Fast minimum-weight double-tree shortcutting for Metric TSP: Is the best one good enough?*

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Abstract. The Metric Traveling Salesman Problem (TSP) is a classical NP-hard optimization problem. The double-tree shortcutting method for Metric TSP yields an exponentially-sized space of TSP tours, each of which approximates the optimal solution within at most a factor of 2. We consider the problem of finding among these tours the one that gives the closest approximation, i.e. the *minimum-weight double-tree shortcutting*. Burkard et al. gave an algorithm for this problem, running in time $O(n^3 + 2^d n^2)$ and memory $O(2^d n^2)$, where d is the maximum node degree in the rooted minimum spanning tree. We give an improved algorithm for the case of small d (including planar Euclidean TSP, where $d \le 4$), running in time $O(4^d n^2)$ and memory $O(4^d n)$. This improvement allows one to solve the problem on much larger instances than previously attempted. Our computational experiments suggest that in terms of the time-quality tradeoff, the minimum-weight double-tree shortcutting method provides one of the best known tour-constructing heuristics.

1 Introduction

The Metric Travelling Salesman Problem (TSP) is a classical combinatorial optimization problem. We represent a set of n points in a metric space by a complete weighted graph on n nodes, where the weight of an edge is defined by the distance between the corresponding points. The objective of Metric TSP is to find in this graph a minimum-weight Hamiltonian cycle (equivalently, a minimum-weight tour visiting every node at least once). The most common example of Metric TSP is the planar Euclidean TSP, where the points lie in the two-dimensional Euclidean plane, and the distances are measured according to the Euclidean metric.

Metric TSP, even restricted to planar Euclidean TSP, is well-known to be NP-hard [14]. Metric TSP is also known to be NP-hard to approximate to within a ratio 1.00456, but polynomial-time approximable to within a ratio 1.5. Fixed-dimension Euclidean TSP is known to have a PTAS (i.e. a family of algorithms with approximation ratio arbitrarily close to 1) [3]; this generalises to any metric defined by a fixed-dimension Minkowski vector norm.

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^{*} The authors acknowledge the support of The University of Warwick's DIMAP (the Centre for Discrete Mathematics and its Applications) during this work.

Two simple methods, double-tree shortcutting [18] and Christofides' [6, 19], allow one to approximate the solution of Metric TSP within a factor of 2 and 1.5, respectively. Both these methods belong to the class of tour-constructing heuristics, i.e. "heuristics that incrementally construct a tour and stop as soon as a valid tour is created" [12]. In both methods, we build an Eulerian graph on the given point set, select an Euler tour of the graph, and then perform shortcutting on this tour by removing repeated nodes, until all node repetitions are removed. In general, it is not prescribed which one of several occurrences of a particular node to remove. Therefore, the methods yield an exponentially-sized space of TSP tours (shortcuttings of a specific Euler tour in a specific Eulerian graph), each of which approximates the optimal solution within at most a factor of 2 (respectively, 1.5).

The two methods differ in the way the initial weighted Eulerian graph is constructed. Both start by finding the graph's minimum-weight spanning tree (MST). The double-tree method then doubles every edge in the MST, while the Christofides method adds to the MST a minimum-weight matching built on the set of odd-degree nodes. The weight of the resulting Euler tour is higher than the optimal TSP tour at most by a factor of 2 (respectively, 1.5), and the subsequent shortcutting can only decrease the tour weight.

While any tour obtained by shortcutting of the original Euler tour approximates the optimal solution within at most a factor of 2 (respectively, 1.5), clearly, it is still desirable to find the shortcutting that gives the closest approximation. Given an Eulerian graph on a set of points, we will consider its minimum-weight shortcutting across all shortcuttings of all possible Euler tours of the graph. We shall correspondingly speak about the minimum-weight double-tree and the minimum-weight Christofides methods.

Unfortunately, for general Metric TSP, both the double-tree and Christofides minimum-weight shortcutting problems are NP-hard. Consider first the double-tree version of the problem. Given an unweighted graphs, any instance of the Hamiltonian cycle problem can be regarded as an instance of Metric TSP with distances 1 and 2. Add an extra node connected to all other nodes by edges of weight 1, and take the newly added edges as the MST. It is easy to see that the minimum-weight shortcutting problem on the resulting instance is equivalent to the original Hamiltonian cycle problem. The minimum-weight double-tree shortcutting problem was also believed for a long time to be NP-hard for planar Euclidean TSP, until a polynomial-time algorithm was given by Burkard et al. [5]. This is the algorithm we improve upon in the current paper. In contrast, the Christofides version of the problem remains NP-hard even for planar Euclidean TSP [15].

In the rest of this paper, we will mainly deal with the rooted MST, which is obtained from the MST by selecting an arbitrary node as the root. In the rooted MST, the terms parent, child, ancestor, descendant, sibling, leaf all have their standard meaning. Let d denote the maximum number of children per node in the rooted MST. Note that in the Euclidean plane, the maximum degree of an unrooted MST is at most 6. Moreover, a node can have degree equal to 6, only

if it is surrounded by six equidistant nodes forming a regular hexagon; we can exclude this degenerate case from consideration by a slight perturbation of the input points. This leaves us with an unrooted MST of maximum degree 5. By choosing a node of degree less than 5 as the root, we obtain a rooted MST with $d \leq 4$.

The minimum-weight double-tree shortcutting algorithm by Burkard et al. [5] applies to the general Metric TSP, and runs in time $O(n^3 + 2^d n^2)$ and memory $O(2^d n^2)$. In this paper, we give an improved algorithm³ for the case of small d, running in time $O(4^d n^2)$ and memory $O(4^d n)$. We then describe our implementation of the new algorithm, which incorporates a couple of additional heuristic improvements designed to speed up the algorithm and to increase its approximation quality. Computational experiments show that the approximation quality and running time of our implementation are among the best known tour-constructing heuristics.

A preliminary version of this paper appeared as [9].

2 The algorithm

Preliminaries. Let G be a weighted graph representing the Metric TSP problem on n points. The double-tree method consists of the following stages:

- construct the minimum spanning tree of G;
- duplicate every edge of the tree, obtaining an *n*-node Eulerian graph;
- select an Euler tour of the double-tree graph;
- reduce the Euler tour to a Hamiltonian cycle by repeated *shortcutting*, i.e. replacing a node sequence a, b, c by a, c, as long as node b appears elsewhere in the current tour.

We say that a Hamiltonian cycle *conforms* to the doubled spanning tree, if it can be obtained from that tree by shortcutting one of its Euler tours. We also extend this definition to paths, saying that a path conforms to the tree, if it is a subpath of a conforming Hamiltonian cycle.

In our minimum-weight double-tree shortcutting algorithm, we refine the bottom-up dynamic programming approach of [5]. Initially, we select an arbitrary node r as the root of the tree. For a node u, we denote by C(u) the set of all children of u, and by T(u) the node set of the maximal subtree rooted at u, i.e. the set of all descendants of u (including u itself). For a set of siblings U, we denote by T(U) the (disjoint) union of all subtrees T(u), $u \in U$. When U is empty, T(U) is also empty.

The characteristic property of a conforming Hamiltonian cycle is as follows: for every node u, upon entering the subtree T(u), the cycle must visit all nodes of T(u) before leaving the subtree, and must not re-enter the subtree afterwards.

³ Note that Burkard et al. [5] also give an $O(2^d n^3)$ algorithm for a more general TSP-type problem, where the set of admissible tours is restricted by an arbitrary PQ-tree. Our algorithm does not improve on the algorithm of [5] for this more general problem.

For an arbitrary node set S, we will say that a path through the graph sweeps S, if it visits all nodes of S consecutively in some order. In this terminology, a conforming Hamiltonian cycle must, for every node u, contain a subpath sweeping the subtree T(u).

In the rest of this section, we denote the metric distance between u and v by d(u,v). We use the symbol \uplus to denote disjoint set union. For brevity, given a set A and an element a, we write $A \uplus a$ instead of $A \uplus \{a\}$, and $A \setminus a$ instead of $A \setminus \{a\}$.

Computing the solution weight (upsweep). The algorithm proceeds by computing minimum-weight sweeping paths in progressively increasing subtrees, beginning with the leaves and finishing with the whole tree T(r). A similar approach is adopted in [5], where in each subtree, all-pairs minimum-weight sweeping paths are computed. In contrast, our algorithm only computes single-source minimum-weight sweeping paths originating at the subtree's root. This leads to substantial savings in time and memory.

A non-root node $v \in C(u)$ is active, if its subtree T(v) has already been processed, but its parent's subtree T(u) has not yet been processed. In every stage of the algorithm, we choose the current node u, so that all children of u (if any) are active. We call T(u) the current subtree. Let $V \subseteq C(u)$, $a \in T(V)$. By $D_V^u(a)$ we denote the weight of the shortest conforming path starting from u, sweeping subtree $u \uplus T(V)$, and finishing at a.

Consider the current subtree T(u). Processing this subtree will yield the values $D_V^u(a)$ for all $V \subseteq C(u)$, $a \in T(V)$. In order to process the subtree, we need the corresponding values for all subtrees rooted at the children of u. More precisely, we need the values $D_W^v(a)$ for every child $v \in C(u)$, every subset $W \subseteq C(v)$, and every destination node $a \in T(W)$. We do not need any explicit information on subtrees rooted at grandchildren and lower descendants of u.

Given the current subtree T(u), the values $D_V^u(a)$ are computed inductively for all sets V of children of u. The induction is on the size of the set V. The base of the induction is trivial: no values D_V^u exist when $V = \emptyset$.

In the inductive step, given a set $V \subseteq C(u)$, we compute the values $D^u_{V \uplus v}(a)$ for all $v \in C(u) \setminus V$, $a \in T(v)$, as follows. By the inductive hypothesis, we have the values $D^u_V(a)$ for all $a \in T(V)$. The main part of the inductive step consists in computing a set of auxiliary values $D^u_{V,W}(v)$, for all subsets $W \subseteq C(v)$. Every such value represents the weight of the shortest conforming path starting from node u, sweeping the subtree $u \uplus T(V)$, then sweeping the subtree $T(W) \uplus v$, and finishing at node v. Suppose the path exits the subtree $u \uplus T(V)$ at node v and enters the subtree $T(W) \uplus v$ at node v. We have

$$D_{V,W}^{u}(v) = \begin{cases} d(u,v) & \text{if } V = \emptyset, W = \emptyset \\ \min_{y \in T(W)} \left[d(u,y) + D_{W}^{v}(y) \right] & \text{if } V = \emptyset, W \neq \emptyset \\ \min_{x \in T(V)} \left[D_{V}^{u}(x) + d(x,v) \right] & \text{if } V \neq \emptyset, W = \emptyset \\ \min_{x \in T(V); y \in T(W)} \left[D_{V}^{u}(x) + d(x,y) + D_{W}^{v}(y) \right] & \text{if } V \neq \emptyset, W \neq \emptyset \end{cases}$$

$$\tag{1}$$

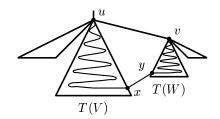


Fig. 1: Computation of $D_{V,W}^u(v)$

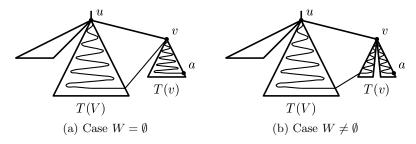


Fig. 2: Computation of $D_{V \uplus v}^u(a)$, $a \in T(v)$

(see Figure 1). The required values $D_W^v(y)$ have been obtained when processing subtrees T(v) for the active nodes $v \in C(u)$. Note that the computed auxiliary set contains $D_{V \uplus v}^u(v) = D_{V,C(v)}^u(v)$.

Now we can compute the values $D^u_{V \uplus v}(a)$ for all $a \in T(v) \setminus v = T(C(v))$. A path corresponding to $D^u_{V \uplus v}(a)$ must sweep $u \uplus T(V)$, and then T(v), finishing at a. While in T(v), the path will first sweep a (possibly single-node) subtree $v \uplus T(\overline{W})$, finishing at v. Then, starting at v, the path will sweep the subtree $v \uplus T(\overline{W})$, where $\overline{W} = C(V) \setminus W$, finishing at a. Considering every possible disjoint bipartitioning $W \uplus \overline{W} = C(V)$, such that $a \in T(\overline{W})$, we have

$$D^u_{V \uplus v}(a) = \min_{W \uplus \overline{W} = C(V): \ a \in T(\overline{W})} \left[D^u_{V,W}(v) + D^v_{\overline{W}}(a) \right] \tag{2}$$

(see Figure 2).

We now have the values $D^u_{V \uplus v}(a)$ for all $a \in T(v)$. The computation (1)–(2) is repeated for every node $v \in C(u) \setminus V$. The inductive step is now completed.

The processing of subtree T(u) completes when all possible choices of subset V and node v have been exhausted.

Eventually, the root r of the tree becomes the current node, and we process the complete tree T(r). This establishes the values $D_S^r(a)$ for all $S \subseteq C(r)$, $a \in T(S)$, which includes the values $D_{C(r)}^r(a)$ for all $a \neq r$. The weight of the minimum-weight conforming Hamiltonian cycle can now be determined as

$$\min_{a \neq r} \left[D_{C(r)}^r(a) + d(a, r) \right] \tag{3}$$

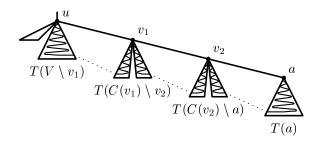


Fig. 3: Computation of $P_V^u(a)$, $a \in T(V)$, k=3

Theorem 1. The upsweep algorithm computes the weight of the minimum-weight tree shortcutting in time $O(4^d n^2)$ and space $O(2^d n)$.

Proof. In computation (1), the total number of quadruples u, v, x, y is at most n^2 (since for every pair x, y, the node u is determined uniquely as the lowest common ancestor of x, y, and the node v is determined uniquely as a child of u and an ancestor of y). In computation (2), the total number of triples u, v, a is also at most n^2 (since for every pair u, a, the node v is determined uniquely as a child of u and an ancestor of y). For every such quadruple or triple, the computation is performed at most 4^d times, corresponding to 2^d possible choices of each of V, W. The cost of computation (3) is negligible. Therefore, the total time complexity of the algorithm is $O(4^d n^2)$.

Since our goal at this stage is just to compute the solution weight, at any given moment we only need to store the values $D_V^u(a)$, where u is either an active node, or the current node (i.e. the node for which these values are currently being computed). When u corresponds to an active node, the number of possible pairs u, a is at most n (since node u is determined uniquely as the root of an active subtree containing a). When u corresponds to the current node, the number of possible pairs u, a is also at most n (since node u is fixed). For every such pair, we need to keep at most 2^d values, corresponding to 2^d possible choices of V. The remaining space costs are negligible. Therefore, the total space complexity of the algorithm is $O(2^d n)$.

Reconstructing the full solution (downsweep). In order to reconstruct the minimum-weight Hamiltonian cycle itself, we must keep all the auxiliary values $D^u_{V,W}(v)$ obtained in the course of the upsweep computation for every parent-child pair u,v. We solve recursively the following problem: given a node u, a set $V \subseteq C(u)$, and a node $a \in T(V)$, find the minimum-weight path $P^u_V(a)$ starting from u, sweeping subtree $u \uplus T(V)$, and finishing at a. To compute the global minimum-weight Hamiltonian cycle, it is sufficient to determine the path $P^r_{C(r)}(a)$, where r is the root of the tree, and a is the node for which the minimum in (3) is attained.

For any $u, V \subseteq C(u)$, $a \in T(V)$, consider the (not necessarily conforming or minimum-weight) path $u = v_0 \to v_1 \to v_2 \to \cdots \to v_k = a$, joining nodes u and

a in the tree (see Figure 3). The conforming minimum-weight path $P_V^u(a)$ first sweeps the subtree $u \uplus T(C(u) \setminus v_1)$. After that, for every node v_i , 0 < i < k, the path $P_V^u(a)$ sweeps the subtree $v_i \uplus T(C(v_i) \setminus v_{i+1})$ as follows: first, it sweeps a subtree $v_i \uplus T(W_i)$, finishing at v_i , and then, starting at v_i , it sweeps the subtree $v_i \uplus T(\overline{W}_i)$, for some disjoint bipartitioning $W_i \uplus \overline{W}_i = C(v_i) \setminus v_{i+1}$. Finally, the path $P_V^u(a)$ sweeps the subtree T(a), finishing at a.

The optimal choice of bipartitionings can be found as follows. We construct a weighted directed graph with a source corresponding to node $u=v_0$, a sink corresponding to node $v_k=a$, and k-1 intermediate layers of nodes, each layer corresponding to a node $v_i, 0 < i < k$. Each intermediate layer consists of at most 2^{d-1} nodes, representing all different disjoint bipartitionings of the set $C(v_i) \setminus v_{i+1}$. The source and the sink represent trivial bipartitionings $\emptyset \uplus (V \setminus v_1) = V \setminus v_1$ and $C(a) \uplus \emptyset = C(a)$, respectively. Every consecutive pair of layers (including the source and the sink) are fully connected by forward edges. In particular, the edge from a node representing the bipartitioning $X \uplus \overline{X}$ in layer i, to the node representing the bipartitioning $Y \uplus \overline{Y}$ in layer i+1, is given the weight $D_{\overline{X},Y}^{v_i}(v_{i+1})$. It is easy to see that an optimal choice of bipartitioning corresponds to the minimum-weight path can be found by a simple dynamic programming algorithm (such as the Bellman–Ford algorithm, see e.g. [7]) in time proportional to the number of edges in the layered graph.

Let $W_1 \uplus \overline{W}_1, \ldots, W_{k-1} \uplus \overline{W}_{k-1}$ now denote the k-1 obtained optimal subtree bipartitionings. The k edges of the corresponding source-to-sink shortest path determine k edges (not necessarily consecutive) in the minimum-weight sweeping path $P_V^u(a)$. These edges are shown in Figure 3 by dotted lines. It now remains to apply the downsweep algorithm recursively in each of the subtrees $u \uplus T(V \setminus v_1), v_1 \uplus T(W_1), v_1 \uplus T(\overline{W}_1), v_2 \uplus T(W_2), v_2 \uplus T(\overline{W}_2), \ldots, v_{k-1} \uplus T(W_{k-1}), v_{k-1} \uplus T(\overline{W}_{k-1}), T(a)$.

Theorem 2. Given the output and the necessary intermediate values of the upsweep algorithm, the downsweep algorithm computes the edges of the minimumweight tree shortcutting in time and space $O(4^d n)$.

Proof. The construction of the auxiliary graph and the minimum-weight path computation runs in time $O(4^d k)$, where k is the number of edges in the tree path $u = v_0 \to v_1 \to v_2 \to \cdots \to v_k = a$ in the current level of recursion. Since the tree paths in different recursion levels are edge-disjoint, the total number of edges in these paths is at most n. Therefore, the time complexity of the downsweep stage is $O(4^d n)$.

By Theorem 1, the space complexity of the upsweep stage is $O(2^d n)$. In addition to the storage used internally by the upsweep stage, we also need to keep all the values $D^u_{V,W}(v)$. The number of possible pairs u,v is at most n (since node u is determined uniquely as the parent of v). For every such pair, we need to keep at most 4^d values, corresponding to 2^d possible choices of each of V, W. The remaining space costs are negligible. Therefore, the total space complexity of the algorithm is $O(4^d n)$.

3 Heuristic improvements and computational experiments

Despite the guaranteed approximation ratio of the double-tree shortcutting and Christofides methods, neither has performed well in previous computational experiments (see [11,17]). However, to our knowledge, none of these experiments explored the minimum-weight double-tree shortcutting approach. Instead, the double-tree shortcutting was performed in some suboptimal, easily computable order, such as a depth-first tree traversal. We shall call this method depth-first double-tree shortcutting.

In particular, Reinelt [17] compares 37 tour-constructing heuristics, including the depth-first double-tree algorithm and the Christofides algorithm, on a set of 24 geometric instances from the TSPLIB database [16]. Although most instances in this experiment are quite small (2000 or fewer points), they still allow us to make some qualitative judgement about the approximation quality of different heuristics. Depth-first double-tree shortcutting turns out to have the lowest quality of all 37 heuristics, while the quality of the Christofides algorithm is somewhat higher, but still far from the top.

Intuitively, it is clear that the reason for the poor approximation quality of the two algorithms may be in the wrong choice of the shortcutting order, especially considering that the overall number of alternative choices is typically exponential. This observation motivated us to implement the minimum-weight double-tree shortcutting algorithm from [5]. It came as no surprise that this algorithm showed higher approximation quality than all the tour constructing heuristics in Reinelt's experiment. Unfortunately, Reinelt's experiment did not account for the running time of the algorithms under investigation. The theoretical time complexity of the algorithm from [5] is $O(n^3 + 2^d n^2)$; in practice, our implementation exhibited quadratic growth in running time on most instances. Both the theoretical and the practical running time were relatively high, which raised some justifiable doubts about the overall superiority of the algorithm.

As it was expected, the introduction of a new efficient minimum-weight tree shortcutting algorithm described in Section 2 significantly improved the running time in our computational experiments. However, this improvement alone was not sufficient for the algorithm to compete against the best existing tour-constructing heuristics. Therefore, we introduced two additional heuristic improvements, one aimed at increasing the algorithm's speed, the other at increasing its approximation quality.

The first heuristic improvement, aimed at speeding up the algorithm, is suggested by the well-known bounded neighbour lists [12, p. 408]. Given a tree, we define the *tree distance* between a pair of nodes a, b, as the number of edges on the unique path from a to b in the tree. Given a parameter k, the *depth-k list* of node u includes all nodes in the subtree T(u) with the tree distance from u not exceeding k. The suggested heuristic improvement is to limit the search across a subtree rooted at u in (1)–(2) to a depth-k list of u for a reasonably low value of k. Our experiments suggest that this approach improves the running time dramatically, while not having a significant effect on the approximation quality.

The second heuristic improvement, aimed at increasing the approximation quality, works by expanding the space of the tours searched, in the hope of finding a better solution in the larger space. Let T be a (not necessarily minimum) spanning tree, and let $\Lambda(T)$ be the set of all tours conforming to T, i.e. the exponential set of all tours considered by the double-tree algorithm. Our goal is to construct a new tree T_1 , such that its node degrees are still bounded by a constant, but $\Lambda(T) \subsetneq \Lambda(T_1)$. We refer to the new set of tours as an enlarged tour neighborhood.

Consider a node u in T, and suppose u has at least one child v which is not a leaf. We construct a new tree T_1 from T by applying the degree increasing operation, which makes node v a leaf, and redefines all children of v to be children of u. It is easy to check that any tour conforming to T also conforms to T_1 . In particular, the nodes of T(v), which are consecutive in any conforming tour of T, are still allowed to be consecutive in any conforming tour of T_1 . Therefore, $\Lambda(T) \subseteq \Lambda(T_1)$. On the other hand, sequence w, u, v, where w is a child of v, is allowed by T_1 but not by T. Therefore, $\Lambda(T) \subsetneq \Lambda(T_1)$.

We apply the above degree-increasing heuristic as follows. Let D be a global parameter, not necessarily related to the maximum node degree in the original tree. The degree-increasing operation is performed only if the resulting new degree of vertex u would not exceed D. Note that when the maximum degree bound would be exceeded, this cannot be avoided by performing the degree increasing operation partially: it would be wrong to reassign only some, instead of all, children of node v to a new parent. To illustrate this statement, suppose that v has two children w_1 and w_2 , which are both leaves. Let w_2 be redefined as a new child of u. The sequence v, w_2, w_1 is allowed by T but not by T_1 , since it violates the requirement for v and w_2 to be consecutive. Therefore, $\Lambda(T) \not\subseteq \Lambda(T_1)$.

Given a tree, the degree increasing operation is applied repeatedly to construct a new tree, obtaining an enlarged tour neighbourhood. In our experiments, we used breadth-first application of the degree increasing operation as follows:

```
Root the minimum spanning tree at a node of degree 1;
Let r' denote the unique child of the root;
Insert all children of r' into queue Q;
while queue Q is not empty do
extract node v from Q;
insert all children of v into Q;
if \deg(parent(v)) + \deg(v) \leq D then
redefine all children of v to be children of parent(v)
```

Incorporating the above two heuristic improvements, the minimum-weight double-tree algorithm from Section 2 was modified to take two parameters: the search depth k, and the degree-increasing limit D. We refer to the double-tree algorithm with fixed parameters k and D as a double-tree heuristic $DT_{D,k}$. We use DT without subscripts to denote the original minimum-weight double-tree algorithm, equivalent to $DT_{1,\infty}$.

Size	1000	3162	10K	31K	100K	316K	1M	3M
DT	7.36	7.82	8.01	8.19	8.39	8.40	8.41	_
$DT_{1,16}$								
$DT_{3,16}$	6.64	6.97	7.04	7.37	7.51	7.53	7.55	7.50
$DT_{3,32}$	6.52	6.84	6.92	7.21	7.31	7.36	7.37	7.31
$DT_{4,16}$	6.00	6.27	6.39	6.69	6.82	6.87	6.85	_
$DT_{4,32}$	5.93	6.22	6.33	6.60	6.74	6.78	6.77	_
$DT_{5,16}$	5.67	5.91	5.97	6.27	6.43	6.51	6.47	_
$DT_{5,32}$	5.62	5.89	5.93	6.23	6.38	6.46	6.43	_

(a) Average excess over the Held-Karp bound (%)

							1M	
							156659	
$DT_{1,16}$	0.04	0.14	0.47	1.57	5.60	20.82	101.09	388.52
							138.41	
- / -							269.73	
							307.31	
$DT_{4,32}$	0.45	2.00	6.93	22.11	74.70	236.33	744.50	_
							807.74	
$\overline{\mathrm{DT}}_{5,32}$	1.11	5.74	20.73	65.96	224.34	695.03	2168.95	_

(b) Average normalised running time (s)

Table 1: Results for DT and $\mathrm{DT}_{D,k}$ on uniform Euclidean distances

We compared experimentally the efficiency of the original algorithm DT with the efficiency of double-tree heuristics $\mathrm{DT}_{D,k}$ for two different search depths k=16,32, and for four different values for the maximum degree parameter D=1 (no degree increasing operation applied), 3, 4, 5. The case D=2 is essentially equivalent to D=1, and therefore not considered. In our computational experiments we used the test data from DIMACS Implementation Challenge [12,2]. These are uniform random Euclidean instances with 1000 points (10 instances), 3162 points (five instances), 10000 points (three instances), 31623 and 100000 points (two instances of each size), 316228, 1000000, and 3168278 points (one instance of each size).

For each heuristic, we consider both its running time and approximation quality. We say that one heuristic *dominates* another, if it is superior in both these respects. The experimental results, presented in Table 1, clearly indicate that nearly all considered heuristics (excluding $DT_{1,16}$) dominate plain DT. Moreover, all these heuristics (again excluding $DT_{1,16}$) dominate DT on each individual instance used in the experiment. For further comparison of the double-tree heuristics with existing tour-constructing heuristics, we chose $DT_{1,16}$ and $DT_{5,16}$.

The main part of our computational experiments consisted in comparing the double-tree heuristics against the most powerful existing tour-constructing heuristics. As a base for comparison, we chose the heuristics analysed in [12],

Size	1000	3162	10K	31K	100K	316K	1M	3M
RA^+	13.96	15.25	15.04	15.49	15.43	15.42	15.48	15.47
Chr-S	14.48	14.61	14.81	14.67	14.70	14.49	14.59	14.51
FI	12.54	12.47	13.35	13.44	13.39	13.43	13.47	13.49
Sav	11.38	11.78	11.82	12.09	12.14	12.14	12.14	12.10
ACh	11.13	11.00	11.05	11.39	11.24	11.19	11.18	11.11
Chr-G	9.80	9.79	9.81	9.95	9.85	9.80	9.79	9.75
Chr-HK	7.55	7.33	7.30	6.74	6.86	6.90	6.79	_
MTS1	6.09	8.09	6.23	6.33	6.22	6.20	-	_
MTS3	5.26	5.80	5.55	5.69	5.60	5.60	_	_
$DT_{1,16}$	8.64	9.24	9.10	9.43	9.74	9.66	9.72	9.66
$DT_{5,16}$	5.67	5.91	5.97	6.27	6.43	6.51	6.47	_

(a) Average excess over the Held-Karp bound (%)

Size	1000	3162	10K	31K	100K	316K	1M	3M
RA^+	0.06	0.23	0.71	1.9	5.7	13	60	222
Chr-S	0.06	0.26	1.00	4.8	21.3	99	469	3636
FI	0.19	0.76	2.62	9.3	27.7	65	316	1301
Sav	0.02	0.08	0.26	0.8	3.1	21	100	386
ACh	0.03	0.12	0.44	1.3	3.8	28	134	477
Chr-G	0.06	0.27	1.04	5.1	21.3	121	423	3326
Chr-HK	1.00	3.96	14.73	51.4	247.2	971	3060	_
MTS1	0.37	2.56	17.21	213.4	1248	11834	_	-
MTS3	0.46	3.55	24.65	989.1	2063	21716	_	_
$DT_{1,16}$	0.04	0.14	0.47	1.57	5.60	20.82	101	389
$DT_{5,16}$	0.62	2.30	7.78	24.48	81.35	254	808	_

(b) Average normalised running time (s)

Table 2: Comparison between established heuristics and DT-heuristics on uniform Euclidean instances

as well as two recent matching-based heuristics from [13]. The experiments were performed on a Sun Systems Enterprise Server E450.

Table 2 shows the results of these experiments. Abbreviations of the heuristics in the table follow [12,13]. As seen from the table, the average approximation quality of $DT_{1,16}$ turns out to be higher than all classical heuristics considered in [12], except Chr-HK. Moreover, heuristic $DT_{1,16}$ dominates heuristics RA^+ , Chr-S, FI, Chr-G. Heuristic $DT_{5,16}$ dominates Chr-HK. Heuristic $DT_{5,16}$ also compares very favourably with MTS heuristics, providing similar approximation quality at a small fraction of the running time. The above results show clearly that double-tree heuristics deserve a prominent place among the best tour-constructing heuristics for Euclidean TSP.

The impressive success of double-tree heuristics must, however, be approached with some caution. Although the normalised time is an excellent tool for com-

Size	1000	3162	10K	31K	100K	316K
RA^+	12.84	13.88	16.08	15.59	16.22	16.33
Chr-S	12.03	12.79	13.08	13.47	13.50	13.45
FI	9.90	11.85	12.82	13.37	13.96	13.92
Sav	13.51	15.97	17.21	17.93	18.20	18.50
ACh	10.21	11.01	11.47	11.78	12.00	11.81
Chr-G	8.08	9.01	9.21	9.47	9.55	9.55
Chr-HK	7.27	7.78	8.37	8.42	8.46	8.56
MTS1	8.90	9.96	11.97	11.61	9.45	-
MTS3	8.52	9.5	10.11	9.72	9.46	_
$DT_{4,16}$	6.37	8.24	8.79	9.40	9.38	9.39
$DT_{5,16}$	5.72	7.17	7.92	8.32	8.46	8.42

(a) Average excess over the Held–Karp bound (%)

Size	1000	3162	10K	31K	100K	316K
RA^+	0.1	0.2	0.7	1.9	5.5	12.7
Chr-S	0.2	0.8	3.2	11.0	37.8	152.8
FI	0.2	0.8	2.9	9.9	30.2	70.6
Sav	0.0	0.1	0.3	0.9	3.4	22.8
ACh	0.0	0.2	0.8	2.1	6.4	54.2
Chr-G	0.2	0.8	3.2	11.0	37.8	152.2
Chr-HK	0.9	3.3	11.6	40.9	197.0	715.1
MTS1	0.78	4.19	45.09	276	1798	_
MTS3	0.84	4.76	49.04	337	2213	_
$DT_{4,16}$	0.2	0.87	3.16	9.55	34.43	120.3
$\overline{\mathrm{DT}_{5,16}}$	1.12	4.85	16.08	53.35	174	569

(b) Average normalised running time (s)

Table 3: Comparison between established heuristics and DT-heuristics on clustered Euclidean instances

paring results reported in different computational experiments, it is only an approximate estimate of the exact running time. According to [12, page 377], "[this] estimate is still typically within a factor of two of the correct time". Therefore, as an alternative way of representing the results of computational experiments, we suggest a graph of the type shown in Figure 4, which compares the heuristics' average approximation quality and running time on random uniform instances with 10000 points. A normalised time t is represented by the interval [t/2, 2t]. The relative position of heuristics in the comparison and the dominance relationships can be seen clearly from the graph. Results for other instance sizes and types are generally similar.

Additional experimental results for clustered Euclidean instances are shown in Table 3 (with $DT_{1,16}$ replaced by $DT_{4,16}$ to illustrate more clearly the overall advantage of DT-heuristics), and for TSPLIB instances in Table 4.

Size	1000	3162	10K	31K	100K
RA^+	17.46	16.28	17.78	19.88	17.39
Chr-S	13.36	14.17	13.41	16.50	15.46
FI	15.59	14.28	13.20	17.78	15.32
Sav	11.96	12.14	10.85	10.87	19.96
ACh	9.64	10.50	10.22	11.83	11.52
Chr-G	8.72	9.41	8.86	9.62	9.50
Chr-HK	7.38	7.12	7.50	6.90	7.42
MTS1	7.0	6.9	5.1	4.7	4.1
MTS3	6.2	5.1	4.0	2.9	2.7
$DT_{1,16}$	6.36	5.99	8.09	9.99	10.02
$DT_{5,16}$	6.13	5.58	7.65	8.98	9.30

(a) Average excess over the Held–Karp bound (%)

Size	1000	3162	10K	31K	100K
RA^+	0.1	0.2	0.8	2.2	5.6
Chr-S	0.1	0.2	1.8	3.9	31.8
FI	0.2	0.8	3.1	9.8	26.4
Sav	0.0	0.1	0.3	0.6	1.4
ACh	0.0	0.1	0.5	1.5	3.9
Chr-G	0.1	0.2	1.8	3.8	29.5
Chr-HK	0.7	2.2	9.7	50.1	177.9
MTS1	-	1.5	34.4	107.3	620.0
MTS3	_	2.1	42.4	135.4	1045.3
$DT_{1,16}$	0.3	0.9	4.1	18.4	49.3
$DT_{5,16}$	0.6	2.1	11.0	57.1	115.1

(b) Average normalised running time (s)

Table 4: Comparison between established heuristics and DT-heuristics on geometric instances from TSPLIB: pr1002, pcb1173, r11304, nrw1379 (size 1000), pr2392, pcb3038, fn114461 (size 3162), pla7397, brd14051 (size 10K), pla33810 (size 31K), pla859000 (size 100K).

While we have done our best to compare the existing and the proposed heuristics fairly, we recognise that our experiments are not, strictly speaking, a "blind test": we had the results of [12] in advance of implementing our method, and in particular of selecting the top DT-heuristics for comparison. However, we never consciously adapted our choices to the previous knowledge of [12], and we believe that any subconscious effect of this previous knowledge on our experimental setup is negligible.

4 Conclusions and open problems.

In this paper, we have presented an improved algorithm for finding the minimumweight double-tree shortcutting approximation for Metric TSP. We challenged

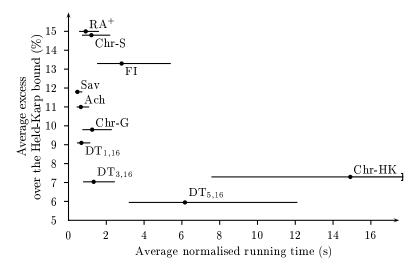


Fig. 4: Comparison between established heuristics and DT-heuristics on uniform Euclidean instances with 10000 points

ourselves to make the algorithm as efficient as possible. The improvement in time complexity from $O(n^3+2^dn^2)$ to $O(4^dn^2)$ (which implies $O(n^2)$ for the Euclidean TSP) placed the minimum-weight double-tree shortcutting method as a peer in the set of the most powerful tour-constructing heuristics. It is known that most powerful tour-constructing heuristics have theoretical time complexity $O(n^2)$, and in practice often exhibit near-linear running time. The minimum-weight double-tree method now also fits this pattern.

While we have not been using the language of parameterised complexity [10], we (and the previous work [5]) have in fact demonstrated that the problem of finding the minimum-weight double-tree tour for Metric TSP is fixed-parameter tractable (where the maximum degree of the MST is the relevant parameter). It would be interesting to see if this connection with parameterised complexity theory can be extended further, e.g. by using any of the established techniques for designing fixed-parameter tractable algorithms.

Our results should be regarded only as a first step in exploring new opportunities. Particularly, the minimum spanning tree is not the only possible choice of the initial tree. Instead, one can choose from a variety of trees, e.g. Held and Karp (1-)trees, approximations to Steiner trees, spanning trees of Delaunay graphs, etc. This variety of choices merits a further detailed exploration.

It is well-known that when the initial tree is a path, the resulting double-tree tour neighborhood is the set of all *pyramidal tours* [5]. In this case, a dozen of conditions on the distance matrix are known (see e.g. [4]), which guarantee that the tour neibborhood contains the absolute minimum-weight tour. It may be possible to generalise this approach by identifying new special types of trees and conditions on the distance matrices, which would guarantee that the minimum-

weight double-tree algorithm finds an absolute minimum-weight tour. For more results on polynomial solvability of TSP with special conditions imposed on the distance matrix, see [4,8].

The minimum-weight shortcutting problem for the Christophides graph remains NP-hard even in the planar Euclidean case. However, our algorithm turns out to be applicable also to this problem on certain classes of instances. It can be shown that if the Christofides graph is a *cactus* (i.e. all its cycles are pairwise edge-disjoint), then the set of all its shortcuttings is a subset of the set of all double-tree shortcuttings. Therefore, our algorithm, as well as the algorithm of [5], can be used to find efficiently the minimum-weight shortcutting when the Christofides graph is a cactus. In particular, such a shortcutting can be found in polynomial time in the planar Euclidean case.

Our efforts invested into theoretical improvements of the algorithm, supported by a couple of additional heuristic improvements, have borne the fruit: computational experiments with the minimum-weight double-tree algorithm show that it becomes one of the best known tour constructing heuristics. It appears that the double-tree method is also well suited for local search improvements based of transformations of trees and searching the corresponding tour neighborhoods. One can easily imagine various tree transformation techniques that could make our method even more powerful.

5 Acknowledgements

The authors thank an anonymous referee of a previous version of this paper, whose detailed comments helped to improve it significantly. The MST subroutine in our code is courtesy of the Concorde project [1].

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