Combinatorial Genericity and Minimal Rigidity

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Abstract

A well studied geometric problem, with applications ranging from molecular structure determination to sensor networks, asks for the reconstruction of a set P of n unknown points from a finite set of pairwise distances (up to Euclidean isometries). We are concerned here with a related problem: which sets of distances are minimal with the property that they allow for the reconstruction of P, up to a finite set of possibilities? In the planar case, the answer is known **generically** via the landmark Maxwell-Laman Theorem from Rigidity Theory, and it leads to a combinatorial answer: the underlying structure of such a generic minimal collection of distances is a **minimally rigid** (aka Laman) graph, for which very efficient combinatorial decision algorithms exist. For non-generic cases the situation appears to be dramatically different, with the best known algorithms relying on exponential-time Gröbner base methods, and some specific instances known to be NP-hard. Understanding what makes a point set **generic** emerges as an intriguing geometric question with practical algorithmic consequences.

Several definitions (some but not all equivalent) of genericity appear in the rigidity literature, and they have either a measure theoretic, topologic or algebraic-geometric flavor. Some generic point sets appear to be highly degenerate. All existing proofs of Laman's Theorem make use at some point of one or another of these **geometric** genericity assumptions.

The main result of this paper is the first purely combinatorial proof of Laman's theorem, together with some interesting consequences. Genericity is characterized in terms of a certain determinant being not identically-zero as a formal polynomial. We relate its monomial expansion to certain colorings and orientations of the graph and show that these terms cannot all cancel exactly when the underlying graph is Laman. As a surprising consequence, genericity emerges as a purely combinatorial concept.

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1 Introduction

Every computational geometer has encountered assumptions of *generic*, *general position* or *non-degenerate* for some algorithm's input data. We know that without these assumptions, one often has to plunge into complicated case analysis, and that in some cases, a comprehensive way of handling non-generic situations may not exist. Some problems become computationally hard without the genericity assumption. To make sure subtle cases do not pop up to ruin correctness claims, different authors may use different notions of what *generic* means, with some of these concepts appearing to be computationally intractable.

In this paper, we focus on what a *generic point set* is for a well-studied problem: two-dimensional point reconstruction from distances, or planar rigidity.

Our main theoretical result is a new proof of the fundamental theorem of planar rigidity which completely demystifies the genericity assumption by turning it into a purely combinatorial concept. Along the way, we generalize this fundamental theorem to handle other types of rigidity and exhibit some very *degenerate*, yet still *generic* situations that would be very hard to sort out without the tools we develop in this paper. In particular, we establish the correctness of a (very simple and elegant) combinatorial algorithm for a natural *generic* rigidity-theoretic problem (*slider pinning*) that we have recently proposed [22], in a very degenerate situation (axis-parallel sliders).

The Point Reconstruction Problem: *Given a set of $m \leq \binom{n}{2}$ pairwise distances among a set $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_n)$ of n unknown points in Euclidean space $\mathbb{R}^d$, find a possible realization* $\mathbf{p}$. This problem arises naturally in many settings, including molecular structure determination [5] and sensor networks [32]. Implicit in the statement is the following relaxation: which sets of distances *allow* reconstruction of $\mathbf{p}$ up to a finite set of possibilities, modulo Euclidean isometries? This is the *bar-and-joint rigidity problem*, formally defined next.

The Rigidity Problem. An abstract **bar-and-joint framework** is a pair $(G, \boldsymbol{\ell})$, where $G = (V, E)$ is a graph with $n = |V|$ vertices and $m = |E|$ edges, and $\boldsymbol{\ell} \in \mathbb{R}^m$ is a vector of non-negative numbers specifying *edge lengths*. A realization $G(\mathbf{p})$ (in some dimension d) of the abstract framework [1] is a mapping of the vertices of G onto a point set $\mathbf{p} \in (\mathbb{R}^d)^n$ achieving the given edge lengths: $||\mathbf{p}_i - \mathbf{p}_j|| = \boldsymbol{\ell}_{ij}$, $\forall ij \in E$. Intuitively, a bar-joint framework models a structure made of fixed-length bars connected by universal joints, allowing (in principle) full rotation of the bars around them. A bar-joint framework is **rigid** if it has only a discrete set of realizations, up to isometries (complete definitions are given in Section 2.1 below). It is *minimally rigid* if it is rigid, but ceases to be so if any bar is removed.

[1] We abbreviate *bar-and-joint* to *bar-joint* and often refer to (G, ℓ) or $G(p)$ simply as *frameworks*.

The purely geometric question of deciding rigidity of a framework seems to be intractable, even for small, fixed dimension d. The best-known algorithms rely on exponential time Gröbner basis techniques, and specific cases are known to be NP-complete [31]. However, for **generic** frameworks in the plane, the landmark Maxwell-Laman theorem states that rigidity has a combinatorial characterization, for which several efficient algorithms are known (more about this later).

Theorem (Maxwell-Laman [18, 27]). *A generic bar-joint framework $G(\mathbf{p})$ is minimally rigid in $\mathbb{R}^2$ if and only if G has $2n-3$ edges, and every non-empty subgraph induced by n' vertices spans at most $2n'-3$ edges.*

A graph satisfying the counting condition of this theorem is called a *minimally rigid graph* or, abstractly (without any reference to rigidity), a **Laman graph**.

As an important consequence for Computational Geometry (one which is in fact paradigmatic), Maxwell-Laman's theorem allows *generic* rigidity questions to be formulated in terms of combinatorial objects (Laman graphs). But what does it mean for a framework to be *generic*?

An analogy. To make our point, we use an analogy with the best studied problem in Computational Geometry: constructing the convex hull of a planar point set. All known convex hull algorithms work in the purely combinatorial setting of an *order type*, relying on a simple primitive for deciding if an ordered triplet of points makes a *left* or *right* turn. To avoid cluttering the algorithm with case analyses, one assumes that the points are in *general position*: no three are collinear. We know that *most* of the point sets are in general position. We know that if a point set is *not* in general position, then a small perturbation of it must be so; if a point is in general position, then so is a small perturbation of it. We have never seen a paper describing a convex hull algorithm that would assume much stronger *general position* assumptions such as, for instance, asking that the points be *algebraically independent*: such assumptions would make the algorithms useless in practice. We also know that such assumptions are not necessary for this problem.

Generic Rigidity. In contrast, various definitions of genericity that appear in the Rigidity Theory literature are not as clearly amenable to combinatorial descriptions, and some are not as geometrically apparent as general position is: whether a set of points is generic depends on the framework's underlying graph, and geometrically degenerate situations such as collinearities or coincident points may be generic enough for rigidity applications. Some authors [24, p. 92] define a *generic framework* as being one where the points $\mathbf{p}$ are algebraically independent. This definition certainly guarantees the correctness of all the known generic rigidity theorems, but, as we said, it is totally unsatisfactory from a practical point of view: it would certainly raise questions about the validity of any fixed-precision arithmetic implementation. Other frequent definitions used in rigidity theory require that generic properties hold *for most of* the point sets (measure-theoretical) [38, p. 1331] or focus on properties which, if they hold for a point set $\mathbf{p}$ (called generic for the property), then they hold for any point in some open neighborhood (topological) [12].

What the correct concept of genericity should be, seems to depend on the problem, and seems to often have a non-computational character, thus affecting clarity and simplicity in proofs and algorithms as well.

Main Contribution and Novelty: a preview of Combinatorial Genericity. The main contribution of this paper is to clarify, and turn into an entirely combinatorial object, the *genericity* concept for planar rigidity. Along the way, we give a new proof of Laman's Theorem in the more general setting of pinning rigidity. A disclaimer, though: we do *not* propose an efficient algorithm for deciding rigidity in non-generic situations; this seems to be a *much* harder problem.

Here is a preview of our approach. We start with the precise mathematical formulation of the minimum rigidity problem, in terms of the rank of the so-called *rigidity matrix*. We treat the point coordinates as unknowns, and formulate the rank in terms of a certain polynomial (arising from a formal determinant) not being identically zero. We remark that we use in fact the appropriate concept from algebraic geometry, where a property is called *generic* if it holds on the complement of an algebraic variety (zero-set of an algebraic system). In this case, the generic point sets would be those for which this determinant would not vanish. This is of course possible if and only if it is not identically zero, in which case the set of non-generic points has measure zero.

The *main idea*, first occurring in this paper, is to associate a set of combinatorial objects to the formal determinant. Monomials in the Laplace expansion of the determinant give rise to *colorings* and *orientations* of the underlying graph of the framework. The colors arise from the two types of coordinates of the unknown points (x or y) and the orientations from the choice of x_i or x_j in the expansion of a product of terms of the form $(x_i - x_j)$. Monomials may appear multiple times and thus may cancel. To prove that a certain determinant is not identically zero *exactly when the graph is Laman*, it suffices to find a monomial with a unique occurrence. We reduce this problem to finding a unique coloring and orientation of the underlying Laman graph, satisfying a specific degree sequence (which captures the power vector of a monomial).

Related work. Our result should be understood in the context of a wide range of previous work. Here are the most relevant references.

Proofs of Laman's theorem and other genericity conditions. The observation that the $(2n-3)$-counts are necessary for minimal rigidity appears in Maxwell's landmark paper of 1864 [27]. Their sufficiency was proven over 100 years later by Laman [18], who employs what are now called Henneberg constructions [16] on minimally rigid graphs. Whiteley [37] simplified this argument with a very elegant, generic, yet *geometrically degenerate*, choice of vertex positions. Lovász and Yemini [24] give a different proof, assuming that the coordinates of $\mathbf{p}$ do not satisfy *any* polynomial relation with integer coordinates (i.e., they are algebraically independent over $\mathbb{Q}$). Whiteley's proof in [39] implicitly makes use of the same condition. Tay [35] gives a proof based on Crapo's [4] so-called 3T2 decompositons of Laman graphs. His approach is to start with a framework with many zero-length edges and then perturb the endpoints to produce the final realization, a generic point set in non-general position. Our recent work on volume rigidity [34] also yields a new proof of Laman's theorem (*completely* different from what we present here); working in a much more general setting, and proving several extensions of Laman's theorem, we need to employ there non-constructive choices of generic values from complements of algebraic varieties. In contrast, in this paper we make extensive use of the richer combinatorial theory of Laman graphs. Another concept of genericity that appears in the rigidity literature [12] is that there is an open neighborhood N of $\mathbf{p}$ so that $\mathbf{q} \in N$ implies that $G(\mathbf{q})$ is a realization of $G(\mathbf{p})$. This paper's definition of genericity implies this condition.

Pinning frameworks. The problem of pinning bar-joint frameworks in the plane (completely immobilizing, removing all motions, including trivial rigid ones) appears in Lovász [25, 26] and, more recently, Fekete [8]. In their model, a framework is immobilized by fixing both coordinates of a vertex or neither of them. This model for pinning is different from the one we use in our paper [22] and here, in which coordinates are fixed separately. Though the problems are related, they induce different underlying combinatorial structures, and algorithmic solutions.

Combinatorial related work. Laman graphs and their generalizations to sparse graphs are very well-studied combinatorially. Our papers [15, 19, 33] provide an introduction to the combinatorial study of sparsity, and the references given there serve as a guide to the large combinatorial literature that we build upon [9, 14, 28, 30, 36, 37, 39]. A specialization of Crapo's [4] 3T2 decomposition of Laman graphs appears as *induced-cut 2-forest* later in this paper.

Algorithmic rigidity. Although the Laman counts seem to require exponential time to check, all the questions about them, and thus about the generic rigidity of a graph are algorithmically tractable. For the **Decision** question, which asks whether the input is a Laman graph, the best known algorithm, which runs in time $O(n\sqrt{n \log n})$, is due to Gabow and Westermann [10]. The other major algorithmic questions of interest involve extracting a maximum-size Laman subgraph for the input, or finding the inclusion-wise maximal rigid subgraphs of the input. *All the known algorithms for these questions require $O(n^2)$ time, even for inputs with $O(n)$ edges.* See [19] for a more complete discussion of algorithmic rigidity problems and references. The most practical family of algorithms for (various problems about) Laman graphs are based on the elegant **pebble games** of Hendrickson and Jacobs [17], which we have generalized and adapted to other rigidity and combinatorial problems in [19, 21, 33]. We make use of these generalizations here and in [22].

From algebra to combinatorics. Combinatorial objects appear naturally in connection with other algebraic-geometric problems. Examples include perfect matchings, arising from the Pfaffian [23, p. 318], and the combinatorics of Newton polytopes [11]. The closest in spirit to our technique is the proof of infinitesimal rigidity for convex polyhedra of Dehn [7], which relies on expanding a minor of the rigidity matrix and proving that it doesn't vanish. Dehn makes use of the combinatorial structure of his framework (a triangulated planar graph) and of the convexity of the embedding. In contrast, our result is an if-and-only-if characterization, and doesn't employ *any* geometrical information about the embedding. We will interpret determinants of pinned-rigidity matrices via graph colorings and orientations.

Overview of the paper. We give the necessary background in rigidity and the related theory of sparse graphs in Section 2. Section 3 develops our new combinatorial results on unique degree sequences associated with certain colorings of Laman graphs and connects them to the more general concept of (combinatorial) pinned rigidity. Finally, in Section 4, we develop the (geometric) rigidity theory for pinned Laman graphs and give the new proof of the Maxwell-Laman theorem, by relating the unique degree sequences to the monomial expansion of a not-identically-zero determinant.

2 Preliminaries

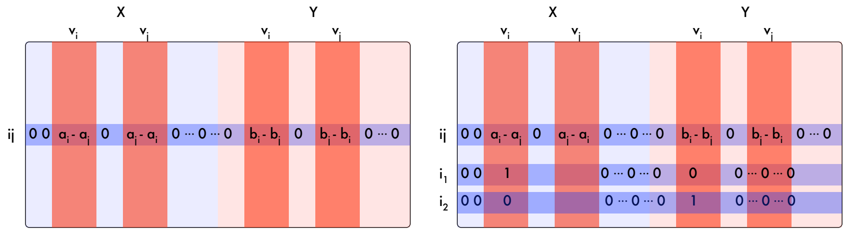
We refer the reader to [13, 38] for an introduction to rigidity theory. For a self-contained presentation, we briefly introduce now the most relevant results.

2.1 Rigidity background

Planar bar-and-joint rigidity relies on three fundamental concepts, built upon one another: continuous, infinitesimal and combinatorial rigidity. We will follow the same paradigm in Section 4, for our new model of pinned rigidity.

Notation. We will identify $\left(\mathbb{R}^2\right)^n$ with $\mathbb{R}^{2n}$ and consider a point set $\mathbf{p} \in \left(\mathbb{R}^2\right)^n$ as either a vector of n points, with $\mathbf{p}_i = (a_i, b_i)$, or as a flattened vector $(a_1, b_1, \ldots, a_n, b_n) \in \mathbb{R}^{2n}$.

Frameworks and continuous rigidity. We are interested in analyzing the rigidity properties of a particular framework $G(\mathbf{p})$. Its **configuration space** $\mathcal{C}(G(\mathbf{p}))$ (shortly $\mathcal{C}$) is the set of all other realizations of the edge lengths of $G(p)$: $\mathcal{C} = \{\mathbf{q} \in \mathbb{R}^{2n} : G(\mathbf{q}) \text{ is a realization of } G(\mathbf{p})\}$. Applying a Euclidean isometry to $\mathbf{p}$ results in a new realization of $G(\mathbf{p})$. To factor out these equivalent realizations, we take the quotient of $\mathcal{C}$ by the group Γ of Euclidean isometries. We say that a framework $G(\mathbf{p})$ is **rigid** if $\mathbf{p}$ is isolated in the quotient topology of $\mathcal{C}/\Gamma$.



(a) (b)

Figure 1: The pattern of the rigidity matrix: (a) the matrix $\mathbf{M}(G)$; (b) the pinned rigidity matrix $\mathbf{M}^\star(G)$; the two rows i_1 and i_2 correspond to fixing both coordinates of the vertex i.

Infinitesimal rigidity. Rigidity of $G(p)$ is a difficult property to establish. Instead, one uses the linearization of the problem. Taking the differential of $\mathcal{C}$ at $\mathbf{p}$ gives rise to the **rigidity matrix**: an $m \times 2n$ matrix with its rows indexed by the edges of G and two columns for each vertex, one for each coordinate. We order the columns so that they form two blocks: the first n correspond to x-coordinates and the second n correspond to y-coordinates. The row for edge ij has $a_i - a_j$ in the column for vertex i's x-coordinate and $a_j - a_i$ in the column for vertex j's x-coordinate; the y-coordinates similarly contain $b_i - b_j$ and $b_j - b_i$; and all the other entries are zero. Figure 1 (a) shows the pattern.

A framework is **infinitesimally rigid** if its **rigidity matrix** $\mathbf{M}(G(\mathbf{p}))$ has corank 3. It is well known [1] that if $G(p)$ is infinitesimally rigid, then it is rigid.

Generic combinatorial rigidity. A framework $G(\mathbf{p})$ is **generic** if the rank of the rigidity matrix $\mathbf{M}(G(\mathbf{p}))$ is maximum over all choices of $\mathbf{p}$. *Combinatorial rigidity* is concerned with finding good characterizations of the *graphs* of generically rigid frameworks. In dimensions $d \geq 3$, no combinatorial characterizations are known, but dimension two is fully understood due to Maxwell-Laman's Theorem.

2.2 Combinatorial rigidity and sparse graphs

We summarize now the relevant combinatorial properties of Laman graphs.

Sparse graphs. Let $G = (V, E)$ be a graph with $n = |V|$ vertices and $m = |E|$ edges; in this paper we will not encounter multiple edges, but we do allow self-loops (shortly, loops). A graph G is (k, ℓ)**-sparse** if every non-empty subgraph induced by n' vertices spans $m' \leq kn - \ell$. If, additionally, G has $kn - \ell$ edges, then G is (k, ℓ)**-tight**.

In particular, the Laman graphs of the Maxwell-Laman theorem are $(2, 3)$-tight. We observe that the sparsity parameters for Laman graphs imply that they are simple (no parallel edges), and that they do not contain loops. We will also employ a characterization of Laman graphs in terms of a special decomposition into forests.

A 2-coloring of the edges of a graph is an **induced-cut 2-forest** if each color forms a forest and any induced subgraph contains a monochromatic cut; graphs admitting such a coloring are exactly Laman graphs [2].

Proposition 1 (**Laman graphs are induced-cut 2-forests** [**33**]). *Let G be a graph with n vertices and $2n-3$ edges. Then G is a Laman graph if and only if it can be colored by an induced-cut 2-forest .*

Proposition 1 is related to another characterization of Laman graphs used in this paper.

Proposition 2 (**Adding one edge to a Laman graph** [**24, 30**]). *Let G be a graph with n vertices and $2n-3$ edges. Then G is a Laman graph if and only if adding any edge to it results in a graph that decomposes into two edge-disjoint spanning trees.*

Haas [14] has generalized this result to all sparse graphs with $k < \ell < 2k$. A further generalization, needed for modeling the slider-pinning problem described in Section 3, is in terms of **map-graphs**. A map-graph is an undirected graph which admits an orientation with out-degree exactly one. Equivalently [29] a map-graph has exactly one cycle per connected component, counting loops as cycles.[3] Map-graphs are known to coincide with $(1,0)$-tight graphs, and graphs which decompose into k edge-disjoint map-graphs (k-map-graphs) coincide with $(k,0)$-tight graphs. In a previous paper we proved a characterization of sparse graphs in terms of map-graphs.

Proposition 3 (**Adding edges and loops to sparse graphs** [**15**]). *Let G be a graph with n vertices and $kn-\ell$ edges. Then G is (k,ℓ)-tight if any only if adding any ℓ edges (including loops) to it results in a k-map-graph.*

Here the added edges come from $K_n^\star$, the complete graph on n vertices with k loops on every vertex and $2k$ copies of each edge. In particular, adding any three loops (not all on one vertex) to a Laman graph results in a 2-map. Our paper [33] gives a more algorithmic treatment of this topic.

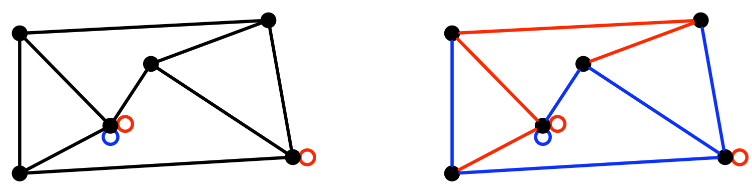
3 Combinatorial pinned rigidity

This section describes the *combinatorial essence* of our results. As previewed in the Introduction, we aim at studying the maximum rank of a generic rigidity matrix

[2] We remind the reader that this concept is a specialization of the 3T2 characterization of Laman graphs given by Crapo [4], and it appeared under that name in our paper [33]. We have changed our terminology to highlight the additional condition that the three trees in a Laman graph must form two forests, which Crapo does not require.

[3] Map-graphs are also known in the matroid literature as pseudotrees, pseudoforests, functional graphs, and bases of the bicycle matroid.

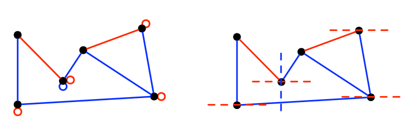
derived from a Laman graph. A slight generalization will come in handy: *pinned Laman graphs*. The monomial expansion of the formal rigidity matrix of a pinned Laman graph will be expressed as a sum of terms that are in one-to-one correspondence with the labeled, colored in-degree sequences of induced-cut 2-forests compatible with the colored loops, defined in this section.



(a) (b)

Figure 2: Pinned Laman graphs: (a) with pre-colored loops; (b) the same pinned graph with a compatible induced-cut 2-forest .

Pinned Laman graphs and mechanisms. Let G be a graph with n vertices, $2n-c$ edges and c loops with specified colors (blue or red). We say that G is an **axis-parallel slider-pinned Laman mechanism** (shortly, a pinned Laman mechanism) if the edges of G can be colored as an induced-cut 2-forest so that each monochromatic tree contains exactly one loop of its color. We observe that this implies that G is a 2-map-graph and that G without the loops is $(2,3)$-sparse. Figure 2 shows an example. We call a pinned Laman mechanism with $2n-3$ edges an **axis-parallel slider-pinned Laman graph** (shortly, a pinned Laman graph). Adding any three loops, not all of the same color, to a Laman graph results in a pinned Laman graph. As above, the added loops come from $K_n^\star$, with the additional restriction that each vertex has exactly one red loop and one blue one.



(a) (b)

Figure 3: Pinned Laman mechanisms and slider pinnning: (a) the combinatorial object; (b) the associated axis-parallel bar-slider framework.

Lemma 4 (**Pinned Laman graphs**)**.** *Let G be a graph with $2n-3$ edges. Then G is a Laman graph if and only if adding any three colored loops to G, not all of the same color, results in a pinned Laman mechanism.*

Proof. If G can be extended by three loops to a pinned Laman graph, then Proposition 1 implies immediately that G is a Laman graph. For the other direction, we suppose that G is a Laman graph. In this case, any coloring of G into two forests is an induced-cut 2-forest , since no subgraph has enough edges to induce two edge-disjoint spanning trees. Now suppose that there are two red loops on necessarily distinct vertices i and j. Add the edge ij, which may be a copy of an existing edge, to G. By Proposition 2 the resulting graph decomposes into two edge-disjoint spanning trees. We may assume that the added edge ij is red. Removing it and keeping the coloring of all the other edges gives the induced-cut 2-forest we need: what is left is a blue spanning tree (and thus incident with the third, blue loop) and two disjoint red trees each containing exactly one red loop. □

Slider pinning. The terminology of pinned Laman mechanisms comes from our previous work on immobilizing bar-joint frameworks by adding **sliders**, which force a vertex to move on a given line [22]. For the specific case of axis-parallel sliders, this amounts to adding an equation to pin down one of the coordinates of the vertex. We introduced pinned Laman mechanisms in [22] as a combinatorial model for these axis-parallel bar-slider structures and studied (a generalization of) their combinatorics in [21]. It is important to note that Lemma 4 does *not* hold for arbitrary pinned Laman mechanisms; the allowed locations and colors for completing a $(2, 3)$-sparse graph to a pinned mechanism depend in a strong way on where the edges are. However, the pinned mechanisms do form the bases of a matroid for which we have developed the combinatorial and algorithmic theory [21, 22]. Corollary 9 below provides a Laman-type theorem for bar-slider frameworks. Figure 3 illustrates the relationship between pinned Laman mechanisms and bar-slider frameworks.

Colored in-degree sequences of pinned Laman mechanims. Let G be a pinned Laman mechanism with n vertices and fix an induced-cut 2-forest coloring of the edges of G that certifies to this (i.e., it is compatible with the colors of the loops). Now fix an orientation of the edges of G; we use the convention that an oriented loop points both into and out of the vertex it is on. This leads to a **labeled colored in-degree sequence** $(\mathbf{r}, \mathbf{b})$, with $\mathbf{r}_i$ and $\mathbf{b}_i$ being the number of red and blue edges pointing at vertex i. In the next section, these will be given an algebraic interpretation as monomials in the expansion of a determinant.

The main result of this section is the following uniqueness result for colored in-degree sequences of Laman mechanisms. This is the *critical* combinatorial step in our proof of the Maxwell-Laman theorem in the next section.

Lemma 5 (**Existence of a unique colored in-degree sequence).** *Let G be a pinned Laman mechanism. Then there is an induced-cut 2-forest compatible with G and an orientation of the edges of G so that the resulting colored in-degree sequence cannot be obtained by any other orientation of G and compatible induced-cut 2-forest .*

Proof. The proof is by induction on n. The base case of $n = 1$ is clear, so we concentrate on the inductive step. Fix an induced-cut 2-forest compatible with the loops of G (by definition, one exists). Then G contains a monochromatic cut of some $c \geq 1$ blue cut edges. Removing these cut edges leaves two disconnected subgraphs $G' = (V', E')$ and $G'' = (V'', E'')$. In each of G' and G'' there are now blue trees that may not contain a blue loop; add a blue loop to each of these at a vertex adjacent to a cut edge. With this modification G' and G'' are smaller pinned Laman mechanisms on n' and n'' vertices, and we may apply the inductive hypothesis to each of them to find colorings and orientations that produce unique in-degree sequences $(\mathbf{r}', \mathbf{b}')$ and $(\mathbf{r}'', \mathbf{b}'')$ in the sense of the lemma.

We now observe that removing the added loops and adding back the cut edges colored blue results in a (possibly new) induced-cut 2-forest that is compatible with G, since this process always joins two disjoint blue trees and removes one of exactly two loops spanned by the larger tree. Since only c loops were added to G' and G'' in total, but together they span at least $c+1$ blue loops, we may assume that after adding back the cut edges and (w.l.o.g.) orienting them all toward V' results in a colored in-degree sequence $(\mathbf{r}, \mathbf{b})$ for G such that $B = \sum_{i \in V'} \mathbf{b}_i > n'$. We argue now that this colored in-degree sequence is the one we want. For contradiction, assume the contrary. Then there is some other compatible coloring of G and an orientation of it with the degree sequence $(\mathbf{r}, \mathbf{b})$. Whatever this new coloring does, the total contribution to the blue in-degree of V' by edges in the induced subgraph is at most n'. This implies that to have $B = \sum_{i \in V'} \mathbf{b}_i$, all of the original cut edges are blue and oriented as by our construction. But then any other way of obtaining the same degree sequence contradicts the uniqueness in the inductive hypothesis. $\square$

Algorithms. Although we are concerned mainly with theory in this extended abstract, we remark that a recent elegant segment tree data structure of Daescu and Kurdia [6] can be used as the basis of an algorithm to obtain one of the unique degree sequences of Lemma 5 by implementing the construction used in the proof. Its overall complexity will be the same as the best algorithm for finding a decomposition into two spanning forests ($O(n\sqrt{n \log n})$, cf. [10]).

We now have the combinatorial tools we will need to prove Laman's theorem. In the next section, we develop the rigidity part of the proof.

4 Main result: Laman's theorem via pinned rigidity

We are ready for our main result: a purely combinatorial approach to generic bar-joint rigidity in the plane. More precisely, we develop, formally, a rigidity theory for structures made from bars, joints and axis-parallel slider-pins (which also serves as the formal setting, not developed anywhere so far, for the more general slider pinning model underlying our algorithms from [22].) The structure of this section echoes our presentation of (unpinned) bar-joint rigidity in Section 2.1, in its development of the three concepts of continuous, infinitesimal and combinatorial rigidity. Although we develop our theory in the general setting of axis-parallel bar-slider frameworks, we use the example of a pinned Laman mechanism throughout, both to establish the connection to Laman rigidity and for intuition.

Pinned rigidity by pinning an edge. Let $G(\mathbf{p})$ be a framework in the plane. We pin down an edge ij by fixing the coordinates of its endpoints. Let $G^\star_{ij}$ be the pinned Laman graph obtained from G by adding three loops (not all of the same color) on i, j. Define the ij**-pinned configuration space** of a pinned framework $G^\star_{ij}(\mathbf{p})$ as $\sigma_{ij}(\mathcal{C}) = \{\mathbf{q} \in \mathbb{R}^{2n} : \mathbf{q}_i = \mathbf{p}_i, \mathbf{q}_j = \mathbf{p}_j, \text{ and } G(\mathbf{q}) \text{ realizes } G(\mathbf{p})\}$. Note that one of the four equations added to pin i and j is made redundant by the equation fixing the distance between i and j. In what follows, we assume that it is omitted.

A pinned framework is **pinned rigid** when $\mathbf{p}$ is an isolated point of $\sigma_{ij}(\mathcal{C})$ and flexible otherwise. We remark that pinned rigidity, unlike rigidity, is defined in terms of an algebraic condition. However, it has an apparent dependency on the choice of edge to pin. We can remove this in the simple case where only the endpoints of an edge are pinned. The next Lemma justifies studying Laman rigidity properties via pinned frameworks and dropping the subscript ij for pinned frameworks.

Lemma 6 (Edge-pinned rigidity concides with rigidity). *A framework $G(\mathbf{p})$ is rigid if and only $G^\star_{ij}(\mathbf{p})$ is pinned rigid for* any *choice of edge ij to pin.*

Proof. Recall that the framework $G(\mathbf{p})$ is rigid if and only if $\mathbf{p}$ is topologically isolated in the quotient $\mathcal{M} = \mathcal{C}/\Gamma$ of the configuration space by the group Γ of Euclidean isometries. Let $\pi : \mathcal{C} \to \mathcal{M}$ be the projection map. Since pinning down an edge removes the isometries from the configuration space, pinning down assigns a representative in $\mathcal{C}$ to every point of $\mathcal{M}$. By abuse of notation, we define $\sigma : \mathcal{M} \to \sigma(\mathcal{C})$ as the corresponding section of π.

To make the rest of the proof easier to read, we summarize the objects related to $G(\mathbf{p})$ in the following diagram:

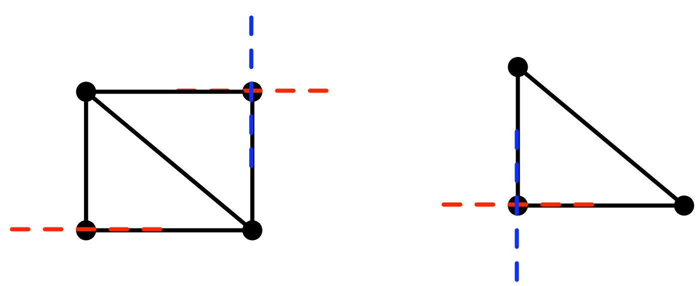
$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] & \sigma(\mathcal{C}) \arrow[l, hook'] \arrow[dl, bend left, "\pi|_{\sigma(\mathcal{C})}"] \\ \mathcal{M} \arrow[ur, bend left, "\sigma"] & \end{tikzcd} \tag{1}$$

We will show that $\sigma(\mathcal{C})$ is homeomorphic to $\mathcal{M}$. This stronger statement implies the lemma, since then $\mathbf{p}$ is isolated in $\mathcal{M}$ if and only if it is isolated in the pinned configuration space. What remains is to check the claimed homeomorphism. By definition π is continuous and onto. Since the quotient space $\mathcal{M}$ contains exactly one point for each equivalence class induced by Euclidean isometries, the restriction $\pi|_{\sigma(\mathcal{C})}$ to pinned configurations is also one-to-one.

To complete the proof, we check that $\sigma = \left(\pi|_{\sigma(\mathcal{C})}\right)^{-1}$ is continuous. We do this by showing that for a closed set $X \subset \mathcal{C}$, $\pi^{-1}(\pi(X))$ is also closed. Let $\mathbf{x}$ be a point in X, and let $\mathbf{x}^i \to \mathbf{x}$ be a sequence in X converging to $\mathbf{x}$. We may assume without loss of generality that $\mathbf{x} \in \sigma(\mathcal{C})$ by applying a Euclidean isometry to the entire sequence. Suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the endpoints of the pinned edge and define the sequence $\mathbf{T}_i = (\mathbf{A}_i, \mathbf{b}_i)$ of Euclidean motions taking $\mathbf{x}_1^i$ and $\mathbf{x}_2^i$ to $\mathbf{x}_1$ and $\mathbf{x}_2$.

Now we observe that, for large enough i, $\mathbf{T}_i$ is in a compact neighborhood of $(\mathbf{I}, 0) \in \Gamma$, where $\mathbf{I}$ is the identity matrix. It follows that $\mathbf{T}_i$ converges to $(\mathbf{I}, 0)$ and, consequently, that $\mathbf{T}_i\mathbf{x}^i \to \mathbf{x}$. Thus $\mathbf{x} \in \pi^{-1}(\pi(X))$, and since $\pi^{-1}(\pi(X))$ contains all of its limit points, it is closed.

Finally, we observe that the choice of edge to pin was arbitrary. $\square$



(a) (b)

Figure 4: Two embeddings of the same axis-parallel bar-slider framework: (a) generic; (b) non-generic. In (b) one triangle is reflected along the diagonal edge, resulting in coincident vertices and overlapping edges.

Pinned rigidity for axis-parallel bar-slider frameworks. We now consider the case of a general axis-parallel bar-slider framework $G^\star(\mathbf{p})$, giving the continuous theory for slider pinning. Here, the configuration space is given by

$$\sigma_{G^\star}(\mathcal{C}) = \\ \{\mathbf{q} \in \mathbb{R}^{2n} : G(\mathbf{q}) \text{ realizes the underlying framework } G(\mathbf{p}), \\ \text{and } \mathbf{q} \text{ agrees with } \mathbf{p} \text{ on pinned coordinates}\}$$

In other words, we add an equation $a_i = \text{const}$ or $b_i = \text{const}$ for each horizontal or vertical slider. As in the case of pinning an edge, $G^\star(\mathbf{p})$ is pinned rigid when $\mathbf{p}$ is an isolated point in $\sigma_{G^\star}(\mathcal{C})$ and flexible otherwise.

We observe that generic point sets for these more general slider-pinned frameworks are only a subset of those for unpinned frameworks. For example, both of the embeddings in Figure 4 are generic for the underlying bar-joint framework, but

Figure 4 (b) is not pinned (it can rotate around the bottom left vertices, which are coincident). Thus an analogue of Lemma 6 does *not* hold for a general pinned Laman graph: there are generic point sets for which not all combinatorial pinned Laman graphs obtained from the underlying framework give rise to a pinned bar-slider framework. However, the same argument used to prove Lemma 6 yields the following relaxation.

Lemma 7 (Pinned rigidity concides with rigidity). *A framework $G(\mathbf{p})$ is rigid if and only if for some pinned Laman graph $G^\star$ obtained by adding three loops to G gives rise to a pinned rigid bar-slider framework $G^\star(\mathbf{p})$.*

We next develop the infinitesimal theory, which is the same for both edge-pinned frameworks and axis-parallel bar-slider frameworks.

Pinned infinitesimal rigidity. Let $G^\star(\mathbf{p})$ be an axis-parallel bar-slider framework with m edges and k sliders, each of which pins down one coordinate of the vertex it constrains. The **pinned rigidity matrix** $\mathbf{M}^\star(G^\star(\mathbf{p}))$ is an $(m+k)\times 2n$ matrix that has one row for each edge $ij \in E$ and three additional rows corresponding to the pinned vertices i and j. The rows corresponding to edges have the same form as those of the rigidity matrix. The row corresponding to pinning the x-coordinate of vertex i has a 1 in the x-coordinate column associated with vertex i and zeros elsewere. Rows for pinning the y-coordinates of vertices i and j are defined similarly. Figure 1 (b) shows the pattern of the pinned rigidity matrix. Like the rigidity matrix, $\mathbf{M}^\star(G^\star(\mathbf{p}))$ arises from the differential of $\sigma_{G^\star}(\mathcal{C})$ at $\mathbf{p}$. We note that the rows of $\mathbf{M}^\star(G^\star(\mathbf{p}))$ span the normal space of $\sigma_{G^\star}(\mathcal{C})$ at $\mathbf{p}$, and the tangent space at $\mathbf{p}$ is identified with the kernel of $\mathbf{M}^\star(G^\star(\mathbf{p}))$.

An axis-parallel bar-slider framework is **infinitesimally rigid** if the rank of $\mathbf{M}^\star(G^\star(\mathbf{p}))$ is $2n$ and infinitesimally flexible otherwise. With the observation above we have the following lemma relating rigidity and pinned rigidity in the case of a pinned Laman graph.

Lemma 8 (Pinned infinitesimal rigidity implies rigidity). *Let $G^\star(\mathbf{p})$ be a pinned Laman graph. If $G^\star(\mathbf{p})$ is infinitesimally pinned rigid, then it is pinned rigid. Moreover, the underlying unpinned framework $G(\mathbf{p})$ is rigid.*

In the proof we will make use of the **complexification** of the configuration space $\mathcal{C}(\mathbb{C})$ as a technical tool. This is defined in the same way as the configuration space, except we interpret the unknowns in the distance equations as complex numbers. Thus $\mathcal{C}(\mathbb{C})$ is an algebraic subset of $\mathbb{C}^{2n}$. The complexification of the pinned configuration space $\sigma_{G^\star}(\mathcal{C}(\mathbb{C}))$ is defined similarly. Both the rigidity matrix and the pinned rigidity matrix have the same form in the complex setting.

Proof. We suppose that $\mathbf{M}^\star(G^\star(\mathbf{p}))$ has rank $2n$. As noted above, we can identify the tangent space of $\sigma_{G^\star}(\mathcal{C}(\mathbb{C}))$ at $\mathbf{p}$ $T_{\mathbf{p}}(\sigma_{G^\star}(\mathcal{C}(\mathbb{C})))$ with the kernel of $\mathbf{M}^\star$. A fundamental result of algebraic geometry says that the dimension of the irreducible components of $\sigma_{G^\star}(\mathcal{C}(\mathbb{C}))$ through $\mathbf{p}$ is bounded by the dimension of $T_{\mathbf{p}}(\sigma_{G^\star}(\mathcal{C}(\mathbb{C})))$ [3, p. 479, Theorem 8].

When the rank of $\mathbf{M}^\star(G^\star(\mathbf{p}))$ is $2n$, the tangent space is zero. By the previous discussion, this implies that $G(\mathbf{p})$ is pinned rigid. By Lemma 7, this implies that $G(\mathbf{p})$ is rigid. $\square$

Remark: Lemma 8 gives an alternative proof of the main theorem of [1] for the case where $d = 2$.

Generic combinatorial pinned rigidity. As in the unpinned case, a pinned framework $G^\star(\mathbf{p})$ is **generic** when the rank of $\mathbf{M}^\star(G^\star(\mathbf{p}))$ is maximum over all choices of $\mathbf{p} \in \mathbb{R}^{2n}$.

Equivalently, we can consider the **generic pinned rigidity matrix $\mathbf{M}^\star(G^\star)$**. This has the same form as $\mathbf{M}^\star(G^\star(\mathbf{p}))$, but has indeterminate entries of the form $a_i - a_j$ and $b_i - b_j$ instead of concrete numbers. The rank of the generic matrix $\mathbf{M}^\star(G^\star)$ is then given by the size of a maximum minor which is not zero as a formal polynomial.

We observe that the generic matrix $\mathbf{M}^\star(G^\star)$ depends only on the underlying graph G, the locations of the sliders, and the coordinate each sliders pins. Thus we will use the pinned Laman mechanisms of the previous section as the combinatorial model of pinned frameworks. Recall that a pinned Laman graph has two red loops and one blue one. We interpret the color of the loops as indicating which coordinate of that vertex to pin, putting the loops in correspondence with the rows of the pinned rigidity matrix associated with pinning.

Comparison to other genericity concepts. At this point, for emphasis, we remind the reader of the stronger concept of genericity that appears in the rigidity literature (to contrast it with ours): the requirement that the coordinates of the vertices be algebraically independent over $\mathbb{Q}$. Frameworks on a variety of *degenerate* point sets, including those having small integer coordinates, would never be generic in this model, making the theory unsuitable for algorithmic purposes.

Proof of Laman's theorem. We now have all the ingredients for a combinatorial proof of Laman's theorem. Here we concentrate on the difficult ("Laman") direction: we will prove that every Laman graph can be realized as a generic, minimally rigid framework. In what follows, we will use the notation $\mathbf{A}[I, J]$ for the submatrix of an $m \times n$ matrix $\mathbf{A}$ induced by the set of rows $I \subset [m]$ and columns $J \subset [n]$.

Proof. Let G be a Laman graph. By Lemma 4, G can be extended to a pinned Laman graph $G^\star$ by adding three loops to the endpoints of any edge. We now consider the generic pinned rigidity matrix $\mathbf{M}^\star$ of $G^\star$. This is a $2n \times 2n$ matrix. We will show that its determinant is non-zero as a formal polynomial, implying that a generic framework with the underlying graph G is minimally rigid by Lemma 8.

Let $X = [n]$ and $Y = [2n]-[n]$. Using the Laplace expansion for the determinant around X we obtain $\det(\mathbf{M}^\star)$ as plus or minus

$$\sum_{B\subset[2n],|B|=n} \det(\mathbf{M}^\star[B,X])\det(\mathbf{M}^\star[R,Y])$$

where the sum is taken over sets of blue edges B and red edges $R = [2n] - B$.

We now interpret each term in the sum combinatorially. If the set of edges corresponding to B does not correspond to a map-graph where the blue loop forms the only cycle, then $\det(\mathbf{M}^\star[B,X])$ is zero since any cycle of edges leads to a dependency and any row corresponding to a red loop induces a row of all zeros. A similar argument applies to $\det(\mathbf{M}^\star[R,Y])$. It follows that combinatorially, the non-zero terms in the expansion correspond to the induced-cut 2-forests of G that are compatible with the added loops. Each of these non-zero terms is of the form

$$\det(\mathbf{M}^\star[B,X])\det(\mathbf{M}^\star[R,Y]) = \pm\left(\prod_{ij\in B}(a_i - a_j)\right)\left(\prod_{ij\in R}(b_i - b_j)\right)$$

To complete the proof, we show that they do not cancel out. The critical observation is that each monomial in the expansion corresponds to picking an orientation of an induced-cut 2-forest and using the blue in-degree as the power of a_i and the red in-degree as the color of b_i. Lemma 5 implies that there is a combinatorially unique monomial, which cannot be canceled symbolically. □

Remark: The rigidity matrix of a Laman graph is not square, and thus it may have many non-singular $(2n-3)\times(2n-3)$ minors. In light of Lemma 4, we can interpret all the possible ways of slider-pinning a Laman graph as picking out a particular minor to test. Since there are only a finite number of these, we have shown that, for a given Laman graph G almost all point sets $\mathbf{p}$ are generic for every extension of G to a pinned Laman graph. The same argument applied to a pinned Laman mechanism establishes a Laman-type theorem for bar-slider frameworks, completing the proof of correctness of our slider pinning algorithms from [22].

Corollary 9 (Generic axis-parallel bar-slider rigidity). *Let $G^\star$ be a graph with n vertices, $2n-k$ edges, and k colored loops. $G^\star$ is realizable as a generic slider-pinned axis-parallel bar-slider framework if and only if $G^\star$ is a pinned Laman mechanism.*

The case for sliders that are differentiable curves follows from this. We do not go into details here. Corollary 9 also implies that for a $(2,3)$-sparse graph, every possible way to extend it to a pinned Laman mechanism can be realized as a generic pinned bar-slider framework.

5 Conclusions and open questions

We gave a new, completely combinatorial, proof of Laman's landmark characterization of planar generic rigidity. Along the way, we introduced a new approach to rigidity and genericity which reduces the problem to elementary combinatorics, completely avoiding complicated geometric arguments.

Although Laman graphs have been well-studied over the past 30 years, our work here introduces oriented colorings of their induced-cut 2-forests as interesting objects of study. In particular, given the close connection between induced-cut 2-forests and the rigidity matrix, enumerating them (and understanding their cancelation patterns) would be interesting.

Some prominent remaining open questions include: (1) finding efficient (combinatorial) algorithms for deciding *rigidity* (as opposed to infinitesimal rigidity, which can be decided by Gaussian elimination) in non-generic cases (alternatively, prove NP-hardness); (2) *extracting* a spanning Laman subgraph from a dense graph in time $o(n^2)$ [20]. In constrast, the problem of *deciding* whether a graph is Laman is known to be $o(n^2)$. This last problem has recently received renewed attention [2, 6], and simplifications for a part of an older $O(n\sqrt{n \log n})$ algorithm of [10] have been proposed, within the same overall asymptotic complexity.

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