Title

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ABSTRACT

We study the contribution of network coding (NC) in improving the multicast capacity of random wireless ad hoc networks. We consider a network with n nodes distributed uniformly in a unit square, with each node acting as a source for independent information to be sent to a multicast group consisting of m randomly chosen destinations. We consider the physical model, and show that the per-session capacity in the presence of arbitrary NC has a tight bound of $\Theta\left(\frac{1}{\sqrt{mn}}\right)$ when $m = O\left(\frac{n}{(log(n))^3}\right)$, and $\Theta\left(\frac{1}{n}\right)$ when $m = \Omega\left(\frac{n}{log(n)}\right)$. Prior work has shown that these same order bounds are achievable on the basis of pure routing, which utilizes only traditional store and forward methods. Therefore, our results demonstrate that the NC gain for multi-source multicast and broadcast is bounded by a constant factor.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design Wireless Communication**]: [Computer-Communication Network]

General Terms

Performance, Theory

Keywords

Capacity, Throughput, Network Coding, Wireless Ad Hoc Networks, Multicast

1. INTRODUCTION

The concept of network coding was first explored by Yeung et. al. [1] and subsequently generalized by Ahlswede et. al. [2] for a single source multicast in arbitrary directed graphs. Since then, the interest in network coding has increased rapidly. A large number of studies have investigated the utility of network coding (NC) for wireless networks, and widely cited experiments [3, 4] have been reported in which NC has been used successfully in combination with other mechanisms to attain large throughput gains compared to approaches based on conventional protocol stacks. These results have led many to believe that a combination of NC with wireless broadcasting can lead to significant improvements in the order throughput of wireless networks. Understandably, there is significant interest in identifying the true impact of NC on the throughput order of wireless networks. However, the exact characterization of network capacity with NC in the presence of multiple access interference is a very hard problem, even for simple networks, and limited results have been reported to date on the subject.

Recent work [5–7] has shown that the throughput gain due to the use of NC in a wireless network is bounded by a constant when the traffic in the network consists of multiple unicast sessions. However, the motivation for the original work by Ahlswede et. al [2] was improving network performance for multicasting, not unicasting. Furthermore, many commercial and defense applications, such as video conferencing, require multicasting of large amounts of information, and the study of the multicast capacity of wireless ad hoc networks is an important research topic in its own right.

Several works [8–15] have studied the multicast and broadcast capacity of wireless networks under conventional routing, and these results show consistently that broadcasting and multicasting significantly alter the throughput order of wireless networks. In the light of these findings, the importance of multicasting and broadcasting, and recent practical results on NC, it is natural to inquire whether the introduction of NC can improve the throughput capacity of multi-pair multicasting. In this work, we undertake the important, and as yet unanswered, task of characterizing the multicast and broadcast throughput order of wireless ad-hoc networks in presence of network coding.

We consider a network consisting of n nodes distributed randomly in the network space, with each node acting as source for m randomly chosen nodes in the network. Our contributions are as follows:

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As our contribution, we show that, in the presence of arbitrary NC, the per-session multicast capacity of random wireless ad hoc network under the physical model has a tight bound of $\Theta\left(\frac{1}{\sqrt{mn}}\right)$ when $m = O\left(\frac{n}{log(n)^2}\right)$, and $\Theta\left(\frac{1}{n}\right)$ when $m = \Omega\left(\frac{n}{log(n)^2}\right)$.

It has already been established in the literature that the above bounds are achievable on the basis of traditional store-and-forward routing methods. Consequently, our analysis demonstrates that the throughout gain due to NC for mutlicast as well as broadcast is bounded by a constant factor!

The remainder of this paper is organized as follows. Section 2 surveys relevant prior work. Section 3 describes the network models and other concepts used proofs. Section 4 deduces the capacity results under the physical model. Section 5 summarizes our conclusions.

2. LITERATURE REVIEW

Gupta and Kumar's original work focused on the unicast capacity of wireless networks [16], an many subsequent contributions have been made on the capacity of wireless networks subject to unicast traffic. However, the focus of this paper, and therefore this section, is on the capacity wireless networks under broadcast and multicast traffic.

Tavli [8] was the first to show that $\Theta\left(n^{-1}\right)$ is a bound on the per-node broadcast capacity of arbitrary networks. Zheng [9] derived the broadcast capacity of power-constrained networks, together with another quantity called "information diffusion rate." The work by Keshavarz et al. [10] is perhaps the most general work on the computation of the broadcast capacity for any number of sources in the network.

Several recent efforts have adressed the multicast capacity of wireless networks, primarily under the protocol model. Jacquet and Rodolakis, [11] proved that the scaling of multicast capacity is decreased by a factor of $O(\sqrt{m})$ compared to the unicast capacity result by Gupta and Kumar [16]. This result implies that multicasting gain, over transmitting the information from each source as m unicasts, is at least $\Theta(\sqrt{m})$. The work by Shakkottai et al [12] assumes there are n^{ϵ} multicast sources and $n^{1-\epsilon}$ destinations per flow for some $\epsilon > 0$. The results from this work are limited in scope, because of its constraints on the number of sources and destinations. Li et al. [13] compute the capacity of wireless ad hoc networks for unicast, multicast, and broadcast applications. Zheng et. al. [14] independently generalized this work and introduced (n, m, k)-casting as a framework for the characterization of all types of information dissemination in wireless networks. Keshavarz et. al. [15] studied the multicast and broadcast capacity of wireless networks, consider the physical model, and generalize the work in [17] to the multicast regime. This prior work has only addressed conventional store-and forward routing for multicast and broadcast traffic.

Since Ahlswede et. al.'s [2] seminal work, most of the theoretical research on NC has focused on directed networks, where each communication link is point to point and has a fixed direction. However, a wireless network is more appropriately modeled by bi-directional links. Li et. al [18, 19] have studied the benefits of NC in undirected networks. The result shows that, for a single unicast or broadcast session, there is no throughput improvement due to NC. In the case of a single multicast session, such an improvement is bounded by a factor of two. Nevertheless, the work by Li et. al does not account for multiple access interference, and hence cannot be directly applied to wireless networks.

As we have stated, there has been prior addressing the unicast capacity of wireless networks that use NC. Liu et. al. [5, 6] have shown that the NC for unicast traffic in a random network (i.e. a network in which the nodes are distributed randomly in a Euclidean space and the sources and desitantions are also placed randomly) is bounded by a constant factor. Keshavarz et. al. [7] extended these conclusions to arbitrary networks and an arbitrary unicast traffic pattern. Furthermore, they also showed that the NC gain for even a single source multicast is bounded by a constant factor in any arbitrary network.

Physical network coding (PNC) [20] and analog network coding (ANC) [21] have been proposed recently, which combine NC with advanced processing at the physical layer that allows receivers to decode multiple concurrent transmissions. ANC was shown [21] to provide throughput gains when compared with digital network coding (i.e., receivers decode at most one packet at a time) and traditional routing (i.e., no NC and receivers decode at most one packet at a time) operating in simple network topologies in which ideal scheduling (i.e., no MAI) is assumed for channel access. Throughput gains have also been reported for PNC in simple topologies [20]. However, we have shown that the order throughput of a wireless network can be increased by embracing interference at the physical layer through multi-packet transmission (MPT) or reception (MPR), without the use of NC [22,23]. Furthermore, we have also shown [24] that using NC together with MPT and MPR does not increase the order throughput of a wireless network for multicasting compared to what MPR and MPT can provide by themsleves.

From the above, it is apparent that prior work has not determined whether NC by itself can provide any gains on the multicast order throughput in wireless networks, which is the subject of this paper.

3. PRELIMINARIES

For a continuous region A, we use $|A|$ to denote its area. We denote the cardinality of a set by $|\mathbb{S}|$, and by $\|X_i - X_j\|$ the distance between nodes i and j. Whenever convenient, we utilize the indicator function $1_{\{P\}}$, which is equal to one if P is true and zero if P is false. $Pr(E)$ represents the probability of event E. We say that an event E occurs with high probability (w.h.p.) if $Pr(E) > (1 - (1/n))$ as $n \to \infty$. We employ the standard order notations O, Ω, and Θ.

We assume that the topology of a network is described by a uniformly random distribution of n nodes in a unit square. Let $V = 1, \ldots, n$ represent the node-set and let X_i be the location of node $i \in V$. To avoid boundary effects, it is typical to assume that the network surface is placed upon a toroid or sphere. However, for mathematical convenience, in this work we ignore edge effects and thus assume that the network is placed in a 2-D plane. Further, in our model, as n goes to infinity, the density of the network also goes to infinity. Therefore, our analysis is applicable only to dense networks. We do not consider mobility of nodes and assume a static stationary distribution of nodes. Our capacity analysis is based on the physical model introduced by Gupta and Kumar [16].

The physical model describes the success of a transmission in terms of Signal-to-Inteference/Noise (SINR) criteria.

Definition 3.1. The Physical Model
All transmissions at all nodes utilize an identical transmission power P. Node i can successfully transmit to node j iff the signal-to-interference/noise ratio (SINR) of the pair transmitter i and receiver j satisfies

$$\mathrm{SINR}_{i \to j} = \frac{Ph_{ij}}{BN_0 + \sum_{k \neq i, k=1}^{n} Ph_{kj}} \geq \beta, \tag{1}$$

where h_{ij} is the channel attenuation factor between nodes i and j, and BN_0 is the total ambient noise power. We assume that the channel attenuation factors are completely determined by the path loss model and hence $h_{ij} = \|X_i - X_j\|^{-\alpha}$. We assume that $\beta \geq 1$ in all our analysis.

We assume that the data rate for each successful transmission is W bits/second, which is a constant value and does not depend on n. Given that W does not change the order capacity of the network, we normalize its value to one.

We focus on the traffic scenario in which each node of the wireless network acts as a multicast source for a randomly chosen set of m distinct destinations.

Definition 3.2. Feasible rate
In a wireless ad hoc network with n nodes in which each source transmits its packets to m destinations, a throughput of $\lambda_m(n)$ bits per second for each multicast session is feasible if there is a spatial and temporal scheme for scheduling and network coding transmissions, such that, by operating the network in a multi-hop fashion and buffering at intermediate nodes when awaiting transmission, every source node can send $\lambda_m(n)$ bits per second on average to its m chosen destination nodes. That is, there is a $T < \infty$ such that in every time interval $[(i-1)T, iT]$ every node can send $T\lambda_m(n)$ bits to its corresponding destination nodes. Let $C_m(n)$ represent the maximum feasible rate.

Definition 3.3. Throughput Order
$C_m(n)$ is said to be of order $\Theta(f(n))$ bits/second if there exist deterministic positive constants c and c' such that

$$\begin{cases} \lim_{n \to \infty} \; Prob\ (C_m(n) = cf(n) \ \text{is feasible}) = 1 \\ \liminf_{n \to \infty} \; Prob\ (C_m(n) = c'f(n) \ \text{is feasible}) < 1. \end{cases} \tag{2}$$

Definition 3.4. Cut
Given a node set V, a cut is the separation of the vertex set V into two disjoint and exhaustive subsets (S, S^C). Here, a vertex partition can be completely described by partitioning the network-area into two region (A, A^c) as shown in Fig. 1, thus we also refer to a closed region A as a cut. The cut-capacity $C(A)$ is defined as the maximum number of simultaneous transmissions that can take place from A^c to A.

Definition 3.5. Multicast Cut-Demand
Given a cut A, a source node in A^c is said to have demand across the cut iff at least one of its destination lies in A. The multicast demand $D(A)$ across the cut is defined as the total number of sources in A^c such that there is at least one destination in the multicast group across the cut.

Definition 3.6. Sparsest Cut
We define the sparsity Γ_A of cut A as the ratio

$$\Gamma_A = \frac{C(A)}{D(A)} \tag{3}$$

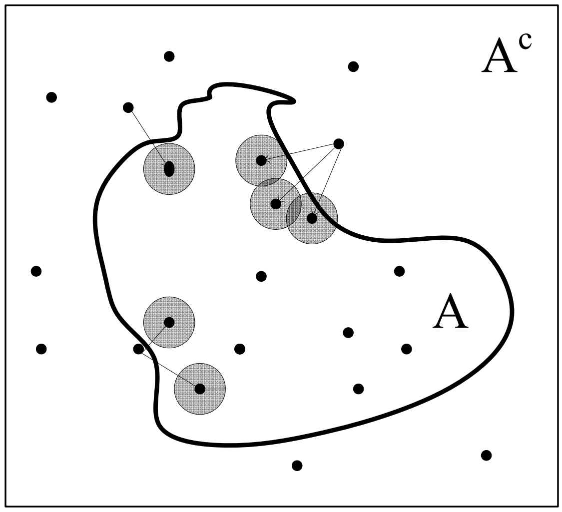

Figure 1: Generalized Sparsity Cut

Hence, the sparsest cut is given by

$$A^* = arg \min_A \Gamma_A \tag{4}$$

where A^* has the least possible sparsity, denoted as Γ_{A^*}.

The conventional definition of Sparsity cut [25]is applicable only to unicast traffic [6]. Our definition generalizes the conventional definition to multicast traffic.

Finally we state the well-known Chernoff Bounds [26], which shall be repeatedly used in the rest of this paper.

Lemma 3.7. Chernoff Bounds: *Consider n i.i.d random variables $Y_i \in \{0,1\}$ with $p = Pr(Y_i = 1)$. Let $Y = \sum_{i=1}^{n} Y_i$. Then for any $1 \geq \delta \geq 0$ and $\delta_2 \geq 0$ we have*

$$Pr(Y \leq (1-\delta_1)np) < 2e^{\frac{-\delta_1^2 np}{3}} \tag{5}$$

$$Pr(Y \geq (1+\delta_2)np) < 2e^{\frac{-\delta_2^2 np}{3}} \tag{6}$$

4. BOUNDS FOR PHYSICAL MODEL

It is well-known that under the conventional definition, the sparsity cut can be used to obtain an upper bound on the unicast traffic flow in a wireless network [6, 25]. In a similar way, our generalized definition provides an upper bound for multicast flows.

Lemma 4.1. *Let $C_m(n)$ be maximum multicast flow-rate in a network and let A^* be the sparsest cut with sparsity Γ_{A^*}, then we have*

$$C_m(n) \leq \Gamma_{A^*} \tag{7}$$

Proof. Let f be the total maximum feasible average rate at which bits can be transmitted from A^c to A, where A is any arbitrary cut. Then by Def. 3.4 we have

$$f \leq C(A) \tag{8}$$

The total information flow across a cut has to be greater than or equal to the sum of the data rates associated with individual multicast sessions that communicate across the cut. Hence,

$$
\begin{aligned}
f &\geq \sum_{i=1}^{n} C_m(n) 1_{\{\text{source } i \text{ in } A^c \text{ has demand across cut } A\}} \\
&= C_m(n) \sum_{i=1}^{n} 1_{\{\text{source } i \text{ in } A^c \text{ has demand across cut } A\}} \\
&= C_m(n) D(A). \qquad (9)
\end{aligned}
$$

Inserting the above equation in Eq.8, we have

$$
C_m(n) \leq \frac{C(A)}{D(A)} = \Gamma_A \leq \Gamma_{A^*}. \qquad (10)
$$

□

In order to prove the upper bounds under the physical we utilize a circular cut, instead of square shaped cut, with radius r_A as shown in Fig. 2. Additionally, we utilize the following property of the physical model. A similar property of "*straight-lined cuts*" has also been utilized by Liu, et. al. [6].

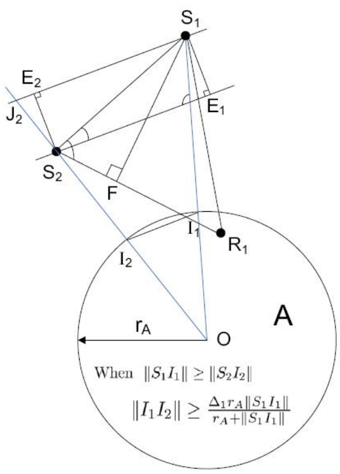

Figure 2: Geometric property of transmissions across the cut

Lemma 4.2. *Consider a circular cut A of radius r_A with its center at point O. Let S_1 and S_2 be two nodes outside A transmitting across the cut in the same slot. We claim that the arc subtended by angle $\angle S_1 O S_2$ on cut A has a length of atleast*

$$
\frac{\Delta_1 r_A max\{L_1, L_2\}}{r_A + max\{L_1, L_2\}} \qquad (11)
$$

where $\Delta_1 = \left(\beta^{\frac{1}{\alpha}} - 1\right)$ *and L_i represents the (minimum) distance of transmitter S_i from cut A.*

Proof. Without loss of generality we can assume that S_1, S_2 are placed as shown in Fig. 2 and $L_1 \geq L_2$. In Fig. 2 the rays OS_1 and OS_2 intersect the cut A at I_1 and I_2 respectively. Therefore, $L_1 = \|S_1 I_1\|$ and $L_1 = \|S_2 I_2\|$. Furthermore, the length of segment $I_1 I_2$ is less than the length of the arc subtended by $\angle S_1 O S_2$. Hence, in order to prove the claim, it is sufficient to show that

$$
\|I_1 I_2\| \geq \frac{\Delta_1 r_A \|S_1 I_1\|}{r_A + \|S_1 I_1\|} \qquad (12)
$$

Consider a receiver R_1 that lies inside A and can successfully decode a transmission from S_1. It follows from Eq. 1 in Definition 3.1 that

$$
\begin{aligned}
&\frac{P\|S_1 R_1\|^{-\alpha}}{BN_o + P\|S_2 R_1\|^{-\alpha}} \geq \beta \\
&\implies \|S_2 R_1\| \geq \beta^{\frac{1}{\alpha}} \|S_1 R_1\| = (1 + \Delta_1)\|S_1 R_1\| \qquad (13)
\end{aligned}
$$

Consider the triangle formed by S_1, S_2 and R_1, as shown in Fig. 2. Now draw a perpendicular from S_1 to F, which is a point on segment $S_2 R_1$. Note that $\|F R_1\| \leq \|S_1 R_1\|$ and hence it is easy to show that $\|S_2 F\| \geq \Delta_1 |S_1 R_1|$. Now draw a line through S_2 parallel to segment $I_1 I_2$ and drop a perpendicular $S_1 E_1$ on this line. Since $\angle S_1 S_2 E_1 \leq \angle S_1 S_2 R_1$, we have $cos\,(\angle S_1 S_2 E_1) \geq cos\,(\angle S_1 S_2 R_1)$, which implies that $|S_2 E_1| \geq |S_2 F|$. Similarly draw a line through S_1 parallel to $I_1 I_2$. Let this line intersect the ray OS_2 at J_2. Drop a perpendicular $S_2 E_2$ on line $S_1 J_2$. Since the triangle $S_1 O J_2$ is isosceles, $\angle S_1 J_2 S_2$ is acute and hence E_2 should lie within the segment $S_1 J_2$. Hence, $\|S_1 J_2\| \geq \|S_1 E_2\|$. Since $S_2 E_1 S_1 E_2$ forms a rectangle we get $\|S_1 J_2\| \geq \Delta_1 |S_1 R_1|$. Finally, we note that $\|S_1 R_1\| \geq \|S_1 I_1\|$ because $S_1 I_1$ is the shortest distance between S_1 and circle A. Hence,

$$
\|S_1 J_2\| \geq \Delta_1 \|S_1 I_1\| \qquad (14)
$$

Consider the triangle $OS_1 J_2$. The Basic Proportionality Theorem implies that

$$
\|I_1 I_2\| = \frac{\|S_1 J_2\| \|S_1 I_1\|}{\|OS_1\|} \qquad (15)
$$

Substituting Eq. 14 in Eq. 15 proves the claim in Eq. 13 □

Theorem 4.3. *In a random geometric network, the multicast capacity under the physical model, with network coding, w.h.p has an upper bound of*

$$
C_m(n) = O\left(\frac{1}{\sqrt{mn}}\right), \qquad (16)
$$

when $m = O\left(\frac{n}{log(n)^2}\right)$ *and* $n \to \infty$.

Proof. Consider a circular cut A with radius $r_A = \frac{1}{4\sqrt{m}}$. Divide the region A^c, as shown in Fig. 3, into sub-region B and $A^c - B$, where the B is an annular region of width $\frac{1}{\sqrt{n}}$. Let n_B and n_{A^C-B} be the maximum number of nodes, from region B and region $A^c - B$ respectively,that can transmit to region A in a single time slot. Hence,

$$
C(A) \leq n_B + n_{A^c-B} \qquad (17)
$$

A transmission from any node in region $A^c - B$ to any node in region A has a minimum hop-length of $\frac{1}{\sqrt{n}}$. Consequently, Lemma 4.2 implies that any two transmitters in $A^c - B$, that transmit in the same slot, have to be separated such that they subtend an arc on A of length at least $\frac{\Delta_1 r_A \frac{1}{\sqrt{n}}}{r_A + \frac{1}{\sqrt{n}}}$. Since the circumference of A is $2\pi r_A$ we have

$$
\begin{aligned}
n_{A^c-B} &\leq 2\pi r_A \times \frac{r_A + \frac{1}{\sqrt{n}}}{\Delta_1 r_A \frac{1}{\sqrt{n}}} \\
&= \frac{2\pi}{\Delta_1}\left(\frac{\sqrt{n}}{4\sqrt{m}} + 1\right) \leq \frac{5\pi\sqrt{n}}{2\Delta_1\sqrt{m}} \qquad (18)
\end{aligned}
$$

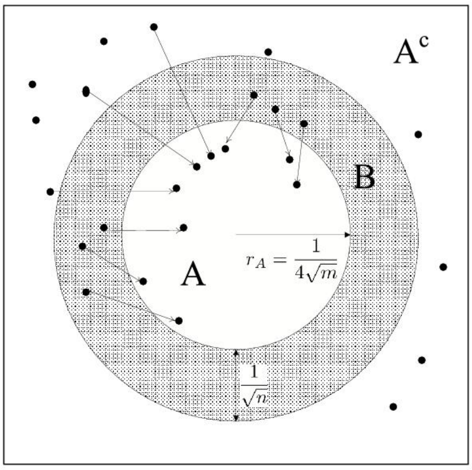

Figure 3: Cut Capacity under Physical Model

To obtain a bound on n_B, observe that the area of region B is given by

$$\begin{aligned} |B| &= \pi\left(r_A + \frac{1}{\sqrt{n}}\right)^2 - \pi r_A^2 \\ &= \frac{2\pi r_A}{\sqrt{n}} + \frac{\pi}{n} \leq \frac{\pi}{2\sqrt{mn}} + \frac{\pi}{\sqrt{mn}} \\ &\leq \frac{3\pi}{2\sqrt{mn}} \end{aligned} \tag{19}$$

If $m = O\left(\frac{n}{(log(n))^2}\right)$, there exists a constant $c_3 \geq 0$ such that

$$|B| \leq \frac{c_3 log(n)}{n} \tag{20}$$

The total number of nodes in B is necessarily greater than n_B. Therefore, the Chernoff Bound of Eq. 5 implies that, for any $\delta_2 \geq 0$, we have

$$\begin{aligned} &\Pr\left(n_B \leq \frac{3\pi(1+\delta_2)\sqrt{n}}{2\sqrt{m}}\right) \leq 2e^{\frac{-\delta_2^2 n|B|}{3}} \\ \leq\ & 2e^{\frac{-\delta_2^2 log(n)}{3c_3}} = \frac{2}{n^{\frac{\delta_2^2}{3c_3}}}. \end{aligned} \tag{21}$$

Consequently, if we choose $\delta_2 \geq 3c_3$, then as $n \rightarrow \infty$ w.h.p we have

$$\begin{aligned} C(A) &\leq \frac{3\pi(1+\delta_2)\sqrt{n}}{2\sqrt{m}} + \frac{5\pi\sqrt{n}}{2\Delta_1\sqrt{m}} \\ &= \frac{\pi(3(1+\delta_2)\Delta_1 + 5)\sqrt{n}}{2\Delta_1\sqrt{m}} \end{aligned} \tag{22}$$

In the previous section, we have already shown that that w.h.p the demand across square shaped cut with area $O(\frac{1}{m})$ is of the order of $\Theta(n)$. Such a property is valid for circular cuts also. Let q_1 be probablity that a source node in A^c has atleast one of its m destinations in the circle A. We can show that

$$\begin{aligned} q_1 &\geq \left(1 - \frac{1}{16}\right)\left(1 - \left(1 - \frac{1}{16m}\right)^m\right) \\ &= \frac{15\left(1 - e^{\frac{1}{16}}\right)}{16} = c_4 \end{aligned} \tag{23}$$

The Chernoff Bound of Eq. 6 implies that there exists a $1 \geq \delta_1 \geq 0$ such that as $n \rightarrow \infty$ w.h.p $D(A) \geq (1-\delta_1)c_4 n$ Therefore, the Sparsity bound from Lemma 4.1, along with Eq. 22 and Eq. 23, implies that w.h.p.

$$C_m(n) \leq \left(\frac{\pi(3(1+\delta_2)\Delta_1 + 5)}{2\Delta_1(1-\delta_1)c_4}\right)\frac{1}{\sqrt{mn}} \tag{24}$$

□

THEOREM 4.4. *Under the physical model, the multicast capacity in a random geometric network with network coding w.h.p. has an upper bound of*

$$C_m(n) = O\left(\frac{1}{mlog(n)}\right) \text{ if } m \leq \frac{n}{log(n)} \tag{25}$$

$$C_m(n) = O\left(\frac{1}{n}\right) \text{ if } m \geq \frac{n}{log(n)} \tag{26}$$

PROOF. Decompose the network into squarelets of side-length $\sqrt{\frac{log(n)}{9n}}$. Let J be an event that there exists a squarelet containing *at least* $\frac{(1-\delta_3)log(n)}{9n}$ nodes, where $1 \geq \delta_3 \geq 0$, with all its eight adjoining squarelets empty. The event J is illustrated in Fig. 4. We are interested in showing that the event J occurs w.h.p. Let η represent the total number of nodes in a squarelet, $p_1 = Pr(\eta = 0)$ and $p_2 = Pr\left(\eta \leq \frac{(1-\delta_3)log(n)}{9n}\right)$, where $1 \geq \delta_3 \geq 0$. p_1 can be computed as

$$p_1 = \left(1 - \frac{log(n)}{9n}\right)^n = e^{\frac{-log(n)}{9}} = n^{\frac{-1}{9}}. \tag{27}$$

We used the fact that $lim_{n\rightarrow\infty}\left(1 - \frac{a}{n}\right)^n = e^{-a}$.

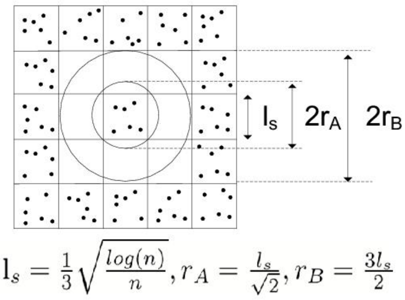

Figure 4: Clustering of nodes

In addition, Eq. 5 implies that

$$\begin{aligned} p_2 &= Pr\left(\eta \leq \frac{(1-\delta_3)log(n)}{9n}\right) \\ &\leq 2e^{\frac{-\delta_3^2 log(n)}{27}} = 2n^{\frac{-\delta_3^2}{27}}. \end{aligned} \tag{28}$$

Therefore, as $n \rightarrow \infty$, in the limit we have

$$\begin{aligned} Pr(J) &\geq 1 - (1 - (1-p_2)p_1^8)^{\frac{9n}{log(n)}} \\ &\geq 1 - (1 - (1 - 2n^{\frac{-\delta_3^2}{27}})n^{\frac{-8}{9}})^{\frac{9n}{log(n)}} \\ &\geq 1 - \left(\left(1 - \frac{n^{\frac{1}{9}}(1 - 2n^{\frac{-1}{27}})}{n}\right)^n\right)^{\frac{9}{log(n)}} \\ &= 1 - e^{-9\frac{n^{\frac{1}{9}}(1-2n^{\frac{-1}{27}})}{log(n)}} = 1. \end{aligned} \tag{29}$$

Note that $e^{-9\frac{n^{\frac{1}{9}}(1-2n^{\frac{-1}{27}})}{log(n)}}$ approaches zero faster than $\frac{1}{n}$ when $n \rightarrow \infty$.

Let us choose a circular cut A of radius $r_A = \frac{l_s}{\sqrt{2}}$ such that A circumscribes a squarelet satisfying property J. Observe that we can draw another circle B of radius $r_B = \frac{3l_s}{2}$ concentric to A, such that all nodes that transmit across the cut A are placed outside B. Therefore the minimum hop-length of any transmission across the cut A is atleast $r_B - r_A$. Therefore Lemma 4.2 implies that

$$\begin{aligned} C(A) &\leq 2\pi r_A \times \frac{r_A + (r_B - r_A)}{\Delta_1 r_A (r_B - r_A)} = \frac{2\pi r_A}{\Delta_1 (r_B - r_A)} \\ &= \frac{2\pi \frac{l_s}{\sqrt{2}}}{\Delta_1 \left(\frac{3l_s}{2} - \frac{l_s}{\sqrt{2}}\right)} = \frac{\pi 2\sqrt{2}}{\Delta_1 (3 - \sqrt{2})} = c_5 \end{aligned} \quad (30)$$

Now let p_3 be the probability that a source has demand across cut A. Observe that all the nodes inside the circle A are within the middle squarelet. Hence the Chernoff Bound can be used to show that as $n \rightarrow \infty$ w.h.p the total number of nodes outside the circle A are at least $n - \frac{(1+\delta_4)log(n)}{9}$, where $\delta_4 \geq 0$. Therefore, as $n \rightarrow \infty$ w.h.p.,

$$\begin{aligned} p_3 &= \left(1 - \frac{(1+\delta_4)log(9n)}{n}\right) \\ &\quad \times \left(1 - \left(1 - \frac{(1-\delta_3)log(n)}{9n}\right)^m\right) \\ &= \left(1 - e^{\frac{-m(1-\delta_3)log(n)}{9n}}\right) \end{aligned} \quad (31)$$

In the above equation we have $p_3 = \Theta(1)$ when $m = \Omega\left(\frac{n}{log(n)}\right)$, while when $m = O\left(\frac{n}{log(n)}\right)$ we have that

$$p_3 \geq \frac{m(1-\delta_3)log(n)}{9n} \quad (32)$$

Therefore, an application of Eq. 6 allows us to show that $D(A) = \Omega(mlog(n))$ when $m = O\left(\frac{n}{log(n)}\right)$, while $D(A) = \Omega(n)$ when $m = \Omega\left(\frac{n}{log(n)}\right)$. We get the final result by calculating the sparsity $\Gamma_A = \frac{C(A)}{D(A)}$ which, as established by Lemma 4.1 provides an upperbound for the capacity $C_m(n)$. $\square$

The upper bounds stated in the above theorem are identical to those of Theorem 2 in [15] and the initial steps in our proof are similar to those in [15]. However, we highlight that our eventual argument utilizes the geometric properties of the cut and hence is distinct from [15]. In particular, the claims and the proof in [15] is applicable only to routing, while our bounds apply to NC.

Keshavarz et. al. [15] have established the following lower bound on the multicast capacity under routing.

THEOREM 4.5. *In a random geometric network the multicast capacity under the physical model, with routing, w.h.p. has an lower bound*

$$\begin{aligned} C_m(n) &= \Omega\left(\tfrac{1}{\sqrt{mn}}\right) && \text{if } m \leq \frac{n}{\log(n)^3} \\ C_m(n) &= \Omega\left(\tfrac{1}{m\sqrt{\log(n)^3}}\right) && \text{if } \frac{n}{\log(n)^3} \leq m \leq \frac{n}{\log(n)^2} \\ C_m(n) &= \Omega\left(\tfrac{1}{\sqrt{mnlog(n)}}\right) && \text{if } \frac{n}{log(n)^2} \leq m \leq \frac{n}{log(n)} \\ C_m(n) &= \Omega\left(\tfrac{1}{n}\right) && \text{if } \frac{n}{log(n)} \leq m \end{aligned} \quad (33)$$

Given that any capacity achieved by routing is necessarily achievable by network coding, putting together the deductions up to this point, we arrive at the following result.

THEOREM 4.6. *Under the physical model, the multicast capacity in a random geometric network with network coding has a tight bound w.h.p. of*

$$C_m(n) = \Theta\left(\tfrac{1}{\sqrt{mn}}\right) \text{ if } m \leq \tfrac{n}{log(n)^3} \quad (34)$$

$$C_m(n) = \Theta\left(\tfrac{1}{n}\right) \text{ if } \tfrac{n}{log(n)} \leq m. \quad (35)$$

Consequently,

COROLLARY 4.7. *In a random geometric network with n nodes and for all values of m, the multicast throughput order gain provided by network coding over routing, under the physical model, is $O(1)$*

5. CONCLUSION

Network coding (NC) has received considerable attention, and recent results for specific instantiations of NC have led many to infer that NC could lead to order throughput gains for multicasting in wireless networks. In this work, we used the physical model to show that the order throughput gain derived from NC for multicasting and broadcasting in wireless networks is bounded by a constant. That is, as the network size increases, NC renders the same order throughput as traditional store-and-forward routing.

Despite this negative result on order throughput for NC, we need to emphasize that, in practice, constant-factor gains should not be ignored, and hence NC may still prove to have much utility in wireless networks. However, together with prior results on the order throughput gains derived from multi-packet transmission and reception (MPTR) [22, 24], the results in this paper indicate that specific implementations of NC should be evaluated against specific implementations of MPTR, not just traditional protocol stacks designed to avoid multiple access interference.

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