# IMPLEMENTATION OF KUMAR'S CORRESPONDENCE 

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#### Abstract

In 1997, N.M. Kumar published a paper which introduced a new tool of use in the construction of algebraic vector bundles. Given a vector bundle on projective $n$-space, a well known theorem of QuillenSuslin guarantees the existence of sections which generate the bundle on the complement of a hyperplane in projective $n$-space. Kumar used this fact to give a correspondence between vector bundles on projective $n$-space and vector bundles on projective ( $n-1$ )-space satisfying certain conditions. He then applied this correspondence to establish the existence of many, previously unknown, rank two bundles on projective fourspace in positive characteristic. The goal of the present paper is to give an explicit homological description of Kumar's correspondence in a setting appropriate for implementation in a computer algebra system.


## 1. Introduction

A fundamental problem in algebraic geometry is the study, classification and construction of varieties, schemes and sheaves. These problems are related in the sense that progress in one area often leads to progress in each of the other areas. For instance, given a sheaf with interesting or unusual properties, one can often obtain correspondingly interesting varieties and schemes as degeneracy loci of the sheaf. A main focus of the present paper is an explicit homological description of a tool of use in the construction of locally free sheaves on $\mathbb{P}^{n}$ over an algebraically closed field, $K$, of arbitrary characteristic. With a slight abuse of language, we will use the term Algebraic Vector Bundle for such a sheaf. A vector bundle $\mathcal{E}$ of rank $r$ on $\mathbb{P}^{n}$ is said to be of low rank if $r<n$. The co-rank of a bundle is the difference $n-r$. It appears that indecomposable low rank vector bundles on $\mathbb{P}^{n}$ are exceedingly rare. In fact, the only known co-rank 2 vector bundles in characteristic zero are the Horrocks-Mumford bundle on $\mathbb{P}^{4}$ and the Horrocks bundle on $\mathbb{P}^{5}[7]$. In characteristic $p>2$ there are the additional co-rank 2 constructions of Kumar, and Kumar et al [10, 11. In characteristic $p=2$ there is a single example of an indecomposable co-rank 3 bundle constructed by Tango [15. It is an open problem to construct other examples or show that they do not exist. In particular, it is unknown if there exist co-rank 2, indecomposable vector bundles on $\mathbb{P}^{n}$ for any value of $n$ greater than 5 . An

[^0]interesting class of problems is concerned with establishing the existence or non-existence of higher co-rank bundles on $\mathbb{P}^{n}$ with prescribed properties.

The first constructions of higher co-rank algebraic vector bundles appeared in the 1970's in the papers of Horrocks-Mumford, Horrocks and Tango. After Horrock's paper in 1978, no fundamentally new, higher corank bundles were shown to exist for 20 years. In 1997, Kumar introduced a completely novel construction method and demonstrated its power by constructing several previously unknown co-rank 2 vector bundles in positive characteristic 10. His method provided fuel for the additional constructions found in [11. Kumar based his construction on the solution, by Quillen and Suslin, of the well-known Serre's conjecture on the existence of finitely generated, non-free $K\left[x_{0}, \cdots, x_{n}\right]$-modules [13, 12, 14]. For a given vector bundle on the $n$-dimensional projective space $\mathbb{P}^{n}$, the theorem of Quillen and Suslin guarantees us the existence of sections that generate the vector bundle on the complement of a hyperplane in $\mathbb{P}^{n}$. The pair of the vector bundle and these sections corresponds to a vector bundle on the hyperplane. Kumar gave necessary and sufficient conditions for a vector bundle on a hyperplane of $\mathbb{P}^{n}$ to be obtained from a vector bundle on $\mathbb{P}^{n}$ in this way. His correspondence between vector bundles on $\mathbb{P}^{n}$ and vector bundles on a hyperplane (satisfying certain conditions) were used to establish the existence of many, previously unknown, rank two vector bundles on $\mathbb{P}^{4}$ in positive characteristic.

The purpose of the present paper is to give an explicit homological description of Kumar's correspondence in a setting appropriate for implementation in a computer algebra system.

## 2. Preliminaries

2.1. Kumar's correspondence. Let $K$ be a field. In 1955, J.P. Serre asked whether there exist finitely generated $K\left[x_{0}, \cdots, x_{n}\right]$-modules which are not free [13. In 1976, Quillen and Suslin independently proved that such modules do not exist, i.e. they showed that every finitely generated projective $K\left[x_{0}, \cdots, x_{n}\right]$-module is free (cf. [12], 14). One can apply the theorem of Quillen and Suslin to vector bundles on $\mathbb{P}^{n}$ as follows. Let $h$ be a linear form in $K\left[x_{0}, \cdots, x_{n}\right]$. Let $H$ be the hyperplane in $\mathbb{P}^{n}$ determined by the zeros of $h$. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^{n}$ of rank $r$. By the theorem of Quillen and Suslin, $\mathcal{E}$ restricted to the complement, $\mathbb{P}^{n} \backslash H$ of $H$, is free. As a consequence, there exist $r$ sections $s_{1}, \ldots, s_{r} \in \mathrm{H}^{0}\left(\mathbb{P}^{n} \backslash H,\left.\mathcal{E}^{\vee}\right|_{\mathbb{P}^{n} \backslash H}\right)$ that generate $\left.\mathcal{E}^{\vee}\right|_{\mathbb{P}^{n} \backslash H}$. It is known that for suitable integers $l_{i}, 1 \leq i \leq r$, the sections $h^{l_{i}} s_{i}$ extend to global sections $\widetilde{s_{i}} \in \mathrm{H}^{0}\left(\mathbb{P}^{n}, \mathcal{E}^{\vee}\left(l_{i}\right)\right)$ (cf. 55). Such sections define an injective morphism of sheaves $\mathcal{E} \rightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}\right)$, which is an injective bundle map outside the divisor defined by $\widetilde{s_{1}} \wedge \cdots \wedge \widetilde{s_{r}} \in$ $\mathrm{H}^{0}\left(\mathbb{P}^{n},\left(\wedge^{r} \mathcal{E}^{\vee}\right)\left(\sum_{i=1}^{r} l_{i}\right)\right) \cong \mathrm{H}^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}\left(\sum_{i=1}^{r} l_{i}-c_{1}(\mathcal{E})\right)\right)$. By construction, this divisor is the $m^{\text {th }}$ infinitesimal neighborhood $H_{m}$ of $H$, where $m=$
$\sum_{i=1}^{r} l_{i}-c_{1}(\mathcal{E})$. In other words, there is an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{E} \rightarrow \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}\right) \rightarrow \mathcal{F} \rightarrow 0 \tag{2.1}
\end{equation*}
$$

where $\mathcal{F}$ is a coherent sheaf whose support is $H$. It is clear that the coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n}$ possesses an $\mathcal{O}_{H_{m}}$-module structure, and from (2.1) it follows that the homological dimension of $\mathcal{F}$ is 1 . Conversely, if there exists a coherent sheaf $\mathcal{F}$ on $\mathbb{P}^{n}$ which has an $\mathcal{O}_{H_{m}}$-structure, has homological dimension 1 and which allows a surjective morphism from a direct sum of $r$ line bundles then there exists a rank $r$ vector bundle $\mathcal{E}$ on $\mathbb{P}^{n}$ and an exact sequence of type (2.1).

Let $\pi$ be the finite morphism $\pi: H_{m} \rightarrow H$ induced by the projection $\mathbb{P}^{n} \backslash P \rightarrow H$ from a point $P \in \mathbb{P}^{n} \backslash H$. Then $\pi_{*}$ induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_{H_{m}}$-modules to the category of quasi-coherent $\mathcal{O}_{H}$-modules having a $\pi_{*} \mathcal{O}_{H_{m}}$-module structure. This correspondence enables us to translate statements about quasi-coherent $\mathcal{O}_{H_{m}}$-modules into statements about quasi-coherent $\mathcal{O}_{H}$-modules.
(1) Since $\pi_{*} \mathcal{O}_{H_{m}} \simeq \bigoplus_{i=0}^{m-1} \mathcal{O}_{H}(-i)$, a quasi-coherent $\mathcal{O}_{H}$-module $\mathcal{Q}$ has a $\pi_{*} \mathcal{O}_{H_{m}}$-module structure if and only if there is a morphism $\phi: \mathcal{Q} \rightarrow \mathcal{Q}(1)$ whose $m^{t h}$ power is zero. Following Kumar, we call such a morphism a nilpotent endomorphism of $\mathcal{Q}$. From the theorem of Auslander and Buchsbaum it follows that a quasi-coherent $\mathcal{O}_{H_{m}}$-module has homological dimension 1 as a coherent sheaf on $\mathbb{P}^{n}$ if and only if the corresponding quasi-coherent $\mathcal{O}_{H}$-module has homological dimension 0 , in other words, if the $\mathcal{O}_{H}$-module is a vector bundle.
(2) Let $\mathcal{M}$ be the direct image sheaf of $\mathcal{F}$ by $\pi$ and let $\phi$ be the corresponding nilpotent endomorphism of $\mathcal{M}$. Since $\pi$ is a finite morphism, there are natural isomorphisms $\mathrm{H}^{0}\left(\mathbb{P}^{n}, \mathcal{F}\left(-l_{i}\right)\right) \simeq \mathrm{H}^{0}\left(H, \mathcal{M}\left(-l_{i}\right)\right)$ for all $1 \leq i \leq r$. We denote the restriction of $\mathcal{G}=\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}\right)$ to $H$ by $\mathcal{G}_{H}$. There is a surjective morphism from $\mathcal{G}$ to $\mathcal{F}$ if and only if the restriction map from $\bigoplus_{i=0}^{m-1} \mathcal{G}_{H}(-i)$ to $\mathcal{M}$ is surjective. The latter condition is equivalent to the condition that there exists a map $\psi: \mathcal{G}_{H} \rightarrow \mathcal{M}$ such that $(\phi, \psi)$ : $\mathcal{M}(-1) \oplus \mathcal{G}_{H} \rightarrow \mathcal{M}$ is surjective.

Theorem 2.1. (Kumar) There is a correspondence between (i) and (ii):
(i) The set of pairs $(\mathcal{E}, s)$, where $\mathcal{E}$ is a rank $r$ vector bundle on $\mathbb{P}^{n}$ and $s$ is a morphism from $\mathcal{E}$ to $\bigoplus_{i=1}^{r} \mathcal{O}\left(l_{i}\right)$ with cokernel $\mathcal{F}$ satisfying:
a) $\mathcal{F}$ is a coherent sheaf on the $m^{\text {th }}$ infinitesimal neighborhood $H_{m}$ of a hyperplane $H$ for some positive integer $m$.
b) The direct image sheaf of $\mathcal{F}$ by the finite morphism $\pi: H_{m} \rightarrow H$ is a vector bundle $\mathcal{M}$ on $H$.
(ii) The set of triples $(\mathcal{M}, \phi, \psi)$, where $\mathcal{M}$ is a vector bundle on $H$, $\phi$ : $\mathcal{M} \rightarrow \mathcal{M}(1)$ is a nilpotent endomorphism and $\psi: \bigoplus_{i=1}^{r} \mathcal{O}_{H}\left(l_{i}\right) \rightarrow \mathcal{M}$
is a morphism such that $(\phi, \psi): \mathcal{M}(-1) \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{H}\left(l_{i}\right) \rightarrow \mathcal{M}$ is surjective.

Proof. See [10] for a detailed proof.
Our goal is to make explicit the procedure for computing the pair $(\mathcal{E}, s)$ corresponding to a given triple $(\mathcal{M}, \phi, \psi)$ and conversely, to make explicit the procedure for computing the triple $(\mathcal{M}, \phi, \psi)$ corresponding to a given pair $(\mathcal{E}, s)$. Let $R$ be the homogeneous coordinate ring of $\mathbb{P}^{n-1}$ and $S$ the homogeneous coordinate ring of $\mathbb{P}^{n}$. Suppose that there exists a morphism $s$ from a rank $r$ vector bundle $\mathcal{E}$ on $\mathbb{P}^{n}$ to $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}\right)$ satisfying the condition in Theorem 2.1. Then $s$ induces a homomorphism from $\mathrm{H}_{*}^{0}\left(\mathbb{P}^{n}, \mathcal{E}\right)$ to $\mathrm{H}_{*}^{0}\left(\mathbb{P}^{n}, \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}\right)\right)=\bigoplus_{i=1}^{r} S\left(l_{i}\right)$. The sheafification of the cokernel of $s$ is the sheaf $\mathcal{F}$. From the cokernel of $s$ we can compute the module $F=\mathrm{H}_{*}^{0}\left(\mathbb{P}^{n}, \mathcal{F}\right)$. Consider the $R$-module ${ }_{R} F$ obtained from $F$ by restriction of scalars. Then the sheaf associated to ${ }_{R} F$ is $\mathcal{M}$. So the key step in each procedure is to compute the $R$-module ${ }_{R} F$ from an $S$-module $F$ or an $S$ module $F$ from an $R$-module $M$ such that ${ }_{R} F=M$. In the following section we will discuss how to carry out these steps.
2.2. Restriction of scalars. Let $S$ be the polynomial ring $K\left[x_{0}, \ldots, x_{n}\right]$ and let $R$ be the polynomial ring $K\left[x_{0}, \ldots, x_{n-1}\right]$. For any graded $S$-module $F$ we denote by ${ }_{R} F$ the $R$-module obtained from $F$ by restriction of scalars. Let $Q$ be the quotient ring $S /\left(x_{n}^{m}\right)$ for some integer $m$. Suppose that $F$ is finitely generated and has a $Q$-module structure (i.e. $F$ is annihilated by the ideal $\left.\left(x_{n}^{m}\right)\right)$. Then ${ }_{R} F$ is also finitely generated and has an ${ }_{R} Q$-module structure. Indeed, the following proposition immediately follows from the definition of restriction of scalars.

Proposition 2.2. Let $F$ be a finitely generated graded $S$-module with minimal generating set $\mathfrak{F}=\left\{f_{i}\right\}_{1 \leq i \leq s}$. Suppose that $F$ has a $Q$-module structure. Then $\mathfrak{M}=\left\{x_{n}^{i} f_{j}\right\}_{\substack{1 \leq i \leq m-1 \\ 1 \leq j \leq}}$ is a generating set for ${ }_{R} F$. Moreover the ${ }_{R} Q$-module structure of ${ }_{R} F$ is determined by the homomorphism $\phi:{ }_{R} F \rightarrow\left({ }_{R} F\right)(1)$ defined by

$$
x_{n}^{i} f_{j} \mapsto \begin{cases}0 & i \geq m-1 \\ x_{n}^{i+1} f_{j} & \text { otherwise } .\end{cases}
$$

Remark 2.3. (1) The homomorphism $\phi:{ }_{R} F \rightarrow\left({ }_{R} F\right)(1)$ corresponds to multiplication $\cdot x_{n}: F \rightarrow F(1)$, and clearly the $m^{\text {th }}$ power of $\phi$ is zero. The homomorphism $\phi:{ }_{R} F \rightarrow\left({ }_{R} F\right)(1)$ obtained in this way will be called the standard nilpotent endomorphism of ${ }_{R} F$.
(2) The generating set $\mathfrak{M}$ of ${ }_{R} F$ is not always minimal. Eliminating redundant elements gives a minimal set $\mathfrak{M}^{\prime}=\left\{g_{1}, \ldots, g_{t}\right\}$ of generators for ${ }_{R} F$. Let

$$
M_{0} \rightarrow{ }_{R} F \rightarrow 0
$$

be the corresponding epimorphism, where $M_{0}$ is a free $R$-module. Note that each $x_{n} g_{i}$ can be written as an $R$-linear combination of $g_{1}, \ldots, g_{t}$ :

$$
x_{n} g_{i}=\sum_{j=1}^{t} a_{i j} g_{j},
$$

where $a_{i j} \in R$. So the matrix $\left(a_{i j}\right)_{1 \leq i, j \leq t}$ defines a lifting $\phi_{0}: M_{0} \rightarrow M_{0}(1)$ of the standard nilpotent endomorphism $\phi$ of $M$, since $\phi$ sends $g_{i}$ to $x_{n} g_{i}$ for $1 \leq i \leq t$. We call the lifting $\phi_{0}$ of $\phi$ given in this way the standard lifting of $\phi$.

A homomorphism from a finitely generated $R$-module $M$ to $M(1)$ is said to be a nilpotent endomorphism of $M$ if its $m^{\text {th }}$ power is zero for some positive integer $m$. The functor ${ }_{R}$. induces an equivalence of categories from the category $\mathfrak{S}_{m}$ of finitely generated $S$-modules having a $Q=S /\left(x_{n}^{m}\right)$ module structure to the category $\mathfrak{R}$ of finitely generated $R$-modules having an ${ }_{R} Q$-module structure (i.e. having a nilpotent endomorphism $\phi$ with $\phi^{m}=$ 0 ). Indeed, for an $R$-module $M=<g_{1}, \ldots, g_{t}>$, we can define a finitely generated $S$-module ${ }^{S} M$ by considering the set of all $S$-linear combinations of the generators of $M$ (i.e. the set $\left\{b_{1} g_{1}+\cdots+b_{t} g_{t} \mid b_{i} \in S\right\}$ ). Its $Q$-module structure is defined by

$$
\begin{equation*}
\phi\left(g_{i}\right)=x_{n} g_{i} \text { for each } i=1, \ldots, t . \tag{2.2}
\end{equation*}
$$

Obviously the functors $R^{\text {. and }}{ }^{S}$. are inverse to each other.
For each $i=1, \ldots, t, x_{n} g_{i}$ can be written as an $R$-linear combination of the $g_{j}$ 's by (2.2), so we can define the standard lifting for $\phi$ in the same way as in Remark 2.3 The following proposition will show us how to compute from $M$ the corresponding module ${ }^{S} M$ :

Proposition 2.4. Let $M$ be an object of $\mathfrak{R}$ and let $\phi$ be a nilpotent endomorphism of $M$ with $\phi^{m}=0$. Suppose that $M$ has a minimal free presentation of type

$$
\begin{equation*}
M_{1} \xrightarrow{\alpha} M_{0} \rightarrow M \rightarrow 0 \tag{2.3}
\end{equation*}
$$

Then the corresponding $S$-module $F$ in $\mathfrak{S}_{m}$ has a presentation

$$
\left(M_{1} \otimes_{R} S\right) \oplus\left(M_{0}(-1) \otimes_{R} S\right) \xrightarrow{\left(\alpha, \phi_{0}(-1)-x_{n}\right)} M_{0} \otimes_{R} S \rightarrow F \rightarrow 0
$$

where $\phi_{0}: M_{0} \rightarrow M_{0}(1)$ is the standard lifting of $\phi$ and $\cdot x_{n}$ is multiplication by $x_{n}$.

Proof. Let $\left\{g_{1}, \ldots, g_{t}\right\}$ be a minimal set of generators for $M$. Then $F=$ $\left\{b_{1} g_{1}+\cdots+b_{t} g_{t} \mid b_{i} \in S\right\}$. Let $\phi_{0}(-1)=\left(a_{i j}\right)_{1 \leq i, j \leq t}$ be the standard lifting of $\phi(-1)$. Then it follows from (2.2) that $\left\{g_{1}, \ldots, g_{t}\right\}$ satisfies the relations

$$
\begin{equation*}
\sum_{j=1}^{t} a_{i j} g_{j}-x_{n} g_{i}=0 \tag{2.4}
\end{equation*}
$$

for all $i=1, \ldots, t$. So $\left(\alpha, \phi_{0}(-1)-x_{n}\right)$ forms part of a presentation matrix of $F$. Suppose that there is a relation on $\left\{g_{1}, \ldots, g_{t}\right\}$ :

$$
c_{1} g_{1}+\cdots+c_{t} g_{t}=0,
$$

where $c_{i} \in S$ for each $i$. Without loss of generality, we may assume that each term $c_{i} g_{i}$ can be rewritten in the form $\left(c_{i}^{\prime} x_{n}+c_{i}^{\prime \prime}\right) g_{i}$, where $c_{i}^{\prime} \in S$ and $c_{i}^{\prime \prime} \in R$. Let $C=\left(c_{1}, c_{2}, \ldots, c_{t}\right), C^{\prime}=\left(c_{1}^{\prime}, c_{2}^{\prime}, \ldots, c_{t}^{\prime}\right), C^{\prime \prime}=\left(c_{1}^{\prime \prime}, c_{2}^{\prime \prime}, \ldots, c_{t}^{\prime \prime}\right)$, $G=\left(g_{1}, g_{2}, \ldots, g_{t}\right)$ and $A=\left(a_{i j}\right)$. By using the relations given in (2.4), we get $c_{1} g_{1}+\cdots+c_{t} g_{t}=C G^{T}=C^{\prime} A G^{T}+C^{\prime \prime} G^{T}$. Set $b_{j}=\Sigma_{i=1}^{t} c_{i}^{\prime} a_{i j}+c_{j}^{\prime \prime}$ and $B=\left[b_{1}, b_{2}, \ldots, b_{j}\right]$. Then $C G^{t}=B G^{T}$. View $c_{i}, b_{i}$ as elements of $R\left[x_{n}\right]$. Let $r=\max \left\{\operatorname{deg}\left(c_{i}\right) \mid 1 \leq i \leq t\right\}$ and $s=\max \left\{\operatorname{deg}\left(b_{i}\right) \mid 1 \leq i \leq t\right\}$. The construction guarantees that $s<r$. If we now repeat the same operation with $b_{1} g_{1}+\cdots+b_{t} g_{t}$ then in a finite number of steps we can decrease the maximum degree of the coefficients of the syzygy until all of the coefficients have degree 0 , i.e. the relation becomes an $R$-linear combination of the $g_{i}$ which is equal to 0 :

$$
d_{1} g_{1}+\cdots+d_{t} g_{t}=0, d_{i} \in R \text { for each } i .
$$

Since we assumed that the presentation of $M$ given in (2.3) is minimal, $\left(d_{1}, \ldots, d_{t}\right)^{t}$ can be generated by column vectors of $\alpha$. Therefore, $\left(\alpha, \phi_{0}(-1)-\right.$ $\left.\cdot x_{n}\right)$ is a presentation matrix of $F$.

## 3. Algorithm

In this section we will develop a procedure for computing a rank $r$ vector bundle on $\mathbb{P}^{n}$ from a given vector bundle on $\mathbb{P}^{n-1}$ satisfying the conditions in Theorem [2.1] The procedure takes as input a triple $(\mathcal{M}, \phi, \psi)$ and produces as output the corresponding pair $(\mathcal{E}, s)$. More specifically, the procedure takes as input:

- The finitely generated $R$-module $M=<g_{1}, \ldots, g_{t}>$ with minimal free presentation

$$
M_{1} \xrightarrow{\alpha} M_{0} \rightarrow M \rightarrow 0
$$

whose associated sheaf, $\mathcal{M}=\widetilde{M}$, is locally free;

- A nilpotent endomorphism $\phi$ of $M$ and its standard lifting $\phi_{0}$;
- A homomorphism $\psi=\left(\psi_{1}, \ldots, \psi_{r}\right)$ from a free module $\bigoplus_{i=1}^{r} R\left(l_{i}\right)$ to $M$ such that the corresponding sheaf morphism from $\bigoplus_{i=1}^{r=1} \mathcal{O}\left(l_{i}\right)$ to $\mathcal{M}$ is a morphism such that $(\phi, \psi): \mathcal{M}(-1) \oplus \bigoplus_{i=1}^{r} \mathcal{O}_{H}\left(l_{i}\right) \rightarrow \mathcal{M}$ is surjective.
The procedure produces as output:
- The finitely generated $S$-module $E$ whose associated sheaf is a rank $r$ vector bundle;
- A homomorphism $s: E \rightarrow \bigoplus_{i=1}^{r} S\left(l_{i}\right)$ such that the coherent sheaf associated to Coker $(s)$ coincides with ${ }^{S} M$.
To get the pair $(\mathcal{E}, s)$ from the triple $(\mathcal{M}, \phi, \psi)$, we take the following steps:
(i) Define a finitely generated $S$-module $F$ by $\left\{a_{1} g_{1}+\cdots+a_{t} g_{t} \mid a_{i} \in\right.$ $R\}$. In practice, this module will be given as the cokernel of the homomorphism $\left(\alpha, \phi_{0}(-1)-x_{n}\right):\left(M_{1} \otimes_{R} S\right) \oplus\left(M_{0}(-1) \otimes_{R} S\right) \rightarrow$ $M_{0} \otimes_{R} S$ (see Proposition [2.4).
(ii) Define the homomorphism from $\bigoplus_{i=1}^{r} S\left(l_{i}\right)$ to $F$ by $\psi=\left(\psi_{1}, \ldots, \psi_{r}\right)$ and compute the syzygy module $\operatorname{Syz}\left(\psi_{1}, \ldots, \psi_{r}\right)$ which represents the desired homomorphism $s: E \rightarrow \bigoplus_{i=1}^{r} S\left(l_{i}\right)$. Note that $\psi_{i}$ can be written as an $R$-linear combination of the $g_{j}$ 's for each $i=1, \ldots, t$. So a simple way of computing $\operatorname{Syz}\left(\psi_{1}, \ldots, \psi_{r}\right)$ is to determine the generating set $\left\{g_{1}, \ldots, g_{t}\right\}$ of $F$ as a $Q=S /\left(x_{n}^{m}\right)$-module by using the presentation matrix of $F$ given in (i). This enables us to compute $\operatorname{Syz}\left(\psi_{1}, \ldots, \psi_{r}\right)$ as a $Q$-module. Indeed, let $N$ be the extension of the module $\operatorname{Syz}\left(\psi_{1}, \ldots, \psi_{r}\right)$ to $S$. Then $\operatorname{Syz}\left(\psi_{1}, \ldots, \psi_{r}\right)$ will be obtained as the quotient of $N$ by $x_{n}^{m} N$.

Remark 3.1. Let $(E, s)$ be the resulting pair. Then we want to check that $\mathcal{E}=\widetilde{E}$ is indeed a rank $r$ vector bundle on $\mathbb{P}^{n}$. By construction, $\mathcal{E}$ can be regarded as a subsheaf of $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}\right)$ :


The entries of the $j^{\text {th }}$ column of $A$ define the scheme of zeros $X_{s_{j}}=\left\{s_{j}=\right.$ $0\}$; the entries of the $i^{\text {th }}$ row of $A$ define the scheme of zeros $X_{\sigma_{i}}=\left\{\sigma_{i}=0\right\}$. Recall that $s$ is an injective bundle map outside the divisor defined by

$$
x_{n}^{m}=\sigma_{1} \wedge \cdots \wedge \sigma_{r} \in \mathrm{H}^{0}\left(\mathbb{P}^{n},\left(\wedge^{r} \mathcal{E}^{\vee}\right)\left(\sum_{i=1}^{r} l_{i}\right)\right) \cong \mathrm{H}^{0}\left(\mathbb{P}^{n}, \mathcal{O}_{\mathbb{P}^{n}}(m)\right),
$$

where $c_{1}$ is the first Chern class of $\mathcal{E}$ and $m=\sum_{i=1}^{r} l_{i}-c_{1}$. The $j^{\text {th }}$ column of $A$ represents the section $t_{j}=s\left(s_{j}\right)$ of $\bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^{n}}\left(l_{i}-m_{j}\right)$. So we have the relation of the form

$$
t_{j_{1}} \wedge \cdots \wedge t_{j_{r}}=x_{n}^{m} \cdot\left(s_{j_{1}} \wedge \cdots \wedge s_{j_{r}}\right),
$$

and hence we can prove that $\widetilde{E}$ is a vector bundle by checking that the ideal quotient ( $I: x_{n}^{m}$ ) defines the empty set in $\mathbb{P}^{n}$, where $I$ is the ideal generated by the maximal minors of $A$.

The following examples will show how the procedure works. The procedure in the first example takes as input the twisted cotangent bundle on $\mathbb{P}^{2}$ and returns as output a stable rank two vector bundle on $\mathbb{P}^{3}$ with Chern classes $\left(c_{1}, c_{2}\right)=(0,1)$. This bundle is the null correlation bundle on $\mathbb{P}^{3}$.

Example 3.2. Let $R=K\left[x_{0}, x_{1}, x_{2}\right]$ and let $S=K\left[x_{0}, x_{1}, x_{2}, x_{3}\right]$. Consider the following Koszul complex:

$$
0 \rightarrow R(-1) \xrightarrow{\alpha_{2}} 3 R \xrightarrow{\alpha_{1}} 3 R(1) \xrightarrow{\alpha_{0}} R(2)
$$

where

$$
\alpha_{0}=\left(x_{0}, x_{1}, x_{2}\right), \quad \alpha_{1}=\left(\begin{array}{ccc}
-x_{1} & -x_{2} & 0 \\
x_{0} & 0 & -x_{2} \\
0 & x_{0} & x_{1}
\end{array}\right) \quad \text { and } \quad \alpha_{2}=\left(\begin{array}{c}
x_{2} \\
-x_{1} \\
x_{0}
\end{array}\right) .
$$

Let $M=\operatorname{Im}\left(\alpha_{1}\right)=<s_{1}, s_{2}, s_{3}>$. Then $\widetilde{M}$ is the twisted cotangent bundle $\Omega^{1}(2)$. The third row, $t_{1}$ of $\alpha_{1}$, induces a map from $\Omega^{1}(2)$ to $\mathcal{O}(1)$ such that $t_{1} \circ s_{1}=0$. So the composite of $s_{1}(1)$ and $t_{1}$ defines a nilpotent endomorphism $\phi$ of $M$, and hence $\widetilde{M}$. In this case, the standard lifting of $\phi$ is

$$
\phi_{0}=\left(\begin{array}{ccc}
0 & x_{0} & x_{1} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right): 3 R \rightarrow 3 R(1) .
$$

This can be summarized in the following sequence of maps

$$
\cdots \rightarrow R \xrightarrow{\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)} 3 R{ }^{\left(\begin{array}{ccc}
-x_{1} & -x_{2} & 0 \\
x_{0} & 0 & -x_{2} \\
0 & x_{0} & x_{1}
\end{array}\right)} 3 R(1)\left(\begin{array}{ll}
0 \\
& 1
\end{array}\right)_{\longrightarrow}(1) \xrightarrow{\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)} 3 R(1) \rightarrow \ldots
$$

The fact that $t_{1} \circ s_{1}=0$ corresponds to

$$
\left(\begin{array}{lll}
0 & 0 & 1
\end{array}\right)\left(\begin{array}{ccc}
-x_{1} & -x_{2} & 0 \\
x_{0} & 0 & -x_{2} \\
0 & x_{0} & x_{1}
\end{array}\right)\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)=0 .
$$

The map $\phi_{0}: 3 R \rightarrow 3 R(1)$ corresponds to

$$
\phi_{0}=\left(\begin{array}{ccc}
0 & x_{0} & x_{1} \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)=\left(\begin{array}{l}
1 \\
0 \\
0
\end{array}\right)\left(\begin{array}{lll}
0 & 0 & 1
\end{array}\right)\left(\begin{array}{ccc}
-x_{1} & -x_{2} & 0 \\
x_{0} & 0 & -x_{2} \\
0 & x_{0} & x_{1}
\end{array}\right) .
$$

By Proposition 2.4 the corresponding $S$-module $F$ in $\mathfrak{S}_{2}$ has the following minimal presentation:

$$
4 S(-1) \xrightarrow{\beta_{0}} 3 S \rightarrow F \rightarrow 0,
$$

where the first column of $\beta_{0}$ is the presentation matrix for $M$ (i.e. $\alpha_{2}$ ) and the next three columns of $\beta_{0}$ are just the columns of the matrix $\phi_{0}(-1)-x_{3} I$ where I is the $3 \times 3$ identity matrix. Thus,

$$
\beta_{0}=\left(\begin{array}{cccc}
x_{2} & -x_{3} & x_{0} & x_{1} \\
-x_{1} & 0 & -x_{3} & 0 \\
x_{0} & 0 & 0 & -x_{3}
\end{array}\right) .
$$

The other generators $s_{2}$ and $s_{3}$ of $M$ define a homomorphism $\psi: 2 R \rightarrow M$, whose lifting is given by the matrix

$$
\psi_{0}=\left(\begin{array}{ll}
0 & 0 \\
1 & 0 \\
0 & 1
\end{array}\right): 2 R \rightarrow 3 R .
$$

This homomorphism together with the nilpotent endomorphism $\phi(-1)$ of $M(-1)$ yields a homomorphism $(\phi(-1), \psi): M(-1) \oplus 2 R \rightarrow M$. The image $N$ is generated by the columns of the matrix

$$
\left(\begin{array}{ccccc}
0 & -x_{0} x_{1} & -x_{1}^{2} & -x_{2} & 0 \\
0 & x_{0}^{2} & x_{0} x_{1} & 0 & -x_{2} \\
0 & 0 & 0 & x_{0} & x_{1}
\end{array}\right): 3 R(-1) \oplus 2 R \rightarrow 3 R(1) .
$$

The first three columns of the matrix come from $\alpha_{1} \phi_{0}(-1)$ (i.e. multiply $\alpha_{1}$ and $\phi_{0}$ ) and the next two columns come from $\alpha_{1} \psi_{0}$ (i.e. multiply $\alpha_{1}$ and $\psi_{0}$ ). The truncated modules $M_{\geq 1}$ and $N_{\geq 1}$ are isomorphic, so the map of sheaves $(\phi(-1), \psi): \Omega^{1}(1) \oplus 2 \mathcal{O} \rightarrow \Omega^{1}(2)$ is surjective. From Theorem 2.1] it follows that there exists a rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^{3}$ with exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{E} \rightarrow 2 \mathcal{O} \rightarrow \widetilde{F} \rightarrow 0 \tag{3.1}
\end{equation*}
$$

Let $\mathfrak{F}=\left\{f_{1}, f_{2}, f_{3}\right\}$ be the minimal generating set of $F$, where for each $i$, $f_{i}$ corresponds to $s_{i}$. By construction, the surjective map from $2 \mathcal{O}$ to $\widetilde{F}$ in Sequence (3.1) is induced by $f_{2}$ and $f_{3}$. Let $Q$ be the quotient ring $S /\left(x_{3}^{2}\right)$. Then $F$, as a $Q$-module, is generated by

$$
f_{1}=\left(\begin{array}{c}
0 \\
x_{1} x_{3} \\
x_{0} x_{3}
\end{array}\right), \quad f_{2}=\left(\begin{array}{c}
x_{0} x_{3} \\
x_{0} x_{1}+x_{2} x_{3} \\
x_{0}^{2}
\end{array}\right), \quad f_{3}=\left(\begin{array}{c}
x_{1} x_{3} \\
x_{1}^{2} \\
x_{0} x_{1}-x_{2} x_{3}
\end{array}\right)
$$

This can be obtained by transposing the matrix that appears in the first step of a free resolution of $\beta_{0}^{T}$ over $Q$ (i.e. find $\left(\operatorname{Syz}\left(\beta_{0}^{T}\right)\right)^{T}$ over $Q$ ). Let $F^{\prime}$ be the module generated by $f_{2}, f_{3}$. The syzygy module $\operatorname{Syz}\left(f_{2}, f_{3}\right)$ over $Q$ is generated by the columns of the matrix

$$
\left(\begin{array}{ccc}
-x_{1} x_{3} & -x_{1}^{2} & -x_{0} x_{1}+x_{2} x_{3} \\
x_{0} x_{3} & x_{0} x_{1}+x_{2} x_{3} & x_{0}^{2}
\end{array}\right) .
$$

Let $N$ be the extension module of $F^{\prime}$ to $S$. Then $F^{\prime}$ is isomorphic to $N / x_{3}^{2} N$, and hence over $S, F^{\prime}$ has the presentation

$$
\gamma_{0}=\left(\begin{array}{ccccc}
-x_{1} x_{3} & -x_{1}^{2} & -x_{0} x_{1}+x_{2} x_{3} & x_{3}^{2} & 0 \\
x_{0} x_{3} & x_{0} x_{1}+x_{2} x_{3} & x_{0}^{2} & 0 & x_{3}^{2}
\end{array}\right) .
$$

This corresponds to the homomorphism $s: E \rightarrow 2 S$, and hence to the injective sheaf morphism $\mathcal{E} \rightarrow 2 \mathcal{O}$. Let $I$ be the ideal generated by the $2 \times 2$ minors of $\gamma_{0}$. Then $\left(I: x_{3}^{2}\right)$ defines the empty set in $\mathbb{P}^{3}$, which implies by Remark 3.1 that $\mathcal{E}$ is a vector bundle on $\mathbb{P}^{3}$.

By resolving $\gamma_{0}$, we get a minimal free resolution of the following type for E:

$$
\begin{equation*}
0 \rightarrow S(-4) \rightarrow 4 S(-3) \rightarrow 5 S(-2) \rightarrow E \rightarrow 0 \tag{3.2}
\end{equation*}
$$

From Sequence (3.2) it follows that the Chern classes of $\widetilde{E}$ are $c_{1}=-2$ and $c_{2}=2$. So the corresponding normalized bundle is a stable rank two vector bundle on $\mathbb{P}^{3}$ with Chern classes $\left(c_{1}, c_{2}\right)=(0,1)$.

Remark 3.3. A construction almost identical to the one outlined in the previous example can be carried out with $\Omega_{\mathbb{P}^{n}}(2)$ whenever $n$ is even. The construction yields a rank $n$ bundle on $\mathbb{P}^{n+1}$.

In the next example, we will discuss the stable rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^{4}$ over an algebraically closed field $K$ of characteristic two constructed by Kumar 10. He proved the existence of this bundle by constructing a rank three vector bundle on $\mathbb{P}^{3}$ over $K$ that satisfies the conditions in Theorem 2.1 Our main goal is to describe $\mathcal{E}$ explicitly by using the algorithm.

Example 3.4. Let $K$ be an algebraically closed field with characteristic two, let $R=K\left[x_{0}, \ldots, x_{3}\right]$ and let $S=K\left[x_{0}, \ldots, x_{4}\right]$. Consider the module $M$ obtained as the cokernel of the map

$$
\alpha_{0}=\left(\begin{array}{cccc}
0 & 0 & x_{0} x_{1}^{2} & x_{1}^{3} \\
0 & 0 & x_{0}^{3} & x_{0}^{2} x_{1} \\
x_{2}^{2} & x_{3}^{2} & 0 & 0 \\
x_{0} & 0 & 0 & x_{3}^{2} \\
0 & x_{1} & x_{2}^{2} & 0 \\
x_{1} & x_{0} & x_{3}^{2} & x_{2}^{2}
\end{array}\right): 2 R(-4) \oplus 2 R(-5) \rightarrow 3 R(-2) \oplus 3 R(-3)
$$

Let $I_{i}(M)$ be the ideal of $i \times i$ minors of $\alpha_{0}$ (i.e. a Fitting invariant of $M$ ). Then $\sqrt{I_{3}(M)}=(1)$ and $I_{4}(M)=0$. By Fitting's Lemma, the corresponding coherent sheaf $\widetilde{M}$ is a rank three vector bundle on $\mathbb{P}^{3}$. Let $\phi_{0}$ be the homomorphism from $3 R(-3) \oplus 3 R(-4)$ to $3 R(-2) \oplus 3 R(-3)$ given by

$$
\left(\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x_{1}^{2} & x_{0}^{2} & x_{0} x_{1} \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

It is easy to check that $\phi_{0}$ induces a nilpotent endomorphism $\phi$ of $M$, and hence of $\widetilde{M}$, whose third power is zero. Therefore, $M$ corresponds to an $S$-module $F$ in $\mathfrak{S}_{3}$. Let $\mathfrak{M}=\left\{g_{i}\right\}_{1 \leq i \leq 6}$ be a minimal generating set of $M$. Then $F$ is obtained as the following set:

$$
F=\left\{a_{1} g_{1}+\cdots+a_{6} g_{6} \mid a_{i} \in S \text { for each } i=1, \ldots, 6\right\}
$$

The relations among $g_{i}$ 's in $S$ are, by Proposition 2.4, given by the matrix

$$
\left(\alpha_{0} \mid \phi_{0}-x_{4}\right)=\left(\begin{array}{cccccccccc}
0 & 0 & x_{0} x_{1}^{2} & x_{1}^{3} & x_{4} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_{0}^{3} & x_{0}^{2} x_{1} & 0 & x_{4} & 0 & 0 & 0 & 0 \\
x_{2}^{2} & x_{3}^{2} & 0 & 0 & 0 & 0 & x_{4} & x_{1}^{2} & x_{0}^{2} & x_{0} x_{1} \\
x_{0} & 0 & 0 & x_{3}^{2} & 0 & 1 & 0 & x_{4} & 0 & 0 \\
0 & x_{1} & x_{2}^{2} & 0 & 1 & 0 & 0 & 0 & x_{4} & 0 \\
x_{1} & x_{0} & x_{3}^{2} & x_{2}^{2} & 0 & 0 & 0 & 0 & 0 & x_{4}
\end{array}\right)
$$

From the "ones" in this matrix, it follows that the minimal set of generators for $F$ consists of $g_{1}, g_{2}, g_{3}$ and $g_{6}$. Eliminating the redundant elements $g_{4}$ and $g_{5}$, we obtain a minimal free presentation of $F$ :

$$
S(-3) \oplus 5 S(-4) \oplus 2 S(-5) \xrightarrow{\beta_{0}} 3 S(-2) \oplus S(-3) \rightarrow F \rightarrow 0
$$

where

$$
\beta_{0}=\left(\begin{array}{cccccccc}
0 & x_{4}^{2} & 0 & 0 & 0 & x_{1} x_{4} & x_{0} x_{1}^{2}+x_{2}^{2} x_{4} & x_{1}^{3} \\
0 & 0 & 0 & x_{4}^{2} & x_{0} x_{4} & 0 & x_{0}^{3} & x_{0}^{2} x_{1}+x_{3}^{2} x_{4} \\
x_{4} & x_{0}^{2} & x_{0} x_{1} & x_{1}^{2} & x_{2}^{2} & x_{3}^{3} & 0 & 0 \\
0 & 0 & x_{4} & 0 & x_{1} & x_{0} & x_{3}^{2} & x_{2}^{2}
\end{array}\right)
$$

Next we define a homomorphism $\psi_{0}$ from $2 R(-2)$ to $3 R(-2) \oplus 3 R(-3)$ by

$$
\left(\begin{array}{llllll}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{array}\right)^{T}
$$

This represents a homomorphism $\psi$ from $2 R(-2)$ to $M$. The cokernel $C$ of $(\phi, \psi)$ has the presentation matrix $\left(\phi_{0}, \psi_{0}, \alpha_{0}\right)$. Minimizing the generators and the corresponding relations, we obtain the following presentation matrix of $C$ :

$$
\left(\begin{array}{ccccccc}
x_{0}^{2} & x_{0} x_{1} & x_{1}^{2} & x_{2}^{2} & x_{3}^{2} & 0 & 0 \\
0 & 0 & 0 & x_{1} & x_{0} & x_{2}^{2} & x_{3}^{2}
\end{array}\right)
$$

Clearly $C$ is an $R$-module of finite length. From Theorem 2.1] it follows that there exist a rank two vector bundle $\mathcal{E}$ on $\mathbb{P}^{4}$ and an exact sequence

$$
\begin{equation*}
0 \rightarrow \mathcal{E} \rightarrow 2 \mathcal{O}(-2) \rightarrow \widetilde{F} \rightarrow 0 \tag{3.3}
\end{equation*}
$$

By construction, the surjective map $2 \mathcal{O}(-2) \rightarrow \widetilde{F}$ in Sequence (3.3) is defined by $g_{1}$ and $g_{2}$. Let $Q$ be the quotient ring $S /\left(x_{4}^{3}\right)$. Let $P=x_{0}^{4} x_{1}^{2}+$ $x_{0}^{2} x_{1} x_{3}^{2} x_{4}+x_{3}^{4} x_{4}^{2} x_{0} x_{1}^{5}+x_{2}^{2} x_{4}\left(x_{0}^{3}+x_{1}^{3}\right)$. Then $\operatorname{Syz}\left(g_{1}, g_{2}\right)$ is generated by the
columns of the matrix $\gamma_{0}$ with:

$$
\gamma_{0}^{T}=\left(\begin{array}{cc}
x_{0}^{2} x_{4}^{2} & x_{1}^{2} x_{4}^{2} \\
x_{0}^{4} x_{4} & x_{0}^{2} x_{1}^{2} x_{4}+x_{0} x_{2}^{2} x_{4}^{2}+x_{1} x_{3}^{2} x_{4}^{2} \\
x_{0}^{3} x_{1} x_{4}+x_{0} x_{3}^{2} x_{4}^{2} & x_{0} x_{1}^{3} x_{4}+x_{1} x_{2}^{2} x_{4}^{2} \\
x_{0}^{2} x_{1}^{2} x_{4}+x_{0} x_{2}^{2} x_{4}^{2}+x_{1} x_{3}^{2} x_{4}^{2} & x_{1}^{4} x_{4} \\
x_{0}^{3} x_{2}^{2}+x_{0}^{2} x_{1} x_{3}^{2}+x_{3}^{4} x_{4} & x_{0} x_{1}^{2} x_{2}^{2}+x_{1}^{3} x_{3}^{2}+x_{2}^{4} x_{4} \\
x_{0}^{4} x_{1}^{2}+x_{0}^{3} x_{2}^{2} x_{4}+x_{0}^{2} x_{1} x_{3}^{2} x_{4} & x_{0}^{2} x_{1}^{4}+x_{2}^{4} x_{4}^{2} \\
x_{0}^{5} x_{1}+x_{0}^{3} x_{3}^{2} x_{4} & x_{0}^{3} x_{1}^{3}+x_{0}^{2} x_{1} x_{2}^{2} x_{4}+x_{2}^{2} x_{3}^{2} x_{4}^{2} \\
x_{0}^{2} & x_{0}^{3} x_{1}^{3}+x_{0}^{2} x_{1} x_{2}^{2} x_{4}+x_{2}^{2} x_{3}^{2} x_{4}^{2} \\
x_{0}^{3} x_{1}^{3}+x_{0} x_{1}^{2} x_{3}^{2} x_{4}+x_{2}^{2} x_{3}^{2} x_{4}^{2} & P \\
x_{0}^{2} x_{1}^{4}+x_{0} x_{1}^{2} x_{2}^{2} x_{4}+x_{1}^{3} x_{3}^{2} x_{4}+x_{2}^{4} x_{4}^{2} & x_{1}^{6}
\end{array}\right) .
$$

Let $N$ denote the extension module of $\operatorname{Syz}\left(g_{1}, g_{2}\right)$ to $S$. Since $\operatorname{Syz}\left(g_{1}, g_{2}\right)$ can be identified with $N / x_{4}^{3} N, \operatorname{Syz}\left(g_{1}, g_{2}\right)$ has, as an $S$-module, the minimal free presentation $\gamma=\left(\begin{array}{ll}\gamma_{0} & \gamma_{1}\end{array}\right)$, where

$$
\gamma_{1}=\left(\begin{array}{cc}
x_{4}^{3} & 0 \\
0 & x_{4}^{3}
\end{array}\right) .
$$

This corresponds to an injective sheaf morphism $s: \mathcal{E} \rightarrow 2 \mathcal{O}(-2)$, whose cokernel equals $\widetilde{F}$.

Let $I$ be the ideal generated by $2 \times 2$ minors of $\gamma$. Then the ideal quotient $\left(I: x_{4}^{3}\right)$ defines the empty subset of $\mathbb{P}^{4}$. By Remark 3.1$] \mathcal{E}$ is a rank two vector bundle on $\mathbb{P}^{4}$. The Chern classes of $\mathcal{E}$ are $c_{1}=-7$ and $c_{2}=16$. These can be computed in the same way as in Example 3.2.

As a final example, we will illustrate how to determine the triple $(\mathcal{M}, \phi, \psi)$ from the pair $(\mathcal{E}, s)$. In general, this direction is easier to carry out with the main difficulty coming from producing the pair $(\mathcal{E}, s)$. We will discuss the Horrocks-Mumford bundle utilizing the ideas of Kaji to produce the sections $s$ required in the correspondence 9 .

Example 3.5. Let $V$ be a five-dimensional vector space with basis $\left\{e_{0}, \ldots, e_{4}\right\}$ over $K$, let $W$ be its dual and let $\mathbb{P}^{4}=\mathbb{P}(V)$ be the projective space of lines in $V$. The homogeneous coordinate ring $K\left[x_{0}, \ldots, x_{4}\right]$ of $\mathbb{P}^{4}$ will be denoted by $S$. Consider the Koszul complex resolving $K=S /\langle W\rangle$ :

$$
0 \rightarrow \bigwedge^{5} W \otimes S(-5) \xrightarrow{\beta_{4}} \cdots \xrightarrow{\beta_{1}} \bigwedge^{1} W \otimes S(-1) \xrightarrow{\beta_{0}} \bigwedge^{0} W \otimes S \rightarrow K \rightarrow 0
$$

Recall that the $i^{t h}$ bundle of differentials $\Omega^{i}=\Omega_{\mathbb{P}^{4}}^{i}$ is obtained as a sheafication of the syzygy module $\mathrm{Syz}_{i+1}(K)$. By choosing appropriate bases for $\bigwedge^{2} W$ and $\bigwedge^{3} W$, we may suppose that $\operatorname{Syz}_{3}(K)$ is generated by the columns
of the following matrix:

$$
\beta_{2}=\left(\begin{array}{cccccccccc}
x_{2} & x_{3} & 0 & 0 & x_{4} & 0 & 0 & 0 & 0 & 0 \\
-x_{1} & 0 & x_{3} & 0 & 0 & x_{4} & 0 & 0 & 0 & 0 \\
x_{0} & 0 & 0 & x_{3} & 0 & 0 & x_{4} & 0 & 0 & 0 \\
0 & -x_{1} & -x_{2} & 0 & 0 & 0 & 0 & x_{4} & 0 & 0 \\
0 & x_{0} & 0 & -x_{2} & 0 & 0 & 0 & 0 & x_{4} & 0 \\
0 & 0 & x_{0} & x_{1} & 0 & 0 & 0 & 0 & 0 & x_{4} \\
0 & 0 & 0 & 0 & -x_{1} & -x_{2} & 0 & -x_{3} & 0 & 0 \\
0 & 0 & 0 & 0 & x_{0} & 0 & -x_{2} & 0 & -x_{3} & 0 \\
0 & 0 & 0 & 0 & 0 & x_{0} & x_{1} & 0 & 0 & -x_{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & x_{0} & x_{1} & x_{2}
\end{array}\right) .
$$

The natural duality $\bigwedge^{p} V \otimes \bigwedge^{p} W \rightarrow K$ extends to a contraction map

$$
\bigwedge^{p} V \otimes \bigwedge^{q} W \rightarrow\left\{\begin{array}{lc}
\bigwedge^{p-q} V & \text { if } p \geq q \\
\bigwedge^{q-p} W & \text { otherwise }
\end{array}\right.
$$

Using this, the linear transformation

$$
\left(\begin{array}{ccccc}
e_{2} \wedge e_{3} & e_{0} \wedge e_{4} & e_{1} \wedge e_{2} & -e_{3} \wedge e_{4} & e_{0} \wedge e_{1} \\
e_{1} \wedge e_{4} & e_{1} \wedge e_{3} & e_{0} \wedge e_{3} & e_{0} \wedge e_{2} & -e_{2} \wedge e_{4}
\end{array}\right)
$$

from $5 \bigwedge^{5} W$ to $2 \bigwedge^{2} W$ induces a sheaf morphism, $A$, from $5 \bigwedge^{5} W \otimes \mathcal{O}(-1)$ to $2 \Omega^{2}(2)$. The matrix representation $A_{0}$ of this morphism with respect to the fixed bases for $\bigwedge^{2} W$ and $\bigwedge^{3} W$ is

$$
\left(\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) .
$$

Let $\beta=\left(\begin{array}{cc}\beta_{2} & 0 \\ 0 & \beta_{2}\end{array}\right)$. One can show that the ideal generated by the maximal minors of the composite of $\beta$ and $A_{0}$ defines the empty set, and thus $A$ is injective as a bundle map. Let $B_{0}=A_{0}^{T} \cdot\left(\begin{array}{cc}0 & I_{10} \\ -I_{10} & 0\end{array}\right)$ where $I_{10}$ is the $10 \times 10$ identity matrix. The matrix $B_{0}$ gives rise to a sheaf morphism $B$ from $2 \Omega^{2}(2)$ to $5 \bigwedge^{0} W \otimes \mathcal{O}$. This sheaf morphism is surjective as a bundle map (since $A$ is injective). $A$ and $B$ can be thought of as the differentials of the following complex:

$$
5 \bigwedge^{5} W \otimes \mathcal{O}(-1) \xrightarrow{A} 2 \Omega^{2}(2) \xrightarrow{B} 5 \bigwedge^{0} W \otimes \mathcal{O}
$$

Since $A$ is an injective bundle map and $B$ is a surjective bundle map the homology, $\mathcal{E}=\operatorname{Ker} B / \operatorname{Im} A$, is a rank two vector bundle on $\mathbb{P}^{4}$. This vector bundle is known as the Horrocks-Mumford bundle, is indecomposable and has Chern classes $c_{1}=-1$ and $c_{2}=4$.

Consider the following $20 \times 1$ matrices $v_{1}$ and $v_{2}$ (discovered by Kaji 9])

$$
v_{1}=\left(0, B_{2} .0,0,0, B_{6}, 0,0,0,0, B_{11}, B_{12}, 0,0, B_{15}, 0,0, B_{18}, 0,0\right)^{T}
$$

$$
v_{2}=\left(0, C_{2}, C_{3}, 0,0, C_{6}, 0,0,0,0, C_{11}, C_{12}, 0,0,0,0,0, C_{18}, 0,0\right)^{T}
$$

where

$$
\begin{array}{ll}
B_{2}=-x_{0}^{5} x_{1}-x_{0} x_{1}^{2} x_{2} x_{3} x_{4}-x_{0}^{3} x_{3} x_{4}^{2} & C_{2}=-x_{0}^{5} x_{3}^{2}-x_{0}^{3} x_{2}^{2} x_{3} x_{4}-x_{0} x_{1} x_{2} x_{3}^{3} x_{4} \\
B_{6}=-x_{0}^{3} x_{1}^{2} x_{2}-x_{0}^{5} x_{4} & C_{3}=-x_{0}^{7} \\
B_{11}=x_{0}^{4} x_{1}^{2}+x_{1}^{3} x_{2} x_{3} x_{4}+x_{0}^{2} x_{1} x_{3} x_{4}^{2} & C_{6}=-x_{0}^{5} x_{2}^{2}-x_{0}^{3} x_{1} x_{2} x_{3}^{2} \\
B_{12}=-x_{1}^{3} x_{2}^{2} x_{4}-x_{0}^{2} x_{1} x_{2} x_{4}^{2} & C_{11}=x_{0}^{6} x_{2}+x_{0}^{4} x_{1} x_{3}^{2}+x_{0}^{2} x_{1} x_{2}^{2} x_{3} x_{4} \\
& \\
& +x_{1}^{2} x_{2} x_{3}^{3} x_{4} \\
B_{15}=x_{0}^{6} & C_{12}=-x_{0}^{2} x_{1} x_{2}^{3} x_{4}-x_{1}^{2} x_{2}^{2} x_{3}^{2} x_{4} \\
B_{18}=x_{0}^{2} x_{1}^{2} x_{2} x_{4}+x_{0}^{4} x_{4}^{2} & C_{18}=x_{0}^{6} x_{3}+x_{0}^{4} x_{2}^{2} x_{4}+x_{0}^{2} x_{1} x_{2} x_{3}^{2} x_{4}
\end{array}
$$

The matrix $v_{1}$ represents a global section $s_{1}$ of $2 \Omega^{2}(9)$; while $v_{2}$ represents a global section $s_{2}$ of $2 \Omega^{2}(10)$. Both $v_{1}$ and $v_{2}$ can be written as $S$-linear combinations of the columns of $\operatorname{Syz}\left(\beta \circ B_{0}\right)$, thus $s_{1}$ and $s_{2}$ correspond to global sections $\widetilde{s}_{1}$ and $\widetilde{s}_{2}$ of $\mathcal{E}(7)$ and $\mathcal{E}(8)$ respectively. Both $\widetilde{s}_{1}$ and $\widetilde{s}_{2}$ are nonzero and together generate $\mathcal{E}$ on $D_{+}\left(x_{0}\right)$. Indeed, if $I$ is the ideal generated by the maximal minors of the matrix $\left(v_{1}, v_{2}, A_{0}\right)$ then the saturation of $I$ with respect to $x_{0}$ determines the locus of points, not on $H$, where $s_{1}$ and $s_{2}$ do not generate $\mathcal{E}\left(H\right.$ is the hyperplane defined by $\left.x_{0}=0\right)$. An easy computation establishes that $V\left(I:\left(x_{0}\right)^{\infty}\right)=V((1))=\emptyset$.

The global sections $\widetilde{s}_{1}$ and $\widetilde{s}_{2}$ can be identified with a sheaf morphism $s=\left(s_{1}, s_{2}\right)$ from $\mathcal{O}(-8) \oplus \mathcal{O}(-7)$ to $\mathcal{E}$. Recall that $\mathcal{E}^{\vee}$ is isomorphic to $\mathcal{E}\left(c_{1}\right)$ (since $\mathcal{E}$ is a rank 2 reflexive sheaf). Taking the transpose of $s$ we obtain the following short exact sequence:

$$
0 \rightarrow \mathcal{E}(-1) \stackrel{s^{\vee}}{\rightarrow} \mathcal{O}(7) \oplus \mathcal{O}(8) \rightarrow \mathcal{F} \rightarrow 0
$$

Since $\widetilde{s}_{1} \wedge \widetilde{s}_{2} \in \mathrm{H}^{0}\left(\mathbb{P}^{4}, \mathcal{E}(7) \wedge \mathcal{E}(8)\right) \simeq \mathrm{H}^{0}\left(\mathbb{P}^{4}, \mathcal{O}(14)\right)$ and since $s_{1}, s_{2}$ generate $\mathcal{E}$ away from $H$, the sheaf $\mathcal{F}$ can be considered as a coherent sheaf on the 14th infinitesimal neighborhood $H_{14}$ of $H$. Let $\pi$ be the finite morphism from $H_{14}$ to $H$ induced by the projection $\mathbb{P}^{4} \backslash P \rightarrow H$ from a point $P$ off $H$. Then the direct image sheaf of $\mathcal{F}$ by $\pi$ is a rank fourteen vector bundle on $H \simeq \mathbb{P}^{3}$. We denote this bundle by $\mathcal{M}$.

Let $R$ be the quotient ring $S /\left(x_{0}\right)$ and let $F$ be the graded $T$-module $\mathrm{H}_{*}^{0} \mathcal{F}$. Then the graded $R$-module $M=\mathrm{H}_{*}^{0} \mathcal{M}$ is the graded $R$-module ${ }_{R} F$ obtained from $F$ by restriction of scalars. It is straightforward to determine that $F$ has a minimal free presentation of the following form:

$$
15 S \xrightarrow{P} S(8) \oplus S(7) \oplus 5 S(1) \rightarrow F \rightarrow 0
$$

Let $\mathfrak{F}=\left\{f_{i}\right\}_{1 \leq i \leq 7}$ be the minimal generating set of $F$. Then it follows from Proposition 2.2 that $\mathfrak{M}=\left\{x_{0}^{i} f_{j} \mid 0 \leq i \leq 13,1 \leq j \leq 7\right\}$ is a set of generators for $M$. The relations among these generators of $M$ can be derived from the presentation matrix $P$ of $F$. Let $P[:, k]$ be the $k^{t h}$ column of $P$ and let $Q$ be the presentation matrix of $M$ with respect to $\mathfrak{M}$. For each
$1 \leq k \leq 15$, we have a syzygy of the form

$$
\sum_{i=1}^{7} P[i, k] f_{i}=0
$$

Then, since

$$
P[i, k]=\sum_{t=0}^{13} Q[7 t+i, k] x_{0}^{t}
$$

we can obtain the entries of $Q[:, k]$ from the entries of $P[:, k]$.
Choosing appropriate bases for $F_{0}$ and $F_{1}$, one can explicitly write $P$. For example, the first column of $P$ is

$$
P[:, 1]=\left(P[1,1] \quad P[2,1] \quad x_{3} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0\right)^{T}
$$

where

$$
\begin{aligned}
P[1,1]= & x_{0}^{6} x_{2}^{2}-x_{1}^{3} x_{2}^{4} x_{3}+2 x_{0}^{2} x_{1} x_{2}^{3} x_{3} x_{4}+x_{0} x_{1}^{4} x_{3}^{2} x_{4}-3 x_{1}^{2} x_{2}^{2} x_{3}^{3} x_{4} \\
& -x_{0}^{2} x_{2} x_{3}^{3} x_{4}^{2}+x_{1} x_{3}^{5} x_{4}^{2}-x_{0} x_{1}^{2} x_{2} x_{3} x_{4}^{3}-x_{2}^{3} x_{3}^{2} x_{4}^{3}+x_{0}^{3} x_{3} x_{4}^{4} \\
P[2,1]= & x_{0}^{4} x_{1}^{2} x_{2}-x_{0}^{3} x_{2}^{3} x_{3}+x_{0} x_{1} x_{2}^{2} x_{3}^{3}+x_{0}^{6} x_{4}+x_{1}^{3} x_{2}^{2} x_{3} x_{4}-x_{0} x_{3}^{5} x_{4} \\
& +2 x_{0}^{2} x_{1} x_{2} x_{3} x_{4}^{2}+x_{1}^{2} x_{3}^{3} x_{4}^{2}-x_{2} x_{3}^{2} x_{4}^{4}
\end{aligned}
$$

We have $P[1,1]=Q[1,1]+Q[8,1] x_{0}+Q[15,1] x_{0}^{2}+Q[22,1] x_{0}^{3}+Q[36,1] x_{0}^{6}$, where

$$
\begin{aligned}
Q[1,1] & =-x_{1}^{3} x_{2}^{4} x_{3}-3 x_{1}^{2} x_{2}^{2} x_{3}^{3} x_{4}+x_{1} x_{3}^{5} x_{4}^{2}-x_{2}^{3} x_{3}^{2} x_{4}^{3} \\
Q[8,1] & =x_{1}^{4} x_{3}^{2} x_{4}-x_{1}^{2} x_{2} x_{3} x_{4}^{3} \\
Q[15,1] & =2 x_{1} x_{2}^{3} x_{3} x_{4}-x_{2} x_{3}^{3} x_{4}^{2} \\
Q[22,1] & =x_{3} x_{4}^{4} \\
Q[36,1] & =x_{2}^{2}
\end{aligned}
$$

Likewise,
$P[2,1]=Q[2,1]+Q[9,1] x_{0}+Q[16,1] x_{0}^{2}+Q[23,1] x_{0}^{3}+Q[30,1] x_{0}^{4}+Q[37,1] x_{0}^{6}$ where

$$
\begin{aligned}
Q[2,1] & =x_{1}^{3} x_{2}^{2} x_{3} x_{4}+x_{1}^{2} x_{3}^{3} x_{4}^{2}-x_{2} x_{3}^{2} x_{4}^{4} \\
Q[9,1] & =x_{1} x_{2}^{2} x_{3}^{3}-x_{3}^{5} x_{4} \\
Q[16,1] & =2 x_{1} x_{2} x_{3} x_{4}^{2} \\
Q[23,1] & =-x_{2}^{3} x_{3} \\
Q[30,1] & =x_{1}^{2} x_{2} \\
Q[37,1] & =x_{4}
\end{aligned}
$$

Finally, $Q[3,1]=x_{3}$ is the remaining nonzero entry in $Q[:, 1]$ (since $P[i, 1]=$ 0 for $4 \leq i \leq 7$ ).

Working our way through the other columns of $P$, the entire matrix $Q$ can be obtained (and has $98=14 \cdot 7$ rows and 15 columns). Upon obtaining $Q$,
one finds that $Q[12,6], Q[10,8], Q[11,10], Q[13,14], Q[14,15], Q[51,12]$ and $Q[57,13]$ are the only entries of $Q$ which are constant and nonzero. Furthermore, each of $\left\{x_{0}^{i} f_{j} \mid i \geq 1, j \geq 3\right\},\left\{x_{0}^{i} f_{1} \mid i \geq 8\right\}$ and $\left\{x_{0}^{i} f_{2} \mid i \geq 7\right\}$, can be written as $R$-linear combinations of

$$
\mathbf{G}=\left\{f_{1}, f_{2}, \ldots, f_{7}\right\} \cup\left\{x_{0}^{i} f_{1} \mid 1 \leq i \leq 7\right\} \cup\left\{x_{0}^{i} f_{2} \mid 1 \leq i \leq 6\right\} .
$$

These linear combinations give rise to the standard nilpotent endomorphism of $M$. Let $g_{j}$ denote the $j^{\text {th }}$ entry of $\mathbf{G}$ for $1 \leq j \leq 20$ and let

$$
M_{0} \xrightarrow{\mathbf{G}} M \rightarrow 0
$$

be the map associated to the minimal set of generators of $M$. Each $x_{0} g_{i}$ can be written as an $R$-linear combination of $g_{1}, \ldots, g_{20}$ :

$$
x_{0} g_{i}=\sum_{j=1}^{20} a_{i j} g_{j}
$$

The matrix $\left(a_{i j}\right)_{1 \leq i, j \leq 20}$ is the standard lifting of the standard nilpotent endomorphism $\phi$ of $M$ (see Remark 2.3). By construction, the first two generators $g_{1}$ and $g_{2}$ of $M$ form a homomorphism $\psi$ from $R(7) \oplus R(8)$ to $M$ such that the cokernel of $(\phi[-1], \psi): M(-1) \oplus R(7) \oplus R(8) \rightarrow M$ is a finite-length $R$-module.

It is interesting to note that the rank fourteen vector bundle $\mathcal{M}$ can be written as the direct sum of nine line bundles and an indecomposable rank five vector bundle.

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