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DDA ERROR ANALYSIS USING SAMPLED DATA TECHNIQUES

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Summary

Sampled data techniques were first applied to digital operations by Linvill and Saltzer[1,2] in order to study the errors resulting from the use of numerical integration techniques. The purpose of this paper is to develop an understanding for the mechanics of errors in the digital differential analyzer and to evolve a conceptually simple error theory. This is accomplished by establishing some of the basic comcepts regarding the applications of sampled data techniques to the integration process; developing the matrix model for the solution of a system of linear differential equations with constant coefficients on a digital differential analyzer and then using the W-transform to finalize the error theory. It is then easily shown that simple adjustments to the coefficient matrix of linear differential equations with constant coefficients will allow one to obtain solutions to these equations the accuracy of which is limited only by round-off errors.

Mechanics of Error in the DDA

Fundamentals

Before progressing directly into the analysis of DDA systems, it is desirable to examine briefly some of the tools to be used. The discrete nature of the variables in digital systems allows the use of sampled data techniques[3] which are already widely used. However, some of the concepts involved in DDA systems are sufficiently different to warrant special attention. Of particular interest is the mechanics of integration.

If y(t) is a continuous function, its sampled counterpart can be represented by y*(t) where

$$y^*(t) = y(t) \cdot \sum_{n=1}^{\infty} \delta(t-nT) \qquad (1)$$

It is to be noticed that given y(t), y*(t) is uniquely determined. However, the inverse relation is not unique. That is, given y*(t), there are an infinite number of functions y(t) that will yield the same sampled function.

Quite frequently it is desirable to convert (by some practical technique) the sampled function back into a continuous function. If a Fourier analysis of the unsampled signal shows that it contains frequencies $f < f_o/2$ where f_o is the sampling frequency, the unsampled function can theoretically be recovered identically by passing the sampled function through an ideal low-pass filter that passes all frequencies below f_o. (This is simply a paraphrase of the sampling theorem.) As a matter of fact, in order to theoretically recover the original signal, the sampling frequency is required only to be at least twice the bandwidth of the unsampled signal.

In the following work, recovery by practical techniques is not the point in question. The concern here is with the sampled function that has resulted and the sampled function that is desired. This will be done by sampling the original continuous function, operating on this sampled function with approximate operators and then comparing the result to the sampled function derived from performing the true operation on the original function and then sampling this true function. Hence the non-uniqueness of the derived sampled function is not a matter of concern nor are any of the previously mentioned recovery restrictions.

The Laplace transform of the sampled y(t) is given by the convolution of Y(s) and the Laplace transform of $\delta(t-nT)$. This can be written in two different ways:

$$Y^*(s) = \frac{1}{2\pi j} \int_c Y(\omega) \cdot \frac{1}{1 - e^{-(s-\omega)T}} \cdot d\omega \quad (2)$$

$$Y^*(s) = \frac{1}{2\pi j} \int_c \frac{1}{1 - e^{-\omega T}} \cdot Y(s-\omega) \cdot d\omega \quad (3)$$

where residues are considered only at the poles of the first function inside the integral sign.

The two forms shown yield considerably different appearing results. Equation (3) yields residues at the poles of $\frac{1}{1 - e^{-\omega T}}$ while (2) yields residues at the poles of $Y(\omega)$. Both equations shed some light on the theory of sampled data systems.

Because of the prevalence of the function e^{-sT} in the equations that follow, this work will use $z=e^{-sT}$ as a substitution where it will simplify the equations. Another substitution that will be used is $a=e^{-\alpha T}$. If these substitutions are used, the following transforms may be obtained.

$y(t)$	$Y(s)$	$Y^*(s)$	
c	$\frac{c}{s}$	$\frac{c}{1-z}$	(4)
ca	$\frac{c}{s+\alpha}$	$\frac{c}{1-az}$	(5)
cta	$\frac{c}{(s+\alpha)^2}$	$\frac{Taz}{(1-az)^2}$	(6)
$\frac{ct^n a}{n!}$	$\frac{c}{(s+\alpha)^{n+1}}$	$\frac{1}{n!}\frac{d^n}{d\omega^n}\left[\frac{1}{1-az\,e^{\omega T}}\right]_{\omega=-\alpha}$	(7)

If concern is maintained for the solution of linear or piece-wise linear systems, these pairs are the only ones needed.

Let $y(t)$ be a continuous function and let

$$x(t) = \int_0^t y(t)\,dt. \tag{8}$$

Let $Y(s)$ and $X(s)$ be the Laplace transforms of $y(t)$ and $x(t)$, respectively, and $Y^*(s)$ and $X^*(s)$ be the Laplace transforms (with z substituted for e^{-sT}) of their sampled counterparts.

Then using the convolution integral, (2),

$$Y^*(s) = \frac{1}{2\pi j}\int_c Y(\omega)\cdot\frac{1}{1-ze^{\omega T}}\cdot d\omega \tag{9}$$

$$X^*(s) = \frac{1}{2\pi j}\int_c X(\omega)\cdot\frac{1}{1-ze^{\omega T}}\cdot d\omega$$

$$= \frac{1}{2\pi j}\int_c \frac{Y(\omega)}{\omega}\cdot\frac{1}{1-ze^{\omega T}}\cdot d\omega. \tag{10}$$

Since sampling and summation are commutative, $y(t)$ can be broken into its exponential components. This is most easily accomplished in the Laplace transform domain by partial fraction expansion.

In general, then $Y(s)$ will be of the form

$$Y(s) = \sum_{r=1}^{m}\frac{A_{o,r}}{s^r} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{A_{i,r}}{(s+\alpha_i)^r}. \tag{11}$$

The reason for making the pole at zero a special case is that integration creates a special case for poles at the origin.

In order to understand the mechanics of integration, consider a typical term from each of the sums of (11).

First, let

$$Y(s) = \frac{A_{o,n}}{s^n} \tag{12}$$

Then

$$Y^*(s) = \frac{1}{2\pi j}\int_c \frac{A_{o,n}}{\omega^n}\left(\frac{1}{1-ze^{\omega T}}\right)d\omega$$

$$= \left[\frac{A_{o,n}}{(n-1)!}\cdot\frac{d^{n-1}}{d\omega^{n-1}}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=0}. \tag{13}$$

and

$$X^*(s) = \frac{1}{2\pi j}\int_c \frac{A_{o,n}}{\omega^{n+1}}\left(\frac{1}{1-ze^{\omega T}}\right)d\omega$$

$$= \left[\frac{A_{o,n}}{n!}\cdot\frac{d^n}{d\omega^n}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=0}. \tag{14}$$

Second, let

$$Y(s) = \frac{A_{1,n}}{(s+\alpha)^n}. \tag{15}$$

Then

$$Y^*(s) = \left[\frac{A_{1,n}}{(n-1)!}\cdot\frac{d^{n-1}}{d\omega^{n-1}}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=-\alpha} \tag{16}$$

and

$$X^*(s) = \left[\frac{A_{1,n}}{(n-1)!}\cdot\frac{d^{n-1}}{d\omega^{n-1}}\left(\frac{1}{\omega(1-ze^{\omega T})}\right)\right]_{\omega=-\alpha}$$

$$+\frac{A_{1,n}}{\alpha^n}\cdot\frac{1}{1-z}. \tag{17}$$

Corresponding to the general term for Y(s) in (11), the general terms for Y*(s) and X*(s) are

$$Y^*(s) = \sum_{r=1}^{m}\left[\frac{A_{o,r}}{(r-1)!}\cdot\frac{d^{r-1}}{d\omega^{r-1}}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=0}$$

$$+\sum_{i=1}^{n}\sum_{r=1}^{k_i}\left[\frac{A_{1,r}}{(r-1)!}\cdot\frac{d^{r-1}}{d\omega^{r-1}}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=-\alpha} \qquad (18)$$

$$X^*(s) = \sum_{r=1}^{m}\frac{A_{o,r}}{r!}\cdot\frac{d^r}{d\omega^r}\left(\frac{1}{1-ze^{\omega T}}\right)_{\omega=0}$$

$$+\sum_{i=1}^{n}\sum_{r=1}^{k_i}\frac{A_{i,r}}{\alpha_1^r}\cdot\frac{1}{1-z}$$

$$+\sum_{i=1}^{n}\sum_{r=1}^{k_i}\left[\frac{A_{i,r}}{(r-1)!}\cdot\frac{d^{r-1}}{d\omega^{r-1}}\left(\frac{1}{\omega(1-ze^{\omega T})}\right)\right]_{\omega=-\alpha} \qquad (19)$$

Note that $\frac{d^n}{d\omega^n}\left[A(\omega)B(\omega)\right]$ yields

$$C_o^n A_n(\omega)B_o(\omega) + C_1^n A_{n-1}(\omega)B_1(\omega)$$

$$+ C_2^n A_{n-2}(\omega)B_2(\omega) \text{ ------ } . \qquad (20)$$

One of the terms of (19) when expanded as in (20) will yield

$$-\frac{1}{\alpha_i} Y^*_{i,r}(s),$$

but the rest of the terms of the expansion are not expressable in terms of Y*(s). This is indicative of the difficulty of finding some difference operator to operate on Y*(s) to achieve, effectively, true integration. In particular note that the Laplace integral operator, 1/s, will definitely not be useable as an operator upon the sampled function in order to yield the sampled counterpart of the integrated continuous function. This is obvious anyway since 1/s operating on an impulse function must yield a continuous function and not a sampled function.

The relatively involved computation indicated in (19) can be expressed much more efficiently by use of the W-transform[4,5]. Those familiar with sampled data technique will recognize that the expressions previously derived are essentially in Z-transform form. (Generally, however, Z is e^{sT} rather than e^{-sT}.) The W-transform can be defined by saying that if a function has a Z-transform equal to F(Z), then the W-transform becomes G(W) = WF(W). The inverse transform becomes

$$f(t) = \frac{1}{2\pi j}\int_c G(W)W^{t/T}dW \qquad (21)$$

where quite amazingly f(t) is the unsampled (continuous) function satisfying the sampling theorem. This becomes an extremely powerful and important relationship. Since f(t) is continuous, it is possible to defferentiate both sides with respect to t, getting

$$f'(t) = \frac{1}{2\pi j}\int_c \frac{\ln W}{T} G(W)W^{t/T}\, dW \qquad (22)$$

Hence in the W-plane $\frac{\ln W}{T}$ becomes a differential operator--something which has not been found for the function in the Z-plane. The result of this discussion on integration can be expressed diagramatically as in Figure 1.

In the DDA, the purpose of the integrator is to accept two inputs, Δx and Δy and to form Δz=yΔx. Δx is then formed into z in the "y" register of another integrator. In the connection of integrators, it is assumed that the z formed is equal to the integral of ydx. It is therefore necessary to examine the methods used for this operation and then compare the results with (19).

There are many methods of approximating the integral of y(t) based on samples[5,6]. However, the primary two upon which operational DDA's have been based are rectangular (Euler) integration and trapezoidal integration. These two methods will be compared with (19). If any other techniques should be implemented, the method of comparison would be the same.

Rectangular Integration (Open Loop Integration)

In "closed type" rectangular integration, the value of x at the time n is assumed to be

$$x_n = x_{n-1} + y_n\Delta t \ . \qquad (23)$$

In a sampled data representation, Δt becomes the sampling interval T, and (23) becomes

$$\left[X^*(s)\right]' = zX^*(s) + TY^*(s)$$

$$\text{or} \quad \left[X^*(s)\right]' = \frac{T}{(1-z)} \cdot Y^*(s) \,.$$

However, an expansion of this last function shows

$$x(0) + x(T)z + x(2T)z^2 + \text{----}$$

$$= T\left[1 + z + z^2 + \text{---}\right]\left[y(0) + y(T) + y(2T) + \text{--}\right]$$

$$= Ty(0) + T\left[y(0) + y(T)\right]z$$

$$+ T\left[y(0) + y(T) + y(2T)\right]z^2 + \text{--}$$

which is in error from the desired equation by an amount

$$Ty(0) + Ty(0)z + Ty(0)z^2 + \text{----} \,.$$

Hence the machine equation must necessarily be

$$\left[X^*(s)\right]' = \frac{T}{(1-z)} \cdot Y^*(s) - \frac{Ty(0)}{(1-z)} \,. \tag{24}$$

The last term of (24) is essentially taken care of in the machine by setting the initial conditions. In the following work y(0) will quite frequently be zero, thereby eliminating the last term from consideration.

Establishing the general term for the output of the machine integrator requires having (24) operate on (18). This yields for the rectangular integration case

$$\left[X^*(s)\right]' = \sum_{r=1}^{m}\left[\frac{A_{o,r}}{(r-1)!}\left(\frac{T}{1-z}\right)\frac{d^{r-1}}{d\omega^{r-1}}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=0}$$

$$- \frac{Ty(0)}{(1-z)} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}$$

$$\left[\frac{A_{i,r}}{(r-1)!}\left(\frac{T}{1-z}\right)\frac{d^{r-1}}{d\omega^{r-1}}\left(\frac{1}{1-ze^{\omega T}}\right)\right]_{\omega=-\alpha} \tag{25}$$

In the interest of compactness, let $d^n/d\omega^n$ be represented by D^n and let $e^{\omega T}$ be represented by Ω. Also let the error function be

$$U^*(s) = X^*(s) - \left[X^*(s)\right]' \,.$$

Then, subtracting (25) from (19) yields

$$U^*(s) = \sum_{r=1}^{m}\left[\frac{A_{o,r}}{r!}\left(D^r - \frac{rT}{1-z}D^{r-1}\right)\left(\frac{1}{1-z\Omega}\right)\right]_{\omega=0}$$

$$- \frac{Ty(0)}{(1-z)} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\left[\frac{A_{i,r}}{(r-1)!} \cdot\right.$$

$$\left.\left(D^{r-1}\frac{1}{\omega} - \frac{T}{1-z}D^{r-1}\right)\left(\frac{1}{1-z\Omega}\right)\right]_{\omega=-\alpha} \tag{27}$$

This function is fairly formidable, but leads to rather interesting results.[7] The errors associated with the integration of some rather simple functions are shown in Table 1. It should be pointed out here, that the process used in finding the error function does not limit y(t) to functions satisfying the sampling theorem.

<u>Trapezoidal Integration</u> (Open Loop Integration)

Trapezoidal integration follows the rule

$$x(nT) = x\left[(n-1)T\right] + T\left[\frac{y\left[(n-1)T\right] + y(nT)}{2}\right] \tag{28}$$

which becomes

$$\left[X^*(s)\right]' = X^*(s)z + \frac{T}{2}\left[Y^*(s)z + Y^*(s)\right]$$

$$\text{or} \left[X^*(s)\right]' = Y^*(s)\left[\frac{T(1+z)}{2(1-z)}\right] - \frac{Ty(0)}{2(1-z)} \tag{29}$$

where the second term of (29) is required to correct for the fact that y(0) may not be zero.

Using the general term (18) for Y(s) yields for the general case

$$\left[X^*(s)\right]' = \sum_{r=1}^{m}\left[\frac{A_{o,r}}{r!}\left(\frac{T(1+z)}{2(1-z)}\right)D^{r-1}\left(\frac{1}{1-z\Omega}\right)\right]_{\omega=0}$$

$$- \frac{Ty(0)}{2(1-z)} + \sum_{i=1}^{n}\sum_{r=1}^{k_i}\left[\frac{A_{i,r}}{(r-1)!}\left(\frac{T(1+z)}{2(1-z)}\right)\right.$$

$$\left.D^{r-1}\left(\frac{1}{1-z\Omega}\right)\right]_{\omega=-\alpha} \,. \tag{30}$$

Subtracting (30) from (19) yields the error function

$$U^*(s) = \sum_{r=1}^{m} \frac{A_{o,r}}{r!} \left[\left(D^r - \frac{rT(1+z)}{2(1-z)} D^{r-1} \right) \left(\frac{1}{1-z\Omega} \right) \right]_{\omega=0}$$

$$- \frac{Ty(0)}{2(1-z)} + \sum_{i=1}^{n} \sum_{r=1}^{k_i} \frac{A_{i,r}}{(r-1)!}$$

$$\left[D^{r-1} \left(\frac{1}{\omega} - \frac{T(1+z)}{2(1-z)} \right) \left(\frac{1}{1-z\Omega} \right) \right]_{\omega=-\alpha} . \quad (31)$$

Application of this formula to simple functions yields the errors shown in Table 1 under Trapezoidal Integration.

Comments Regarding Open Loop Integration

By straightforward, but rather lengthy computations, it can be shown[7] that the maximum per unit error (magnitude) in the integration of a function with a pole at $s=-\alpha$ (regardless of the order of the pole) is approximated by $\alpha T/2$ for rectangular and $\alpha^2 T^2/2\cdot 3!$ for trapezoidal integration. (Absolute errors are approximately $T/2$ and $T^2/2\cdot 3!$ respectively.) It is interesting to notice that there is no phase shift in the integration of a cosine wave when trapezoidal integration is used. This is a valuable asset in closed loop DDA operation.

Mechanization of the DDA

The digital differential analyzer is sufficiently different, conceptually, in its operation that, before proceeding, a few words regarding its mechanization are in order. The parallel type of operation will be considered first.

Consider the set of simultaneous differential equations

$$\dot{X} = AX + F(t). \quad (32)$$

The machine representation is of the form

$$\Delta X = \left[AX + F(t) \right] \Delta t . \quad (33)$$

To mechanize this equation, the machine utilizes $x_i(nT)$ to produce the values of $\Delta X(nT)$, then uses both of these to produce $x_i\ (n+1)T$. At first glance, this would seem to preclude the use of closed form integration techniques. [If $x(nT) = \int_0^{nT} y(t)dt$, open form integration techniques are those that attempt to produce the true integral from knowledge of $x(mT)$ for m from zero to n-1. Closed form integration techniques are those that attempt to produce the true integral from knowledge of $x(mT)$ for m from zero to n.] However, the machine variable is discrete in nature. Since $\Delta x_i(nT)$ is added to $x_i(nT)$ to produce $x_i(n+1)T$, the next value of x_i is actually known and so it is possible to produce the "integral increment" in closed form. Hence one operation utilizes $x_i(nT)$ to form Δx_i and then forms

$$X\left[(n+1)T\right] = X(nT) + \Delta X(nT).$$

The formation of $\Delta X(nT)$ establishes the rule of integration used by the machine.

Before continuing, consider the following simple example. A system of equations

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2$$

is represented on the machine by

$$\Delta x_1 = a_{11}x_1\Delta t + a_{12}x_2\Delta t$$

$$\Delta x_2 = a_{21}x_1\Delta t + a_{22}x_2\Delta t .$$

The schematic for the machine set-up is shown in Figure 2.

The solution is initiated by establishing initial conditions for $x_i(0)$. The $x_i(0)$ are operated on to produce the $\Delta x_i(0)$. When the first Δt occurs, x_i is updated to produce $x_2(T)\Delta t$ which in turn forms the new $\Delta x_2(T)$. This completes one full cycle. The computer solution evolves as this cycle is continuously repeated.

As previously mentioned, integration in the DDA occurs in the formulation of the function $\Delta X(nT)$. The mechanized scheme may use either open or closed form numerical techniques. Consider, for example, trapezoidal integration. The quantity $\Delta X(nT)$ is developed by

$$\Delta X(nT) = A\left[X(nT) + \frac{\Delta X(nT)}{2}\right]\Delta t + \left[\frac{f(nT) + \Delta f(nT)}{2}\right]\Delta t.$$

This is the equivalent of the mathematical expression

$$\int_{nT}^{(n+1)T} y\,dt \cong \Delta x(nT) = \left[\frac{y[(n+1)T] + y(nT)}{2}\right]\Delta t$$

$$= \left[\frac{y(nT) + y(nT) + \Delta y(nT)}{2}\right]\Delta t$$

$$= \left[y(nT) + \frac{\Delta y(nT)}{2}\right]\Delta t .$$

Hence

$$\int_{nT}^{(n+1)T} \dot{X}dt = \int_{nT}^{(n+1)T}\left[AX + F\right]dt$$

becomes

$$\Delta X(nT) = A\left[\frac{X[(n+1)T] + X(nT)}{2}\right]\Delta T$$

$$+\left[\frac{F(nT) + F(n+1)T}{2}\right]\Delta t .$$

This in turn becomes

$$\Delta X(nT) = A\left[X(nT) + \frac{\Delta X(nT)}{2}\right]\Delta t$$

$$+\left[\frac{F(nT) + F[(n+1)T]}{2}\right]\Delta t$$

$$= \left[1 - A\frac{\Delta t}{2}\right]^{-1}\left[AX(nT) + \left(\frac{F(nT) + F[(n+1)T]}{2}\right)\right]\Delta t .$$

In the machine, Δt is represented by T, so that the last equation can be put in the form

$$X\left[\frac{2(1-z)}{T(1+z)}\right] = (AX + F).$$

Therefore, the machine's differential operator becomes

$$D^*(z) = \frac{2(1-z)}{T(1+z)} .$$

In the serial type DDA, only one integrator is updated at one time. This means that the mechanization takes on the following form (for rectangular integration without a driving function).

$$\Delta x_1(nT) = \left[a_{11}x_1(nT) + a_{12}x_2(nT) + a_{13}x_3(nT) + \text{--}\right]\Delta t$$

$$\Delta x_2(nT) = \left[a_{21}x_1\left[(n+1)T\right] + a_{22}x_2(nT) + a_{23}x_3(nT) + \text{--}\right]\Delta t$$

$$\Delta x_3(nT) = \left[a_{31}x_1\left[(n+1)T\right] + a_{32}x_2\left[(n+1)T\right] + a_{33}x_3(nT) + \text{--}\right]\Delta t\text{ , etc.}$$

This complicates the error theory somewhat but the analysis follows the same pattern as that established in the following work.

Analysis of DDA Techniques

In analyzing solutions to differential equations on the DDA, it is desireable to use the W-transform because one may use the differential operator in the W-plane.

If the W-transform is applied to the set of equations

$$\dot{X} = AX \tag{34}$$

the result in the W-domain is

$$\frac{\ln w}{T}X(w) = AX(w) \tag{35}$$

or

$$\left[\frac{\ln w}{T}I - A\right]X(w) = 0. \tag{36}$$

The determinant of the square matrix in (36) yields a polynomial in $(\ln w)/T$ which when solved for yields the poles in the W-plane.

For example, if $sX-A$ yields a pole at $s=-\alpha$, then a root of the polynomial would be $(\ln w)/T = -\alpha$ which yields a pole at $w=e^{-\alpha T}$. This is in accordance with the theory of differential equations and W-transform theory.

The problem, then, is what sort of error occurs in the location of these poles if one uses an approximate integration operator rather than the true operator.

Substitution of the W-transform for the Z form in (24) and (29) shows that the W-transform of the rectangular and trapezoidal integration techniques are,

for rectangular:

$$\left[X(w)\right]' = \left[\frac{T}{w-1}\right]\left[Y(w)\right]' - \frac{Ty(0)}{w-1} \tag{37}$$

for trapezoidal:

$$\left[X(w)\right]' = \left[\frac{T(w+1)}{2(w-1)}\right]\left[Y(w)\right]' - \frac{Ty(0)}{2(w-1)} . \tag{38}$$

The differential operators, therefore, are for the rectangular case :

$$\left[Y(w)\right]' = \left[\frac{w-1}{T}\right]\left[X(w)\right]' + y(0) \tag{39}$$

for the trapezoidal case:

$$\left[Y(w)\right]' = \left[\frac{2(w-1)}{T(w+1)}\right]\left[X(w)\right]' + \frac{Y(0)}{w+1} . \tag{40}$$

1. The Rectangular Case. Substitution of (39) into (34) yields

$$\left(\frac{w-1}{T}\right)\left[X(w)\right]' = A\left[X(w)\right]' - X(0)$$

or

$$\left[\left(\frac{w-1}{T}\right)I - A\right]\left[X(w)\right]' = - X(0)$$

or

$$\left[X(w)\right]' = \left[\left(\frac{w-1}{T}\right)I - A\right]^{-1} X(0). \tag{41}$$

The poles of $\left[X(w)\right]'$ are hence the roots of the determinant

$$\left|\left(\frac{w-1}{T}\right)I - A\right|.$$

If the true roots of the original equation lie at $w_i = e^{-\alpha_i T}$, then the resulting polynomial in $(w-1)/T$ will yield roots at

$$\left(\frac{w-1}{T}\right) = -\alpha, \text{ so that } w_i = 1 - \alpha_i T.$$

Comparison with the expansion of $e^{-\alpha T}$ shows the error in the location of the roots to be

$$\frac{(\alpha_i T)^2}{2!} - \frac{(\alpha_i T)^3}{3!} + \cdots \approx \frac{a_i T^2}{2!} . \tag{42}$$

Therefore the product αT must be kept small if the solution is to closely resemble the true solution. This requirement is similar to the one discovered for small error in open loop integration in the first part of this paper.

An interesting observation may be made regarding the solution of the equation $x'' + \omega^2 x = 0$. Here the poles in the W-plane should lie at $w = e^{\pm j\omega T}$. (Poles on the unit circle yield undamped oscillations.) However, if one uses rectangular integration, the poles will lie at $1 \pm \omega T$ (see Figure 3). These poles lie outside the unit circle and hence the solution will yield exponentially increasing oscillations. The further they are from the origin, the greater will be the negative damping and the greater the error in the resulting frequency.

2. The Trapezoidal Case. Substitution of (40) into (34) yields

$$\left(\frac{2(w-1)}{T(w+1)}\right)\left[X(w)\right]' = A\left[X(w)\right]' + \frac{X(0)}{w+1}$$

$$\text{or } \left[\left(\frac{2(w-1)}{T(w+1)}\right)I - A\right]\left[X(w)\right]' = \frac{X(0)}{w+1}$$

$$\text{and } \left[X(w)\right]' = \left[\left(\frac{2(w-1)}{T(w+1)}\right) I - A\right]^{-1} \frac{X(0)}{w+1} \tag{43}$$

In this case the poles occur for

$$\left(\frac{2(w-1)}{T(w+1)}\right) = -\alpha , \text{ or for}$$

$$w = \frac{1-\alpha T/2}{1+\alpha T/2} = 1-\alpha T + \frac{(\alpha T)^2}{2^1} - \frac{(\alpha T)^3}{2^2} + \cdots .$$

This shows that the position of the pole will be in error by an amount

$$\frac{(3!-2^2)(\alpha T)^3}{2^2 \cdot 3!} - \frac{(4!-2^3)(\alpha T)^4}{2^3 \cdot 4!} + \cdots \approx \frac{(\alpha T)^3}{12} . \tag{44}$$

Again it is seen that trapezoidal integration produces less error than rectangular integration. There appears to be an introduction of a pole at $w = -1$, see (43). However, this pole will be cancelled when the inverse of the matrix is taken. This will always be the case regardless of the integration operator. Hence the only frequencies present will be the negative roots of the matrix (except for those introduced by round-off errors).

Considering again the solution of the equation $x'' + \omega^2 = 0$, where the poles should lie at $w = e^{\pm j\omega T}$, it is seen that the poles lie instead at

$$w = \frac{1 \pm j\omega T/2}{1 \pm j\omega T/2} .$$

Since the magnitude of the numerator and denominator are the same, the poles will lie on the unit circle. Hence the solution will be undamped and is at least without error in this respect. The actual location of the poles, however, is at

$$e^{\pm 2 \tan^{-1} \omega T/2},$$

which means the location of the poles will be in error by approximately

$$e^{\pm j \frac{(\omega T)^2}{12}}$$

This means the angle per unit error in W-plane $\approx \frac{(\omega T)^2}{12}$. (Comparison with Table 1 shows that this is the same as the per unit error in the magnitude of an integrated cos function.) For an error of about 1 percent the product ωT should be about 0.33. This is rather interesting since it indicates that approximately ten samples per cycle will yield 1 percent accuracy in the resultant frequency. For an error of 0.1 percent, thirty samples should be taken; for 0.01, one hundred samples; etc. In the closed loop system, establishment of proper initial conditions will preclude any error in magnitude since poles lie on the unit circle. (This of course neglects errors caused by rounding.)

It can be said, then, that theory regarding sensitivity[7] and errors due to parameter perturbation[8] can be applied to DDA systems in the W-plane. The perturbation of roots, however, is more distinct since the differential operators are of a precise nature.

Function Generation

The ability to determine errors precisely immediately suggests the possibility of modifying the original differential equation to accurately produce the desired function--or the desired solution to the original differential equation. This indeed is possible for those functions that can be described by linear differential equations with constant coefficients. Here the roots in the w-Plane are given precisely by $w_i = e^{s_i t}$, where the s_i are the s-plane poles of the desired function. If the poles of the modified differential equation are γ_i, then

for rectangular integration:

$$\gamma_i = \frac{w_i - 1}{T} = \frac{e^{s_i t} - 1}{T}$$

and for the trapezoidal integration:

$$\gamma_i = \frac{2(w_i - 1)}{T(w_i + 1)} = \frac{2(e^{s_i t} - 1)}{T(e^{s_i t} + 1)} .$$

The modified differential equation may then be derived from

$$\prod_{i=1}^{n} (s - \gamma_i).$$

The resulting differential equation is then put in matrix form and solved using the desired operator. The matrix will generally consist of the constants of the type tangent φ or $e^{\alpha T}$. The accuracy is limited only by truncation of these terms and round-off in the computer. Hence, this technique could be invaluable for accurate generation of simple functions.

Round-Off Errors

Round-off errors are particularly difficult to resolve. Since these errors are unpredictable without prior knowledge of the solution, it would seem that deterministic methods would fail to produce any useful results. However, a few concepts are presented here which may be useful in some considerations.

Consider the system of linear differential equations

$$\dot{X} = AX.$$

In the W-transform domain, this becomes

$$\frac{\ln w}{T} Z(w) = AX(w).$$

The machine equation is actually set up to perform

$$\left[X(w)\right]' = O(w)\left[A\right]\left[X(w)\right]' - \left(\frac{k}{w-1}\right)AX(0) + \left(\frac{1}{w-1}\right) X(0) \tag{45}$$

where O(w) is the mechanized integration operator, and the k is a constant necessary to achieve the proper initial conditions, mathematically. The round-off process can be considered as the addition of some function, $\varepsilon(w)$ to the function after the "integration" operation has been performed.

Equation (45) then becomes

$$\left[X(w)\right]' = O(w)A\left[X(w)\right]' - \left(\frac{k}{w-1}\right)AX(0) + \left(\frac{1}{w-1}\right)X(0) + E(w). \tag{46}$$

$$\text{So } \left[X(w)\right]' = \left[I-O(w)A\right]^{-1}\left\{\left[1-kA\right]\left(\frac{1}{w-1}\right)X(0) + E(w)\right\}. \tag{47}$$

Supposedly the poles of E(w) will be more or less randomly distributed in the normal case. However, it is observed from (47) that should poles of E(w) coincide with poles of the "unrounded" $[X(w)]'$, it would be possible to obtain almost any kind of error from the round-off procedure. However, E(w) cannot have poles of higher order than one by nature of its origin. This means that the worst effect that could occur would be to increase the order of some "natural" pole by one. The worst type of pole would be one lying on the unit circle in the W-plane since this would give a response of $t \sin \omega t$ in the equivalent unsampled domain. If one should know the location of the poles in the "unrounded" $[X(w)]'$, it would be possible to set a bounds on the error by assuming the worst case, namely one pole in E(w) lying on each pole of $[X(w)]'$.

It is believed by the author to be not possible to extend the theory of error caused by round-off any further than has been done in the previous discussion if deterministic methods are to be used. However, it is possible that some statistical studies might show that this type of error has some predictable distribution.

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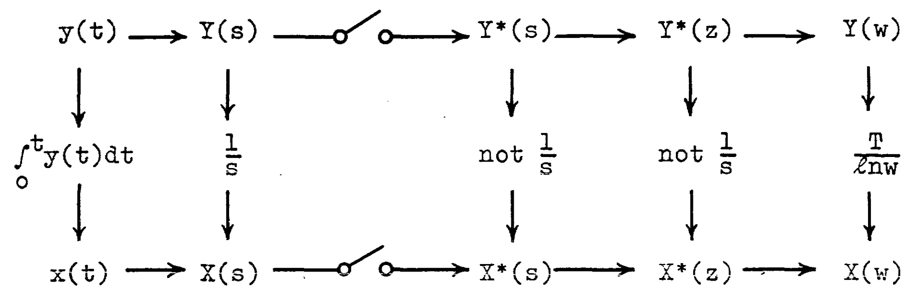

Figure 1. Integration Operations.

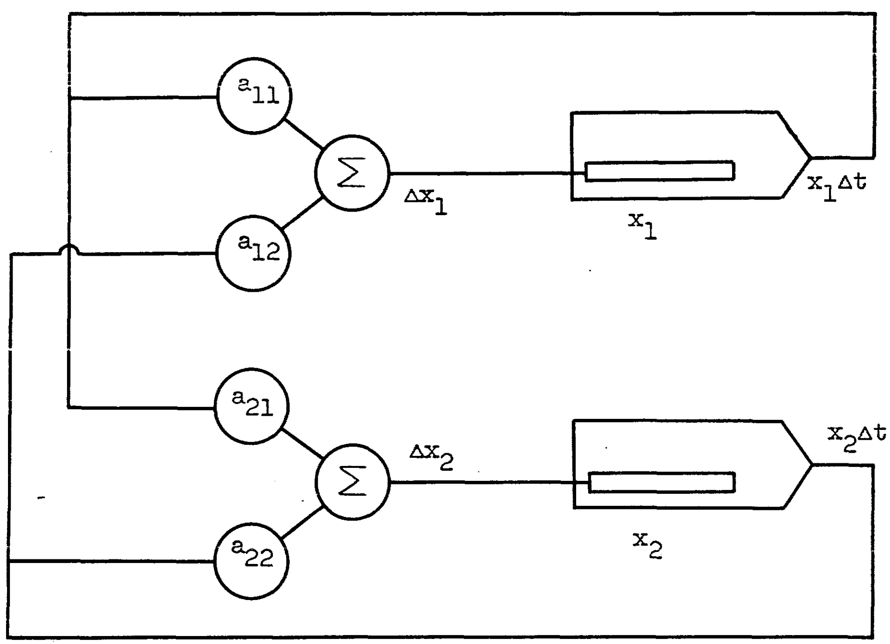

Figure 2. Machine Representation of X = AX.

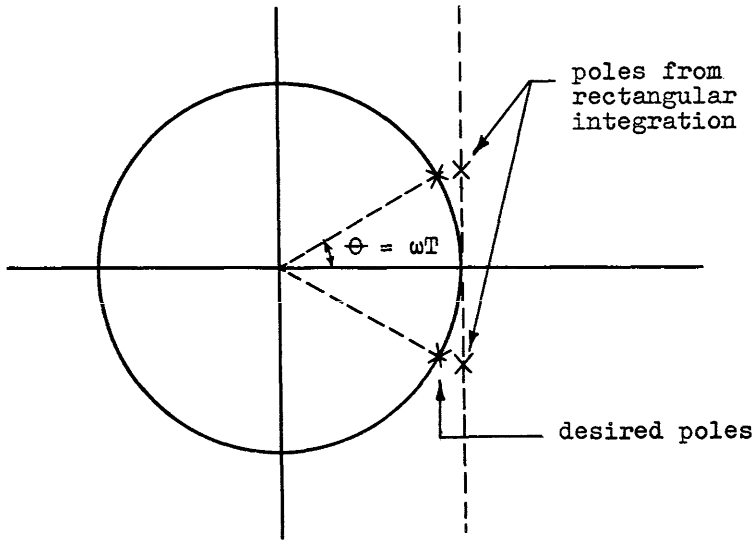

Figure 3. Location of Poles $x'' + \omega^2 x = 0$.

y(t)	Absolute Error (Equivalent Continuous Function)		True Integral
	Rectangular	Trapezoidal	
1	0	0	t
t	$-\frac{Tt}{2}$	0	$\frac{t^2}{2!}$
$\frac{t^2}{2!}$	$-\frac{Tt^2}{4} - \frac{T^2t}{12}$	$-\frac{T^2t}{6}$	$\frac{t^3}{3!}$
$\frac{t^3}{3!}$	$-\frac{Tt^3}{12} - \frac{T^2t^2}{24}$	$-\frac{T^2t^2}{12}$	$\frac{t^4}{4!}$
$\frac{t^4}{4!}$	$-\frac{Tt^4}{48} - \frac{T^2t^3}{12} - \frac{T^4t}{6!}$	$-\frac{T^2t^3}{72} - \frac{T^4t}{6!}$	$\frac{t^5}{5!}$
$e^{-\alpha t}$	$-\frac{T}{2}(1 - e^{-\alpha t})$	$-\frac{\alpha T^2}{12}(1 - e^{-\alpha t})$	$-\frac{1}{\alpha}(1 - e^{-\alpha t})$
$te^{-\alpha t}$	$+\frac{T}{2} te^{-\alpha t} - \frac{T^2}{12}(1 - e^{-\alpha t})$	$+\frac{\alpha T^2}{12} te^{-\alpha t} - \frac{T^2}{12}(1 - e^{-\alpha t})$	$-\frac{1}{\alpha} te^{-\alpha t} + \frac{1}{\alpha^2}(1 - e^{-\alpha t})$
$\cos \omega t$	$+\frac{T}{2} + \frac{\omega T}{12} \sin \omega t - \frac{T}{2} \cos \omega t$	$+\frac{\omega T^2}{12} \sin \omega t$	$+\frac{1}{\omega} \sin \omega t$
	Actual Function Developed		
$\cos \omega t$	$-\frac{T}{2} + \frac{T}{2 \sin \omega T/2} \sin(\omega t + \omega T/2)$	$+\frac{T \cos \omega T/2}{2 \sin \omega T/2} \sin \omega t$	

Table I. Absolute Errors.