement. The goal of this paper is to determine how small $k$ can be made such that a polynomial amount of information about $H$ can be obtained from a POVM on $k$ coset states. More precisely, we want to know how small $k$ can be so that there exists a POVM on $k$ coset states that gives polynomially large total variation distance between every pair of candidate hidden subgroups. Note that this POVM can have many classical outcomes, and it may have to be repeated several times if we want to identify the actual hidden subgroup $H$ with constant probability.

In this paper, we show that for many groups $G$ this number $k$ has to be quite large, sometimes as large as $\Omega(\log |G|)$. This matches the information theoretic upper bound of $O(\log |G|)$ for general groups [EHK99b]. Our result can be viewed as a negative result because highly entangled measurements seem hard to implement in general. Note that the time required to perform a generic measurement entangled across $k$ states increases exponentially with $k$.

For abelian groups the picture simplifies dramatically. Indeed, in this case a POVM operating on one coset state (i. e., $k=1$ ) exists that gives a polynomial amount of information about the hidden subgroup. Moreover, this measurement is efficiently implementable using the quantum Fourier transform over the group. The Fourier based approach extends to some non-abelian groups as well, e. g., dihedral, affine and Heisenberg groups, and shows that for these groups there are measurements on single coset states that give polynomially large information about the hidden subgroup [EH00 MRRS04, RRS05].

Other than the general information-theoretic upper bound, only a few examples of measurements operating on more than one coset state (i. e., $k>1$ ) are known that give a polynomial amount of information about the hidden subgroup. Kuperberg [Kup03] gave a measurement for the dihedral group operating on $2^{O(\sqrt{\log |G|}}$ coset states that also takes $2^{O(\sqrt{\log |G| \mid})}$ time to implement. Bacon et al. [BCD05] gave an efficiently implementable measurement for the Heisenberg group operating on two coset states, and similar efficient measurements for some other groups operating on a constant number of coset states.

The case of the symmetric group $S_{n}$ has been much harder to understand. First it was shown that some restricted measurements related to the abelian case cannot solve the problem [HRT03]. Next the non-abelian
aspects of the group were attacked by Grigni et al. [GSVV04] who showed that for hidden subgroups in $S_{n}$, measuring the Fourier transform of a single coset state using random choices of bases for the representations of $S_{n}$ gives exponentially little information. They left open the question whether a clever choice of basis for each representation space can indeed give enough information about the hidden subgroup. Recently, a breakthrough has been made by Moore, Russell and Schulman [MRS05] who answered this question in the negative for $k=1$ by showing that any measurement on a single coset state of $S_{n}$ gives exponentially little information, i. e., any algorithm for the HSP in $S_{n}$ that measures one coset state at a time requires at least $\exp (\Omega(n))$ coset states. Subsequently, Moore and Russell MR05] extended this result by showing that any algorithm that jointly measures two coset states at a time requires at least $\exp (\Omega(\sqrt{n} / \log n))$ coset states. However, their techniques fail for algorithms that jointly measure three or more coset states at a time, and they left the $k \geq 3$ case open.

In this paper, we show that no quantum measurement on $k=o(n \log n)$ coset states can extract polynomial amount of information about the hidden subgroup in $S_{n}$. Thus, any algorithm operating on coset states that solves the hidden subgroup problem in $S_{n}$ in polynomial time has to make joint measurements on $k=\Omega(n \log n)$ coset states, matching the information theoretic upper bound. Our results apply to the hidden subgroups arising out of the reduction from isomorphism of rigid graphs, and rules out any efficient quantum algorithm that tries to solve graph isomorphism via the standard reduction to the HSP in $S_{n}$ using measurements that act jointly on less than $n \log n$ coset states at a time.

Our lower bound holds for a more general setting: Given a group $G$, suppose we want to decide if the hidden subgroup is a conjugate of an a priori known order two subgroup $H$, or the identity subgroup. We show a lower bound on the total number of coset states required by any algorithm that jointly measures at most $k$ states at a time and that distinguishes between the above two cases. Our main theorem uses only properties of $G$ that can be read off from the values of the characters at the two elements of $H$. We also prove a transfer lemma that allows us to transfer lower bounds proved for subgroups and quotient groups to larger groups. Using our main theorem and the transfer lemma, we show lower bounds on the order of entangled measurements required to efficiently solve the HSP using coset states in groups PSL $\left(2, \mathbb{F}_{p^{m}}\right)$, $\mathrm{GL}\left(n, \mathbb{F}_{p^{m}}\right)$, and groups of the form $G^{n}$, where $G$ a constant-sized group satisfying a suitable property, including all groups $\left(S_{m}\right)^{n}$ where $m \geq 4$ is a constant. The case of $\left(S_{4}\right)^{n}$ is interesting, because there is an efficient algorithm for the HSP making joint measurements on $n^{O(1)}$ states using the orbit coset techniques of [FIM $\left.{ }^{+} 03\right]$. However, the orbit coset approach creates coset states not just for the hidden subgroup $H$, but also for various subgroups of the form $H N$, where $N \unlhd\left(S_{4}\right)^{n}$. This example suggests that one way to design efficient algorithms for the HSP making highly entangled measurements may be to use coset states for subgroups of $G$ other than just the hidden subgroup $H$.

Recently, Childs and Wocjan [CW05] proposed a hidden shift approach to graph isomorphism. They established a lower bound for the total number of hidden shift states required and also showed that a single hidden shift state contains exponentially little information about the isomorphism. Our results generalize both their bounds and imply that $o(n \log n)$ hidden shift states contain exponentially little information about the isomorphism.

The chief technical innovation required to prove our main theorem is an improved upper bound for the second moment of the probability of observing a particular measurement outcome as we vary over different candidate hidden subgroups. In particular, we give a new and improved analysis of the projection lengths of vectors of the form $\mathbf{b} \otimes \mathbf{b}$ onto homogeneous spaces of irreducible representations of a group. The earlier works MRS05 MR05 tried to bound these projection lengths using simple geometric methods. As a result, their methods failed beyond $k=2$ for the symmetric group. Instead, we make crucial use of the representation-theoretic structure of the projection operators as well as the structure of the vectors, in order
to prove upper bounds on the projection lengths better than those obtainable by mere geometry．This allows us to prove a general theorem that applies with large $k$ for many groups．

Finally，we also prove a simple lower bound on the total number of coset states required by any algorithm to solve the HSP in a group $G$ ．This lower bound gives a simple proof of the fact that distinguishing a hidden reflection from the identity subgroup in the dihedral group $D_{n}$ requires $\Omega(\log n)$ coset states．

## 2 Preliminaries

## 2．1 Graph isomorphism and HSP

The usual reduction of deciding isomorphism of two $n$－vertex graphs to HSP in $S_{2 n}$ actually embeds the problem into a proper subgroup of $S_{2 n}$ ，namely，$S_{n} \backslash S_{2}$［EHK99a］．The elements of $S_{n}$ 2 $S_{2}$ are tuples of the form $(\pi, \sigma, b)$ where $\pi, \sigma \in S_{n}$ and $b \in \mathbb{Z}_{2}$ with the multiplication rule $\left(\pi_{1}, \sigma_{1}, 0\right) \cdot\left(\pi_{2}, \sigma_{2}, b\right):=$ $\left(\pi_{1} \pi_{2}, \sigma_{1} \sigma_{2}, b\right)$ and $\left(\pi_{1}, \sigma_{1}, 1\right) \cdot\left(\pi_{2}, \sigma_{2}, b\right):=\left(\pi_{1} \sigma_{2}, \sigma_{1} \pi_{2}, 1 \oplus b\right)$ ．The embedding of $S_{n} 乙 S_{2}$ in $S_{2 n}$ treats $\{1, \ldots, 2 n\}$ as a union of $\{1, \ldots, n\} \cup\{n+1, \ldots, 2 n\}$ with $\pi, \sigma$ permuting the first and second sets respectively when $b=0$ ，and $\pi$ permuting the first set onto the second and and $\sigma$ permuting the second set onto the first when $b=1$ ．There is an element of the form $\left(\pi, \pi^{-1}, 1\right)$ ，called an involutive swap，in the hidden subgroup iff the two graphs are isomorphic．

Additionally，if the two graphs are rigid，i．e．，have no non－trivial automorphisms，then the hidden sub－ group is trivial if they are non－isomorphic，and is generated by $\left(\pi, \pi^{-1}, 1\right)$ if they are isomorphic where $\pi$ is the unique isomorphism from the first graph onto the second．This element $\left(\pi, \pi^{-1}, 1\right)$ is of order two，and is a conjugate in $S_{n} \backslash S_{2}$ of $h:=(e, e, 1)$ where $e \in S_{n}$ is the identity permutation．Viewed as an element of $S_{2 n}, h=(1, n+1)(2, n+2) \cdots(n, 2 n)$ ．The set of conjugates of $h$ in $S_{n}$ 久 $S_{2}$ is the set of all involu－ tive swaps $\left(\pi, \pi^{-1}, 1\right), \pi \in S_{n}$ ，and corresponds exactly to all the isomorphisms possible between the two graphs．Also $S_{n} 乙 S_{2}$ is the smallest group containing all involutive swaps as a single conjugacy class．This algebraic property makes $S_{n}$ 乙 $S_{2}$ ideal for the study of isomorphism of rigid graphs as a hidden subgroup problem．Note that graph automorphism，i．e．，deciding if a given graph has a non－trivial automorphism，is Turing equivalent classically to isomorphism of rigid graphs［KST93］．

In this paper，we consider the following problem：Given that the hidden subgroup in $S_{n}$ 2 $S_{2}$ is either generated by an involutive swap or is trivial，decide which case is true．Graph automorphism as well as rigid－graph isomorphism reduces to this problem．We show that any efficient algorithm using coset states that solves this problem needs to make measurements entangled across $\Omega(n \log n)$ states（Corollary 14）． Note that any lower bound for this problem for a coset state based algorithm holds true even when the involutive swaps are considered as elements of $S_{2 n}$ rather than $S_{n} \ell S_{2}$ ．This is because of the following general transfer lemma．

Lemma 1 （Transfer lemma）．Let $G$ be a finite group and suppose that either $G \leq \tilde{G}$ or $G \cong \tilde{G} / N, N \unlhd \tilde{G}$ holds．Then lower bounds for coset state based algorithms for the HSP in $G$ transfer to the same bounds for the HSP in $\tilde{G}$ and vice versa，as long as the hidden subgroups involved are contained in $G$ ．

Proof．Let $H \leq G$ ．The case $G \leq \tilde{G}$ follows from the observation that $\mathbb{C}[\tilde{G}]=\bigoplus_{\tilde{g} \in \tilde{G} / G} L_{\tilde{g}} \cdot \mathbb{C}[G]$ ， where $\tilde{G} / G$ denotes a system of left coset representatives of $G$ in $\tilde{G}$ and $L_{\tilde{g}}$ stands for left multiplication by $\tilde{g}$ ．Then，$\sigma_{H}^{\tilde{G}}=\bigoplus_{\tilde{g} \in \tilde{G} / G} L_{\tilde{g}} \cdot \sigma_{H}^{G} \cdot L_{\tilde{g}}^{\dagger}$ ，and so any coset state based algorithm without loss of generality performs the same operations on each block of the orthogonal direct sum．The case $G \cong \tilde{G} / N$ follows from the observation that $\mathbb{C}[G]$ is isometric to the subspace of $\mathbb{C}[\tilde{G}]$ spanned by coset states of $N$ namely states of
the form $|\tilde{g} N\rangle, \tilde{g} \in \tilde{G}$. There is a subgroup $\tilde{H} \leq \tilde{G}, N \unlhd \tilde{H}$ such that $\tilde{H} / N \cong H$. Hence, $\sigma_{H}^{G} \cong \sigma_{\tilde{H}}^{\tilde{G}}$. Thus, any coset state based algorithm without loss of generality performs the same operations on $\sigma_{H}^{G}$ and $\sigma_{\tilde{H}}^{\tilde{G}}$.

Childs and Wocjan [CW05] showed an $\Omega(n)$ lower bound for the total number of hidden shift states required to solve graph isomorphism, and also proved that a single hidden shift state contains exponentially little information about the isomorphism. However, their results do not rule out an algorithm that makes joint measurements on, say, two states at a time and uses a total of $O(n)$ hidden shift states. Since the hidden shift state corresponding to the shift $\left(\pi, \pi^{-1}\right)$, where $\pi \in S_{n / 2}$ is exactly the coset state for the hidden subgroup generated by the involutive swap $\left(\pi, \pi^{-1}, 1\right)$ in $S_{n / 2}$ l $S_{2}$, Lemma 1 and Corollary 14 of our paper show that any efficient algorithm using hidden shift states to solve the graph isomorphism problem needs to make measurements entangled across $\Omega(n \log n)$ states, generalizing their results.

### 2.2 Quantum Fourier transform and POVMs

We collect some standard facts from representation theory of finite groups; see e.g. the book by Serre [Ser77] for more details. We use the term irrep to denote an irreducible unitary representation of a finite group $G$ and denote by $\widehat{G}$ a complete set of inequivalent irreps. For any unitary representation $\rho$ of $G$, let $\rho^{*}$ denote the representation obtained by entry-wise conjugating the unitary matrices $\rho(g)$, where $g \in G$. Note that the definition of $\rho^{*}$ depends upon the choice of the basis used to concretely describe the matrices $\rho(g)$. If $\rho$ is an irrep of $G$ so is $\rho^{*}$, but in general $\rho^{*}$ may be inequivalent to $\rho$. Let $V_{\rho}$ denote the vector space of $\rho$, define $d_{\rho}:=\operatorname{dim} V_{\rho}$, and notice that $V_{\rho}=V_{\rho^{*}}$. The group elements $|g\rangle$, where $g \in G$ form an orthonormal basis of $\mathbb{C}^{|G|}$. Since $\sum_{\rho \in \widehat{G}} d_{\rho}^{2}=|G|$, we can consider another orthonormal basis called the Fourier basis of $\mathbb{C}^{|G|}$ indexed by $|\rho, i, j\rangle$, where $\rho \in \widehat{G}$ and $i, j$ run over the row and column indices of $\rho$. The quantum Fourier transform over $G, \mathrm{QFT}_{G}$ is the following linear transformation:

$$
|g\rangle \mapsto \sum_{\rho \in \widehat{G}} \sqrt{\frac{d_{\rho}}{|G|}} \sum_{i, j=1}^{d_{\rho}} \rho_{i j}(g)|\rho, i, j\rangle
$$

It follows from Schur's orthogonality relations (see e.g. [Ser77] Chapter 2, Proposition 4, Corollary 3]) that QFT $_{G}$ is a unitary transformation in $\mathbb{C}^{|G|}$.

For a subgroup $H \leq G$ and $\rho \in \widehat{G}$, define $\rho(H):=\frac{1}{|H|} \sum_{h \in H} \rho(h)$. It follows from Schur's lemma (see e.g. [Ser77] Chapter 2, Proposition 4]) that $\rho(H)$ is an orthogonal projection to the subspace of $V_{\rho}$ consisting of vectors that are point-wise fixed by every $\rho(h), h \in H$. Define $r_{\rho}(H):=\operatorname{rank}(\rho(H))$; then $r_{\rho}(H)=$ $\frac{1}{|H|} \sum_{h \in H} \chi_{\rho}(h)$, where $\chi_{\rho}$ denotes the character of $\rho$. Notice that $r_{\rho}(H)=r_{\rho^{*}}(H)$. For any subset $S \leq G$ we define $|S\rangle:=\frac{1}{\sqrt{|S|}} \sum_{s \in S}|s\rangle$ to be the uniform superposition over the elements of $S$. The standard method of attacking the HSP in $G$ using coset states [GSVV04] starts by forming the uniform superposition $\frac{1}{\sqrt{|G|}} \sum_{g \in G}|g\rangle|0\rangle$. It then queries $f$ to get the superposition $\frac{1}{\sqrt{|G|}} \sum_{g \in G}|g\rangle|f(g)\rangle$. Ignoring the second register the reduced state on the first register becomes the density matrix $\sigma_{H}^{G}=\frac{1}{|G|} \sum_{g \in G}|g H\rangle\langle g H|$, that is the reduced state is a uniform mixture over all left coset states of $H$ in $G$. It can be easily seen that applying $\mathrm{QFT}_{G}$ to $\sigma_{H}^{G}$ gives us the density matrix $\frac{|H|}{|G|} \bigoplus_{\rho \in \widehat{G}} \bigoplus_{i=1}^{d_{\rho}}|\rho, i\rangle\langle\rho, i| \otimes \rho^{*}(H)$, where $\rho^{*}(H)$ operates on the space of column indices of $\rho$. When measuring this state, we obtain an irrep $\rho$ with probability $\frac{d_{\rho}|H| r_{\rho(H)}}{|G|}$. Conditioned on measuring $\rho$ we obtain a uniform distribution $1 / d_{\rho}$ on the row indices. The reduced state on the space of column indices after having observed an irrep $\rho$ and a row index $i$ is then given by the state
$\rho^{*}(H) / r_{\rho}(H)$, and a basic task for a hidden subgroup finding algorithm is how to extract information about $H$ from it.

If the the hidden subgroup is the trivial subgroup $\{1\}$, the probability of measuring $\rho$ is given by the so-called Plancherel distribution $\mathcal{P}(\rho):=\frac{d_{\rho}^{2}}{|G|}$. This distribution will be useful to us later on in the proof of the main theorem.

POVMs are the most general way to obtain classical information from quantum states [NC00]. The elements of a POVM $\mathcal{M}$ in $\mathbb{C}^{n}$ are positive operators $E_{i} \geq 0$ which have to satisfy the completeness condition $\sum_{i} E_{i}=\mathbb{1}_{n}$. If the state of the quantum system is given by the density matrix $\sigma$, then the probability $p_{i}$ to observe outcome labeled $i$ is given by the Born rule $p_{i}=\operatorname{Tr}\left(\sigma E_{i}\right)$. The following observation is crucial for the HSP case: since the states $\sigma_{H}^{G}$ are simultaneously block diagonal in the Fourier basis for any $H \leq G$, the elements of any POVM $\mathcal{M}$ operating on these states can without loss of generality be assumed to have the same block structure. From this it is clear that any measurement to identify $H$ without loss of generality first applies the quantum Fourier transform $\mathrm{QFT}_{G}$ to $\sigma_{H}^{G}$, measures the name $\rho$ of an irrep, the index $i$ of a row, and then measures the reduced state on the column space of $\rho$ using a POVM $\mathcal{M}_{\rho}$ in $\mathbb{C}^{d_{\rho}}$. This POVM $\mathcal{M}_{\rho}$ may depend on $\rho$ but is independent of $i$.

Furthermore, $\mathcal{M}_{\rho}$ can be assumed to be a frame, i. e., a collection $\mathcal{B}_{\rho}:=\left\{\left(a_{b}, b\right)\right\}$, where $b \in \mathbb{C}^{d_{\rho}}$ with $\|b\|=1$ and $0 \leq a_{b} \leq 1$ such that $\sum_{b \in \mathcal{B}} a_{b}|b\rangle\langle b|=\mathbb{1}_{d_{\rho}}$ i.e. a frame is a POVM with rank one elements. Orthonormal bases are special cases of frames in which $a_{b}=1$ for all $b \in \mathcal{B}_{\rho}$. We can assume that the POVM on the column space is a frame because any POVM can be refined to a frame such that for any quantum state, the probabilities according to the original POVM are certain sums, independent of the state measured, of probabilities according to the frame.

If the the hidden subgroup is the trivial subgroup $\{1\}$, after observing an irrep $\rho$ and a row index $i$, the reduced state on the space of column indices of $\rho$ is the totally mixed state $\frac{\mathbb{1}_{d_{\rho}}}{d_{\rho}}$. The probability of observing a vector $b$ in frame $\mathcal{B}_{\rho}$ is given by the so-called natural distribution on $\mathcal{B}_{\rho}$ defined by $\mathcal{N}(b \mid \rho):=\frac{d_{\rho}^{2}}{|G|}$. This distribution will be useful to us later on in the proof of the main theorem.

The above description was for single register quantum Fourier sampling. Fourier sampling on $k$ registers can be defined analogously. Here one starts off with $k$ independent copies of the coset state $\sigma_{H}^{G}$, i. e., with the state $\left(\sigma_{H}^{G}\right)^{\otimes k} \cong \sigma_{H^{k}}^{G^{k}}$ and applies $\operatorname{QFT}_{G}^{\otimes k}$ to it. Here $G^{k}, H^{k}$ denote the $k$-fold direct product of $G, H$ respectively. Note that since $\widehat{G^{k}} \cong \widehat{G}^{\otimes k}$, we have that $\mathrm{QFT}_{G^{k}}=\mathrm{QFT}_{G}^{\otimes k}$. We can express an irrep $\boldsymbol{\rho}$ of $G^{k}$ as $\boldsymbol{\rho}=\otimes_{i=1}^{k} \rho_{i}, \rho_{i} \in \widehat{G}$; observe that $V_{\boldsymbol{\rho}}=\otimes_{i=1}^{k} V_{\rho_{i}}$. We adopt the convention that multiregister vectors and representations are denoted in boldface type. After applying $\mathrm{QFT}_{G}^{\otimes k}$, we measure the name $\rho$ of an irrep of $G^{k}$, i.e, irreps $\rho_{1}, \ldots, \rho_{k}$ of $G$. After that, we measure a row index of $\rho$ i. e., row indices of $\rho_{1}, \ldots, \rho_{k}$, and then measure the resulting reduced state in the column space of $\rho$ using a frame $\mathcal{B}$ of $V_{\rho}$. The frame $\mathcal{B}$ used depends on the observed $\rho$ but not on the observed row indices. Notice that only the application of the frame $\mathcal{B}$ may be an entangled measurement, the application of $\mathrm{QFT}_{G^{k}}$ and measurement of $\rho$ together with a row index of $\rho$ are single register operations.

## 3 The main theorem

Let $G$ be a group and $h \in G$ be an involution, that is, $H:=\{1, h\}$ is an order two subgroup of $G$. We let $H^{g}:=g H^{-1}$ denote the conjugate of $H$ by $g \in G$. Let $k$ be a positive integer. Fix a POVM $\mathcal{M}$ on $\mathbb{C}[G]^{\otimes k} \cong \mathbb{C}\left[G^{k}\right]$. Let $\mathcal{M}_{H^{g}}, \mathcal{M}_{\{1\}}$ denote the classical probability distributions obtained by measuring the states $\sigma_{H^{g}}^{\otimes k}, \sigma_{\{1\}}^{\otimes k}$ respectively according to $\mathcal{M}$. We will show that the average total variation distance
between $\mathcal{M}_{H^{g}}$ and $\mathcal{M}_{\{1\}}$ over conjugates $H^{g}, g \in G$ is at most $2^{k}$ times a quantity that depends purely on the pair $(G, H)$. In the next section, we will show that this quantity is exponentially small for many pairs $(G, H)$ of interest, including when $G=S_{n} \swarrow S_{2}$ and $H$ is generated by an involutive swap, i. e., the case relevant to isomorphism of rigid graphs.

Theorem 2 (Main theorem). Let $G$ be a finite group and $H:=\{1, h\}$ be an order two subgroup of $G$. Let $k \geq 1$ be an integer. Fix a POVM $\mathcal{M}$ on $\mathbb{C}[G]^{\otimes k}$ and let $\mathcal{M}_{H^{g}}, \mathcal{M}_{\{1\}}$ denote the classical probability distributions obtained by measuring the states $\sigma_{H}^{\otimes k}, \sigma_{\{1\}}^{\otimes k}$ respectively according to $\mathcal{M}$. For $\varepsilon>0$, define the set

$$
\mathcal{S}_{\varepsilon}:=\left\{\tau \in \widehat{G}: \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}} \geq \varepsilon\right\} .
$$

Suppose that $2 k \varepsilon<1$ holds. Define

$$
\delta_{1}:=\varepsilon+\frac{1}{|G|} \cdot\left(\sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}\left|\chi_{\tau}(h)\right|\right) \cdot\left(\sum_{\nu \in \widehat{G}} d_{\nu}\right) \leq \varepsilon+\left(\sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}\left|\chi_{\tau}(h)\right|\right) \cdot\left(\frac{|\widehat{G}|}{|G|}\right)^{1 / 2},
$$

and

$$
\delta_{2}:=2^{k}(1+2 k \varepsilon) \delta_{1}^{1 / 2}+3 k \varepsilon+\frac{3 k}{|G|} \cdot \sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}^{2} .
$$

Then

$$
\mathrm{E}_{g}\left[\left\|\mathcal{M}_{H^{g}}-\mathcal{M}_{\{1\}}\right\|_{1}\right] \leq \delta_{2},
$$

where the expectation is taken over the uniform distribution on $g \in G$.
By a $k$-entangled POVM $\mathcal{F}$ on $t$ coset states, we mean that $\mathcal{F}$ consists of a sequence of POVM's $\left(\mathcal{M}_{i}\right)_{i \in\left[t^{\prime}\right]}$, where each $\mathcal{M}_{i}$ operates on a fresh set of at most $k$-coset states and $t^{\prime} \leq t$. The number of coset states operated upon by $\mathcal{F}$ is at most $t$. The outcome of $\mathcal{F}$ is a sequence of length $t^{\prime}$ corresponding to the outcomes of $\mathcal{M}_{i}$. The choice of $\mathcal{M}_{i}$ may depend on the observed outcomes of $\mathcal{M}_{1}, \ldots, \mathcal{M}_{i-1}$. If required, further classical postprocessing may be done on the outcome of $\mathcal{F}$. We now prove the following corollary of Theorem 2

Corollary 3. Suppose $\mathcal{F}$ is a $k$-entangled POVM on $t$ coset states. Then for at least a fraction of $1-\sqrt{t \delta_{2}}$ conjugate subgroups $H^{g}, g \in G$,

$$
\left\|\mathcal{F}_{H^{g}}-\mathcal{F}_{\{1\}}\right\|_{1} \leq \sqrt{t \delta_{2}} .
$$

Proof. Using Theorem 2 and triangle inequality, it is easy to see that $\mathrm{E}_{g}\left[\left\|\mathcal{F}_{H^{g}}-\mathcal{F}_{\{1\}}\right\|_{1}\right] \leq t \delta_{2}$. Applying Markov's inequality to the expectation over $g \in G$ finishes the proof.

The remainder of the section is devoted to proving Theorem 2 We first give some notation that will be useful for the proofs of various lemmas. Our notation and setup is inspired to a large extent by the notation in (MR05].

As argued in the previous section, we can assume without loss of generality that $\mathcal{M}$ first applies $\mathrm{QFT}_{G}^{\otimes k}$ to $\sigma_{H^{g}}^{\otimes k}$, measures the name of an irrep of $G^{k}, \rho^{*}$ together with a row index of $\boldsymbol{\rho}^{*}$, and then measures the resulting reduced state in the column space of $\rho^{*}$ using a frame $\mathcal{B}$ of $V_{\rho^{*}}=V_{\rho}$. If $\boldsymbol{\rho}=\otimes_{i=1}^{k} \rho_{i}, \rho_{i} \in \widehat{G}$, then $\boldsymbol{\rho}^{*}=\otimes_{i=1}^{k} \rho_{i}^{*}$. The frame $\mathcal{B}$ used depends on the observed $\boldsymbol{\rho}^{*}$ but not on the observed row indices.

Suppose the hidden subgroup is $H^{g}$ for some $g \in G$. It is easy to see that the probability that $\mathcal{M}$ measures $\rho^{*}$ is given by

$$
\mathcal{M}_{H^{g}}\left(\boldsymbol{\rho}^{*}\right)=\frac{d_{\boldsymbol{\rho}^{*}}\left|H^{g}\right|^{k} \cdot r_{\boldsymbol{\rho}^{*}}\left(\left(H^{g}\right)^{k}\right)}{|G|^{k}}=\frac{2^{k} d_{\boldsymbol{\rho}} r_{\boldsymbol{\rho}}\left(H^{k}\right)}{|G|^{k}}
$$

Notice that $\mathcal{M}_{H^{g}}\left(\boldsymbol{\rho}^{*}\right)=\mathcal{M}_{H}(\boldsymbol{\rho})$. Let $\mathcal{B}=\left\{a_{\mathbf{b}}, \mathbf{b}\right\}$, where $0 \leq a_{\mathbf{b}} \leq 1$ and $\sum_{\mathbf{b}} a_{\mathbf{b}}|\mathbf{b}\rangle\langle\mathbf{b}|=\mathbb{1}_{V_{\boldsymbol{\rho}}}$. Then the reduced state in the column space of $\rho^{*}$ is $\frac{\rho\left(\left(H^{g}\right)^{k}\right)}{r_{\rho}\left(H^{k}\right)}$, if $r_{\rho} \neq 0$. Hence, the probability of observing a particular b conditioned on having observed $\rho^{*}$ is

$$
\mathcal{M}_{H^{g}}\left(\mathbf{b} \mid \boldsymbol{\rho}^{*}\right)=\frac{a_{\mathbf{b}}\langle\mathbf{b}| \boldsymbol{\rho}\left(\left(H^{g}\right)^{k}\right)|\mathbf{b}\rangle}{r_{\boldsymbol{\rho}}\left(H^{k}\right)}
$$

if $r_{\rho}\left(H^{k}\right) \neq 0$, and 0 otherwise. Similarly, if the hidden subgroup is the identity subgroup then

$$
\mathcal{M}_{\{1\}}\left(\boldsymbol{\rho}^{*}\right)=\frac{d_{\boldsymbol{\rho}}^{2}}{|G|^{k}}=\mathcal{P}(\boldsymbol{\rho})
$$

where $\mathcal{P}(\cdot)$ is the Plancherel distribution on irreps of $G^{k}$. Also

$$
\mathcal{M}_{\{1\}}\left(\mathbf{b} \mid \rho^{*}\right)=\frac{a_{\mathbf{b}}}{d_{\boldsymbol{\rho}}}=\mathcal{N}\left(\mathbf{b} \mid \rho^{*}\right)
$$

where $\mathcal{N}\left(\cdot \mid \rho^{*}\right)$ is the natural distribution corresponding to the frame $\mathcal{B}$.
For a non-empty subset $I \subseteq[k]$, define $\boldsymbol{\rho}^{I}:=\left(\otimes_{i \in I} \rho_{i}\right) \otimes\left(\otimes_{i^{\prime} \in[k] \backslash I} \mathbb{1}_{d_{\rho_{i^{\prime}}}}\right)$, where $\mathbb{1}_{d_{\rho_{i^{\prime}}}}$ denotes the identity representation of $G$ of degree equal to that of $\rho_{i^{\prime}}$. For non-empty subsets $I_{1}, I_{2} \subseteq[k]$, define $\boldsymbol{\rho}^{I_{1}, I_{2}}:=\boldsymbol{\rho}^{I_{1}} \otimes \boldsymbol{\rho}^{I_{2}}$. For a representation $\boldsymbol{\theta}=\otimes_{i=1}^{n} \theta_{i}$ of $G^{n}, \theta_{i}$ representation of $G$, we use $\boldsymbol{\theta}(g)$ as a shorthand for $\otimes_{i=1}^{n} \theta_{i}(g)$. For an irrep $\tau \in \widehat{G}$, we use $a_{\tau}^{\boldsymbol{\theta}}$ to denote the multiplicity of $\tau$ in the ClebschGordan decomposition of $\boldsymbol{\theta}$, i. e. the number of times $\tau$ occurs in $\boldsymbol{\theta}$ when $\boldsymbol{\theta}$ is viewed as a representation of $G$ embedded as the diagonal subgroup of $G^{n}$. We let $\Pi_{\tau}^{\theta}$ denote the orthogonal projection from $V_{\boldsymbol{\theta}}$ onto the homogeneous component of $\tau$ in the above decomposition. We use the following shorthand for expectations: $\mathrm{E}_{\boldsymbol{\rho}}[\cdot], \mathrm{E}_{\mathbf{b}}[\cdot]$ and $\mathrm{E}_{g}[\cdot]$ denote expectations over the Plancherel distribution on irreps, natural distribution on frame vectors and uniform distribution on elements of $G$ respectively.

We define a function $X: \widehat{G}^{\otimes k} \times \mathcal{B} \times G \rightarrow[-1,1]$ as

$$
X(\boldsymbol{\rho}, \mathbf{b}, g):=\langle\mathbf{b}| \boldsymbol{\rho}\left(\left(H^{g}\right)^{k}\right)|\mathbf{b}\rangle-\frac{1}{2^{k}}
$$

where $\mathcal{B}$ is a frame for $V_{\rho}$. The importance of $X$ will become clear in Lemma 11 below, which shows that $\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}[|X(\boldsymbol{\rho}, \mathbf{b}, g)|]$ is closely related to the total variation distance between $\mathcal{M}_{H^{g}}$ and $\mathcal{M}_{\{1\}}$.

We start by proving the following lemma, which is similar to [MR05, Lemma 11]. The lemma gives us a way to express the second moment of $X$ in terms of projections of 'coupled' frame vectors $\mathbf{b} \otimes \mathbf{b}$ onto homogeneous components corresponding to irreps $\tau \in \widehat{G}$. The advantage of doing this is that we can now distinguish between 'good' irreps, namely those with $\frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}$ small, and 'bad' irreps, namely those where $\frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}$ is large. The contribution of 'good' irreps to the second moment of $X$ is small. This idea of distinguishing between 'good' and 'bad' irreps goes back to MRS05.
Lemma 4.

$$
\mathrm{E}_{g}\left[X(\boldsymbol{\rho}, \mathbf{b}, g)^{2}\right]=\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \sum_{\tau \in \widehat{G}} \frac{\chi_{\tau}(h)}{d_{\tau}}\left\|\Pi_{\tau}^{\boldsymbol{\rho}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}
$$

Proof. Since

$$
\begin{aligned}
X(\boldsymbol{\rho}, \mathbf{b}, g) & =\frac{1}{2^{k}}\left(\langle\mathbf{b}| \mathbb{1}_{d_{\rho}}|\mathbf{b}\rangle+\sum_{I \neq\{ \}}\langle\mathbf{b}| \boldsymbol{\rho}^{I}\left(g h g^{-1}\right)|\mathbf{b}\rangle\right)-\frac{1}{2^{k}} \\
& =\frac{1}{2^{k}} \sum_{I \neq\{ \}}\langle\mathbf{b}| \boldsymbol{\rho}^{I}\left(g h g^{-1}\right)|\mathbf{b}\rangle,
\end{aligned}
$$

we get

$$
\begin{aligned}
& \mathrm{E}_{g}\left[X(\boldsymbol{\rho}, \mathbf{b}, g)^{2}\right]=\mathrm{E}_{g}\left[\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}}\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}\left(g h g^{-1}\right)|\mathbf{b}\rangle \cdot\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}\left(g h g^{-1}\right)|\mathbf{b}\rangle\right] \\
& =\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \mathrm{E}_{g}\left[\langle\mathbf{b} \otimes \mathbf{b}| \boldsymbol{\rho}^{I_{1}, I_{2}}\left(g h g^{-1}\right)|\mathbf{b} \otimes \mathbf{b}\rangle\right] \\
& =\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \mathrm{E}_{g}\left[\langle\mathbf{b} \otimes \mathbf{b}| \bigoplus_{\tau \in \widehat{G}} a_{\tau}^{\rho_{1}^{I_{1}, I_{2}}} \tau\left(g h g^{-1}\right)|\mathbf{b} \otimes \mathbf{b}\rangle\right] \\
& =\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}}\langle\mathbf{b} \otimes \mathbf{b}| \bigoplus_{\tau \in \widehat{G}} a_{\tau}^{\rho_{1}^{I_{1}, I_{2}}} \mathrm{E}_{g}\left[\tau\left(g h g^{-1}\right)\right]|\mathbf{b} \otimes \mathbf{b}\rangle \\
& =\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}}\langle\mathbf{b} \otimes \mathbf{b}| \bigoplus_{\tau \in \widehat{G}} a_{\tau}^{\rho_{\tau}^{I_{1}, I_{2}}} \frac{\chi_{\tau}(h)}{d_{\tau}} \mathbb{1}_{V_{\tau}}|\mathbf{b} \otimes \mathbf{b}\rangle \\
& =\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \sum_{\tau \in \widehat{G}} \frac{\chi_{\tau}(h)}{d_{\tau}}\left\|\Pi_{\tau}^{\rho_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2} .
\end{aligned}
$$

The fifth equality above follows from Schur's lemma.
Lemma 4 takes care of the 'good' irreps. However for 'bad' irreps $\tau$, we have to do something to bound $\left\|\Pi_{\tau}^{\rho_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}$. The papers MRS05 MR05] tried to bound it using the following simple geometric argument: If $\mathcal{B}$ is an orthonormal basis for $V_{\boldsymbol{\rho}}$, then $\{\mathbf{b} \otimes \mathbf{b}\}_{\mathbf{b} \in \mathcal{B}}$ is an orthonormal set in $V_{\boldsymbol{\rho}} \otimes V_{\boldsymbol{\rho}}$. Hence the expectation, over the uniform distribution on $\mathcal{B}$, of the above quantity is upper bounded by $\frac{\operatorname{rank}\left(\Pi_{\rho}^{\rho} \rho_{1}, I_{2}\right.}{d_{\rho}}$. If $\mathcal{B}$ is a POVM rather than an orthonormal basis, a similar argument can be made. This simple method works for $k=1,2$ for the symmetric group, but fails for $k \geq 3$. This is because $\operatorname{rank}\left(\Pi_{\tau}^{\rho_{1}, I_{2}}\right)$ becomes larger than $d_{\rho}$. The problem with the simple method is that $\operatorname{rank}\left(\Pi_{\tau}^{\rho_{1}, I_{2}}\right)$ can be potentially as large as $d_{\boldsymbol{\rho}}^{2}$. This is where we need new ideas as compared to those in [MRS05, MR05]. We use the fact that the projection $\Pi_{\tau}^{\rho_{1}, I_{2}}$ is not arbitrary, but is rather the projection onto the homogeneous component corresponding to an irrep of $G$. There is an explicit representation-theoretic formula for such a projection operator (see e.g. [Ser77] Chapter 2, Theorem 8]). Using this formula allows us to 'decouple' $\Pi_{\tau}^{\rho_{1}, I_{2}}(\mathbf{b} \otimes \mathbf{b})$ into an expression involving only $\rho^{I_{1}}$ and $\mathbf{b}$, and $\boldsymbol{\rho}^{I_{2}}$ and $\mathbf{b}$, that is, it allows us to remove the tensor product. This 'decoupling' gets around the problem that the rank of the projector can be larger than $d_{\boldsymbol{\rho}}$ whereas the size of the basis $\boldsymbol{\mathcal { B }}$ is only $d_{\boldsymbol{\rho}}$. It allows us to apply a standard corollary of Schur's orthogonality relations and finally bound the length of the projection of $\mathbf{b} \otimes \mathbf{b}$ by a small quantity.

We now state a few facts that will be used in our 'decoupling' arguments. The next fact is easy to show and was used in the simple geometric approach of [MRS05, MR05] to bound $\left\|\Pi_{\tau}^{\boldsymbol{\rho}_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}$.
Fact 5. Let $W$ be a subspace of $V$. Let $\mathcal{B}:=\left\{a_{b}, b\right\}$ be a frame for $V$. Let $\Pi_{W}^{V}$ denote the orthogonal projection from $V$ onto $W$. Then

$$
\mathrm{E}_{b}\left[\left\|\Pi_{W}^{V}(b)\right\|^{2}\right]=\frac{\operatorname{dim} W}{\operatorname{dim} V}
$$

where the expectation is taken over the natural distribution on $\mathcal{B}$.
The following fact is a special case of [MR05] Lemma 12], and can be easily proved by considering the regular representation of $G^{n}$.

Fact 6. Let $\boldsymbol{\theta}:=\left(\bigotimes_{i=1}^{n} \theta_{i}\right) \otimes\left(\bigotimes_{i^{\prime}=1}^{n^{\prime}} \mathbb{1}_{d_{i^{\prime}}}\right)$ be a representation of $G^{n+n^{\prime}}$, where $\theta_{i} \in \widehat{G}$ and $\mathbb{1}_{d_{i^{\prime}}}$ is the identity representation of $G$ of dimension $d_{i^{\prime}}$. Suppose each $\theta_{i}$ is chosen independently from the Plancherel distribution on $\widehat{G}$. Fix $\tau \in \widehat{G}$. Let $a_{\tau}^{\boldsymbol{\theta}}$ denote the multiplicity of $\tau$ in the Clebsch-Gordan decomposition of $\boldsymbol{\theta}$ i. e. viewing $\boldsymbol{\theta}$ as a representation of $G$ embedded as the diagonal subgroup of $G^{n+n^{\prime}}$. Then

$$
\mathrm{E}_{\boldsymbol{\theta}}\left[\frac{a_{\tau}^{\boldsymbol{\theta}}}{d_{\boldsymbol{\theta}}}\right]=\frac{d_{\tau}}{|G|} .
$$

The following fact is a standard result in representation theory (see e.g. [Ser77] Chapter 2, Proposition 4, Corollary 3]), and follows from Schur's orthogonality relations.
Fact 7. Suppose $\tau \in \widehat{G}$ and $b \in V_{\tau},\|b\|=1$. Then,

$$
\left.\left.\mathrm{E}_{g}[|\langle b| \tau(g)| b\rangle\right|^{2}\right]=\frac{1}{d_{\tau}} .
$$

We start off the 'decoupling' process by the following lemma.
Lemma 8. Fix $I_{1}, I_{2} \subseteq[k], I_{1}, I_{2} \neq\{ \}, \boldsymbol{\rho} \in \widehat{G}^{\otimes k}, \tau \in \widehat{G}$ and $\mathbf{b} \in V_{\rho}$. Then,

$$
\left.\left\|\Pi_{\tau}^{\rho_{\tau}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2} \leq \frac{d_{\tau}^{2}}{2}\left(\left.\mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]+\left.\mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right] .
$$

Proof.

$$
\begin{aligned}
\left\|\Pi_{\tau}^{\boldsymbol{\rho}_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2} & \left.=\left|\langle\mathbf{b} \otimes \mathbf{b}| \Pi_{\tau}^{\boldsymbol{\rho}_{1} I_{1}, I_{2}}\right| \mathbf{b} \otimes \mathbf{b}\right\rangle \mid \\
& \left.=\left|\langle\mathbf{b} \otimes \mathbf{b}| d_{\tau} \mathrm{E}_{g}\left[\chi_{\tau}(g)^{*} \boldsymbol{\rho}^{I_{1}}(g) \otimes \boldsymbol{\rho}^{I_{2}}(g)\right]\right| \mathbf{b} \otimes \mathbf{b}\right\rangle \mid \\
& =d_{\tau}\left|\mathrm{E}_{g}\left[\chi_{\tau}(g)^{*}\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}(g)|\mathbf{b}\rangle \cdot\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}(g)|\mathbf{b}\rangle\right]\right| \\
& \left.\leq d_{\tau}^{2} \mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}(g)\right| \mathbf{b}\right\rangle|\cdot|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}(g)|\mathbf{b}\rangle \mid\right] \\
& \left.\left.\leq \frac{d_{\tau}^{2}}{2}\left(\left.\mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]+\left.\mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]\right) .
\end{aligned}
$$

The second equality follows from a standard result in representation theory describing the projection operator onto a homogeneous component corresponding to an irrep of $G$ (see e.g. [Ser77] Chapter 2, Theorem 8]), the first inequality follows by bounding a character value by the dimension of the representation, and the second inequality follows from the fact that $|x y| \leq \frac{|x|^{2}+|y|^{2}}{2}$ for any pair of complex numbers $x, y$.

We now prove a crucial lemma that allows us to prove good upper bounds on $\left\|\Pi_{\tau}^{\rho_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}$.
Lemma 9. Fix $I \subseteq[k], I \neq\{ \}$. Then, $\left.\left.\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right] \leq \sum_{\tau \in \widehat{G}} \frac{d_{\tau}}{|G|}$.
Proof. We use the notation $\tau \prec \rho^{I}$ to denote a single copy of $\tau \in \widehat{G}$ occurring in the Clebsch-Gordan decomposition of $\boldsymbol{\rho}^{I}$ i.e. treating $\boldsymbol{\rho}^{I}$ as a representation of $G$ embedded in the diagonal of $G^{k}$. A given $\tau \in \widehat{G}$ can occur more than once in the decomposition, or not at all. We let $\mathbf{b}_{\tau}$ denote the orthogonal projection of $\mathbf{b}$ onto this copy of $\tau$. Note that if $\tau$ occurs more than once, then there will be several orthogonal vectors $\mathbf{b}_{\tau}$. If $\left\|\mathbf{b}_{\tau}\right\|>0$, define $\widehat{\mathbf{b}}_{\tau}$ to be $\mathbf{b}_{\tau}$ normalized; otherwise, let $\widehat{\mathbf{b}}_{\tau}$ be an arbitrary unit vector in the copy of $\tau$ under consideration. We now have

$$
\begin{aligned}
& \left.\left.\left.\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I}(g)\right| \mathbf{b}\right\rangle\left.\right|^{2}=\left|\langle\mathbf{b}| \bigoplus_{\tau \prec \boldsymbol{\rho}^{I}} \tau(g)\right| \mathbf{b}\right\rangle\left.\right|^{2}=\left|\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\langle\mathbf{b}_{\tau}\right| \tau(g)\right| \mathbf{b}_{\tau}\right\rangle\left.\right|^{2} \\
& \left.=\left|\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\|\mathbf{b}_{\tau}\right\| \cdot\left\|\mathbf{b}_{\tau}\right\|\left\langle\widehat{\mathbf{b}}_{\tau}\right| \tau(g)\right| \widehat{\mathbf{b}}_{\tau}\right\rangle\left.\right|^{2} \\
& \left.\leq\left.\left(\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\|\mathbf{b}_{\tau}\right\|^{2}\right) \cdot\left(\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\|\mathbf{b}_{\tau}\right\|^{2}\left|\left\langle\widehat{\mathbf{b}}_{\tau}\right| \tau(g)\right| \widehat{\mathbf{b}}_{\tau}\right\rangle\right|^{2}\right) \\
& \left.=\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\|\mathbf{b}_{\tau}\right\|^{2}\left|\left\langle\widehat{\mathbf{b}}_{\tau}\right| \tau(g)\right| \widehat{\mathbf{b}}_{\tau}\right\rangle\left.\right|^{2} .
\end{aligned}
$$

The inequality above follows from Cauchy-Schwartz, and the last equality is because $\sum_{\tau \prec \rho^{I}}\left\|\mathbf{b}_{\tau}\right\|^{2}=$ $\|\mathbf{b}\|^{2}=1$. Now,

$$
\begin{aligned}
& \left.\left.\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}| |\langle\mathbf{b}| \boldsymbol{\rho}^{I}(g)|\mathbf{b}\rangle\right|^{2}\right] \\
& \left.\quad \leq\left.\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}\left[\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\|\mathbf{b}_{\tau}\right\|^{2}\left|\left\langle\widehat{\mathbf{b}}_{\tau}\right| \tau(g)\right| \widehat{\mathbf{b}}_{\tau}\right\rangle\right|^{2}\right] \\
& \left.\quad=\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}}\left[\left.\sum_{\tau \prec \boldsymbol{\rho}^{I}}\left\|\mathbf{b}_{\tau}\right\|^{2} \mathrm{E}_{g}\left[\left|\left\langle\widehat{\mathbf{b}}_{\tau}\right| \tau(g)\right| \widehat{\mathbf{b}}_{\tau}\right\rangle\right|^{2}\right]\right] \\
& \quad=\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}}\left[\sum_{\tau \prec \boldsymbol{\rho}^{I}} \frac{\left\|\mathbf{b}_{\tau}\right\|^{2}}{d_{\tau}}\right]=\mathrm{E}_{\boldsymbol{\rho}}\left[\sum_{\tau<\boldsymbol{\rho}^{I}} \frac{\mathrm{E}_{\mathbf{b}}\left[\left\|\mathbf{b}_{\tau}\right\|^{2}\right]}{d_{\tau}}\right] \\
& \quad=\mathrm{E}_{\boldsymbol{\rho}}\left[\sum_{\tau<\boldsymbol{\rho}^{I}} \frac{d_{\tau}}{d_{\tau} d_{\boldsymbol{\rho}}}\right]=\mathrm{E}_{\boldsymbol{\rho}}\left[\sum_{\tau \in \widehat{G}} \frac{a_{\tau}^{\boldsymbol{\rho}^{I}}}{d_{\boldsymbol{\rho}}}\right]=\sum_{\tau \in \widehat{G}} \mathrm{E}_{\boldsymbol{\rho}}\left[\frac{a_{\tau}^{\boldsymbol{\rho}^{I}}}{d_{\boldsymbol{\rho}}}\right] \\
& =\sum_{\tau \in \widehat{G}} \frac{d_{\tau}}{|G|} .
\end{aligned}
$$

The second equality follows from Fact 7 the fourth equality follows from Fact 5 and the last equality follows from Fact 6

The next lemma ties up the above threads to prove an upper bound on the second moment of the function $X$ independent of $k$.

Lemma 10. $\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}\left[X(\boldsymbol{\rho}, \mathbf{b}, g)^{2}\right]<\varepsilon+\frac{1}{|G|} \cdot\left(\sum_{\nu \in \widehat{G}} d_{\nu}\right) \cdot\left(\sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}\left|\chi_{\tau}(h)\right|\right)$.
Proof. First, note that

$$
\begin{aligned}
& \mathrm{E}_{g}\left[X(\boldsymbol{\rho}, \mathbf{b}, g)^{2}\right] \\
& =\left|\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \sum_{\tau \in \widehat{G}} \frac{\chi_{\tau}(h)}{d_{\tau}}\left\|\Pi_{\tau}^{\rho_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}\right| \\
& \\
& \leq \frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \sum_{\tau \in \widehat{G}} \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}\left\|\Pi_{\tau}^{\rho_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2} \\
& \\
& <\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}}\left(\varepsilon \cdot \sum_{\tau \in \widehat{G} \backslash \mathcal{S}_{\varepsilon}}\left\|\Pi_{\tau}^{\rho_{1}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}+\sum_{\tau \in \mathcal{S}_{\varepsilon}} \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}\left\|\Pi_{\tau}^{\boldsymbol{\rho}_{1}, I_{2}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}\right) \\
& \quad<\varepsilon+\frac{1}{4^{k}} \sum_{I_{1}, I_{2} \neq\{ \}} \sum_{\tau \in \mathcal{S}_{\varepsilon}} \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}\left\|\Pi_{\tau}^{\rho_{1}, I_{2}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2} .
\end{aligned}
$$

The equality follows from Lemma 4 and the fact that the quantity in the absolute value sign is non-negative, and the last inequality follows from the fact that $\sum_{\tau \in \widehat{G} \backslash \mathcal{S}_{\varepsilon}}\left\|\Pi_{\tau}^{\rho_{\tau}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2} \leq\|\mathbf{b} \otimes \mathbf{b}\|^{2}=1$.

Fix $I_{1}, I_{2} \subseteq[k], I_{1}, I_{2} \neq\{ \}$. Then,

$$
\begin{aligned}
& \mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}}\left[\sum_{\tau \in \mathcal{S}_{\varepsilon}} \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}\left\|\Pi_{\tau}^{\boldsymbol{\rho}^{I_{1}, I_{2}}}(\mathbf{b} \otimes \mathbf{b})\right\|^{2}\right] \\
& \left.\left.\quad \leq \mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}}\left[\sum_{\tau \in \mathcal{S}_{\varepsilon}} \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}} \cdot \frac{d_{\tau}^{2}}{2}\left(\left.\mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]+\left.\mathrm{E}_{g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]\right)\right] \\
& \left.\left.\quad=\left(\sum_{\tau \in \mathcal{S}_{\varepsilon}} \frac{d_{\tau}\left|\chi_{\tau}(h)\right|}{2}\right)\left(\left.\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{1}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]+\left.\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}, g}\left[\left|\langle\mathbf{b}| \boldsymbol{\rho}^{I_{2}}(g)\right| \mathbf{b}\right\rangle\right|^{2}\right]\right) \\
& \quad \leq \frac{1}{|G|} \cdot\left(\sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}\left|\chi_{\tau}(h)\right|\right) \cdot\left(\sum_{\nu \in \widehat{G}} d_{\nu}\right) .
\end{aligned}
$$

The first inequality is due to Lemma 8 and the second inequality is due to Lemma 9 Combining the above two upper bounds proves the present lemma.

We now connect the function $X$ to the total variation distance between $\mathcal{M}_{H^{g}}$ and $\mathcal{M}_{\{1\}}$.
Lemma 11. Define $\mu_{g}:=\mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}}[|X(\boldsymbol{\rho}, \mathbf{b}, g)|]$. Suppose $2 k \varepsilon<1$. Then,

$$
\left\|\mathcal{M}_{H^{g}}-\mathcal{M}_{\{1\}}\right\|_{1}<2^{k}(1+2 k \varepsilon) \mu_{g}+3 k \varepsilon+\frac{3 k}{|G|} \cdot \sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}^{2} .
$$

Proof. If the hidden subgroup is $H^{g}$ for some $g \in G$, the probability of observing an irrep $\rho^{*} \in \widehat{G}^{\otimes k}$, row index $\mathbf{i} \in\left[d_{\rho}\right]$ and frame vector $\mathbf{b} \in \mathcal{B}$ is given by

$$
\mathcal{M}_{H^{g}}(\boldsymbol{\rho}, \mathbf{i}, \mathbf{b})=\mathcal{M}_{H}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{M}_{H^{g}}\left(\mathbf{b} \mid \boldsymbol{\rho}^{*}\right)
$$

If the hidden subgroup is $\{1\}$, the probability of observing an irrep $\boldsymbol{\rho}^{*} \in \widehat{G}^{\otimes k}$, row index $\mathbf{i} \in\left[d_{\boldsymbol{\rho}}\right]$ and frame vector $\mathbf{b} \in \mathcal{B}$ is given by

$$
\mathcal{M}_{\{1\}}(\boldsymbol{\rho}, \mathbf{i}, \mathbf{b})=\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{N}(b \mid \boldsymbol{\rho}) .
$$

Define a new probability vector $\mathcal{M}_{H^{g}}^{\prime}$ as

$$
\mathcal{M}_{H^{g}}^{\prime}(\boldsymbol{\rho}, \mathbf{i}, \mathbf{b}):=\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{M}_{H^{g}}\left(b \mid \rho^{*}\right)
$$

Define a set $\mathcal{S}_{\varepsilon}:=\left\{\boldsymbol{\rho} \in \widehat{G}^{\otimes k}: \exists i \in[k], \rho_{i} \in \mathcal{S}_{\varepsilon}\right\}$. Define another new vector $\mathcal{M}_{\{1\}}^{\prime \prime}$ with non-negative entries as

$$
\mathcal{M}_{\{1\}}^{\prime \prime}(\boldsymbol{\rho}, \mathbf{i}, \mathbf{b}):=\left\{\begin{array}{ll}
\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\rho}} \cdot \frac{a_{\mathrm{b}}}{2^{k} r_{\rho}\left(H^{k}\right)}, & \text { if } \boldsymbol{\rho} \notin \mathcal{S}_{\varepsilon}, \\
0 & \text { otherwise }
\end{array} .\right.
$$

Note that $\mathcal{M}_{\{1\}}^{\prime \prime}$ may not be a probability vector.
Define $D_{\varepsilon}:=\sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}^{2}$. Let $\mathcal{P}\left(\mathcal{S}_{\varepsilon}\right), \mathcal{P}\left(\mathcal{S}_{\varepsilon}\right)$ denote the probabilities of $\mathcal{S}_{\varepsilon}, \mathcal{S}_{\varepsilon}$ under the Plancherel distributions on $\widehat{G}, \widehat{G}^{\otimes k}$ respectively. Then, $\mathcal{P}\left(\mathcal{S}_{\varepsilon}\right) \leq k \mathcal{P}\left(\mathcal{S}_{\varepsilon}\right)=\frac{k D_{\varepsilon}}{|G|}$. Also since

$$
\frac{d_{\boldsymbol{\rho}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)}=\prod_{i=1}^{k} \frac{d_{\rho_{i}}}{2 r_{\rho_{i}}}=\prod_{i=1}^{k} \frac{d_{\rho_{i}}}{d_{\rho_{i}}+\chi_{\rho_{i}}(h)}=\prod_{i=1}^{k}\left(1+\frac{\chi_{\rho_{i}}(h)}{d_{\rho_{i}}}\right)^{-1},
$$

we have

$$
(1+\varepsilon)^{-k} \leq \frac{d_{\boldsymbol{\rho}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)} \leq(1-\varepsilon)^{-k}, \quad \text { for } \boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \boldsymbol{\mathcal { S }}_{\varepsilon} .
$$

By the convexity of the function $y=x^{-k}$, we have that $\left|(1-\varepsilon)^{-k}-1\right| \geq\left|(1+\varepsilon)^{-k}-1\right|$. Since $2 k \varepsilon<1$, it can be shown by induction that $(1-\varepsilon)^{-k} \leq 1+2 k \varepsilon$. Hence,

$$
\left|1-\frac{d_{\boldsymbol{\rho}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)}\right| \leq 2 k \varepsilon, \quad \text { for } \boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \mathcal{S}_{\varepsilon} .
$$

Now,

$$
\begin{aligned}
&\left\|\mathcal{M}_{\{1\}}^{\prime \prime}-\mathcal{M}_{\{1\}}\right\|_{1} \\
&= \sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \mathcal{S}_{\varepsilon}} \sum_{\mathbf{i}=1}^{d_{\boldsymbol{\rho}}} \sum_{\mathbf{b} \in \mathcal{B}}\left|\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \frac{a_{\mathbf{b}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)}-\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \frac{a_{\mathbf{b}}}{d_{\boldsymbol{\rho}}}\right|+\sum_{\boldsymbol{\rho} \in \mathcal{S}_{\varepsilon}} \sum_{\mathrm{i}=1}^{d_{\boldsymbol{\rho}}} \sum_{\mathbf{b} \in \mathcal{B}} \mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \frac{a_{\mathbf{b}}}{d_{\boldsymbol{\rho}}} \\
&=\sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \mathcal{S}_{\varepsilon}} \sum_{\mathbf{b} \in \mathcal{B}} \mathcal{P}(\boldsymbol{\rho}) \cdot \frac{a_{\mathbf{b}}}{d_{\boldsymbol{\rho}}}\left|\frac{d_{\boldsymbol{\rho}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)}-1\right|+\sum_{\boldsymbol{\rho} \in \mathcal{S}_{\varepsilon}} \mathcal{P}(\boldsymbol{\rho}) \\
& \leq \sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \boldsymbol{\mathcal { S }}_{\varepsilon}} \mathcal{P}(\boldsymbol{\rho}) \cdot 2 k \varepsilon+\frac{k D_{\varepsilon}}{|G|} \\
& \leq 2 k \varepsilon+\frac{k D_{\varepsilon}}{|G|} .
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\left\|\mathcal{M}_{H^{g}}^{\prime}-\mathcal{M}_{\{1\}}^{\prime \prime}\right\|_{1} \\
&= \sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k}\left\langle\mathcal{S}_{\varepsilon}\right.} \sum_{\mathbf{i}=1}^{d_{\boldsymbol{\rho}}} \sum_{\mathbf{b} \in \mathcal{B}}\left|\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{M}_{H^{g}}\left(\mathbf{b} \mid \rho^{*}\right)-\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \frac{a_{\mathbf{b}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)}\right| \\
&+\sum_{\boldsymbol{\rho} \in \mathcal{S}_{\varepsilon}} \sum_{\mathbf{i}=1}^{d_{\boldsymbol{\rho}}} \sum_{\mathbf{b} \in \mathcal{B}} \mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{M}_{H^{g}}\left(\mathbf{b} \mid \rho^{*}\right) \\
&= \sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \mathcal{S}_{\varepsilon}} \mathcal{P}(\boldsymbol{\rho}) \sum_{\mathbf{b} \in \mathcal{B}}\left|\frac{a_{\mathbf{b}}\langle\mathbf{b}| \boldsymbol{\rho}\left(\left(H^{g}\right)^{k}\right)|\mathbf{b}\rangle}{r_{\boldsymbol{\rho}}\left(H^{k}\right)}-\frac{a_{\mathbf{b}}}{2^{k} r_{\boldsymbol{\rho}}\left(H^{k}\right)}\right|+\sum_{\boldsymbol{\rho} \in \mathcal{\mathcal { S }}_{\varepsilon}} \mathcal{P}(\boldsymbol{\rho}) \\
& \leq\left.\sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \mathcal{S}_{\varepsilon}} \mathcal{P}(\boldsymbol{\rho}) \cdot \frac{d_{\boldsymbol{\rho}}}{r_{\boldsymbol{\rho}}\left(H^{k}\right)} \sum_{\mathbf{b} \in \mathcal{B}} \frac{a_{\mathbf{b}}}{d_{\boldsymbol{\rho}}}\left|\langle\mathbf{b}| \boldsymbol{\rho}\left(\left(H^{g}\right)^{k}\right)\right| \mathbf{b}\right\rangle-\frac{1}{2^{k}} \left\lvert\,+\frac{k D_{\varepsilon}}{|G|}\right. \\
& \leq 2^{k}(1+2 k \varepsilon) \sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k} \backslash \mathcal{S}_{\varepsilon}} \mathcal{P}(\boldsymbol{\rho}) \cdot \mathrm{E}_{\mathbf{b}}[|X(\boldsymbol{\rho}, \mathbf{b}, g)|]+\frac{k D_{\varepsilon}}{|G|} \\
& \leq 2^{k}(1+2 k \varepsilon) \mathrm{E}_{\boldsymbol{\rho}, \mathbf{b}}[|X(\boldsymbol{\rho}, \mathbf{b}, g)|]+\frac{k D_{\varepsilon}}{|G|}=2^{k}(1+2 k \varepsilon) \mu_{g}+\frac{k D_{\varepsilon}}{|G|} .
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
& \left\|\mathcal{M}_{H^{g}}^{\prime}-\mathcal{M}_{H^{g}}\right\|_{1}=\sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k}} \sum_{\mathbf{i}=1}^{d_{\rho}} \sum_{\mathbf{b} \in \mathcal{B}}\left|\mathcal{P}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{M}_{H^{g}}\left(\mathbf{b} \mid \rho^{*}\right)-\mathcal{M}_{H}(\boldsymbol{\rho}) \cdot \frac{1}{d_{\boldsymbol{\rho}}} \cdot \mathcal{M}_{H^{g}}\left(\mathbf{b} \mid \rho^{*}\right)\right| \\
& =\sum_{\boldsymbol{\rho} \in \widehat{G}^{\otimes k}}\left|\mathcal{P}(\boldsymbol{\rho})-\mathcal{M}_{H}(\boldsymbol{\rho})\right| \leq k \cdot \sum_{\tau \in \widehat{G}}\left|\frac{d_{\tau}^{2}}{|G|}-\frac{d_{\tau}|H| r_{\tau}(H)}{|G|}\right|=k \cdot \sum_{\tau \in \widehat{G}}\left|\frac{d_{\tau}^{2}}{|G|}-\frac{d_{\tau}\left(d_{\tau}+\chi_{\tau}(h)\right)}{|G|}\right| \\
& =k \cdot \sum_{\tau \in \widehat{G}} \frac{d_{\tau}\left|\chi_{\tau}(h)\right|}{|G|} \leq k \cdot\left(\sum_{\tau \in \widehat{G} \backslash \mathcal{S}_{\varepsilon}} \frac{d_{\tau}^{2}}{|G|} \cdot \frac{\left|\chi_{\tau}(h)\right|}{d_{\tau}}+\sum_{\tau \in \mathcal{S}_{\varepsilon}} \frac{d_{\tau}^{2}}{|G|}\right)<k \cdot\left(\varepsilon \sum_{\tau \in \widehat{G} \backslash \mathcal{S}_{\varepsilon}} \frac{d_{\tau}^{2}}{|G|}+\frac{D_{\varepsilon}}{|G|}\right) \\
& \leq k \varepsilon+\frac{k D_{\varepsilon}}{|G|} .
\end{aligned}
$$

The first inequality follows from $k$ applications of the triangle inequality. Finally,

$$
\begin{aligned}
\left\|\mathcal{M}_{H^{g}}-\mathcal{M}_{\{1\}}\right\|_{1} & \leq\left\|\mathcal{M}_{H^{g}}-\mathcal{M}_{H^{g}}^{\prime}\right\|_{1}+\left\|\mathcal{M}_{H^{g}}^{\prime}-\mathcal{M}_{\{1\}}^{\prime \prime}\right\|_{1}+\left\|\mathcal{M}_{\{1\}}^{\prime \prime}-\mathcal{M}_{\{1\}}\right\|_{1} \\
& \leq 2^{k}(1+2 k \varepsilon) \mu_{g}+3 k \varepsilon+\frac{3 k}{|G|} \cdot \sum_{\tau \in \mathcal{S}_{\varepsilon}} d_{\tau}^{2}
\end{aligned}
$$

We are now ready to prove the main theorem of the paper.
Proof of Theorem 2 The theorem follows from Lemmas 10 and 11 using the convexity of the square function. The upper bound on $\delta_{1}$ follows from the observation that Cauchy-Schwartz implies that $\sum_{\nu \in \widehat{G}} d_{\nu} \leq$ $|\widehat{G}|^{1 / 2}\left(\sum_{\nu \in \widehat{G}} d_{\nu}^{2}\right)^{1 / 2}=|\widehat{G}|^{1 / 2}|G|^{1 / 2}$.

Finally, we prove a simple lower bound, irrespective of the order of entanglement, on the total number of coset states $t$ required to distinguish a hidden subgroup $H^{g}$ from the identity hidden subgroup. For that, we need the following theorem.
Theorem 12. Let $G$ be a finite group and $H:=\{1, h\}$ be an order two subgroup of $G$. Let $t \geq 1$ be an integer. Then,

$$
\left\|\mathrm{E}_{g}\left[\sigma_{H^{g}}^{\otimes t}\right]-\sigma_{\{1\}}^{\otimes t}\right\|_{\mathrm{tr}}<\frac{2^{t}}{|G|} \sum_{\tau \in \widehat{G}} d_{\tau}\left|\chi_{\tau}(h)\right| .
$$

Proof. Let $\boldsymbol{\rho} \in \widehat{G}^{\otimes t}, I \subseteq[t], I \neq\{ \}$. Using arguments similar to those above, it is easy to see that

$$
\begin{aligned}
& \left\|\mathrm{E}_{g}\left[2^{t} \boldsymbol{\rho}\left(\left(H^{g}\right)^{t}\right)\right]-\boldsymbol{\rho}\left(\{1\}^{t}\right)\right\|_{\mathrm{tr}}=\left\|\mathrm{E}_{g}\left[\mathbb{1}_{d_{\rho}}+\sum_{I \neq\{ \}} \boldsymbol{\rho}^{I}\left(g h g^{-1}\right)\right]-\mathbb{1}_{d_{\rho}}\right\|_{\mathrm{tr}}=\left\|\sum_{I \neq\{ \}} \mathrm{E}_{g}\left[\boldsymbol{\rho}^{I}\left(g h g^{-1}\right)\right]\right\|_{\mathrm{tr}} \\
& \leq \sum_{I \neq\{ \}}\left\|\mathrm{E}_{g}\left[\boldsymbol{\rho}^{I}\left(g h g^{-1}\right)\right]\right\|_{\mathrm{tr}}=\sum_{I \neq\{ \}}\left\|\bigoplus_{\tau \in \widehat{G}} \frac{\chi_{\tau}(h)}{d_{\tau}} \bigoplus_{j=1}^{a_{\tau}^{\rho^{I}}} \mathbb{1}_{d_{\tau}}\right\|_{\mathrm{tr}}=\sum_{I \neq\{ \}} \sum_{\tau \in \widehat{G}} a_{\tau}^{\boldsymbol{\rho}^{I}}\left|\chi_{\tau}(h)\right| .
\end{aligned}
$$

Writing the density matrices in the Fourier basis and using Fact 6we get,

$$
\begin{aligned}
& \left\|\mathrm{E}_{g}\left[\sigma_{H^{g}}^{\otimes t}\right]-\sigma_{\{1\}}^{\otimes t}\right\|_{\mathrm{tr}} \\
& =\| \mathrm{E}_{g}\left[\frac{2^{t}}{|G|^{t}} \bigoplus_{\boldsymbol{\rho}} \bigoplus_{\mathbf{i}=1}^{d_{\rho}}\left|\boldsymbol{\rho}^{*}, \mathbf{i}\right\rangle\left\langle\boldsymbol{\rho}^{*}, \mathbf{i}\right| \otimes \boldsymbol{\rho}\left(\left(H^{g}\right)^{t}\right)\right]-\frac{1}{|G|^{t}} \bigoplus_{\boldsymbol{\rho}} \bigoplus_{\mathbf{i}=1}^{d_{\rho}}\left|\boldsymbol{\rho}^{*}, \mathbf{i}\right\rangle\left\langle\boldsymbol{\rho}^{*}, \mathbf{i}\right| \otimes \boldsymbol{\rho}\left(\{1\}^{t}\right) \|_{\mathrm{tr}} \\
& =\| \frac{1}{|G|^{t}} \bigoplus_{\boldsymbol{\rho}} \bigoplus_{\mathbf{i}=1}^{d_{\rho}}\left|\boldsymbol{\rho}^{*}, \mathbf{i}\right\rangle\left\langle\boldsymbol{\rho}^{*}, \mathbf{i}\right| \otimes\left(\mathrm{E}_{g}\left[2^{t} \boldsymbol{\rho}\left(\left(H^{g}\right)^{t}\right)\right]-\boldsymbol{\rho}\left(\{1\}^{t}\right)\right) \|_{\mathrm{tr}} \\
& =\frac{1}{|G|^{t}} \sum_{\boldsymbol{\rho}} d_{\boldsymbol{\rho}}\left\|\mathrm{E}_{g}\left[2^{t} \boldsymbol{\rho}\left(\left(H^{g}\right)^{t}\right)\right]-\boldsymbol{\rho}\left(\{1\}^{t}\right)\right\|_{\mathrm{tr}} \leq \frac{1}{|G|^{t}} \sum_{\boldsymbol{\rho}} d_{\boldsymbol{\rho}} \sum_{I \neq\{ \}} \sum_{\tau \in \widehat{G}} a_{\tau}^{\rho^{I}}\left|\chi_{\tau}(h)\right| \\
& =\sum_{I \neq\{ \}} \sum_{\tau \in \widehat{G}}\left|\chi_{\tau}(h)\right|\left(\sum_{\boldsymbol{\rho}} \frac{d_{\boldsymbol{\rho}}^{2}}{|G|^{t}} \frac{a_{\tau}^{\boldsymbol{\rho}^{I}}}{d_{\boldsymbol{\rho}}}\right)=\sum_{I \neq\{ \}} \sum_{\tau \in \widehat{G}} \frac{d_{\tau}\left|\chi_{\tau}(h)\right|}{|G|} \\
& <\frac{2^{t}}{|G|} \sum_{\tau \in \widehat{G}} d_{\tau}\left|\chi_{\tau}(h)\right| .
\end{aligned}
$$

Corollary 13. Any algorithm using a total of $t$ coset states that distinguishes with constant probability between the case when the hidden subgroup is trivial and the case when the hidden subgroup is $H^{g}$ for some $g \in G$ must satisfy $t=\Omega(\log (1 / \eta))$.

Proof. The algorithm can be viewed as a two-outcome POVM that outputs 1 with probability at least $2 / 3$ if the hidden subgroup is non-trivial, and 0 with probability at least $2 / 3$ if the hidden subgroup is trivial. Thus, the POVM distinguishes between the states $\mathrm{E}_{g}\left[\sigma_{H^{g}}^{\otimes t}\right]$ and $\sigma_{\{1\}}^{\otimes t}$ with constant total variation distance. Since the trace distance is always an upper bound on the total variation distance, invoking Theorem 12] completes the proof.

The above corollary shows, for example, that any coset state based algorithm solving the HSP in $S_{n}$ 乙 $S_{2}$ needs a total number of $\Omega(n \log n)$ coset states. In the next section, we apply Theorem 2 to show a stronger result, namely, any algorithm solving the HSP in $S_{n} 2 S_{2}$ using polynomially many coset states needs to make measurements entangled across $\Omega(n \log n)$ coset states. However, Corollary 13 can sometimes prove nontrivial lower bounds on the total number of coset states for solving the HSP in groups $G$ where Theorem 2 can only prove a constant lower bound on the order of entanglement. For example, the HSP in groups $G:=A \rtimes \mathbb{Z}_{2}$, where $A$ is an abelian group and $\mathbb{Z}_{2}$ acts on $A$ by inversion can be solved by an algorithm using a total number of $O(\log |G|)$ coset states that measures one coset state at a time [EH00]. Using Corollary [13], one can show a matching $\Omega(\log |G|)$ lower bound on the total number of coset states when $A$ is the cyclic group $\mathbb{Z}_{n}$, i. e., $G$ is the dihedral group $D_{n}$. Using a different technique, Childs and Wocjan [CW05] in fact show an $\Omega(\log |G|)$ lower bound on the total number of coset states for the above groups for all abelian $A$.

## 4 Limitations of quantum coset states for HSP: Examples

### 4.1 The wreath product $S_{n} \backslash S_{2}$ and graph isomorphism

The representation theory of the wreath product $G=S_{n} \swarrow S_{2}$ is well-known. The following is a summary of the necessary results, for more details we refer to Appendix the wreath product has irreps $\kappa_{\lambda, \lambda^{\prime}}$ of dimension $2 d_{\lambda} d_{\lambda^{\prime}}$, where $\lambda, \lambda^{\prime} \in \widehat{S_{n}}, \lambda \neq \lambda^{\prime}$. Define $h:=(e, e, 1) \in G$, where $e$ is the identity permutation in $S_{n}$. The character value of $h$ on these irreps is zero. Furthermore, there are irreps $\vartheta_{\lambda}$ and $\vartheta_{\lambda}^{\prime}$ of dimension $d_{\lambda}^{2}$, where $\lambda \in \widehat{S_{n}}$. The character values of $\vartheta_{\lambda}$ and $\vartheta_{\lambda}^{\prime}$ on $h$ are given by $d_{\lambda}$ and $-d_{\lambda}$, respectively. The total number of irreps of $G$ is $|\widehat{G}|=\binom{p(n)}{2}+2 p(n) \leq p(n)^{2}$, where $p(n)$ denotes the number of partitions of $n$.

In order to apply Theorem 2 we choose $\varepsilon=n^{-\alpha n}$ for some constant $\alpha>0$ to be determined later. Then $\mathcal{S}_{\varepsilon}=\left\{\sigma \in \widehat{G}: \frac{\left|\chi_{\sigma}(h)\right|}{d_{\sigma}} \geq \varepsilon\right\}=\left\{\vartheta_{\lambda}, \vartheta_{\lambda}^{\prime}: d_{\lambda} \leq n^{\alpha n}\right\}$. Hence we obtain that

$$
\sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma} \cdot\left|\chi_{\sigma}(h)\right| \leq 2 \sum_{\lambda \in \widehat{S_{n}}, d_{\lambda} \leq n^{\alpha n}} d_{\lambda}^{2} \cdot d_{\lambda} \leq p(n) n^{2 \alpha n} \cdot n^{\alpha n} \leq n^{3 \alpha n} e^{\nu \sqrt{n}}
$$

Here we have estimated the partition number as $p(n)=O\left(e^{\nu \sqrt{n}}\right)$, where $\nu=\pi \sqrt{\frac{2}{3}}$. We also compute that

$$
\sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma}^{2} \leq 2 \sum_{\lambda \in \widehat{S_{n}}, d_{\lambda} \leq n^{\alpha n}} d_{\lambda}^{4} \leq p(n) n^{4 \alpha n} \leq n^{4 \alpha n} e^{\nu \sqrt{n}}
$$

In order to apply Theorem 2 we now define $\alpha:=1 / 4$ and obtain that

$$
\begin{aligned}
\delta_{1} & \leq \varepsilon+\left(\sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma}\left|\chi_{\sigma}(h)\right|\right)\left(\frac{|\widehat{G}|}{|G|}\right)^{1 / 2} \leq n^{-\alpha n}+n^{3 \alpha n} e^{\nu \sqrt{n}}\left(\frac{p(n)^{2}}{2(n!)^{2}}\right)^{1 / 2} \\
& \leq n^{-1 / 4 n}+\frac{n^{3 / 4 n} e^{2 \nu \sqrt{n}}}{\sqrt{2} n!}=n^{-\Omega(n)},
\end{aligned}
$$

where we have used the fact that $n!\geq(n / e)^{n}$ for large $n$. For the parameter $\delta_{2}$ in Theorem we obtain

$$
\begin{aligned}
\delta_{2} & =2^{k}(1+2 k \varepsilon) \delta_{1}^{1 / 2}+3 k \varepsilon+\frac{3 k \sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma}^{2}}{|G|} \\
& \leq 2^{k}\left(1+2 k n^{-1 / 4 n}\right) n^{-\Omega(n)}+3 k n^{-1 / 4 n}+3 k \frac{n^{n} e^{\nu \sqrt{n}}}{2(n!)^{2}}=2^{k} n^{-\Omega(n)} .
\end{aligned}
$$

Hence, we have proved the following corollary to Theorem 2
Corollary 14. Any algorithm operating on coset states that solves the hidden subgroup problem in $G=$ $S_{n}$ 乙 $S_{2}$ in polynomial time has to make joint measurements on $k=\Omega(n \log n)$ coset states. The same is true for any algorithm that solves the hidden subgroup problem in $S_{n}$ using coset states. Also, any efficient algorithm for isomorphism of two n-vertex graphs that uses the standard reduction to HSP in $S_{2 n}$ and then uses coset states to solve the HSP needs to make measurements entangled across $k=\Omega(n \log n)$ coset states.

Finally, we remark that if we apply Theorem 2 to all the full-support involutions in $S_{2 n}$, we only get a lower bound of $k=\Omega(n)$. This is because we use Roichman's Roi96] upper bound on the normalized characters of $S_{2 n}$ in order to define $\mathcal{S}_{\epsilon}$, as in [MRS05], and Roichman's bound is always at least $e^{-O(n)}$. Since the involutive swaps form an exponentially small fraction of all the full-support involutions, it is possible that an average hidden full-support involution may be distinguishable from the hidden identity subgroup by an $O(n)$-entangled POVM acting on $n^{O(1)}$-coset states. However, no such POVM is known and the best upper bound for this problem continues to be the $k=O(n \log n)$ information-theoretic one.

### 4.2 The projective linear groups $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$

The representation theory of the projective linear groups $G=\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ over any finite field $\mathbb{F}_{q}$ is wellknown. The following is a summary of the necessary results, for more details we refer to Appendix B We treat the cases $q$ even and $q$ odd separately. In case $q$ odd we have that $\left|\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)\right|=\frac{q\left(q^{2}-1\right)}{2}$. There is one conjugacy class of $\frac{q(q \pm 1)}{2}$ involutions (depending on whether $q \equiv 1$ or 3 modulo 4); let $h$ denote a fixed member of this conjugacy class. The degrees of the irreps are given by $1, q, q \pm 1$, and $\frac{q \pm 1}{2}$. The character values $|\chi(h)|$ can be upper bounded by $1,1,2$, and 1 , respectively. There is a total number of $|\widehat{G}|=\frac{q+5}{2}$ irreps.

In order to apply Theorem we choose $\varepsilon=\frac{2}{q-1}$. Then

$$
\mathcal{S}_{\varepsilon}=\left\{\sigma \in \widehat{G}: \frac{\left|\chi_{\sigma}(h)\right|}{d_{\sigma}} \geq \varepsilon\right\}=\{\mathbb{1}\}
$$

contains only the trivial irrep. With this choice of the parameter $\varepsilon$ we have that

$$
\sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma} \cdot\left|\chi_{\sigma}(h)\right|=1, \quad \sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma}^{2}=1, \quad \text { and } \quad\left(\frac{|\widehat{G}|}{|G|}\right)^{1 / 2}=\left(\frac{(q+5) / 2}{q\left(q^{2}-1\right) / 2}\right)^{1 / 2}=O\left(q^{-1}\right) .
$$

Hence, we can bound the parameter $\delta_{1}$ used in Theorem 2 as follows:

$$
\delta_{1} \leq \varepsilon+\left(\sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma}\left|\chi_{\sigma}(h)\right|\right)\left(\frac{|\widehat{G}|}{|G|}\right)^{1 / 2} \leq \frac{2}{q-1}+1 \cdot O\left(q^{-1}\right)=O\left(q^{-1}\right)
$$

For the parameter $\delta_{2}$ we obtain

$$
\begin{aligned}
\delta_{2} & =2^{k}(1+2 k \varepsilon) \delta_{1}^{1 / 2}+3 k \varepsilon+\frac{3 k \sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\sigma}^{2}}{|G|} \\
& \leq 2^{k}\left(1+2 k \frac{2}{q-1}\right) O\left(q^{-1 / 2}\right)+3 k \frac{2}{q-1}+3 k \frac{1}{q\left(q^{2}-1\right) / 2} \leq 2^{k} O\left(q^{-1 / 2}\right) .
\end{aligned}
$$

The case $q=2^{n}$, where $\left|\operatorname{PSL}\left(2, \mathbb{F}_{2^{n}}\right)\right|=\left|\operatorname{SL}\left(2, \mathbb{F}_{2^{n}}\right)\right|=q\left(q^{2}-1\right)$, can be treated similarly. There we use $\varepsilon=\frac{1}{q-1}$ which implies that $\delta_{2} \leq 2^{k} O\left(q^{-1 / 2}\right)$. Hence, using Theorem 2 we have shown the following result:

Corollary 15. Let $q$ be a prime power. Then any algorithm operating on coset states that solves the hidden subgroup problem in $G=\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ in polynomial time has to make joint measurements on $k=\Omega(\log |G|)=\Omega(q)$ coset states.

### 4.3 Special and general linear groups

Corollary 16. Any algorithm solving the HSP in $\mathrm{SL}\left(2, \mathbb{F}_{q}\right)$ or $\mathrm{GL}\left(2, \mathbb{F}_{q}\right)$ efficiently using coset states needs to make measurements entangled across $k=\Omega(\log q)$ registers.

Proof. By Corollary 15 any algorithm solving the $\operatorname{HSP}$ in $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ efficiently using coset states needs to make measurements entangled across $k=\Omega(\log q)$ registers. The statement now follows from Lemma by using the facts that $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right) \cong \operatorname{SL}\left(2, \mathbb{F}_{q}\right) / \zeta\left(\operatorname{SL}\left(2, \mathbb{F}_{q}\right)\right)$ and that $\operatorname{SL}\left(2, \mathbb{F}_{q}\right) \leq \operatorname{GL}\left(2, \mathbb{F}_{q}\right)$.

Corollary 17. Any algorithm solving the HSP in $\mathrm{GL}\left(n, \mathbb{F}_{p^{m}}\right)$ efficiently using coset states needs to make measurements entangled across $k=\Omega(n(m \log p+\log n))$ registers.

Proof. Since $\operatorname{GL}\left(n, \mathbb{F}_{p^{m}}\right)$ contains all $n \times n$ permutation matrices, a lower bound of $k=\Omega(n \log n)$ follows from Corollary 14 and Lemma Also, we can use the embedding of GL $\left(2, \mathbb{F}_{p^{n m}}\right) \leq \operatorname{GL}\left(2 n, \mathbb{F}_{p^{m}}\right)$ via $\left(\begin{array}{cc}a & b \\ c & d\end{array}\right) \mapsto\left(\begin{array}{cc}M_{a} & M_{b} \\ M_{c} & M_{d}\end{array}\right)$, where for each $x \in \mathbb{F}_{p^{n m}}$ the matrix $M_{x} \in \operatorname{GL}\left(n, \mathbb{F}_{p^{m}}\right)$ realizes multiplication by $x$ with respect to a fixed basis of $\mathbb{F}_{p^{n m}}$ over $\mathbb{F}_{p^{m}}$. Hence by Lemma 1 we obtain that for the HSP in $\mathrm{GL}\left(2 n, \mathbb{F}_{p^{m}}\right)$ at least as much entanglement is necessary as in case of $\operatorname{GL}\left(2, \mathbb{F}_{p^{n m}}\right)$. The latter has been bounded by $\Omega(n m \log p)$ in Corollary 16

### 4.4 Direct products of the form $G^{n}$

In this section we show that for a large class of finite groups $G$, efficient algorithms for HSP for direct products of the form $G^{n}$, where $n \geq 1$, require entangled measurements on at least $k=\Omega(n)$ coset states. Let $G$ be a finite group and let $\widehat{G}=\left\{\sigma_{1}, \ldots, \sigma_{m}\right\}$ denote the irreducible representations of $G$. Recall that the centralizer $C(g)$ of an element $g \in G$ is the subgroup $C(g):=\{c \in G: c g=g c\}$. Let $h$ be an involution in $G$, and let $\sigma \in \widehat{G}$. Then either $\left|\chi_{\sigma}(h)\right|=d_{\sigma}$ or $\frac{\left|\chi_{\sigma}(h)\right|}{d_{\sigma}}<1-\frac{2|C(h)|}{|G|}$ holds [Gal94]. We define $\varepsilon:=\left(1-\frac{2|C(h)|}{|G|}\right)^{t}$, where $t=t(n)$ is a function of $n$ to be determined later.

The irreps of $G^{n}$, where $n \geq 1$, are given by $\sigma:=\sigma_{1} \otimes \ldots \otimes \sigma_{n}$, where $\sigma_{i} \in \widehat{G}$. We let $\Lambda:=\{\sigma \in \widehat{G}:$ $\left.\left|\chi_{\sigma}(h)\right|=d_{\sigma}\right\}, \lambda:=\sum_{\sigma \in \Lambda} d_{\sigma}^{2}$, and $\mu:=\sum_{\sigma \in \widehat{G} \backslash \Lambda} d_{\sigma}^{2}=|G|-\lambda$. The following property of the set

$$
\mathcal{S}_{\varepsilon}:=\left\{\boldsymbol{\sigma} \in \widehat{G}^{n}: \frac{\left|\chi_{\boldsymbol{\sigma}}(h, \ldots, h)\right|}{d_{\sigma}} \geq \varepsilon\right\}
$$

holds for our choice of the parameter $\varepsilon$ : if $\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}$ then necessarily at least $n-t$ positions $\sigma_{i}$ have to be from $\Lambda$, i. e., have to satisfy $\left|\chi_{\sigma_{i}}(h)\right|=d_{\sigma_{i}}$. Indeed, otherwise we would have more than $t$ positions $\sigma_{j}$ in each of which $\frac{\left|\chi_{\sigma_{j}}(h)\right|}{d \sigma_{j}} \leq 1-\frac{2|C(h)|}{|G|}$, making the product less than $\varepsilon$. We next give an estimate for the quantity $\sum_{\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}^{2}$ appearing in Theorem 2 For that we require the following lemma for estimating the tail of the binomial distribution.

Lemma 18. Let $\alpha, \beta>0$, let $n \geq 1$, and let $t=n / c$, where $c>\frac{\alpha+\beta}{\beta}$. Then

$$
\sum_{\ell=n-t}^{n}\binom{n}{\ell} \alpha^{\ell} \beta^{n-\ell} \leq\left(\alpha\left(\frac{c e(\alpha+\beta)}{\alpha}\right)^{1 / c}\right)^{n} .
$$

Proof. We have that

$$
\begin{aligned}
& \sum_{\ell=n-t}^{n}\binom{n}{\ell} \alpha^{\ell} \beta^{n-\ell}=(\alpha+\beta)^{n} \sum_{\ell=n-t}^{n}\binom{n}{\ell}\left(\frac{\alpha}{\alpha+\beta}\right)^{\ell}\left(\frac{\beta}{\alpha+\beta}\right)^{n-\ell} \\
& \leq(\alpha+\beta)^{n}\binom{n}{n-t}\left(\frac{\alpha}{\alpha+\beta}\right)^{n-t} \\
& =\alpha^{n}\binom{n}{t}\left(\frac{\alpha+\beta}{\alpha}\right)^{t} \leq \alpha^{n}\left(\frac{n e(\alpha+\beta)}{t \alpha}\right)^{t}=\left(\alpha\left(\frac{c e(\alpha+\beta)}{\alpha}\right)^{1 / c}\right)^{n},
\end{aligned}
$$

where the first inequality follows from the union bound on probabilities and the second one from $\binom{n}{t} \leq$ $\left(\frac{n e}{t}\right)^{t}$.

Suppose we fix $\ell \geq n-t$ locations for putting in irreps from $\Lambda$. The contribution of this configuration to $\sum_{\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}^{2}$ is the sum of products of squares of dimensions of $\ell$ irreps from $\Lambda$ and $n-\ell$ irreps from $\widehat{G} \backslash \Lambda$, which simplifies to $\lambda^{\ell} \mu^{n-\ell}$. Letting $\alpha:=\lambda, \beta:=\mu$, and $t=n / c$, with some constant $c$ to be determined later, we obtain the following bound from Lemma 18

$$
\sum_{\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}^{2} \leq \sum_{\ell=n-t}^{n}\binom{n}{\ell} \lambda^{\ell} \mu^{n-\ell} \leq \lambda^{n}\left(\left(\frac{c e|G|}{\lambda}\right)^{1 / c}\right)^{n}
$$

Hence, for any given $\kappa>0$ we can find a constant $c>0$ such that $\sum_{\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}^{2} \leq \lambda^{n}(1+\kappa)^{n}$ holds for all $n \geq c$. Note that the same upper bound applies to $\sum_{\sigma \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}\left|\chi_{\boldsymbol{\sigma}}(h, \ldots, h)\right|$. Also, observe that $\sum_{\boldsymbol{\rho} \in \widehat{G}^{n}} d_{\boldsymbol{\rho}}=\left(\sum_{\rho \in \widehat{G}} d_{\rho}\right)^{n}$. Now, we can bound the parameter $\delta_{1}$ used in Theorem[2]

$$
\begin{aligned}
\delta_{1} & \leq \varepsilon+\frac{1}{|G|^{n}}\left(\sum_{\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}\left|\chi_{\boldsymbol{\sigma}}(h, \ldots, h)\right|\right)\left(\sum_{\boldsymbol{\rho} \in \widehat{G}^{n}} d_{\boldsymbol{\rho}}\right) \\
& \leq\left(\left(1-\frac{2|C(h)|}{|G|}\right)^{1 / c}\right)^{n}+\frac{\lambda^{n}(1+\kappa)^{n}}{|G|^{n}}\left(\sum_{\rho \in \widehat{G}} d_{\rho}\right)^{n} .
\end{aligned}
$$

For the following we make the assumption that $|G|>\lambda(1+\kappa)\left(\sum_{\rho \in \widehat{G}} d_{\rho}\right)$ holds. This implies that there exists a constant $\gamma_{1}>0$ such that $\delta_{1} \leq \gamma_{1}^{n}$. For the parameter $\delta_{2}$ in Theorem [2]e obtain

$$
\begin{aligned}
\delta_{2} & =2^{k}(1+2 k \varepsilon) \delta_{1}^{1 / 2}+3 k \varepsilon+\frac{3 k}{|G|^{n}} \sum_{\boldsymbol{\sigma} \in \mathcal{S}_{\varepsilon}} d_{\boldsymbol{\sigma}}^{2} \\
& \leq 2^{k}\left(1+2 k\left(1-\frac{2|C(h)|}{|G|}\right)^{n / c}\right) \gamma_{1}^{n / 2}+3 k\left(1-\frac{2|C(h)|}{|G|}\right)^{n / c}+3 k\left(\frac{\lambda(1+\kappa)}{|G|}\right)^{n} .
\end{aligned}
$$

Now, since our assumption implies that $|G|>\lambda(1+\kappa)$, we obtain that there exists a constant $\gamma_{2}>0$ such that $\delta_{2} \leq \gamma_{2}^{n}$. Hence, we have proved the following corollary to Theorem 2

Corollary 19. Let $G$ be a finite group and let $h \in G$ be an involution. Let $\widehat{G}$ denote the set of irreps of $G$ and let $\Lambda:=\left\{\sigma \in \widehat{G}:\left|\chi_{\sigma}(h)\right|=d_{\sigma}\right\}$. Suppose that $|G|>\left(\sum_{\sigma \in \Lambda} d_{\sigma}^{2}\right)\left(\sum_{\rho \in \widehat{G}} d_{\rho}\right)$ holds. Then any efficient algorithm operating on coset states that distinguishes between the case when the hidden subgroup is a conjugate of the subgroup $\langle(h, \ldots, h)\rangle \leq G^{n}$, and the case when the hidden subgroup is the identity subgroup in $G^{n}$, needs to make measurements entangled across $\Omega(n)$ registers.

Recently, Alagic, Moore and Russell AMR05] showed that any measurement on a single coset state gives exponentially little information about a hidden subgroup in the group $G^{n}$, where $G$ is fixed and satisfies a suitable condition. Their condition on $G$ is weaker than our condition in Corollary 19 but they only prove lower bounds for algorithms measuring one coset state at a time. They also give several examples of families of groups satisfying their condition, including all non-abelian finite simple groups. In fact, the condition of Corollary 19 holds for all families of groups $G$ considered in their paper, showing that efficient coset state based algorithms solving the HSP for their families of groups $G^{n}$ need to make measurements entangled across $\Omega(n)$ registers.

From Corollary 19 it is easy to prove Corollary 20 via the Cauchy-Schwartz inequality.
Corollary 20. Let $G$ be a finite group and let $h \in G$ be an involution. Let $\widehat{G}$ denote the set of irreps of $G$ and let $\Lambda:=\left\{\sigma \in \widehat{G}:\left|\chi_{\sigma}(h)\right|=d_{\sigma}\right\}$. Suppose that $|G|^{1 / 2}>|\widehat{G}|^{1 / 2}\left(\sum_{\sigma \in \Lambda} d_{\sigma}^{2}\right)$ holds. Then any efficient algorithm operating on coset states that distinguishes between the case when the hidden subgroup is a conjugate of the subgroup $\langle(h, \ldots, h)\rangle \leq G^{n}$, and the case when the hidden subgroup is the identity subgroup in $G^{n}$, needs to make measurements entangled across $\Omega(n)$ registers.

Using Corollary 20, we prove the following result.
Corollary 21. Any efficient algorithm operating on coset states that distinguishes between the case when the hidden subgroup is a conjugate of the subgroup $\langle(h, \ldots, h)\rangle \leq\left(S_{m}\right)^{n}$ where $h \in S_{m}$ is any involution and $m \geq 5$ is fixed, and the case when the hidden subgroup is the identity subgroup in $\left(S_{m}\right)^{n}$, needs to make measurements entangled across $\Omega(n)$ registers. The same holds also when $m=4$ and $h=(1,2) \in S_{4}$.

Proof. Let $G=S_{m}$, where $m \geq 5$, and let $h$ be any involution in $G$. Recall that for $m \geq 5$ all irreps of $S_{m}$ of degree greater than 1 are faithful [JK81, Theorem 2.1.13], and that the center of $S_{m}$ is trivial. Since for faithful $\sigma \in \widehat{S_{m}}$ we have that $\left|\chi_{\sigma}(h)\right|=d_{\sigma}$ implies that $h$ is in the center, we obtain that $\left|\chi_{\sigma}(h)\right|<d_{\sigma}$ for all $\sigma \in \widehat{S_{m}}$ with $d_{\sigma}>1$. Hence $\Lambda=\{\mathbb{1}$, alt $\}$ consists of the trivial and the alternating character only and we obtain that $\sum_{\sigma \in \Lambda} d_{\sigma}^{2}=2$. Since for $m \geq 5$ we have that $|G|^{1 / 2}=\sqrt{m!}>2 \sqrt{p(m)}=|\widehat{G}|^{1 / 2} \sum_{\sigma \in \Lambda} d_{\sigma}^{2}$, where $p(m)$ denotes the partition number of $m$, the statement for $m \geq 5$ follows from Corollary 20

For $m=4$ and $h=(1,2)$ we observe that the set $\Lambda$ is again given by $\Lambda=\{\mathbb{1}$, alt $\}$. We verify that the condition $\left|S_{4}\right|^{1 / 2}=\sqrt{24}>2 \sqrt{5}=\left|\widehat{S_{4}}\right|^{1 / 2} \sum_{\sigma \in \Lambda} d_{\sigma}^{2}$ holds. Hence the statement for this case also follows from Corollary 20

## Acknowledgments

We thank Andrew Childs, Frédéric Magniez and Umesh Vazirani for helpful discussions and comments.

## References

[AMR05] G. Alagic, C. Moore, and A. Russell. Strong Fourier sampling fails over $G^{n}$. ArXiv preprint quant-ph/0511054, 2005.
[AT03] D. Aharonov and A. Ta-Shma. Adiabatic quantum state generation and statistical zero knowledge. In Proceedings of the 35th Annual ACM Symposium on Theory of computing, pages 20-29, 2003. Also: ArXiv preprint quant-ph/0301023.
[BBBV97] C. Bennett, E. Bernstein, G. Brassard, and U. Vazirani. Strengths and weaknesses of quantum computing. SIAM J. Comput., 26(5):1510-1523, 1997.
[BCD05] D. Bacon, A. Childs, and W. van Dam. From optimal measurement to efficient quantum algorithms for the hidden subgroup problem over semidirect product groups. In Proceedings of the 46th Annual IEEE Symposium on Foundations of Computer Science, 2005. Also: ArXiv preprint quant-ph/0504083.
[Bea97] R. Beals. Quantum computation of Fourier transforms over the symmetric groups. In Proceedings of the Symposium on Theory of Computing (STOC'97), pages 48-53, El Paso, Texas, 1997.
[BH97] G. Brassard and P. Høyer. An exact polynomial-time algorithm for Simon's problem. In Proceedings of Fifth Israeli Symposium on Theory of Computing and Systems, pages 12-33. ISTCS, IEEE Computer Society Press, 1997. Also: ArXiv preprint quant-ph/9704027.
[BZ99] Y. G. Berkovich and E. M. Zhmud. Characters of finite groups, part 2, volume 181 of Translations of Mathematical Monographs. American Mathematical Society, 1999.
[CW05] A. Childs and P. Wocjan. On the quantum hardness of solving isomorphism problems as nonabelian hidden shift problems. ArXiv preprint quant-ph/0510185, 2005.
[EH00] M. Ettinger and P. Høyer. On quantum algorithms for noncommutative hidden subgroups. Advances in Applied Mathematics, 25(3):239-251, 2000.
[EHK99a] M. Ettinger, P. Høyer, and E. Knill. A quantum observable for the graph isomorphism problem. ArXiv preprint quant-ph/9901029, 1999.
[EHK99b] M. Ettinger, P. Høyer, and E. Knill. Hidden subgroup states are almost orthogonal. ArXiv preprint quant-ph/9901034, 1999.
[FH91] W. Fulton and J. Harris. Representation theory: A first course, volume 129 of Graduate Texts in Mathematics. Springer, 1991.
$\left[\mathrm{FIM}^{+} 03\right]$ K. Friedl, G. Ivanyos, F. Magniez, M. Santha, and P. Sen. Hidden translation and orbit coset in quantum computing. In Proceedings of the 35th Annual ACM Symposium on Theory of Computing, pages $1-9,2003$. Also: ArXiv preprint quant-ph/0211091.
[Ga194] P. X. Gallagher. Character values at involutions. Proceeedings of the American Mathematical Society, 120(3):657-659, 1994.
[Gro96] L. Grover. A fast quantum mechanical algorithm for database search. In Proceedings of the 28th Annual ACM Symposium on Theory of Computing, pages 212-219, 1996. Also: ArXiv preprint quant-ph/9605043.
[GSVV04] M. Grigni, L. Schulman, M. Vazirani, and U. Vazirani. Quantum mechanical algorithms for the nonabelian hidden subgroup problem. Combinatorica, pages 137-154, 2004.
[Hal02] S. Hallgren. Polynomial-time quantum algorithms for Pell's equation and the principal ideal problem. In Proceedings of the 34th Annual ACM Symposium on Theory of computing, pages 653-658, 2002.
[Hal05] S. Hallgren. Fast quantum algorithms for computing the unit group and class group of a number field. In Proceedings of the 37th Annual ACM Symposium on Theory of Computing, pages 468474, 2005.
[HRT03] S. Hallgren, A. Russell, and A. Ta-Shma. The hidden subgroup problem and quantum computation using group representations. SIAM Journal on Computing, 32(4):916-934, 2003.
[IMS03] G. Ivanyos, F. Magniez, and M. Santha. Efficient quantum algorithms for some instances of the non-abelian hidden subgroup problem. International Journal of Foundations of Computer Science, pages 723-740, 2003. Also: ArXiv preprint quant-ph/0102014.
[Isa76] I. M. Isaacs. Character theory of finite groups. Academic Press, 1976.
[JK81] G. James and A. Kerber. The representation theory of the symmetric group. Addison-Wesley, Reading, 1981.
[Kit95] A. Yu. Kitaev. Quantum measurements and the abelian stabilizer problem. ArXiv preprint quant-ph/9511026, 1995.
[KST93] J. Köbler, U. Schöning, and J. Torán. The graph isomorphism problem. Birkhäuser, 1993.
[Kup03] G. Kuperberg. A subexponential-time quantum algorithm for the dihedral hidden subgroup problem. ArXiv preprint quant-ph/0302112, 2003.
[LR92] John D. Lafferty and Daniel Rockmore. Fast Fourier analysis for $\mathrm{SL}_{2}$ over a finite field and related numerical experiments. Experimental Mathematics, 1(2):115-139, 1992.
[ME98] M. Mosca and A. Ekert. The hidden subgroup problem and eigenvalue estimation on a quantum computer. In Quantum Computing and Quantum Communications, volume 1509 of Lecture Notes in Computer Science, pages 174-188. Springer-Verlag, 1998.
[MR05] C. Moore and A. Russell. The symmetric group defies strong Fourier sampling: Part II. ArXiv preprint quant-ph/0501066, 2005.
[MRRS04] C. Moore, D. Rockmore, A. Russell, and L. Schulman. The power of basis selection in fourier sampling: Hidden subgroup problems in affine groups. In Proceedings of the Fifteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1113-1122, 2004. Journal version in preparation. Also: ArXiv preprint quant-ph/0503095.
[MRS05] C. Moore, A. Russell, and L. Schulman. The symmetric group defies strong Fourier sampling. In Proceedings of the 46th Annual IEEE Symposium on the Foundations of Computer Science, pages 479-488, 2005. Also: ArXiv preprint quant-ph/0501056.
[NC00] M. Nielsen and I. Chuang. Quantum computation and quantum information. Cambridge University Press, 2000.
[Roi96] Y. Roichman. Upper bound on the characters of the symmetric groups. Inventiones Mathematicae, 125:451-485, 1996.
[RRS05] J. Radhakrishnan, M. Rötteler, and P. Sen. On the power of random bases in Fourier sampling: Hidden subgroup problem in the Heisenberg group. In Proceedings of the 32nd International Colloquium on Automata, Languages and Programming, Lecture Notes in Computer Science, vol. 3580, pages 1399-1411. Springer-Verlag, 2005. Also: ArXiv preprint quant-ph/0503114.
[Ser77] J. P. Serre. Linear representations of finite groups. Springer, 1977.
[Sho97] P. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM Journal on Computing, 26(5):1484-1509, 1997.
[Sim94] D. R. Simon. On the power of quantum computation. In Proceedings of the 35th Annual Symposium on Foundations of Computer Science, pages 116-123, Los Alamitos, CA, 1994. Institute of Electrical and Electronic Engineers Computer Society Press.
[SV05] A. Schmidt and U. Vollmer. Polynomial time quantum algorithm for the computation of the unit group of a number field. In Proceedings of the 37th Annual ACM Symposium on Theory of Computing, pages 475-480, 2005.

## A Representations of the wreath product $S_{n} \backslash S_{2}$

We describe the irreducible representations of the wreath product $S_{n} 2 S_{2}$, i. e., the group $\left(S_{n} \times S_{n}\right) \rtimes Z_{2}$. We will also get formulas for the character values under these representations in terms of the character values of irreducible representations of $S_{n}$.

Let $\widehat{S_{n}}=\left\{\sigma_{i}: i=1, \ldots, p(n)\right\}$ denote the irreducible representations of $S_{n}$, where $p(n)$ denotes the number of partitions of $n$. Denote the degree of $\sigma_{i} \in \widehat{S_{n}}$ by $d_{i}$. Letting $N:=\left(S_{n} \times S_{n}\right)$ and $G:=$ $\left(S_{n} \times S_{n}\right) \rtimes Z_{2}$ we have that $N \triangleleft G$ is a normal subgroup of index 2 . The irreducible representations of $N$ are given by $\widehat{N}=\left\{\sigma_{i} \otimes \sigma_{j}: i, j=1, \ldots, p(n)\right\}$ and we define the shorthand $\phi_{i, j}:=\sigma_{i} \otimes \sigma_{j}$. Define $t:=(e, e, 1) \in G$, where $e$ is the identity permutation in $S_{n}$. A transversal of $N$ in $G$ is given by $T=\{(e, e, 0), t\}$. Then $t$ acts on $\widehat{N}$ as $\left(\sigma_{i} \otimes \sigma_{j}\right)^{t}=\left(\sigma_{j} \otimes \sigma_{i}\right)$. Hence we have that $\phi_{i, j}^{t}=\phi_{j, i}$. Since all $\phi_{i, j}$ are pair-wise inequivalent, we obtain the following two cases from Clifford's Theorem [Isa76].
(i) $i=j$. Then $\phi_{i, j} \cong \phi_{i, j}^{t}$. Hence $\phi_{i, j}$ has precisely 2 pairwise inequivalent extensions to $G$. One of these extensions is $\vartheta_{i}=\overline{\phi_{i, i}}$ in which the image of $t$ permutes the tensor factors of $\mathbb{C}^{d_{i}} \otimes \mathbb{C}^{d_{i}}$, where $d_{i}=\operatorname{deg}\left(\sigma_{i}\right)$. Hence if $\left\{e_{k}: k=1, \ldots, d_{i}\right\}$ denotes the standard basis of $\mathbb{C}^{d_{i}}$ then $\vartheta_{i}(t)$ is given by the matrix $\mathrm{SWAP}_{d_{i}}$ which maps $e_{k} \otimes e_{\ell} \mapsto e_{\ell} \otimes e_{k}$. The other extension $\vartheta_{i}^{\prime}$ of $\phi_{i, i}$ to $G$ is given by defining the image of $t$ to be $\vartheta^{\prime}(t):=-\vartheta_{i}(t)$. Note that both extensions have degree $d_{i}^{2}$. The character value $\operatorname{tr}\left(\vartheta_{i}(t)\right)$ is given by the number of invariant tensors under the swap operation, i. e., $\operatorname{tr}\left(\vartheta_{i}(t)\right)=d_{i}$ and $\operatorname{tr}\left(\vartheta_{i}^{\prime}(t)\right)=-d_{i}$.
(ii) $i \neq j$. Then $\phi_{i, j} \not \not \phi_{i, j}^{t}=\phi_{j, i}$. Hence $\kappa_{i, j}:=\phi_{i, j} \uparrow_{T} G$ is irreducible. Moreover, we have that $\left(\phi_{i, j} \uparrow_{T} G\right) \downarrow N=\phi_{i, j} \oplus \phi_{j, i}$ and

$$
\left(\phi_{i, j} \uparrow_{T} G\right)(t)=\left(\begin{array}{cc}
\mathbf{0}_{d_{i} d_{j}} & \mathbb{1}_{d_{i} d_{j}} \\
\mathbb{1}_{d_{i} d_{j}} & \mathbf{0}_{d_{i} d_{j}}
\end{array}\right) .
$$

We summarize the facts relevant for this paper in the following table by showing the images of elements of the form $(\pi, \mu, e)$ and $t=(e, e, 1)$ under the irreducible representations of $G=\left(S_{n} \times S_{n}\right) \rtimes Z_{2}$ :

| Irrep | Irrep on $(\pi, \mu, e)$ | Char. on $(\pi, \mu, e)$ | Irrep on $t$ | Char. on $t$ |
| :---: | :---: | :---: | ---: | :---: |
| $\vartheta_{i}$ | $\sigma_{i}(\pi) \otimes \sigma_{i}(\mu)$ | $\chi_{i}(\pi) \chi_{i}(\mu)$ | SWAP $_{d_{i}}$ | $d_{i}$ |
| $\vartheta_{i}^{\prime}$ | $\sigma_{i}(\pi) \otimes \sigma_{i}(\mu)$ | $\chi_{i}(\pi) \chi_{i}(\mu)$ | $-\operatorname{SWAP}_{d_{i}}$ | $-d_{i}$ |
| $\kappa_{i, j}$ | $\left(\begin{array}{cc}\sigma_{i}(\pi) \otimes \sigma_{j}(\mu) & \mathbf{0}_{d_{i} d_{j}} \\ \mathbf{0}_{d_{i} d_{j}} & \sigma_{j}(\pi) \otimes \sigma_{i}(\mu)\end{array}\right)$ | $\chi_{i}(\pi) \chi_{j}(\mu)+\chi_{j}(\pi) \chi_{i}(\mu)$ | $\left(\begin{array}{cc}\mathbf{0}_{d_{i} d_{j}} & \mathbb{1}_{d_{i} d_{j}} \\ \mathbb{1}_{d_{i} d_{j}} & \mathbf{0}_{d_{i} d_{j}}\end{array}\right)$ | 0 |

Overall, there are $\binom{p(n)}{2}$ pairwise inequivalent irreducible representations $\kappa_{i, j} \in \widehat{G}$, one for each pair $i, j$ such that $i \neq j$. We have that the degree of $\kappa_{i, j}$ is given by $2 d_{i} d_{j}$. The character $\chi_{i, j}$ of $\kappa_{i, j}$ satisfies $\kappa_{i, j}(t)=0$ for all $i \neq j$. Furthermore, there are $2 p(n)$ pairwise inequivalent irreducible representations $\vartheta_{i}$ and $\vartheta_{i}^{\prime}$.

## B Representations of the projective linear groups $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$

We briefly recall some facts from the representation theory of the projective linear groups $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$, where $q$ is a prime power. Good references on the complex representation theory of these groups are available, see e.g, [BZ99, FH91, LR92]. We treat the cases $q$ odd and $q=2^{n}$ separately and begin by describing the conjugacy classes of involutions and the irreducible representations of $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ for $q$ odd. Recall that for $q$ odd, the center of $\operatorname{SL}\left(2, \mathbb{F}_{q}\right)$ consists only of the identity matrix and the matrix

$$
c:=\left(\begin{array}{rr}
-1 & 0 \\
0 & -1
\end{array}\right) .
$$

Once the characters of $\operatorname{SL}\left(2, \mathbb{F}_{q}\right)$ are known, we therefore have to filter out only those characters $\chi$ for which $\chi(c)=\chi(1)$ holds in order to obtain the irreducible representations of $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$.

## B. 1 The case $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ where $q \equiv 1 \bmod 4$

The involutions are given by conjugates of the residue class of

$$
h=\overline{\left(\begin{array}{rr}
0 & 1 \\
-1 & 0
\end{array}\right)} \in \operatorname{PSL}\left(2, \mathbb{F}_{q}\right),
$$

where the bar denotes the fact that we are using coset representatives with respect to the center $\langle c\rangle$ of $\mathrm{SL}\left(2, \mathbb{F}_{q}\right)$. There is a total of $\frac{q(q-1)}{2}$ many involutions that are conjugates of $h$. The characters and their values on $h$ are summarized in the following table.

| Irrep name | Parameters | Number of irreps | Degree | Character value at $h$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbb{1}$ | - | 1 | 1 | 1 |
| $\psi$ | - | 1 | $q$ | 1 |
| $\theta_{k}$ | $k=2,4, \ldots, \frac{q-1}{2}$ | $\frac{q-1}{4}$ | $q-1$ | 0 |
| $\chi_{j}$ | $j=2,4, \ldots, \frac{q-5}{2}$ | $\frac{q-5}{4}$ | $q+1$ | $2(-1)^{k / 2}$ |
| $\zeta_{\ell}$ | $\ell=1,2$ | 2 | $\frac{q+1}{2}$ | $(-1)^{(q-1) / 4}$ |

## B. 2 The case $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ where $q \equiv 3 \bmod 4$

Similar to the previous case all involutions are conjugate to the element $h$ defined as above. However, now there are $\frac{q(q+1)}{2}$ involutions conjugate to $h$. The characters and their values on $h$ are summarized in the following table.

| Irrep name | Parameters | Number of irreps | Degree | Character value at $h$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbb{1}$ | - | 1 | 1 | 1 |
| $\psi$ | - | 1 | $q$ | -1 |
| $\theta_{k}$ | $k=2,4, \ldots, \frac{q-3}{2}$ | $\frac{q-3}{4}$ | $q-1$ | $2(-1)^{k / 2+1}$ |
| $\chi_{j}$ | $j=2,4, \ldots, \frac{q-3}{2}$ | $\frac{q-3}{4}$ | $q+1$ | 0 |
| $\eta_{\ell}$ | $\ell=1,2$ | 2 | $\frac{q-1}{2}$ | $(-1)^{\frac{q+1}{4}+1}$ |

## B. 3 The case $\operatorname{PSL}\left(2, \mathbb{F}_{q}\right)$ where $q=2^{n}$

This case behaves quite differently from the case $q$ odd. First, observe that in this case the center is trivial, i. e., $\operatorname{PSL}\left(2, \mathbb{F}_{2^{n}}\right)=\operatorname{SL}\left(2, \mathbb{F}_{2^{n}}\right)$. All involutions in $\operatorname{SL}\left(2, \mathbb{F}_{2^{n}}\right)$ are conjugate to the element

$$
h=\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right) \in \mathrm{SL}\left(2, \mathbb{F}_{q}\right),
$$

and there is a total number of $q^{2}-1$ of such involutions. The characters and their values on $h$ are summarized in the following table.

| Irrep name | Parameters | Number of irreps | Degree | Character value on $h$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathbb{1}$ | - | 1 | 1 | 1 |
| $\psi$ | - | 1 | $q$ | 0 |
| $\theta_{k}$ | $k=1,2, \ldots, \frac{q}{2}$ | $\frac{q}{2}$ | $q-1$ | -1 |
| $\chi_{j}$ | $j=1,2, \ldots, \frac{q-2}{2}$ | $\frac{q-2}{2}$ | $q+1$ | 1 |

