# The strength of replacement in weak arithmetic

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#### **Abstract**

The replacement (or collection or choice) axiom scheme  $BB(\Gamma)$  asserts bounded quantifier exchange as follows:

$$\forall i < |a| \exists x < a \phi(i, x) \rightarrow \exists w \forall i < |a| \phi(i, [w]_i)$$

where  $\phi$  is in the class  $\Gamma$  of formulas. The theory  $S_2^1$  proves the scheme  $\mathrm{BB}(\Sigma_1^b)$ , and thus in  $S_2^1$  every  $\Sigma_1^b$  formula is equivalent to a strict  $\Sigma_1^b$  formula (in which all non-sharply-bounded quantifiers are in front). Here we prove (sometimes subject to an assumption) that certain theories weaker than  $S_2^1$  do not prove either  $\mathrm{BB}(\Sigma_1^b)$  or  $\mathrm{BB}(\Sigma_0^b)$ . We show (unconditionally) that  $V^0$  does not prove  $\mathrm{BB}(\Sigma_0^B)$ , where  $V^0$  (essentially  $\mathrm{I}\Sigma_0^{1,b}$ ) is the two-sorted theory associated with the complexity class  $\mathrm{AC}^0$ . We show that PV does not prove  $\mathrm{BB}(\Sigma_0^b)$ , assuming that integer factoring is not possible in probabilistic polynomial time. Johannsen and Pollet introduced the theory  $C_2^0$  associated with the complexity class  $\mathrm{TC}^0$ , and later introduced an apparently weaker theory  $\Delta_1^b - \mathrm{CR}$  for the same class. We use our methods to show that  $\Delta_1^b - \mathrm{CR}$  is indeed weaker than  $C_2^0$ , assuming that RSA is secure against probabilistic polynomial time attack.

Our main tool is the KPT witnessing theorem.

### 1 Introduction

We are concerned with the strength of various theories of bounded arithmetic associated with the complexity classes P, TC<sup>0</sup>, and AC<sup>0</sup>. Our goal is to show that some of these theories cannot prove replacement, which is the axiom scheme

$$\forall i < |a| \exists x < a \,\phi(i, x) \to \exists w \,\forall i < |a| \,\phi(i, [w]_i). \tag{1}$$

(where  $\phi(i,x)$  can have other free variables). We use BB( $\Gamma$ ) to denote replacement for all formulas  $\phi$  in a class  $\Gamma$  (usually  $\Sigma_0^b$  or  $\Sigma_1^b$ ). Replacement is also sometimes known as "collection" (eg. [11]) or "choice" (eg. [20]). We begin by briefly describing the main theories of interest.

The language of first order arithmetic that we use is

$$\{0, 1, +, \cdot, <, |x|, (x)_i, [x]_i, x \# y\}.$$

Here |x| is the length of x in binary notation,  $(x)_i$  is the ith bit of x,  $[x]_i$  is the ith element of the sequence coded by x, and x # y is  $2^{|x| \cdot |y|}$ . All our theories in this language are assumed to include a set of axioms BASIC fixing the algebraic properties of these symbols; see [2, 11] for more detail.

In the first order setting we will look at  $BB(\Sigma_0^b)$ , or "sharply bounded replacement". A sharply bounded or  $\Sigma_0^b$  formula is one in which every quantifier is bounded by a term of the form |t|. A  $\Sigma_1^b$  formula is a sharply bounded formula preceded by a mixture of bounded existential and sharply bounded universal quantifiers. A strict  $\Sigma_1^b$  formula is a sharply bounded formula preceded by a block of bounded existential quantifiers.

The strongest theory we look at is  $S_2^1$  [2], defined as BASIC together with "length induction", that is the LIND axiom

$$\phi(0) \land \forall x < |a| (\phi(x) \to \phi(x+1)) \to \phi(|a|)$$
 (2)

for all  $\Sigma_1^b$  formulas  $\phi$ .

 $S_2^1$  proves BB( $\Sigma_1^b$ ), and hence for every  $S_2^1$ -formula  $\phi$  there is a strict- $\Sigma_1^b$  formula  $\phi'$  such that  $S_2^1$  proves ( $\phi \leftrightarrow \phi'$ ). This fact may have influenced Buss's [2] original decision not to choose strict  $\Sigma_i^b$  as the standard definition of  $\Sigma_i^b$ . The general definition allows Buss to prove [2] Thm 2.2 showing that if a theory  $T^+$  extends T by adding  $\Sigma_1^b$ -defined function symbols then  $\Sigma_1^b$  formulas in the extended language are provably equivalent to  $\Sigma_1^b$  formulas in the original language. This result may not hold if  $\Sigma_1^b$  is taken to be strict  $\Sigma_1^b$  and T does not prove replacement. We show here that certain weaker

theories (likely) do not prove replacement. For these theories, strict  $\Sigma_1^b$  is a more appropriate definition, and extensions by  $\Sigma_1^b$ -defined functions must be handled with care.

The first order theory we will use most often is PV [4] (called  $PV_1$  in [11] and QPV in [5]). This is defined by expanding our language to include a function symbol for every polynomial time algorithm, introduced inductively by Cobham's limited recursion on notation. These are called PV functions, and quantifier free formulas in this language are PV formulas. One way to axiomatize PV is BASIC plus universal axioms defining the new function symbols plus the induction scheme IND

$$\phi(0) \land \forall x < a (\phi(x) \to \phi(x+1)) \to \phi(a)$$

for open formulas  $\phi(x)$ . However it is an important fact that PV is a universal theory, and can be axiomatized by its universal consequences [2, 5].

PV and  $S_2^1$  are closely linked to the complexity class P. The provably total  $\Sigma_1^b$  (or even strict  $\Sigma_1^b$ ) functions in these theories are precisely the polynomial time functions.  $S_2^1$  is  $\Sigma_1^b$ -conservative over PV [2], but PV cannot prove the  $\Sigma_1^b$ -LIND axiom scheme (2) for  $S_2^1$  unless the polynomial hierarchy (provably) collapses [13, 3, 20].

First order theories are unsuitable for dealing with very weak complexity classes such as  $AC^0$ , in which we cannot even define multiplication of strings. In this setting it is more natural to work with a two-sorted or "second order" theory.  $V^0$  is the theory described in the Notes [6], page 56. It is based on  $\Sigma_0^p$ -comp [20] and is essentially the same as  $I\Sigma_0^{1,b}$ . The two sorts are numbers and strings (finite sets of numbers). The axioms consist of number axioms giving the basic properties of  $0, 1, +, \cdot, \leq$ , two axioms defining the "length" |X| of a finite set X to be 1 plus the largest element in X, or 0 if X is empty, and the comprehension scheme for  $\Sigma_0^B$  formulas. The  $\Sigma_0^B$  formulas allow bounded number quantifiers, but no string quantifiers, and represent precisely the uniform  $AC^0$  relations on their free string variables.

If we add to  $V^0$  a function  $X \cdot Y$  for string multiplication, we get a theory equivalent to the first order theory  $\Sigma_0^b$  – LIND. The number sort would correspond to sharply bounded numbers and the string sort to "large" numbers; the  $\Sigma_0^B$  induction available in  $V^0$  would correspond to  $\Sigma_0^b$  – LIND.

With this correspondence (known as RSUV isomorphism [18, 17]) in mind, we consider  $V^0$  and the first order fragments of  $S_2^1$  as fitting naturally into one hierarchy of theories of bounded arithmetic. The only differences

between the two approaches will be in the notation for strings and sequences.  $(z)_i = 1$  in the first order setting corresponds to Z(i) or  $i \in Z$  in the second order setting;  $[z]_i$  corresponds to  $Z^{[i]}$  (see next paragraph).

In second order bounded arithmetic the replacement scheme (1) becomes

$$\forall i < n \; \exists X < n \; \phi(i, X) \rightarrow \exists W \; \forall i < n \; \phi(i, W^{[i]}).$$

Here  $\exists X < n \phi$  stands for  $\exists X(|X| < n \wedge \phi)$  and  $W^{[i]}(u)$  is formally  $W(\langle i, u \rangle)$  where  $\langle i, u \rangle$  is a standard pairing function (so  $W^{[i]}$  is row i in the two-dimensional bit array W).

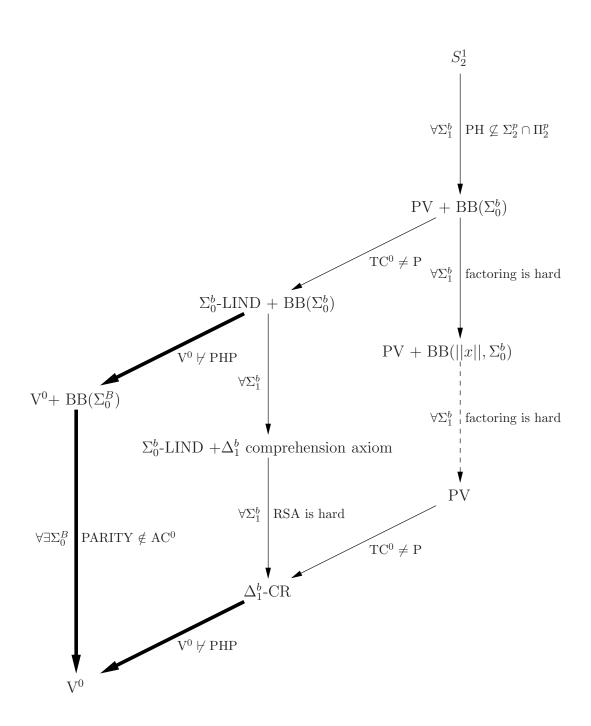
Our main results are that  $V^0$  does not prove  $\Sigma_0^B$  replacement (unconditionally) and that, unless integer factoring is possible in probabilistic polynomial time, PV does not prove  $\Sigma_0^b$  replacement. (As mentioned above,  $S_2^1$  does prove  $\Sigma_0^b$  replacement.)

We summarize our results with a picture of the structure of theories between  $S_2^1$  and  $V_0$ . An arrow on the diagram represents inclusion. To the right of an arrow we give a sufficient condition for the two theories to be distinct. A bold arrow indicates that this condition is true, and that the theories in fact are distinct. To the left of an arrow we show the conservativity between the two theories.

We will begin with the bottom of the diagram. We have already talked about  $V^0$  and PV.  $\Delta_1^b - CR$  was introduced by Johannsen and Pollett in [10] to correspond to the complexity class  $TC^0$  of constant-depth circuits with threshold gates. The  $\Sigma_1^b$  functions provably total in  $\Delta_1^b - CR$  are precisely the uniform  $TC^0$  functions. The theory is defined as the closure of the BASIC axioms and the LIND axioms for open formulas under the normal rules of logical deduction together with the  $\Delta_1^b$ -comprehension rule: if we can prove that a  $\Sigma_1^b$  formula  $\phi(x)$  is equivalent to a  $\Pi_1^b$  formula  $\psi(x)$ , then are allowed to introduce comprehension for  $\phi$ ,

$$\exists w \, \forall i < |a|, (w)_i = 1 \leftrightarrow \phi(i).$$

 $\Delta_1^b$  – CR proves induction for sharply bounded formulas, so we can think of  $V^0$  as a subtheory of it. In fact [14] defines an extension VTC<sup>0</sup> of  $V^0$  by adding an axiom for the function NUMONES(X) (which counts the number of 1's in the string X) and proves VTC<sup>0</sup> is RSUV isomorphic to  $\Delta_1^b$  – CR. But VTC<sup>0</sup> proves the pigeonhole principle, as represented by a  $\Sigma_0^B$  formula PHP(X, n) [14], and  $V^0$  does not [6]. Hence  $\Delta_1^b$  – CR is strictly stronger than  $V^0$ .



The  $\Delta_1^b$ -comprehension rule is a derived rule of PV. This is because by results in [2] if a formula  $\phi$  is provably  $\Delta_1^b$  in PV, then PV proves that the characteristic function of  $\phi$  is computable in polynomial time, and hence that comprehension holds for  $\phi$ . Thus PV is an extension of  $\Delta_1^b$  – CR.

PV is separated from  $\Delta_1^b - CR$  by the circuit value principle, which says that "for all circuits C and all inputs  $\bar{x}$ , there exists a computation of C on  $\bar{x}$ ". This is provable in PV, but under the assumption that P does not equal uniform  $TC^0$  it is not provable in  $\Delta_1^b - CR$ .

Turning now to the top of the diagram, [2] proves the  $\forall \Sigma_1^b$ -conservativity of  $S_2^1$  over PV. If PV + BB( $\Sigma_0^b$ ) proves  $S_2^1$ , then PV  $\vdash S_2^1$  [20] and hence the bounded arithmetic hierarchy collapses to PV and the polynomial hierarchy PH collapses to  $\Sigma_2^p \cap \Pi_2^p$  [20, 3].

The  $\forall \exists \Sigma_0^B$ -conservativity of  $V^0 + \mathrm{BB}(\Sigma_0^B)$  over  $V^0$  is from Zambella [20].  $\Sigma_0^b - \mathrm{LIND} + \mathrm{BB}(\Sigma_0^b)$  was introduced in [9] by Johannsen and Pollett (where they call it  $C_2^0$ ), and proved to be  $\forall \Sigma_1^b$  conservative over  $\Delta_1^b - \mathrm{CR}$  in [10]. From these conservativity results it follows that  $V^0 + \mathrm{BB}(\Sigma_0^B)$  does not prove the pigeonhole principle and  $\Delta_1^b - \mathrm{CR} + \mathrm{BB}(\Sigma_0^b)$  does not prove the circuit value principle (unless P equals uniform  $\mathrm{TC}^0$ ), which gives us the separations between the three theories with replacement.

In the body of the paper we show the separations between the theories with and without various kinds of replacement, using a similar argument in all cases.

In section 2 we describe how our general argument goes. In section 3 we use it together with the fact that parity is not computable in nonuniform  $AC^0$  to separate  $V^0$  from  $V^0 + BB(\Sigma_0^b)$ .

In section 4 we show that if PV proves  $\Sigma_0^b$ -replacement, then factoring is possible in probabilistic polynomial time. (This strengthens a result in [19] where the weaker conclusion "RSA is insecure" was proved.) We observe that this is true even if we look at weak versions of  $\Sigma_0^b$ -replacement, where we code very short sequences of witnesses; for example BB( $\Sigma_0^b$ , ||x||) in the diagram is the scheme of replacement for sequences of double-log length:

$$\forall i < ||a|| \exists y < a \, \phi(i, y) \rightarrow \exists w \, \forall i < ||a|| \, \phi(i, [w]_i).$$

The dotted line in the diagram represents the fact that if factoring is hard, then all the theories  $BB(\Sigma_0^b,|x|)$ ,  $BB(\Sigma_0^b,||x||)$ ,  $BB(\Sigma_0^b,||x||)$ ,  $BB(\Sigma_0^b,||x||)$ , ... are distinct (in fact we show something slightly stronger than this). By a similar argument, all these theories are distinct over  $V^0$  (in place of PV), without

any assumptions, but for the sake of tidiness we have not put this on the diagram.

The theory of strong  $\Delta_1^b$  comprehension is like  $\Delta_1^b$ -CR, except that rather than having a rule that if a formula is provably  $\Delta_1^b$  then comprehension holds for it, we have the " $\Delta_1^b$  comprehension axiom scheme"

$$\forall x (\phi(x) \leftrightarrow \neg \psi(x)) \to \exists w \, \forall i < |a| (\phi(i) \leftrightarrow (w)_i = 1)$$
 (3)

where  $\phi, \psi \in \Sigma_1^b$  (and may contain other parameters); so comprehension holds for  $\phi$  in a structure, if  $\phi$  is  $\Delta_1^b$  in that structure. The question is raised in [10], whether this theory is strictly stronger than  $\Delta_1^b - CR$ . We show that it is, under a cryptographic assumption. We consider a principle not shown on the diagram, which we call "unique replacement". We show that if RSA is secure against probabilistic polynomial time attack then PV does not prove unique replacement, and that it follows that PV, and hence  $\Delta_1^b - CR$ , does not prove the  $\Delta_1^b$  comprehension axiom scheme.

We have not looked for a separation between this last theory and  $\Sigma_0^b$  – LIND + BB( $\Sigma_0^b$ ).

A preliminary version of this paper appears in [7].

# 2 Witnessing with an interactive computation

First we recall a standard lemma.

**Lemma 1** Over BASIC,  $\Sigma_0^b$ -replacement is equivalent to strict  $\Sigma_1^b$ -replacement. Hence over PV,  $\Sigma_0^b$ -replacement is equivalent to replacement for PV formulas, since PV proves that every PV formula is equivalent to a strict  $\Sigma_1^b$  formula.

Similarly over  $V^0$ ,  $\Sigma_0^B$ -replacement is equivalent to  $\Sigma_1^B$ -replacement, where a  $\Sigma_1^B$  formula is a  $\Sigma_0^B$  formula preceded by a block of bounded existential string quantifiers.

Our main tool in this paper is the KPT witnessing theorem. We state it here for PV, although it holds in a much more general form.

**Theorem 2** [13] Let  $\phi$  be a PV formula and suppose PV  $\vdash \forall x \exists y \forall z \phi(x, y, z)$ .

Then there exists a finite sequence  $f_1, \ldots, f_k$  of PV function symbols such that

$$PV \vdash \forall x \, \forall \overline{z}, \, \phi(x, f_1(x), z_1) \vee \phi(x, f_2(x, z_1), z_2) \\ \vee \ldots \vee \phi(x, f_k(x, z_1, \ldots, z_{k-1}), z_k).$$

**Proof** Let  $b, c_1, c_2, ...$  be a list of new constants, and let  $t_1, t_2, ...$  be an enumeration of all terms built from symbols of PV together with  $b, c_1, c_2, ...$ , where the only new constants in  $t_k$  are among  $\{b, c_1, ..., c_{k-1}\}$ . It suffices to show that

$$PV \cup {\neg \phi(b, t_1, c_1), \neg \phi(b, t_2, c_2), \dots, \neg \phi(b, t_k, c_k)}$$

is unsatisfiable for some k.

Suppose otherwise. Then by compactness

$$PV \cup \{\neg \phi(b, t_1, c_1), \neg \phi(b, t_2, c_2), ...\}$$
 (4)

has a model M. Since PV is universal, the substructure M' consisting of the denotations of the terms  $t_1, t_2, ...$  is also a model for (4). It is easy to see that

$$M' \models PV + \forall y \exists z \neg \phi(b, y, z)$$

and hence PV  $\not\vdash \forall x \exists y \forall z \phi(x, y, z)$ .

Now choose a function f which can be computed in polynomial time but which is hard to invert. Suppose PV proves the following instance of replacement (which has a and y as parameters, and m = |a|):

$$\forall i < m \ \exists u < a \ f(u) = [y]_i \to \exists w \ \forall j < m \ f([w]_j) = [y]_j.$$

We can rewrite this as

$$\exists i < m \ \exists w \ \forall u < a, \ f(u) = [y]_i \to \forall j < m \ f([w]_j) = [y]_j.$$

Applying our witnessing theorem, we get  $k \in N$  and functions  $g_1, \ldots, g_k$  and  $h_1, \ldots, h_k$  (which have a as a suppressed argument), such that

$$PV \vdash \forall \bar{z} < a,$$

$$(f(z_1) = [y]_{g_1(y)} \to \forall j < m \ f([h_1(y)]_j) = [y]_j)$$

$$\lor (f(z_2) = [y]_{g_2(y,z_1)} \to \forall j < m \ f([h_2(y,z_1)]_j) = [y]_j)$$

$$\lor \dots$$

$$\lor (f(z_k) = [y]_{g_k(y,z_1,\dots,z_{k-1})} \to$$

$$\forall j < m \ f([h_k(y,z_1,\dots,z_{k-1})]_j) = [y]_j)$$

This allows us to write down an algorithm which given an input y (considered as a sequence  $[y]_0, \ldots, [y]_{m-1}$ ) will ask for a pre-image of f on at most k elements of y, and with this information will output a number w coding a sequence of pre-images of all m elements of y.

The algorithm is as follows. Let  $w = h_1(y)$ . If  $\forall j < m f([w]_j) = [y]_j$  then output w and halt. Otherwise calculate  $g_1(y)$  and ask for a pre-image of  $[y]_{g_1(y)}$ ; store the answer as  $z_1$ . Then let  $w = h_2(y, z_1)$ . If  $\forall j < m f([w]_j) = [y]_j$  then output w and halt. Otherwise calculate  $g_2(y, z_1)$  and ask for a pre-image of  $[y]_{g_2(y,z_1)}$ ; store the answer as  $z_2$ , and so on. By our assumption the algorithm will run for at most k steps of this form before it outputs a suitable w.

Now fix a such that |a| = m > k, and choose a sequence  $[x]_0, \ldots, [x]_{m-1}$  of numbers less than a. Let y encode the pointwise image of x under f. Run the algorithm above, and reply to queries with elements of x. We will end up with w encoding a sequence of pre-images of y, which will clash in some way with our assumption that f is hard to invert. If f is an injection, w will be the same as x; we use this in section 3. If f is not an injection and x was chosen at random, then w is probably different from x; we use this in sections 4 and 5.

The important properties of PV used in the argument above are that it is universal and can define functions by cases (needed for the KPT witnessing theorem) and that it can manipulate sequences. We show now how to make  $V^0$  into a universal theory in which we can carry out the same argument.

We start by referring to [6], pp 66–73. A relation  $R(\bar{x}, \bar{Y})$  is in (uniform)  $AC^0$  iff it is defined by some  $\Sigma_0^B$  formula  $A(\bar{x}, \bar{Y})$ . A number function  $f: \mathbb{N}^k \times (\{0,1\}^*)^\ell \longrightarrow \mathbb{N}$  is an  $AC^0$  function iff there is an  $AC^0$  relation R and a polynomial p such that

$$f(\bar{x}, \bar{Y}) = \min z < p(\bar{x}, |\bar{Y}|) \ R(z, \bar{x}, \bar{Y})$$

$$\tag{5}$$

A string function  $F(\bar{x}, \bar{Y})$  is an  $AC^0$  function iff  $|F(\bar{x}, \bar{Y})| \leq p(\bar{x}, |\bar{Y}|)$  for some polynomial p, and the bit graph

$$B_F(i, \bar{x}, \bar{Y}) \equiv F(\bar{x}, \bar{Y})(i)$$

is an  $AC^0$  relation.

We denote by  $V^0(\text{FAC}^0)$  a conservative extension of  $V^0$  obtained by adding a set  $\text{FAC}^0$  of function symbols with universal defining axioms for all  $\text{AC}^0$  functions, based on the above characterizations.  $\text{FAC}^0$  is essentially

 $\mathcal{R}-def$  in [20].) This can be done in such a way that  $V^0(\text{FAC}^0)$  is a universal theory. In particular, the  $\Sigma_0^B$  comprehension axioms follow since for every  $\Sigma_0^B$  formula  $\phi$  there is a FAC<sup>0</sup> string function whose range is the set of strings asserted to exist by the the comprehension axiom for  $\phi$ . Further, from (5) it is clear that for every  $\Sigma_0^B$  formula  $\phi$  there is a quantifier-free formula  $\phi'$  in the language of  $V^0(\text{FAC}^0)$  such that

$$V^0(\text{FAC}^0) \vdash (\phi \leftrightarrow \phi')$$

From these remarks, it is clear that the usual proof of the KPT witnessing theorem can be adapted to show the following:

**Theorem 3** Let  $\phi(X, Y, Z)$  be a  $\Sigma_0^B$  formula such that  $V^0 \vdash \forall X \exists Y \forall Z \phi(X, Y, Z)$ . Then there are FAC<sup>0</sup> functions  $F_1, ..., F_k$  such that

$$V^{0}(FAC^{0}) \vdash \forall X \forall \overline{Z},$$
  

$$\phi(X, F_{1}(X), Z_{1}) \lor \phi(X, F_{2}(X, Z_{1}), Z_{2})$$
  

$$\lor \dots \lor \phi(X, F_{k}(X, Z_{1}, \dots, Z_{k-1}), Z_{k}).$$

Using this we can show that if  $V^0$  proves  $\Sigma_0^B$ -replacement, then for any  $AC^0$  function F there exists  $k \in N$  and a uniform  $AC^0$  algorithm that will find a pre-image under F of any sequence  $Y^{[0]}, \ldots, Y^{[m-1]}$  of strings by asking at most k queries of the form "what is a pre-image of  $Y^{[i]}$ ?"

# 3 Replacement in $V^0$ and parity

Let PARITY be the set of all strings over  $\{0,1\}$  with an odd number of 1s. By a (nonuniform)  $AC^0$  circuit family we mean a polynomial size bounded depth family  $\langle C_n : n \in \mathbb{N} \rangle$  of Boolean circuits over  $\wedge, \vee, \neg$  such that  $C_n$  has n inputs and one output. Ajtai's theorem [1, 8] states that no such circuit family accepts PARITY.

We show that if  $V^0$  proves the  $\Sigma_0^B$  replacement scheme, then (using KPT witnessing) there exists a (uniform) randomized  $AC^0$  algorithm for PARITY. This algorithm shows the existence of a (uniform)  $AC^0$  circuit family such that each circuit has a vector  $\bar{r}$  of random input bits in addition to the standard input bits, and with probability p > 2/3 the circuit correctly determines whether the standard input is in PARITY and with probability 1-p the circuit produces an output indicating failure. From this a standard

argument shows the existence of a nonuniform  $AC^0$  circuit family for parity, violating the above theorem.

Let PAR be the function that maps a binary string of length m to its parity vector. That is, PAR(m,Y) = X if |X| < m and, for each i < m, X(i) is the parity of the string  $Y(0) \dots Y(i)$ . In what follows we take m to be a parameter, assume Y is an m-bit string, and suppress the argument m from PAR(m,Y).

Plainly PAR(Y) cannot be computed in  $AC^0$ . However its inverse, which we will call UNPAR, is in uniform  $AC^0$ : the *i*th bit of UNPAR(X) is given by the  $\Sigma_0^B$  formula  $(i = 0 \land X(i)) \lor (i > 0 \land X(i-1) \oplus X(i))$ . Here UNPAR has an argument m, which we suppress.

Notice also that for all m-bit strings A, B, C, writing  $\oplus$  for bitwise XOR, if  $A = B \oplus C$  then  $PAR(A) = PAR(B) \oplus PAR(C)$ .

**Theorem 4**  $V^0$  does not prove  $BB(\Sigma_0^B)$ .

**Proof** Suppose  $V^0 \vdash BB(\Sigma_0^B)$ . Then applying the argument of section 2 to the function UNPAR, for some fixed k there is a uniform  $AC^0$  algorithm which, for any sequence  $Y^{[0]}, \ldots, Y^{[m-1]}$  of binary strings of length m makes k queries of the form "what is  $PAR(Y^{[i]})$ ?" and outputs the sequence of parity vectors of Y.

We will show how to use this algorithm to compute the parity of a single string in uniform randomized  $AC^0$ . Suppose  $m \geq 3k$  and let I be the input string of length m which we want to compute the parity of.

Choose m strings  $U_0, \ldots, U_{m-1}$  in  $\{0,1\}^m$  at random, and for each i compute  $V_i = UNPAR(U_i)$ . Choose a number  $r, 0 \le r < m$ , uniformly at random. Define the string Y (thought of as an  $m \times m$  binary matrix) by the condition

$$Y^{[i]} = \begin{cases} V_i & \text{if } i \neq r \\ I \oplus V_r & \text{if } i = r. \end{cases}$$

Since for each m the function UNPAR defines a bijection from the set  $\{0,1\}^m$  to itself, and since for each I with |I| < m the map  $X \mapsto I \oplus X$  also defines a bijection from that set to itself, it follows that the string Y defined above, interpreted as an  $m \times m$  bit matrix, is uniformly distributed over all such matrices.

Now run our interactive  $AC^0$  algorithm on Y. If the algorithm queries "what is  $PAR(Y^{[i]})$ ?" for  $i \neq r$ , reply with  $U_i$  (which is the correct answer). If the algorithm queries "what is  $PAR(Y^{[r]})$ ?", then abort the computation.

Since at most k different values of i are compared to r and since for each input I each pair (Y, r) is equally likely to have been chosen, it follows that the computation will be aborted with probability at most  $k/m \le 1/3$ .

Hence with probability at least 2/3 the algorithm is not aborted, we are able to answer all the queries correctly, and we obtain W such that  $W^{[r]} = PAR(Y^{[r]}) = PAR(I \oplus V_r)$ . But  $I = V_r \oplus (I \oplus V_r)$  and hence

$$PAR(I) = PAR(V_r) \oplus PAR(I \oplus V_r)$$
$$= U_r \oplus W^{[r]}$$

We use this to compute PAR(I) and use bit m-1 of PAR(I) to determine whether  $I \in PARITY$ .

For each input I the algorithm succeeds with probability at least 2/3, where the probability is taken over its random input bits.

Since no such  $AC^0$  algorithm exists, it follows that  $V^0$  does not prove the  $\Sigma_0^B$  replacement scheme.

## 4 Replacement in PV and factoring

We adapt the proof [16] that cracking Rabin's cryptosystem based on squaring modulo n is as hard as factoring.

Let n be the product of distinct odd primes p and q. Suppose  $0 < x_1 < n$  and  $gcd(x_1, n) = 1$ . Let  $c = x_1^2$ . Then c has precisely four square roots  $x_1, x_2, x_3, x_4$  modulo n, as follows.

Let  $x_p = (x_1 \mod p)$  and  $x_q = (x_1 \mod q)$ . By the Chinese remainder theorem there are uniquely determined numbers  $x_1, x_2, x_3, x_4$  with  $0 < x_i < n$  such that

$$x_1 \equiv x_p \pmod{p} \qquad x_1 \equiv x_q \pmod{q}$$

$$x_2 \equiv x_p \pmod{p} \qquad x_2 \equiv -x_q \pmod{q}$$

$$x_3 \equiv -x_p \pmod{p} \qquad x_3 \equiv x_q \pmod{q}$$

$$x_4 \equiv -x_p \pmod{p} \qquad x_4 \equiv -x_q \pmod{q}$$

Now  $x_1 - x_2 \equiv 0 \pmod{p}$  and  $x_1 - x_2 \equiv 2x_q \not\equiv 0 \pmod{q}$ , so  $\gcd(x_1 - x_2, n) = p$ . So from  $x_1$  and  $x_2$  we can recover p, and similarly from  $x_1$  and  $x_3$  we can recover q.

Hence if we have one square root of c, and are then given a square root at random, we can factor n with probability  $\frac{1}{2}$ .

**Theorem 5** If PV proves replacement for sharply bounded formulas, then factoring (of products of two odd primes) is possible in probabilistic polynomial time.

**Proof** We will use our standard argument, taking squaring modulo n as our function f (so f has n as a parameter).

If PV proves BB( $\Sigma_0^b$ ) then there is polynomial time algorithm which, for some fixed  $k \in \mathbb{N}$ , given any sequence  $y_0, \ldots, y_{m-1}$  of squares (modulo n), makes at most k queries of the form "what is the square root of  $y_i$ ?" and, if these are answered correctly, outputs square roots of all the  $y_i$ s.

Now suppose n is large enough that m = |n| > k. Choose numbers  $x_0, \ldots, x_{m-1}$  uniformly at random with  $0 < x_i < n$ . We may assume that  $\gcd(x_i, n) = 1$  for all i, since otherwise we can immediately find a factor of n.

For each i let  $y_i = (x_i^2 \mod n)$ . Let y code the sequence  $y_0, \ldots y_{m-1}$ , so  $[y]_i = y_i$ . Notice that each  $x_i$  is distributed uniformly amongst the four square roots of  $[y]_i$ .

Run our algorithm, and to each query "what is the square root of  $[y]_i$ ?", answer with  $x_i$ . We will get as output w coding a sequence  $[w]_0, \ldots, [w]_{m-1}$  of square roots of  $[y]_0, \ldots, [y]_{m-1}$ .

If we think of n as fixed, the value of w depends only on the inputs given to the algorithm, namely y and the k many numbers  $x_i$  that we gave as replies. Let i be some index for which  $x_i$  was not used. Then  $x_i$  is distributed at random among the square roots of  $[y]_i$ , and  $[w]_i$  is a square root of  $[y]_i$  that was chosen without using any information about which square root  $x_i$  is. Hence  $\gcd(x_i - [w]_i, n)$  is a factor of n with probability  $\frac{1}{2}$ .

Notice that the only property of the function | | we used was that we could find some n with |n| > k. So any nondecreasing, not eventually constant function would do in the place of | |. Hence if PV only proves replacement for very short sequences, that is still enough to give us factoring.

In fact under the assumption that factoring is hard we can show that these replacement schemes form a hierarchy. For any  $\alpha$  with one argument, let BB( $\alpha$ , PV) be the axiom scheme:

$$\forall i \!<\! \alpha(b) \, \exists y \!<\! b \, \phi(i,y) \rightarrow \exists w \, \forall i \!<\! \alpha(b) \, \phi(i,[w]_i)$$

for all PV formulas  $\phi$ . We will assume that our base theory proves that  $\alpha(x) < |x|$  and that  $\alpha$  is increasing.

We need a generalization of a result of Zambella, lemma 3.3 of [20]. The lemma there is presented for a two-sorted system similar to  $V^0$  and with |x| rather than  $\alpha(x)$ .

An  $\exists^b PV$  formula is a PV formula preceded by a bounded existential quantifier; modulo PV this is the same as a strict  $\Sigma_1^b$  formula.

**Lemma 6** Any model  $N \models \text{PV}$  has an  $\exists^b \text{PV}$ -elementary extension to a model  $M \models \text{PV} + \text{BB}(\alpha, \text{PV})$  such that every element of M is of the form  $f(a, \bar{b})$  for some  $f \in \text{PV}$ ,  $a \in N$  and  $\bar{b} \subseteq \alpha(M)$ , where  $\alpha(M) = \{x \in M : x < \alpha(y), \text{ some } y \in M\}$ . Informally, M is formed from N by only adding new " $\alpha$ -small" elements and closing under PV functions.

**Proof** Let L be the language of PV with the addition of a name for every element of N, and let T be the universal theory of N in this language, so every model of T will be an  $\exists$ -elementary, and hence  $\exists^b PV$ -elementary, extension of N. Enumerate as  $(t_1, \phi_1(x, y)), (t_2, \phi_2(x, y)), \ldots$  all pairs consisting of closed terms in L and binary PV formulas with parameters from L. We will use this to construct a chain  $T = T_0 \subseteq T_1 \subseteq T_2 \subseteq \ldots$  of theories.

Suppose that  $T_i$  has been constructed and is a consistent, universal theory. If  $T_i \vdash \forall x < \alpha(t_{i+1}) \exists y \ \phi_{i+1}(x,y)$  then put  $T_{i+1} = T_i$ . Otherwise introduce a new constant symbol c and put

$$T_{i+1} = T_i \cup \{c < \alpha(t_{i+1})\} \cup \{\forall y \, \neg \phi_{i+1}(c, y)\}.$$

Note that  $T_{i+1}$  is consistent and universal.

Let  $T^*$  be the union of this chain of theories, and let  $L^*$  be L together with all the new constant symbols that were added in the construction of  $T^*$ . Enumerate all pairs of closed terms and binary formulas in  $L^*$ , and repeat the above construction to get a theory  $T^{**}$  and a language  $L^{**}$ . Repeat this step  $\omega$  times, and let  $T^+$  be the union of the theories and  $L^+$  its language.

 $T^+$  is consistent and universal, so there is a model  $M \models T^+$  each element of which is named by some closed  $L^+$ -term.  $M \models T$ , so M is an  $\exists^b PV$ -elementary extension of N. Also, each time a new constant c was introduced to  $L^+$ ,  $c < \alpha(t)$  was introduced to  $T^+$  for some term t. So M is the closure of elements of N and new " $\alpha$ -small" elements, as required.

To show that M is a model of  $BB(\alpha, PV)$ , suppose that a is an element of M and  $\phi(x, y)$  is a PV formula with parameters from M, and

$$M \vDash \forall x < \alpha(a) \exists y \, \phi(x, y).$$

Then by the construction of M, we may assume that a is named by some closed  $L^+$  term t and that  $\phi(x,y)$  is a parameter-free  $L^+$  formula; and by the construction of  $T^+$  we must have that  $T^+ \vdash \forall x < \alpha(t) \exists y \phi(x,y)$ , since  $T^+$  either proves this or its negation. But  $T^+$  is a universal theory, so by using Herbrand's theorem and the properties of PV we can find a PV function symbol f (with parameters) such that  $T^+ \vdash \forall x < \alpha(t) \phi(x, f(x))$ . Now by the comprehension available in PV, we can find some  $w \in M$  such that  $M \models \forall x < \alpha(t) \phi(x, [w]_x)$ , as required.

We can now adapt the proof of the KPT witnessing theorem to get the following:

#### Theorem 7 Suppose

$$PV + BB(\alpha, PV) \vdash \forall x \exists y \forall z \phi(x, y, z)$$

for an  $\exists^b PV$  formula  $\phi$ . Then there exist  $k \in \mathbb{N}$ , a term  $s(x, \bar{z})$  and functions  $f_1, \ldots, f_k$  such that

PV 
$$\vdash \forall x \, \forall \bar{z}, \, \exists i < \alpha(s)^k \, \phi(x, [f_1(x)]_i, [z_1]_i)$$
  
 $\vee \, \exists i < \alpha(s)^k \, \phi(x, [f_2(x, z_1)]_i, [z_2]_i)$   
 $\vee \dots \vee \, \exists i < \alpha(s)^k \, \phi(x, [f_k(x, z_1, \dots, z_{k-1})]_i, [z_k]_i)$ 

(we include the exponent k here because the range of  $\alpha$  might not be closed under multiplication).

**Proof** Enumerate all pairs of PV functions as  $(s_1, f_1), (s_2, f_2), \ldots$  with infinite repetitions in such a way that for each k both  $s_k$  and  $f_k$  take k or fewer arguments. Assume that the conclusion of the theorem is false, and let T be the theory

$$PV + \{ \forall i < \alpha(s_1(b, c_1))^1 \neg \phi(b, [f_1(b)]_i, [c_1]_i), \\ \forall i < \alpha(s_2(b, c_1, c_2))^2 \neg \phi(b, [f_2(b, c_1)]_i, [c_2]_i), \ldots \}$$

where b and  $c_1, c_2, \ldots$  are new constant symbols. Then T is finitely satisfiable (we can take the term s in the statement of the theorem as the sum of our finite set of terms  $s_1, \ldots, s_k$ ).

Let N be a model of T, and let  $N' \subseteq N$  be the substructure consisting of all the elements named by terms. Since T is universal,  $N' \models T$ . Let M

be the extension of N given by lemma 6 to a model of BB( $\alpha$ , PV). By  $\exists^b PV$  elementariness, M is also a model of T.

Now let a be any element of M. By the construction of M, for some  $\bar{d} \subseteq \alpha(M)$ , some  $e \in N'$  and some PV function g we have  $a = g(\bar{d}, e)$ . Furthermore by the construction of N' we know that  $\bar{d} < \alpha(h_1(b, c_1, \ldots, c_k))$  and  $e = h_2(b, c_1, \ldots, c_k)$  for some k and some PV functions  $h_1$  and  $h_2$ .

In this paragraph we identify a number  $i < \alpha(h_1(b, \bar{c}))^k$  with the sequence  $\bar{i} = i_1 \dots i_k$  of numbers less than  $\alpha(h_1(b, \bar{c}))$  that it codes. We can find l > k such that  $f_l$  is the PV function symbol that takes as input  $b, c_1, \dots, c_l$  and outputs (as a single number) the sequence  $w_1 \dots w_{\alpha(h_1(b,c_1,\dots,c_k))^k}$  where  $w_i = g(\bar{i},h_2(b,c_1,\dots,c_k))$ . Then  $a = [f_l(b,c_1,\dots,c_l)]_d$  and since  $M \models T$  we have  $M \models \neg \phi(b,a,[c_{l+1}]_d)$ . Here a was chosen arbitrarily, so we have shown that  $M \models \text{PV} + \text{BB}(\alpha,\text{PV}) + \neg \forall x \exists y \forall z \phi(x,y,z)$ .

Corollary 8 Suppose that factoring is not possible in probabilistic polynomial time. Then  $BB(\alpha, PV)$  is not provable in  $PV + BB(\beta, PV)$ , for terms  $\alpha, \beta$  where  $\alpha(x), \beta(x) < |x|$  and  $\alpha$  grows faster than any polynomial in  $\beta$ .

**Proof** Our standard argument is that if replacement is provable in PV, then there is a polynomial time interactive algorithm that queries k square roots and outputs |n| square roots, for some fixed  $k \in \mathbb{N}$ .

By theorem 7 we can show, by a similar argument, that if  $PV+BB(\beta, PV) \vdash BB(\alpha, PV)$  then we have a polynomial time interactive algorithm that queries  $k\beta(n)^k$  square roots modulo n and outputs  $\alpha(n)$  square roots, for some fixed  $k \in \mathbb{N}$ .

So if n is sufficiently large that  $\alpha(n) > k\beta(n)^k$ , we can use the argument of theorem 5 to factor n.

This gives a hierarchy of theories

$$PV + BB(|x|, PV) \supset PV + BB(||x||, PV) \supset \dots$$

The same argument goes through in  $V^0$ . One way to see this is to notice that the important difference between PV and  $V^0$  is that the PV functions are closed under polynomial time iteration, and no such iteration is used in the proof here. So we have the unconditional separation result

**Theorem 9** BB( $\alpha, \Sigma_0^B$ ) is not provable in  $V^0 + BB(\beta, \Sigma_0^B)$ , for terms  $\alpha, \beta$  where  $\alpha(n), \beta(n) < n$  and  $\alpha$  grows faster than any polynomial in  $\beta$ .

**Proof** If the theorem is false, then there is  $k \in \mathbb{N}$  and an interactive algorithm that, given  $\alpha(n)$  many vectors  $v_1, \ldots, v_{\alpha(n)}$ , each of length n, will make  $k\beta(n)^k$  queries of the form "what is the parity vector of  $v_i$ ?" and then output the parity vectors of all the  $v_i$ s. So if  $\alpha(n) \geq 3k\beta(n)^k$ , then by adapting the argument of section 3 we get a probabilistic uniform  $AC^0$  algorithm which computes parity.

## 5 Unique replacement in PV and RSA

We define "unique replacement" to be the scheme

$$\forall i < |a| \exists !x < b \phi(i, x) \rightarrow \exists w \forall i < |a| \phi(i, [w]_i).$$

**Theorem 10** If PV proves unique replacement for sharply bounded formulas, then the injective WPHP for PV formulas can be witnessed in probabilistic polynomial time (and hence in particular we can crack RSA [12]).

**Proof** (Simplified from the model-theoretic proof in [19].) First notice that it is sufficient to show that PV does not prove unique replacement for some PV formula  $\phi$ . For suppose that  $\phi$  is decided by the polynomial time machine with code e, and that for some fixed i there is a unique x such that  $\phi(i,x)$ . Then there is a unique pair (z,x) such that z is an accepting computation of the machine e on input (i,x), and the property of being an accepting computation is sharply bounded.

In the rest of this proof x and y will code sequences of |n| numbers each of size  $< n^{|n|}$  and with elements  $[x]_i, [y]_i$ , and z will code a sequence of |n| numbers each of size < n and with elements  $\langle z \rangle_i$ .

Suppose that h is a PV function from  $n^{|n|}$  to n. Note that from any PV function  $g: 2n \to n$  we can derive such a function h with the property that a witness to WPHP for h yields in polynomial time a witness to WPHP for g ([15], or see [19] for an explicit polynomial time construction).

Choose  $x < n^{|n|^2}$  at random and let  $z < n^{|n|}$  be such that  $\langle z \rangle_0 = h([x]_0), \ldots, \langle z \rangle_{|n|-1} = h([x]_{|n|-1}).$ 

Assume that PV proves the following instance of unique replacement:

$$\exists i < |n| \, \forall u < n^{|n|} \, h(u) \neq \langle z \rangle_i$$

$$\vee \, \exists i < |n| \, \exists u_1 < u_2 < n^{|n|} \, h(u_1) = h(u_2)$$

$$\vee \, \exists y < n^{|n|^2} \, \forall i < |n| \, h([y]_i) = \langle z \rangle_i.$$

Then by our witnessing theorem, for some k (independent of n) there is a deterministic interactive computation which takes n and z as its initial input. Then for k steps it gives us an index i < |n| and expects an input  $y < n^{|n|}$ ; if we can guarantee that for each such step we have  $h(y) = \langle z \rangle_i$ , then the computation outputs either  $u_1$  and  $u_2$  mapping to the same thing, in which case we are done (and this case is the only one that is different from normal replacement), or  $y < n^{|n|^2}$  satisfying  $\forall i < |n| h([y]_i) = \langle z \rangle_i$ .

Run the computation, and to each index i queried respond with  $[x]_i$ . The computation must output some y satisfying  $\forall i < |n| \, h([y]_i) = \langle z \rangle_i$ . Now the computation is deterministic, and if we think of n as fixed, there were  $n^{|n|(k+1)}$  possible different inputs to the machine: namely  $n^{|n|}$  different possibilities for z and  $(n^{|n|})^k$  different possibilities for the k responses  $[x]_i$ . Hence there are at most  $n^{|n|(k+1)}$  possible outputs y. However x was originally chosen at random from  $n^{|n|^2}$  possibilities. So if k < n-1 then with high probability x is not a possible output of the machine, so  $x \neq y$  and for some i < |n| we have  $[x]_i \neq [y]_i$  but  $h([x]_i) = \langle z \rangle_i = h([y]_i)$ .

Notice that part of this argument can be formalized in PV, to show that if PV proves unique replacement, then PV proves that the surjective WPHP for PV functions implies the injective WPHP for PV functions. In the proof above randomness was used to find some x outside the range of a given polynomial time algorithm; in the formal PV proof we would use the surjective WPHP to provide such an x.

Corollary 11 Suppose PV proves the  $\Delta_1^b$  comprehension axiom scheme (3). Then PV proves unique replacement for PV formulas and by theorem 10 we can crack RSA.

**Proof** Let  $\phi(i, x)$  be any PV formula (with parameters) and suppose that the hypothesis of the theorem holds. Let  $M \models \text{PV}$ ,  $a, b \in M$  and suppose  $M \models \forall i < |b| \exists ! x < a \phi(i, x)$ . Then

$$M \models \forall i < |b| \, \forall j < |a|,$$
$$\exists x < a \, (\phi(i, x) \land x_j = 1) \leftrightarrow \forall x < a \, (\phi(i, x) \rightarrow x_j = 1).$$

Over PV,  $\phi$  is equivalent to both a  $\Sigma_1^b$  and a  $\Pi_1^b$  formula, so we can apply comprehension and get some w such that

$$M \models \forall i < |b| \, \forall j < |a|,$$
  
$$([w]_i)_j = 1 \leftrightarrow \exists x < a \, (\phi(i, x) \land x_j = 1).$$

Here we assume without loss of generality that a is a power of 2, so that we can switch easily between thinking of w as a binary sequence of length |b||a| and as a sequence of |b| many binary numbers  $[w]_1 \dots [w]_{|b|}$ , each of length |a|. We also use the fact that in PV the formula  $\phi(i,x)$  can be written in both a strict  $\Sigma_1^b$  and a strict  $\Pi_1^b$  way, which we need to apply comprehension.

Now pick any i < |b|. There is some unique  $x \in M$  such that  $\phi(i, x)$ ; and by the construction of w, for each j < |a| we know  $([w]_i)_j = 1$  if and only if  $x_j = 1$ . Hence  $[w]_i = x$ .

So 
$$M \models \forall i < |b| \phi(i, [w]_i)$$
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