SEPARATOR THEOREMS AND TURÁN-TYPE RESULTS FOR PLANAR INTERSECTION GRAPHS

JACOB FOX AND JANOS PACH

Abstract. We establish several geometric extensions of the Lipton-Tarjan separator theorem for planar graphs. For instance, we show that any collection C of Jordan curves in the plane with a total of m crossings has a partition into three parts $C = S \cup C_1 \cup C_2$ such that $|S| = O(\sqrt{m})$, $\max\{|C_1|, |C_2|\} \le \frac{2}{3}|C|$, and no element of C_1 has a point in common with any element of C_2. These results are used to obtain various properties of intersection patterns of geometric objects in the plane. In particular, we prove that if a graph G can be obtained as the intersection graph of n convex sets in the plane and it contains no complete bipartite graph $K_{k,k}$ as a subgraph, then the number of edges of G cannot exceed $c_k n$, for a suitable constant c_k.

1. Introduction

Given a collection $C = \{\gamma_1, \ldots, \gamma_n\}$ of compact connected sets in the plane, their *intersection graph* $G = G(C)$ is a graph on the vertex set C, where γ_i and γ_j $(i \ne j)$ are connected by an edge if and only if $\gamma_i \cap \gamma_j \ne \emptyset$. For any graph H, a graph G is called H*-free* if it does not have a subgraph isomorphic to H. Pach and Sharir [11] started investigating the maximum number of edges an H-free intersection graph $G(C)$ on n vertices can have.

If H is not bipartite, then the assumption that G is an intersection graph of compact convex sets in the plane does not significantly effect the answer. The *extremal number* $\mathrm{ex}(H, n)$ is defined as the maximum number of edges over all H-free graph on n vertices. According to the Erdős-Stone theorem, we have

$$\mathrm{ex}(n, H) = (1 - \frac{1}{\chi(H) - 1} + o(1))\frac{n^2}{2},$$

where $\chi(H)$ is the chromatic number of H. This bound is asymptotically tight if H is not bipartite, as shown by the complete $(\chi(H)-1)$-partite graph whose vertex classes are of roughly equal size. On the other hand, this graph can be obtained as the intersection graph of a collection of segments in the plane, where the segments in each of the vertex classes are parallel.

The problem becomes more interesting if H is bipartite. By a slight modification of the classical bound of Kővári, Sós, and Turán (see Bollobás [2]), we obtain

$$\mathrm{ex}(K_{k,k}, n) \le n^{2-1/k}, \tag{1}$$

provided that $n \ge 2k$.

Pach and Sharir [11] proved that for any positive integer k there is a constant c_k such that, if $G(C)$ is a $K_{k,k}$-free intersection graph of n convex bodies in the plane, then it has at most $c_k n \log n$ edges. In other words, for every collection of n convex bodies in the plane with no k of them intersecting k others, there are at most $c_k n \log n$ intersecting pairs.

In the case $k = 2$, they improved their bound by a $\log n$ factor to $O(n)$. They further conjectured that if H is any bipartite graph, then there is a constant c_H such that every intersection graph of n convex bodies in the plane that does not contain H as a subgraph has at most $c_H n$ edges. Radoičić and Tóth [12] used discharging methods to prove this conjecture for $H \in \{C_6, C_8, K_{2,3}, K_{2,4}\}$. The aim of this paper is to prove the conjecture in its full generality.

Theorem 1. *For any bipartite graph H, there is a constant c_H such that every H-free intersection graph of n convex bodies in the plane has at most $c_H n$ edges.*

Obviously, it is sufficient to prove the theorem for *balanced* complete bipartite graphs $H = K_{k,k}$. It follows from our proof that in this case c_H can be taken to be $2^{O(k)}$. Of course, this yields that $c_H = 2^{O(k)}$ for any bipartite graph H with k vertices.

Theorem 1, as well as Theorem 4, is stated for intersection graphs of *convex bodies* in the plane, that is, compact convex sets with nonempty interior. It is not hard to argue that these statements remain true for any finite family C of plane convex sets. To see this, pick a point in the intersection of any two sets belonging to C, and replace each set $\gamma \in C$ by the convex hull of all points selected in γ.

A graph G is called *d-degenerate* if every subgraph of G has a vertex of degree at most d. Every d-degenerate graph has chromatic number at most $d+1$. Theorem 1 implies that every H-free intersection graph of convex bodies is $2c_H$-degenerate. Therefore, we obtain

Corollary 2. *For any bipartite graph H, the chromatic number of every H-free intersection graph of n convex bodies in the plane is at most $2c_H + 1$. (Here c_H denotes the same constant as in Theorem 1.)*

We present two proofs for Theorem 1, based on two different geometric separator theorems of independent interest that can be regarded as generalizations of the Lipton-Tarjan separator theorem for planar graphs [6].

Given a graph $G = (V, E)$ with vertex set V and edge set E, a *weight function* $w : V \to \mathbb{R}_{\geq 0}$ is a nonnegative function on the vertex set such that the sum of the weights is at most 1. For any subset $S \subset V$, the weight $w(S)$ is defined to be $\sum_{v \in S} w(v)$.

A *separator* in a graph $G = (V, E)$ with respect to a weight function w is a subset $S \subset V$ of vertices such that there is a partition $V = S \cup V_1 \cup V_2$ such that $w(V_1), w(V_2) \leq 2/3$ and there are no edges between V_1 and V_2. If the weight function is not specified, it is assumed that $w(v) = \frac{1}{|V|}$ for every vertex v.

The original Lipton-Tarjan separator theorem [6] states that for every planar graph G with n vertices and for every weight function w for G, there is a separator of size $O(n^{1/2})$.

In the sequel, if two members of a family of curves have a point in common, we call this point a *crossing*. If k members of the family pass through the same point, then we count this crossing with multiplicity $\binom{k}{2}$. Equivalently, in our statements we can and will assume without loss of generality that no three curves of the family pass through the same point.

In Section 2, we prove the following separator theorem for intersection graphs of *Jordan regions*, that is, for simply connected *compact* regions in the plane, bounded by a closed Jordan curve.

Theorem 3. *Let C be a collection of compact Jordan regions in the plane with at most m crossings between their boundaries and with at most m containments, and let w be a weight function. Then $G(C)$, the intersection graph of C, has a separator of size $O(\sqrt{m})$ with respect to w.*

By Koebe's representation theorem [5], the vertex set of any planar graph can be represented by a packing of disks in the plane with two disks touching each other if and only if the corresponding vertices are adjacent. Applying Theorem 3 to the system of disks (with $m \leq 3n - 6$), we immediately obtain the Lipton-Tarjan separator theorem.

Moreover, Theorem 3 also generalizes the $d = 2$ special case of the following result of Miller, Teng, Thurston, and Vavasis [9]: The intersection graph of any collection of n balls in d-dimensional space with the property that no k of them have a point in common has a separator of size $O(dk^{1/d}n^{1-1/d})$. To see this, it is enough to notice that such an intersection graph is $O_d(k)$-degenerate, thus its number of edges is $O_d(kn)$. (Just consider the smallest ball.)

In Section 3, we establish a separator theorem for families of plane convex bodies.

Theorem 4. *For any weight function, every K_k-free intersection graph of convex bodies in the plane with m edges has a separator of size $O(\sqrt{km})$.*

Since planar graphs are K_5-free, this result can also be regarded as a generalization of the Lipton-Tarjan separator theorem.

A family of graphs is *hereditary* if it is closed under taking induced subgraphs. By a simple recursive argument, in Section 4 we show that if all members of a hereditary family of graphs have a small separator, then the number of edges of these graphs is at most linear in the number of vertices.

Theorem 5. *Let $\epsilon > 0$, and let F be a hereditary family of graphs such that every member of F with n vertices has a separator of size $O(n/(\log n)^{1+\epsilon})$. Then every graph in F on n vertices has at most $c_F n$ edges, where c_F is a suitable constant.*

In the last section, we combine this result with Theorems 3 and 4 to deduce Theorem 1, and some similar statements. We also raise some open problems.

2. Separator Theorem for Jordan Regions

The aim of this section is to prove Theorem 3. By slightly perturbing the sets in C, if necessary, we can assume that no three elements of C share a boundary point. We may also assume without loss of generality that every element of C intersects at least one other element.

Let C_0 consist of all $\gamma \in C$ whose weight satisfies $w(\gamma) \geq 1/m^{1/2}$ or γ is involved in at least $\frac{1}{3}m^{1/2}$ containments with other elements of C. Let $C_1 = C \setminus C_0$. Notice that $|C_0| \leq 7\sqrt{m}$, since otherwise we would obtain that $w(C_0) > 1$ or that the number of containments exceeds m, which is impossible.

Let U_1 be the set of all crossings that lie on the boundary of at least one element of C_1. Let U_2 be a collection of $3|C_1|$ points not in U_1 such that the boundary of each $\gamma \in C_1$ contains precisely three points in U_2. Let $U = U_1 \cup U_2$, so that $|U| = O(m)$.

Consider the planar graph $P = (U, F)$, whose every edge is a piece of the boundary of an element of C_1 between two consecutive points that belong to U. By definition, P is a *simple* graph, that is, it has no loops or double edges. Obviously, the boundaries of the elements of C_1 correspond to a decomposition of the edges of P into $|C_1|$ edge-disjoint cycles. Every vertex of P belongs to the boundary of at most two elements of C_1.

For any $\gamma \in C_1$, let $d(\gamma)$ denote the number of points on the boundary of γ that belong to U. For any vertex v of P that belongs to the boundary of two elements $\gamma_1, \gamma_2 \in C_1$, assign to v the weight

$$w'(v) = \frac{w(\gamma_1)}{d(\gamma_1)} + \frac{w(\gamma_2)}{d(\gamma_2)}.$$

If a vertex v is one of the three points in U_2 that have been selected on some $\gamma \in C_1$, then let

$$w'(v) = \frac{w(\gamma)}{d(\gamma)}.$$

Notice that $w'(U) = w(C_1) \leq 1$.

Let us triangulate P by adding extra edges, if necessary, and denote the resulting triangulated plane graph by P'. Applying a variant of the Lipton-Tarjan separator theorem for triangulated planar graphs, proved by Alon, Seymour, and Thomas [1], there exists partition $U = A_0 \cup A_1 \cup A_2$ such that

(1) the elements of A_0 form a cycle in P', of length at most $\sqrt{6}|U|^{1/2}$;

(2) the elements of A_1 and A_2 lie in the exterior and in the interior of this cycle, respectively; and
(3) $w'(A_1), w'(A_2) \leq \frac{2}{3}w(C_1)$.

Let
$$V_0 = C_0 \cup \{\gamma : \gamma \in C_1 \text{ and the boundary of } \gamma \text{ contains a point in } A_0\}.$$
Since each point of A_0 belongs to the boundary of at most two elements of C_1, we have $|V_0| \leq |C_0| + 2|A_0| = O(m^{1/2})$. If $w(V_0) \geq 1/3$, then, trivially, V_0 separates C into two sets of weight at most $2/3$ (the sets $C \setminus V_0$ and $\emptyset$). Thus, we can assume that $w(V_0) < 1/3$.

Let
$$V_1 = \{\gamma : \gamma \in C_1 \text{ and all points in } U \text{ on the boundary of } \gamma \text{ belong to } A_1\}.$$
Let $V_2 = C \setminus (V_0 \cup V_1)$. We may assume without loss of generality that $w(V_2) \geq w(V_1)$, so that $w(V_2) \geq 1/3$. For $i \in \{1, 2\}$, we have
$$w(V_i) \leq w'(A_i) \leq \frac{2}{3}w(C_1) \leq \frac{2}{3}.$$

If there is a $\gamma \in V_1$ which is not disjoint from all the elements of V_2, then γ must contain in its interior all elements of V_2. Clearly, the number of elements of V_2 is at least the ratio of $w(V_2)$ to the maximum weight of the elements in V_2, so that we have
$$|V_2| \geq \frac{1}{3}m^{1/2}.$$
However, this implies that γ contains at least $\frac{1}{3}m^{1/2}$ elements of C, contradicting our assumption that $\gamma \in C_1$ (because, by definition, now $\gamma \in C_0$).

Therefore, every element of V_1 is disjoint from every element of V_2, and V_0 is a separator of size $O(m^{1/2})$ for the intersection graph $G(C)$ with respect to the weight function w. This completes the proof of Theorem 3.

3. Separator Theorem for Plane Convex Bodies

In this section we prove Theorem 4. Let C be a finite collection of convex bodies in the plane with m pairs intersecting and no k elements pairwise intersecting. For any pair of convex bodies in C that intersect, pick a *witness* point that belongs to their intersection. Replace each convex body $\gamma \in C$ by the convex hull $\pi = \pi(\gamma)$ of its witness points. Notice that the intersection graph of the resulting convex polygons $\pi(\gamma), \gamma \in C$ is precisely the same as the intersection graph of C.

The number of sides of a convex polygon π is at most the number of other convex polygons that intersects it. We may slightly enlarge each convex polygon, if necessary, without creating any new intersecting pairs so as to ensure that no two of them share more than a finite number of boundary points.

The following lemma guarantees that the total number of crossings between the boundaries of the convex polygons $\pi(\gamma)$ is $O(km)$. Since the number of containments among these polygons is bounded by m, by Theorem 3, the intersection graph of C has a separator of size $O(\sqrt{km})$, completing the proof of Theorem 4.

Lemma 6. *Let π be a convex d-gon, and let $C = \{\pi_1, \ldots, \pi_D\}$ be a collection of $D \geq d$ convex polygons such that the boundary of π_i intersects the boundary of π in a finite number of points. If C has no k pairwise intersecting elements, then the number of intersections between the boundary of π and the boundaries of the elements of C is $O(kD)$.*

Proof. A *convex geometric graph* is a graph that can be drawn in the plane so that its vertices form the vertex set of a convex polygon and its edges are straight-line segments. We need the following theorem of Capoyleas and Pach [3] (see also [10]): The maximum number of edges that a convex geometric graph with n vertices can have without containing k pairwise crossing edges is $2(k-1)n - \binom{2k-1}{2}$, provided that $n \geq 2k-1$.

Suppose without loss of generality that the polygons π and π_i are in “general position” in the sense that no three of them have a boundary point in common and no vertex of a polygon lies on a side of another. Let $v_1, \ldots, v_d$ denote the (open) sides of the polygon π. Construct a graph $G = (V, E)$ on the vertex set $V = \{v_1, \ldots, v_d\}$, by recursively adding straight-line segments (“edges”) connecting certain sides v_i and v_j of π so that no two sides are connected by more than one segment and at the end of the procedure the edge set E can be partitioned into D parts $E_1, \ldots, E_D$ satisfying the following condition: For any i, no two edges (segments) in E_i intersect, and if an edge in E_i intersects an edge in E_j for some $i \neq j$, then π_i intersects π_j.

For $i = 1, \ldots, D$, let $V_i \subseteq V$ be the set of all sides of π that meet the boundary of π_i. Let E_1 be a maximum system of disjoint segments whose endpoints belong to distinct elements in V_1 so that no two sides of π are connected by more than one segment. Notice that all segments in E_1 lie in the polygon π_1. If $E_1, \ldots, E_{i-1}$ have already been determined for some $i \leq D$, then let E_i be a maximum system of disjoint segments that can be selected within the polygon π_i such that

(1) the endpoints of each segment belong to distinct elements of V_i, and
(2) any pair of sides of π can be connected by at most one segment in $E_1 \cup \ldots \cup E_i$.

Finally, let $E = E_1 \cup \ldots \cup E_D$.

Collapsing the (open) sides of π into single points, we can see that that the graph $G = (V, E)$ is a convex geometric graph. It follows directly from the definition that the edge set E of G has the partition property described above. Therefore, in view of the fact that C does not have k pairwise intersecting elements, we obtain that G has no k pairwise crossing edges.

Assume now $|V_i| \geq 7k$ for some $1 \leq i \leq D$. Applying the Capoyleas-Pach theorem, we obtain that the subgraph of $(V, \cup_{j<i} E_j)$ induced by V_i has at most

$$2(k-1)|V_i| - \binom{2k-1}{2} \leq \frac{1}{2}\binom{|V_i|}{2}$$

edges. Therefore, there is a vertex $v \in V_i$ whose degree in this graph is at most $\frac{|V_i|-1}{2}$. Clearly, the corresponding side of π can be connected to all other sides in V_i not adjacent to v, which yields that $|E_i| \geq \frac{|V_i|-1}{2}$. Thus, the total number of edges of G is at least

$$\sum_{i=1}^{D} \frac{|V_i|-1}{2} - \frac{7k-2}{2} D = \frac{1}{2}\sum_{i=1}^{D} |V_i| - \frac{7k-1}{2} D.$$

On the other hand, applying the Capoyleas-Pach theorem once more, we conclude that the last quantity cannot exceed $2(k-1)d - \binom{2k-1}{2}$. Rearranging the terms, we get

$$\sum_{i=1}^{D} |V_i| \leq 4(k-1)d - 2\binom{2k-1}{2} + (7k-1)D < 11kD.$$

Since any convex polygon π_i intersects each edge of π in at most two points, the number of intersections between the boundary of π and the boundaries of the polygons π_i $(1 \leq i \leq D)$ is at most $2\sum_{i=1}^{D} |V_i| < 22kD$, which completes the proof. □

Using Theorem 3, it is not hard to establish the following variant of Theorem 4, which gives a smaller separator in the case $k > m^{1/3}$.

Corollary 7. *Every intersection graph of convex bodies in the plane with m edges has a separator of size $O(m^{2/3})$.*

Proof. Let C be a finite collection of convex bodies in the plane with m intersecting pairs. As in the proof of Theorem 4, replace the elements of C by convex polygons π with a total of $O(m)$ sides such that no two polygons share more than a finite number of boundary points, and the intersection graph of the polygons remains the same as the intersection graph of the original convex bodies.

We define a separator $S \subseteq C$ as follows. Include in S every polygon that intersects at least $m^{1/3}$ others. Since there are m intersecting pairs, the number of such polygons is at most $2m^{2/3}$. Each remaining polygon has fewer than $m^{1/3}$ sides, so any two of them share fewer than $2m^{1/3}$ boundary points. Since there are m intersecting pairs of polygons, the total number of boundary crossings between them is smaller than $2m^{4/3}$. Thus, we can apply Theorem 3 to the intersection graph of the remaining polygons to obtain a separator of size $O(m^{2/3})$. Putting together the elements of this separator with the polygons already selected, the resulting collection S is a separator of size $O(m^{2/3})$ for the intersection graph of C. □

4. Bounding the Number of Edges Using Small Separators

The aim of this section is to prove Theorem 5. In fact, we establish a slightly stronger statement.

Given a nonnegative function f defined on the set of positive integers, we say that a family F of graphs is *f-separable*, if every graph in F with n vertices has a separator of size at most $f(n)$.

Theorem 8. *Let $\phi(n)$ be a monotone decreasing nonnegative function defined on the set of positive integers, and let n_0 and C be positive integers such that*

$$\phi(n_0) \leq \frac{1}{12} \quad \textit{and} \quad \prod_{i=0}^{\infty} \left(1 + \phi\left((4/3)^i n_0\right)\right) \leq C.$$

If F is an $n\phi(n)$-separable hereditary family of graphs, then every graph in F on $n \geq n_0$ vertices has fewer than $\frac{Cn_0}{2}n$ edges.

Proof. Let $G_0 = (V, E)$ be a member of the family F with n vertices and average degree d. By definition, there is a partition $V = V_0 \cup V_1 \cup V_2$ with $|V_0| \leq n\phi(n)$, $|V_1|, |V_2| \leq \frac{2}{3}n$, such that no vertex in V_1 is adjacent to any vertex in V_2.

Let d' and d'' denote the average degree of the vertices in the subgraphs of G_0 induced by $V_0 \cup V_1$ and $V_0 \cup V_2$. Clearly, we have

$$d'(|V_0| + |V_1|) + d''(|V_0| + |V_2|) \geq 2|E| = d|V|,$$

so that

$$d' \frac{|V_0| + |V_1|}{|V| + |V_0|} + d'' \frac{|V_0| + |V_2|}{|V| + |V_0|} \geq d \frac{|V|}{|V| + |V_0|}.$$

Consequently, d' or d'' is at least

$$d \frac{|V|}{|V| + |V_0|} \geq d \frac{1}{1 + \phi(n)}.$$

Suppose without loss of generality that d' is at least as large as this number, and let G_1 denote subgraph of G induced by $V_0 \cup V_1$. By assumption, we have that $\phi(n) \leq \frac{1}{12}$ and $|V_0| \leq n\phi(n)$. Therefore, G_1 has $|V_0| + |V_1| \leq \frac{1}{12}n + \frac{2}{3}n = \frac{3}{4}n$ vertices.

Proceeding like this, we find a sequence of induced subgraphs $G_0 \supset G_1 \supset G_2 \supset \ldots$ with the property that, if G_i has n_i vertices and average degree d_i, then G_{i+1} has at most $\frac{3}{4}n_i$ vertices and average degree at least $\frac{1}{1+\phi(n_i)}d_i$. We stop with G_j if the number of vertices of G_j is at most n_0. Notice that the average degree of G_j is at least $\frac{1}{C}d$.

Suppose that $d \geq Cn_0$. Then the average degree in G_j is at least as large as the number of vertices of G_j, which is a contradiction. Hence, we have $d < Cn_0$, and the number of edges of G is at most $\frac{dn}{2} < \frac{Cn_0}{2}n$, completing the proof. □

Taking logarithms and approximating $\ln(1+x)$ by x, we obtain that $\prod_{i=0}^{\infty}(1+\phi((4/3)^i)) \neq \infty$ if and only if $\sum_{i=0}^{\infty}\phi((4/3)^i) \neq \infty$ if and only if $\sum_{i=0}^{\infty}\phi(2^i) \neq \infty$. Therefore, Theorem 8 has the following corollary.

Corollary 9. *Let F be an $n\phi(n)$-separable hereditary family of graphs, where $\phi(n)$ is a monotone decreasing nonnegative function such that $\sum_{i=0}^{\infty}\phi(2^i) \neq \infty$. Then every graph in F on n vertices has at most $O(n)$ edges.*

Since $\sum_{i=1}^{\infty} 1/i^{1+\epsilon}$ converges for all $\epsilon > 0$, Theorem 5 is an immediate consequence of Corollary 9.

5. Proof of Theorem 1 and Concluding Remarks

Now we can simply combine the results of the previous sections to obtain a proof of Theorem 1. Consider the family F of $K_{k,k}$-free intersection graphs of convex bodies in the plane. Obviously, this is a hereditary family. Since every $K_{k,k}$-free graph is also K_{2k}-free, by Theorem 4, every member graph in F with n vertices and m edges has a separator of size $O(\sqrt{km})$.

On the other hand, by the Kővári-Sós-Turán theorem (1), we know that in every such graph $m = O(n^{2-1/k})$. Consequently, there is a separator of size $O\left(\sqrt{k}n^{1-1/(2k)}\right)$. Thus, we can apply Theorem 5 to conclude that every graph in F on n vertices has at most $O(n)$ edges. This completes the proof of Theorem 1.

For the proof of Theorem 1 in the special case $H = K_{2,2}$, Pach and Sharir [11] developed a vertical decomposition argument that allowed them to reduce the problem from intersection graphs of *convex bodies* to intersection graphs of *segments*. Radoičić and Tóth [12] generalized this reduction argument to every forbidden bipartite graph H. Following this approach, we can also obtain Theorem 1 as an immediate corollary of

Theorem 10. *For any two positive integers k and t, there is a constant $c_{k,t}$ such that every $K_{k,k}$-free intersection graph of n Jordan arcs in the plane, with no pair intersecting in more than t points, has at most $c_{k,t}n$ edges.*

Notice that Theorem 10 can be proved in exactly the same way as Theorem 1, except that instead of Theorem 4 now we can apply Theorem 3 to the family of compact Jordan regions that can be obtained from the Jordan arcs by slightly "fattening" them. Since no pair of arcs are allowed to cross more than t times, if a $K_{k,k}$-free intersection graph G of n arcs has m edges, we have that the number of crossings between the boundaries of the fattened objects is $O(tm)$. We obtain that G has a separator of size $O(\sqrt{tm}) = O\left(\sqrt{t}n^{1-1/(2k)}\right)$, and Theorem 10 follows.

If F is the family of $K_{k,k}$-free intersection graphs of segments, we have $t = 1$, so that there exist separators of size $O\left(n^{1-1/(2k)}\right)$. In this case, we can apply Theorem 8 with $\phi(n) = O\left(n^{-1/(2k)}\right)$ to obtain that $n_0 < 2^{O(k)}$. This implies that Theorem 5, and hence Theorem 1, hold with $c_F = c_H = 2^{O(k)}$.

Theorem 10 and Theorem 3 can be easily generalized to other orientable surfaces. For instance, it is not hard to see that, for all positive integers k, t, and g, there is a constant $c_{k,t,g}$ such that for every $K_{k,k}$-free intersection graph of n Jordan arcs embedded in an orientable surface of genus g, with no pair intersecting in more than t points, has at most $c_{k,t,g}n$ edges. In order to prove this, one has to establish

Theorem 11. *There is a positive constant c such that the intersection graph of any collection of arcs embedded in an orientable surface of genus with m crossings has a separator of size at most $c\sqrt{gm}$.*

The only difference in the proof is that, instead of the Lipton-Tarjan separator theorem, we have to use its generalization due to Gilbert, Hutchinson, and Tarjan [4]: Every graph with n vertices that can be drawn without crossing in an orientable surface of genus g has a separator of size $O(\sqrt{gn})$.

Notice that Theorem 11 can be regarded as a generalization of the theorem of Gilbert, Hutchinson, and Tarjan. Indeed, their theorem is trivial for $g \geq n$, since every graph has a separator of size n. Thus, we may assume $g < n$. From Euler's formula, every graph with $n \geq 3$ vertices, embeddable in a surface of genus g, has at most $3(n-2+g) < 6n$ edges. If a graph G is embedded in a surface, one can cut each edge at some point in its interior, and replace each vertex v by the union $U(v)$ of the portions of the edges between v and the cut points (including v). By tracing the sets $U(v)$ along their exteriors, we obtain a collection of arcs with no pair intersecting in more than two points. The intersection graph of these arcs is isomorphic to G.

Several interesting questions remain unsolved. What is the right order of magnitude of the constants appearing in Theorem 1? In particular, what is the maximum number of edges that a $K_{2,2}$-free intersection graph of n plane convex bodies can have? We have a construction showing that this number is at least $2n-4$.

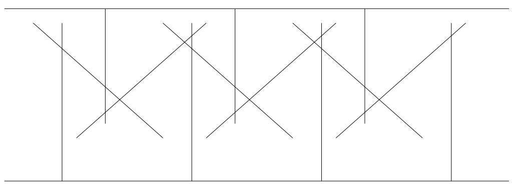

Figure 1. $K_{2,2}$-free intersection graph of n segments

Are there any separator theorems in higher dimensions, other than the result of Miller et al. [9]? Since every finite graph is the intersection graph of a system of three-dimensional convex bodies [13], any nontrivial result of this kind may hold only for special classes of geometric objects. Is it true that any $K_{k,k}$-free intersection graph of n segments in $\mathbb{R}^3$ has at most $O_k(n)$ edges? Do they have bounded chromatic number? Is this true at least for $k=2$?

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E-mail address: jacobfox@math.princeton.edu

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E-mail address: pach@cims.nyu.edu