On the Complexity of Nash Dynamics and Sink equilibria

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Abstract

Studying Nash dynamics is an important approach for analyzing the outcome of games with repeated selfish behavior of self-interested agents. Sink equilibria has been introduced by Goemans, Mirrokni, and Vetta for studying social cost on Nash dynamics over pure strategies in games. However, they do not address the complexity of sink equilibria in these games. Recently, Fabrikant and Papadimitriou initiated the study of the complexity of Nash dynamics in two classes of games. In order to completely understand the complexity of Nash dynamics in a variety of games, we study the following three questions for various games: (i) given a state in game, can we verify if this state is in a sink equilibrium or not? (ii) given an instance of a game, can we verify if there exists any sink equilibrium other than pure Nash equilibria? and (iii) given an instance of a game, can we verify if there exists a pure Nash equilibrium (i.e, a sink equilibrium with one state)?

In this paper, we almost answer all of the above questions for a variety of classes of games with succinct representation, including anonymous games, player-specific and weighted congestion games, valid-utility games, and two-sided market games. In particular, for most of these problems, we show that (i) it is PSPACE-complete to verify if a given state is in a sink equilibrium, (ii) it is NP-hard to verify if there exists a pure Nash equilibrium in the game or not, (iii) it is PSPACE-complete to verify if there exists any sink equilibrium other than pure Nash equilibria. To solve these problems, we illustrate general techniques that could be used to answer similar questions in other classes of games.

Keywords: Nash equilibria, potential games, sink equilibria

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1 Introduction

A standard approach in studying the outcome of a system involving self-interested behavior of agents is to investigate the Nash dynamics of the corresponding games. In Nash dynamics, agents repeatedly respond to the current state of the game by playing a best-response strategy. Studying such dynamics is very important for understanding the behavior of a system throughout time, and the outcome of the game after many repeated game play. Similar to the recent efforts in studying the complexity of game theoretic concepts such as mixed Nash equilibria [8, 4], and pure NE [10, 21], studying the complexity of Nash dynamics can help us better understand the outcome of a game.

In an attempt to study such dynamics for pure strategies, Goemans, Mirrokni, and Vetta [15] introduced the concept of sink equilibria in games: sink equilibria are strongly connected components of a strategy profile graph associated with the game with no outgoing edges. Equivalently, sink equilibria characterize all states for which the probability of reaching that state after a sufficiently large random best-response sequence is nonzero. Also any random best-response sequence will converge to a sink equilibrium with probability one. Moreover, sink equilibria generalize pure Nash equilibria in that a pure Nash equilibrium is a single-state sink equilibrium of the game.

Goemans et al. [15] studied sink equilibria for their social cost in two classes of games. However, they did not consider the complexity of sink equilibria or Nash dynamics in those games. Recently, Fabrikant and Papadimitriou [11] initiated the study of the complexity of sink equilibria. by studying the problem of verifying if a state is in a sink equilibria for two classes of games. Extending on these ideas, we formalize several questions related to Nash dynamics of various games and completely study the complexity of the Nash dynamics and sink equilibria in these games.

Sink equilibria characterize all strategy profiles in the game with a nonzero probability of reaching them after a long enough best-response walk. Therefore, given a strategy profile, in order to verify if there is a non-zero probability of reaching this state after a sufficiently long random best-response walk we need to verify if this state is in a sink equilibrium or not. This problem has been considered by Fabrikant and Papadimitriou [11] for two classes of games, and is as follows:

IN A SINK **problem.** Given an instance of a game and a strategy profile in this game, can we verify if this strategy profile belongs to any sink equilibria or not?

For a given state in a game, an interesting problem is to estimate the probability of reaching this state after a long random best-response walk. Note that a hardness result for IN A SINK problem implies that for a given state, even approximating this probability is a computationally hard problem, (since distinguishing the probability of zero and nonzero is hard). Fabrikant and Papadimitriou showed that IN A SINK problem is PSPACE-hard for graphical games and a BGP next-hop routing game [11]. We show that this problem is PSPACE-complete for weighted/player-specific congestion games, valid-utility games, two-sided market games, and anonymous games. The proofs for all the above games except anonymous games are similar and based on a reduction from halting problem of a space bounded Turing machine. The proof for anonymous games has unique features and is different from the rest.

Given an instance of a game, it is very helpful to know if the random repeated self-interested actions of the agents in the game can cycle forever or such dynamics will converge to a pure Nash equilibria with probability one. This problem is related to characterizing the structure of sink equilibria in a game, and in particular the existence of non-singleton sink equilibria. Having such a sink equilibrium indicates that even random Nash dynamics may also converge to an everlasting cycle. As a result, we formalize the following problem in games:

HAS A NON-SINGLETON SINK **problem.** Given an instance of a game, can we verify if this game possesses a non-singleton sink equilibrium, i.e., sink equilibria other than pure Nash equilibria.

Pure Nash equilibria (if they exist) are local optima of the Nash dynamics. Other than the problem of computing a pure Nash equilibrium in various games, the problem of verifying if such equilibria exist has been studied for various classes of games. We complement the previous questions with the following problem:

HAS A SINGLETON SINK **problem.** Given an instance of a game, can we verify if this game possesses a pure Nash equilibrium (singleton sink equilibrium)?

Answering all the above questions for a game gives a thorough understanding of the complexity of Nash dynamics and the complexity of characterizing sink equilibria in that game.

Our Results. We study the above four problems in a variety of games with succinct representation including player-specific and weighted congestion games, anonymous games, valid-utility games, and two-sided market games. All of these games are well-studied for their existence of pure Nash equilibria, complexity of mixed and pure NE, or/and their price of anarchy for different social functions [13, 19, 18, 16, 6]. To solve these problems, we illustrate general techniques that could be used as tools to answer similar questions for other classes of games.

Fabrikant and Papadimitriou showed that IN A SINK problem is PSPACE-hard for graphical games and a BGP next-hop routing game [11]. They posed this problem as an open question for weighted congestion games, and valid-utility games. We show that this problem is PSPACE-complete for weighted/player-specific congestion games, valid-utility games, two-sided market games, and anonymous games. The proofs for all the above games except anonymous games are similar and based on a reduction from halting problem of a space bounded Turing machine. The proof for anonymous games has unique features and is different from the rest. The hardness of the IN A SINK problem in anonymous games is despite the fact that approximate pure Nash equilibria can be computed in these games in polynomial time [?].

For HAS A NON-SINGLETON SINK problem, we prove that it is PSPACE-complete for weighted/player-specific congestion games, valid-utility games, two-sided market games, and anonymous games. The reductions for HAS A NON-SINGLETON SINK problem extend the proofs for the IN A SINK problem.

HAS A SINGLETON SINK problem has been well-studied for all games in this paper except for valid-utility games and two-sided market games. We show that HAS A SINGLETON SINK PROBLEM is NP-hard for these games as well. Our results for two-sided markets characterize the complexity of existence of a stable matching in many-to-one two-sided matching markets; an extensively studied problem in the economics literature [13, 20, 17]. Existing results for many-to-one two-sided markets give sufficient conditions for existence of stable matchings (or pure Nash equilibria) in different variants of the problem [13, 20, 17], but they have not explored the complexity of verifying the existence of stable matchings (or pure Nash equilibria) in these games.

Related Work. Prior to this paper, the HAS A NON-SINGLETON SINK problem has not been studied for any of the above games. In a sink problem has been studied only for graphical games [11]. HAS SINGLETON SINK problem, however, has been studied extensively for all the above games except validutility games and two-sided market games. In fact, it has been shown that HAS A SINGLETON SINK problem is NP-hard for weighted congestion games and local-effect games[9], player-specific congestion games [2], graphical games [11], and action-graph games [16]. For anonymous games it has been shown that hat an approximate NE are computable in polynomial time[7] and that HAS A SINGLETON SINK is TC⁰-complete[3].

There has been a recent significant progress in understanding the complexity of equilibria in games. The complexity of mixed Nash equilibria is now well-understood by the recent results on PPAD-hardness of computing mixed NE[8, 4], and even for computing approximate mixed NE[5]. The complexity of pure Nash equilibria in various games (especially congestion games) have also been well-studied by recent results on PLS-completeness of computing a pure Nash equilibrium[10, 1], and even for

2 Preliminaries

2.1 General Definitions

Strategic games. A strategic game (or a normal-form game) $\Lambda = \langle N, (\Sigma_i), (u_i) \rangle$ has a finite set $N = \{1, \ldots, n\}$ of players. Player $i \in N$ has a set Σ_i of strategies (or strategies). The whole strategy set is $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n$ and a strategy profile $S \in \Sigma$ is also called a profile or state. The utility function of player i is $u_i : \Sigma \to \mathbb{R}$, which maps the joint strategy $S \in \Sigma$ to a real number. Let $S = (s_1, \ldots, s_n)$ denote the profile of strategies taken by the players, and let $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ denote the profile of strategies taken by all players other than player i. Note that $S = (s_i, s_{-i})$. An improvement move s_i' for a player i in a profile S is a move for which $u_i(s_{-i}, s_i') \geq u_i(S)$. A best response move S_i'' for a player i in a profile S is an improvement move that has the maximum utility. Note that in cost minimizing games, each player i wants to minimize the cost $c_i(S) = -u_i(S)$ in strategy profile S. This type of games include congestion games with delay functions on edges which will be defined later.

Nash equilibria (NE): A strategy profile $S \in \Sigma$ is a pure Nash equilibrium if no player $i \in N$ can benefit from unilaterally deviating from his strategy to another strategy, i.e., $\forall i \in N \ \forall s_i' \in \Sigma_i$: $u_i(s_{-i}, s_i') \leq u_i(S)$. We can also define α -Nash equilibria as follows. For $1 > \alpha > 0$, a state S is an α -Nash equilibrium if for every player i, $c_i(s_{-i}, s_i') \geq (1 - \alpha)c_i(S)$ for all $s_i' \in \Sigma_i$.

State graph. Given any game Λ , the state graph $G(\Lambda)$ is an arc-labeled directed graph as follows. Each vertex in the graph represents a joint strategy S. There is an arc from state S to state S' with label i iff there exists player i and strategy $s'_i \in \Sigma_i$ such that $S' = (s_{-i}, s'_i)$, i.e., S' is obtained from S by a move of a single player i that improves his utility from S to S'.

Nash dynamics. A Nash dynamics or best-response dynamics is equivalent to a walk in the state graph.

Sink equilibria. Given any game Λ , a sink equilibrium is a subset of states T that form a strongly connected component of the state graph such that there is no outgoing edge from states in T to any state outside T. As a result, any pure Nash equilibrium of a game is a single-state sink equilibrium, and a game may have several sink equilibria.

2.2 Definition of games

(Unweighted) Congestion Games. An (unweighted) congestion game is defined by a tuple $\langle N, E, (\Sigma_i)_{i \in N}, (d_e)_{e \in E} \rangle$ where E is a set of resources, $\Sigma_i \subseteq 2^E$ is the strategy space of player i, and $d_e : \mathbb{N} \to \mathbb{Z}$ is a delay function associated with resource e. For a strategy profile $S = (s_1, \ldots, s_n)$, we define the congestion $n_e(S)$ on resource e by $n_e(S) = |\{i|e \in s_i\}|$, that is $n_e(S)$ is the number of players that selected an strategy containing resource e in S. The cost (or delay) $c_i(S)$ of player i in a strategy profile S is $c_i(S) = -u_i(S) = \sum_{e \in s_i} d_e(n_e(S))$.

In weighted congestion games, player i has weighted demand w_i . In this game, the congestion (load) on resource e in a state S, denoted by by $l_e(S)$ is as follows $l_e(S) = \sum_{i|e \in s_i} w_i$. The cost or delay of players is defined the same way as the congestion games. A player-specific congestion game is defined by a tuple $\langle N, E, (\Sigma_i)_{i \in N}, (d_{e,i})_{e \in E, i \in N} \rangle$ where E and $\Sigma_i \subseteq 2^E$ are the same as congestion games, and $d_{e,i} : \mathbb{N} \to \mathbb{Z}$ is a delay function associated with resource e and player i. The congestion $n_e(S)$ on resource e is defined the same as congestion games. The cost (delay) $c_i(S)$ of player i in a strategy profile S is $c_i(S) = -u_i(S) = \sum_{e \in s_i} d_{e,i}(n_e(S))$.

Many-to-one Two-sided Markets. We model the many-to-one two-sided market $(\mathcal{X}, \mathcal{Y})$ between two sides of active agents \mathcal{X} and passive agents \mathcal{Y} as a game $G(\mathcal{X}, \mathcal{Y})$ among active agents $x \in \mathcal{X}$. The strategy set of each active agent $x \in \mathcal{X}$ is a lower-ideal ¹ family of subsets of passive agents. Each agent $x \in \mathcal{X}$ also has a preference (a.k.a social choice) over its strategies. This preference is capture by a utility function $u_x : 2^{\mathcal{Y}} \to \mathbb{R}$ which assigns a utility, $u_x(T)$, to each subset $T \subseteq \mathcal{Y}$. Each agent $y \in \mathcal{Y}$ has a strict preference list over the set of agents $x \in \mathcal{X}$ that can play this set, i. e., x is preferred to x' by y iff $u_y(x) > u_y(x')$. We assume that $u_y(x) \neq u_y(x')$ for any two agents $x \in \mathcal{X}$ in the preference list of agent y such that $y \in s_x$. In this case, we say that x is the winner of agent y, or equivalently, agent x wins agent y. The goal of each active agent x is to maximize the utility of the set of passive agents that she wins. Given a strategy profile x is equal to x be the set of passive agents that agent x wins. The utility of player x in strategy profile x is equal to x. We possible this utility.

It is not see that pure Nash equilibria of the above game correspond to stable matchings for many-to-one two-sided markets as defined by ...

Valid-utility Games. Here we briefly define the class of valid-utility games; see [22] for more details. In valid-utility games, for each player i, there exists a ground set of markets V_i . We denote by V the union of ground sets of all players, i.e., $V = \bigcup_{i \in U} V_i$. The feasible strategy set F_i of player i is a subset of the power set, 2^{V_i} , of V_i . Thus, a strategy s_i of player i is a subset of V_i ($s_i \subseteq V_i$). The empty set, denoted \emptyset_i for player i, corresponds to player i taking no action.

Let $\mathcal{G}(U, \{F_i | i \in U\}, \{u_i() | i \in U\})$ be a non-cooperative strategic game where $F_i \subseteq 2^{V_i}$ is a family of feasible strategies for player i. Let $V = \bigcup_{i \in U} V_i$ and let the social function be $\gamma : \Pi_{i \in U} 2^V \to \mathbb{R}^+ \cup \{0\}$. Then \mathcal{G} is a valid-utility game if it satisfies the following properties: (1) The social function γ is submodular and non-decreasing, (2) The utility of a player is at least the difference in the social function when the player participates versus when it does not participate. and (3) For any strategy profile, the sum of the utilities of players should be less than or equal to the social function for that strategy profile.

This framework encompasses a wide range of games including the facility location games, traffic routing games, auctions [22], market sharing games [14], and distributed caching games [12]. In [22] it was shown that the price of anarchy (for mixed Nash equilibria) in valid-utility games is at most 2. **Anonymous games.** Anonymous games [6] are games in which players have the *same strategy sets*, but different utilities for the same strategies; however, these utilities do not depend on the identity of the other players, but only on the *number* of other players taking each action. An interesting subclass of these games is anonymous games with a *constant-size strategy set* in which the size of the strategy set of players is a fixed constant.

3 Existence of Pure Nash Equilibria

In this section, we study the HAS A PURE problem for succinct games. This problem has been already considered and resolved for weighted congestion games [] and player-specific congestion games []. We resolve this problem for many-to-one two-sided markets and valid-utility games. The result for two-sided markets imply that given an instance of the many-to-one stable matching problem, verifying if there exists a stable matching is NP-hard.

¹A family \mathcal{F} of subsets is lower-ideal if and only if for any subset $S \in \mathcal{F}$ and $S' \in S$, then $S' \in \mathcal{F}$.

Theorem 1. Has a singleton Sink is NP-hard for (i) uniform utility-based two-sided market games, (ii) many-to-one two-sided market games, and (iii) valid-utility games.

Proof. To prove NP-completeness, we give a reduction from the 3SAT problem. Given an instance of the 3SAT problem, we construct an instance of the utility-based two-sided market game as follows: for each variable x_i , we put a player X_i with a one and a zero strategy. For each clause c_j , we put two players C_j and K_j each with a one and a zero strategy. We construct the game such that C_j and K_j have a cycle of best responses if and only if the clause is not satisfied. In other words, if the X-players choose a strategy profile that satisfies all clauses, all clause players eventually reach a stable solution.

The zero strategy of C_j is $\{a_j, b_j\}$ and the one strategy is $\{c_j\}$. The zero strategy of K_j is $\{a_j\}$ and the one strategy is $\{b_j\} \cup \{r_{j,i}|\text{for all variables}x_i \text{ in clause }c_j\}$. The a-markets have utility 305 and prefer the K-players. The b-markets have utility 8 and prefer the K-players. The c-markets have utility 310. The r-markets have utility 100 and prefer the K-players. Note that there is a best response cycle of C_j and K_j if and only if none of the three $r_{i,j}$ -markets is allocated by an K-player.

The zero strategy of a player X_i is $\{r_{i,j}|x_i \in c_j\} \cup \{p_{i,j}|\bar{x}_i \in c_j\}$. The one strategy of a player X_i is $\{r_{i,j}|\bar{x}_i \in c_j\} \cup \{p_{i,j}|x_i \in c_j\}$. The p-markets have utility 100. Note that both strategies have the same utility for a X-player independent of the strategy profile of other players. Furthermore, X_i gets the utility from $r_{i,j}$, if and only if it satisfies clause c_j ,

The above theorem implies that given an instance of the many-to-one stable matching problem, the problem of verifying if this game has a stable matching or not is NP-hard. Known results in the economic literature for many-to-one two-sided markets discuss necessary and sufficient conditions for existence of stable matchings (or pure Nash equilibria) for different variants of two-sided markets [13, 20, 17], however, before our results, the known results have not addressed the complexity of verifying the existence of stable matchings (or pure Nash equilibria) given an instance of these markets.

4 Sink Equilibria and Weighted Congestion Games

In this section, we study the complexity of the IN A SINK and HAS A SINK problem for weighted congestion games. The interesting aspect of this proof is that we can use similar reductions for a variety of games with succinct representation. Applying this proof on many examples shows the strength of the proof technique.

Theorem 2. In a Sink is PSPACE-hard for weighted congestion games.

Proof. We give a reduction from the space-bounded halting problem for Turing machines. First, we reduce an instance of this problem (a TM M, an input x and a tape bound t) to the halting problem for a TM $M' = (Q, \Sigma, b, \Gamma, \delta, q_0, \{q_h\})$ which simulates M on x without its own input. Let $\Sigma = \{0, 1\}$ and $\Gamma = \{0, 1, b\}$. Starting from an empty tape, M' halts if and only if M rejects x. Furthermore, M' uses additional tape cells and states for a counter that counts up to the total number of configurations of M. When M accepts, the counter overflows, or M exceeds the tape bound t, M' erases the whole tape, moves the head to the initial position and returns to state q_0 . M' uses tape cells only right of its initial position and at most t' tape cells. Note that starting from every total configuration M' never stops only if M rejects x.

To complete the proof, we construct a congestion game $G_{M'}$ that simulates Turing machine M'. A strategy profile s which we define later is in a sink equilibrium if and only if M' runs forever. The game consists of three types of *configuration* players, a *transition player*, a set of *control* players, and a *clock* player. The first type of configuration players is a *state* player with |Q| strategies. The second

type of configuration players is a position player for the position of the head with t' strategies; and the third type of configuration players is a set of cell players cell_i for each tape cell $0 \le i \le t'$ with the $|\Gamma|$ strategies for the content of the tape cell i. There is a simple bijective mapping between the strategy profiles of the configuration players and the configurations of M'.

The game is constructed in such a way that every sequence of improvement steps can be divided in rounds. At the end of a round i, let c_i be the configuration obtained from the strategy profile of the configuration players. For every sequence of improvement steps, $c_1 \vdash c_2 \vdash c_3 \vdash \dots$ denotes the run of M' starting from c_1 .

We now describe our construction in more details. The strategies of the configuration players are described in Figure 1. Every strategy of a configuration player has two unique resources, an α resource and a β resource. The α resources have delay 0 if allocated by one player and delay 1 otherwise. The β resources have delay 0 if allocated by one player and delay M otherwise.

state player				
strategies	resources	delays		
$q \in Q$	α^q	0/1		
	eta^q	0/M		

position player			
strategies	resources	delays	
$0 \le i \le t'$	α^i	0/1	
	eta^i	0/M	

player cell _i with $0 \le i \le t'$			
strategies	resources	delays	
$\sigma \in \Gamma$	α_i^{σ}	0/1	
	eta_i^σ	0/M	

Figure 1: Definition of strategies of the three types of configuration players

	Player Co	on	$\operatorname{trol}_{W,q,i,i',\sigma}$
ĺ	Strategy	v	Resources

Strategy	Resources	Delays
Zero	$\beta^0_{W,q,i,i',\sigma}$	0/M
	$\alpha^0_{W,q,i,i',\sigma}$	0/1
One	$\beta^1_{W,q,i,i',\sigma}$	0/M
	$\alpha^1_{W,q,i,i',\sigma}$	0/1

Player	$\operatorname{Control}_{V,q,i,i',\sigma}$
G.	ъ

·	, 4,0,0,0	
Strategy	Resources	Delays
Zero	$\beta^0_{V,q,i,i',\sigma}$	0/M
	$\alpha^0_{V,q,i,i',\sigma}$	0/1
One	$\beta^1_{V,q,i,i',\sigma}$	0/M
	$\alpha^1_{V,q,i,i',\sigma}$	0/1

Strategy	Resources	Delays
Zero	eta_D^0	0/M
	α_D^0	0/1
One	β_D^1	0/M
	α_D^1	0/1

Figure 2: Strategies of the control players, for each $q \in Q$, $0 \le i \le n$, $i' \in \{i-1, 1, i+1\}$, and $\sigma \in \Gamma$

Each control player has two strategies, Zero and One, which are constructed in the same manner like strategies of configuration players (see Figure 2). The transition player has the following strategies Wait, Done, Halt, and several strategies $\operatorname{Read}_{q,i,\sigma}$, $\operatorname{Write}_{q',i',i,\sigma'}$, and $\operatorname{Verify}_{q',i',i,\sigma'}$ (for each $i,i' \in$ $\{1,\ldots,t'\},\ q,q'\in Q,\ \mathrm{and}\ \sigma,\sigma'\in\Sigma$). The details of theses strategies and the resources they contain are listed in Figure 3. The clock player has two strategies, Trigger and Wait. Trigger contains the two resources, TriggerMain and TriggerClock. The strategy Wait contains one resource with constant delay of 110.

Let us remark that each α - or β -resource is allocated by at most two players; the transition player and one of the configuration or control players. The general idea is that the improvement steps for the transition player is determined by the strategy profile of the configuration and control players. That is, the transition player never deviates to a strategy that contains a β -resource which is allocated by another player. On the other hand, the transition player determines the improvement steps for configuration and control players if he allocates α -resources. Note that each α -resource is associated with exactly one strategy of exactly one configuration or control player.

Now, we are ready to describe the aforementioned sequence of improvement steps that corresponds to one round in more details. Consider any strategy profile in which the clock players are on Trigger, the transition player is on Wait and all control players except control_D are on One. Let q be the strategy of the state player, i the strategy of the position player and $\sigma_0, \ldots, \sigma_{t'}$ the strategies played

Strategy	Resources	Delays
Wait	$\beta_{W,q',i',i,\sigma'}^1, \beta_V^1 q', i', i, \sigma'$ for all q', i', i, σ'	0/M
	$\alpha_D^{1,1,1,1}$	0/1
	TriggerMain	0/100/100
$Read_{q,i,\sigma}$	β^p for all $p \in Q \setminus q$	0/M
for each $q \in Q$,	β^j for all $j \neq i$	0/M
$0 \le i \le t' \text{ and } \sigma \in \Gamma$	$\beta_i^{\sigma'}$ for all $\sigma' \in \Gamma \setminus \sigma$	0/M
	$\mid eta_D^1 \mid$	0/M
	$\alpha_{W,q',i',i,\sigma'}^{0}$ with $\delta(q,\sigma)=(q',\sigma',d)$ and $i'=i+d$	0/1
	N.N.	80
$Write_{q',i',i,\sigma'}$	α^p for all $p \in Q' \setminus q'$	0/1
for each $q' \in Q$, $0 \le i \le t'$,	α^j for all $j \neq i'$	0/1
$i' \in \{i-1, i, i+1\},$	α_i^{σ} for all $\sigma \in \Gamma \setminus \sigma'$	0/1
and $\sigma' \in \Gamma$	$\alpha_{V,q',i',i,\sigma'}^{0}$	0/1
	$egin{array}{c} eta^{0,T,T,\sigma}_{W,q',i',i,\sigma'} \end{array}$	0/M
	N.N.	60
$Verify_{q',i',i,\sigma'}$	β^p for all $p \in Q \setminus q'$	0/M
for each $q' \in Q$, $0 \le i \le t'$,	β^j for all $j \neq i'$	0/M
$i' \in \{i-1, i, i+1\},$	β_i^{σ} for all $\sigma \in \Gamma \setminus \sigma'$	0/M
and $\sigma' \in \Gamma$	$igg eta_{V,q',i',i,\sigma'}^{ar{0}}$	0/M
	$\alpha_D^{0,1,1,2,3,3}$	0/1
	N.N.	40
Done	triggerClock	0/0/20
	$\mid eta_D^0 \mid$	0/M
	$\alpha_{W,q',i',i,\sigma'}^{\vec{1}}, \alpha_{V,q',i',i,\sigma'}^{1} \text{ for all } q',i',i,\sigma'$	0/1
	N.N.	20
Halt	β^q for all $q \in Q \setminus q_h$	0/M

Figure 3: Definition of strategies of the transition player. Resources that are denoted by N.N. are used by the transition player only and have a constant delay.

by the players $cell_0, \ldots, cell_{t'}$. Figure 4 describes the sequence of improvement steps emerging from this strategy profile. The strategy profile at the end of the round differs from the initial one only in the choices of the configuration players. The deviations of the configuration players corresponds to a step of the Turing machine M'. Note that this sequence is essentially unique as there are no other improving deviations. If and only if the state player is on q_h , the transition player may deviate to the strategy Halt. This is a Nash equilibrium of $G_{M'}$. Now let s be a strategy profile in which the clock players is on Trigger, the transition player on Wait, and all control players except control p0 on One. Let the configuration players' choice in s correspond to the initial configuration of p1. Then, p2 is in a sink equilibrium if and only if p1 does not halt.

We now consider the problem HAS A NON-SINGLETON SINK for weighted congestion games.

Theorem 3. Has a non-singleton Sink is PSPACE-hard for weighted congestion games.

This results follows from the proof of Theorem 2 and the following Lemma. The lemma implies that there is at most one unique sink equilibrium in the constructed game.

- (1) The transition player deviates from Wait to Read_{q,i,σ_i}.
- (2) Player $\operatorname{control}_{W,q',i',i,\sigma'}$ deviates to Zero.
- (3) The transition player deviates to Write_{g',i',i,σ'}.
- (4) The configuration players deviate to the new configuration and the player control $_{V,q',i',i,\sigma'}$ deviates to Zero.
- (5) The transition player deviates to $Verify_{q',i',i,\sigma'}$.
- (6) The player control_D deviates to One.
- (7) The transition player deviates to Done.
- (8) The clock player deviates to Wait and the controll players except control_D deviate to Zero
- (9) The transition player deviates to Wait.
- (10) The clock player deviates to Trigger and the player control_D deviates to Zero

Figure 4: Description of a round.

Lemma 4. Every Sink equilibrium contains a strategy profile in which the clock player is on Trigger, the main player on Wait and all controll players on their Zero strategy.

Proof. If no player has delay M or greater, the game converges as described in Figure 4 and eventually reaches a strategy profile in which the clock player is on Trigger, the main player on Wait and all controll players on their Zero strategy. Note that no strategy profile with a player having delay M or greater is reachable. If players have delay of M or greater, there is a sequence of improvement steps such that no player has delay of M or more, e.g. each control or configuration player with delay of M changes to another strategy.

Thus, every sink equilibrium also contains the strategy profile that corresponds to the initial configuration of M'. Therefore, there is a unique sink equilibrium if and only if M rejects x.

5 Sink Equilibria and Player-Specific Congestion Games

Theorem 5. In a Sink is PSPACE-hard for player-specific congestion games.

One can easily replace the clock player in the construction which is the only player with non-uniform weight by a player with weight 1 and modify the (player-specific) delay functions as follows. For the transition player the resource TriggerMain has delay 0 if one player allocates it and delay 100 otherwise. For the clock player the resource TriggerMain has always delay 100. The delay functions of the resource TriggerClock is identical for both players. It has delay 0 if one player allocates the resource and delay 20 for two or more players. For each strategy profile the delay for each player is identical to the delay in the previous example.

Theorem 6. Has a non-singleton Sink is PSPACE-hard for player-specific congestion games.

Proof. This result follows by the same argument as for Theorem 3. \Box

6 Sink Equilibria and Anonymous Games

Next, we consider anonymous games with constant-size strategy set and show that IN A SINK for this game is also PSPACE-complete.

Theorem 7. In a Sink is PSPACE-hard for anonymous games with constant-size strategy sets.

We give a reduction from the halting problem of a space bounded Turing machine M' as defined in the proof of Theorem 2. Additionally, we assume that states of M' are denoted by q_0, \ldots, q_m where q_m is the halting state. We construct an anonymous game with a constant number of strategies. Each player has a set of (allowed) strategies. Every strategy that is not allowed always has utility 0. The only other utility values in the game are 1 and 2. Given a strategy profile $s = (s_1, \ldots, s_k)$, let $|s_i|$ denotes the number of players that play strategy s_i .

The game consists of the three types of configuration players and five types of auxiliary players and two control players. The strategy choices of the configuration players can be mapped to configurations of the TM M'. Every sequence of improvement steps can be partitioned into rounds. Each round simulates one step of M'. At the end of a round i, let c_i be the configuration obtained from the strategy profile of the configuration players. For every sequence of improvement steps, $c_1 \vdash c_2 \vdash c_3 \vdash \dots$ equals the run of M' starting from c_1 .

We first describe the configuration players before we describe the remaining players and the process that simulates one step of M'. The first type of configuration players are |Q| identical state players that choose between the two actions state¹ and state⁰. For $j = |\text{state}^1|$ corresponds to M' being in state q_j . The second type are t' identical position players that choose between the two actions position¹ and position⁰. For $p = |\text{position}^1|$ corresponds to the head of M' being in position p. The third type are the cell players $\text{cell}_0, \ldots, \text{cell}_{t'}$ which choose between the actions cell^0 , cell^1 , cell^b , and change. Unlike the previous two types of players, the cell players are non-identical, i.e., each player has a different payoff function. For each $1 \le i \le t'$, player cell_i on action cell^0 (cell^1 or cell^b) corresponds to the fact that tape cell_i contains the symbol 0 (1 or blank).

Players	allowed strategies
$\operatorname{cell}_1,\ldots,\operatorname{cell}_{t'}$	$\text{cell}^0, \text{cell}^1, \text{ cell}^b, \text{ change}$
$position_1, \dots, position_{t'}$	position ¹ , position ⁰
$\mathrm{state}_1,\ldots,\mathrm{state}_m$	state ¹ , state ⁰
$tape_1, \dots, tape_{t'}$	$tape^0$, $tape^1$, $tape^b$
symbol	symbol0, symbol1, symbolb
new-sym	$\text{new-sym}^0, \text{new-sym}^1, \text{ new-sym}^b$
$\text{new-pos}_1, \dots, \text{new-pos}_{t'}$	new-pos ¹ , new-pos ⁰
$new-state_1, \dots, new-state_m$	new-state ¹ , new-state ⁰
transition1	init, tape-change, eval-tape, new-sym, new-sym2, new-pos,
	new-pos2, new-state, new-state2, halt
transition2	Xinit, Xtape-change, Xeval-tape, Xfnew-sym, Xnew-sym2,
	Xnew-pos Xnew-pos2, Xnew-state, Xnew-state2

Figure 5: Players and their strategies

There are five types of *auxiliary players* and two *control players*. All players and their allowed strategies are listed in Figure 5. The utility functions for each player are described in Appendix B.

The players $tape_1, \ldots, tape_{t'}$ have identical payoff functions. They are used to evaluate symbol at the current position. The player symbol saves this symbol. The players new-sym, new-pos₁, ..., new-pos_{t'}, new-state₁, ..., new-state_m calculate the changes to the configuration. The control players ensure that strategy changes happen in a certain order that corresponds to one step.

Lemma 8. Let c be a configuration of M' and c' the successor configuration. Every sequence of improvement steps from a strategy profile in which the configuration players play corresponding to c and the first control player is on init, reaches a strategy profile in which the configuration players play corresponding to c' and the first control player is on init.

Proof. We now describe this sequence of improvement steps which we call a round. It is listed in Figure 1 in detail. One can easily check for each of the strategy profiles that the next one is essentially unique.

In a round, the first control player successively changes through his strategies (c.f. steps (2),(4),...). The second control player follows his choices in his corresponding strategies. By construction of the payoff function, this ensures that the control players only change their strategies in a certain order. Each of these steps of the first control player is interrupted by improvement steps of subsets of configuration or auxiliary players. The utility functions (cf. Figure 7) are designed in such a way that these improvement steps are possible if and only if the control player plays the corresponding strategy. Additionally, the control player may only continue with his next step after these other player have changed their strategies (cf. Figure 8).

We now describe the improvement steps of the configuration and auxiliary players only. Consider any strategy profile of the configuration players and assume the first control player is on init (strategy profile (1) in Figure 1 in Appendix B. The t' tape players change to a strategy profile in that the number of players on tape⁰, tape¹, and tape^b equals the number of players on cell⁰, cell¹, and cell^b (2). The player cell_i with $i = |position^i|$ changes to his strategy to change (4). The symbol player changes to symbol⁰, symbol¹, or symbol^b depending on which strategy was left by the player cell_i (6). This can be coded into the utility function by evaluating the difference of number of players in the cell and tape strategies. The player new-symbol changes to the strategy new-symbol σ' where σ' corresponds to the new symbol (8). This can be coded as a function as from number of players on symbol⁰, symbol¹, symbol^b, and state¹. The player cell_i changes to the strategy cell^{σ'} (10). Exactly i' players choose new-pos¹ where i' is the new position of M' (12). The players position change their strategies such that $|position^1| = |new-pos^1| = i'$ (14). Exactly q' players new-state choose new-state¹ where $q_{q'}$ is the new state of M' (16). The players state change their strategies such that $|\text{state}^1| = |\text{new-state}^1| = q'$ (18). The configuration players' strategy profile now corresponds to the new configuration after one step of M'.

Theorem 9. HAS A NON-SINGLETON SINK is PSPACE-hard for anonymous games.

Proof. By construction of M' and the proof of Theorem 7, it suffices to show that every infinite sequence of improvement steps contains a strategy profile with player control 1 on init, i.e. a profile listed in the first row of Table 1.

The strategy changes of control1 have to occur in the same order as listed in Table 1. Therefore, every sequence with infinite strategy changes of control1 contains a profile with control1 on init. We, therefore, show that there is no infinite sequence that contains no strategy change of control1. Thus, fix any strategy choice for player control1. Observe that the utility functions of the remaining players (cf. Figure 7) do not allow an infinite sequence.

7 Sink Equilibria in other Games

Theorem 10. In a Sink is PSPACE-hard for (i) uniform utility-based two-sided market games, (ii) many-to-one two-sided market games, and (iii) valid-utility games.

Theorem 11. Has a non-singleton Sink is PSPACE-hard for (i) uniform utility-based two-sided market games, (ii) many-to-one two-sided market games, and (iii) valid-utility games.

The proof is a rework of the proof for Theorem 2 and is shifted to Appendix A. The Nash dynamics of the uniform utility-based two-sided market game that we describe there is isomorphic to the Nash dynamics of the congestion game in the proof for Theorem 2.

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A Proof of Theorem 10

The Nash dynamics of the uniform utility-based two-sided market game that we describe here is isomorphic to the Nash dynamics of the congestion game in the proof for Theorem 2. Thus, all properties easily transfer. The strategies of the transition player and the preferences of the markets can be found in Figure 6. The strategies of the remaining players can be obtained from the previous proof.

Strategy	Markets	Utilities (Preference)
Wait	$\beta^1_{W,q',i',i,\sigma'}$ for all q',i',i,σ'	M (Control _{W,q',i',i,\sigma'} , transition player)
	$\beta_V^1 q', i', i, \sigma'$ for all q', i', i, σ'	M (Control _{V,q',i',i,σ'} , transition player)
	α_D^1	1 (transition player, $Control_D$)
	TriggerMain	100 (clock player, transition player)
$Read_{q,i,\sigma}$	β^p for all $p \in Q \setminus q$	M (state player, transition player)
for each $q \in Q$,	β^{j} for all $j \neq i$	M (position player, transition player)
$0 \le i \le t'$ and $\sigma \in \Gamma$	$\beta_i^{\sigma'}$ for all $\sigma' \in \Gamma \setminus \sigma$	M (cell _i , transition player)
	$\mid eta_D^1 \mid$	M (Control _D , transition player)
	$\alpha_{W,q',i',i,\sigma'}^{0}$ with $\delta(q,\sigma) = (q',\sigma',d)$	1 (transition player, Control _{W,q',i',i,σ'})
	and $i' = i + d$	
	N.N.	$N - (Q + t' + \Gamma - 1)M + 20$
$Write_{q',i',i,\sigma'}$	α^p for all $p \in Q' \setminus q'$	1 (transition player, state player)
for each $q' \in Q$, $0 \le i \le t'$,	α^j for all $j \neq i'$	1 (transition player, position player)
$i' \in \{i-1, i, i+1\},$	α_i^{σ} for all $\sigma \in \Gamma \setminus \sigma'$	1 (transition player, cell_i
and $\sigma' \in \Gamma$	$egin{array}{c} lpha_{V,q',i',i,\sigma'}^0 \ lpha_0^0 \end{array}$	1 (transition player, $Control_{V,q',i',i,\sigma'}$)
	$\mid eta_{W,q',i',i,\sigma'}^0 \mid$	M (Control _{W,q',i',i,σ'} , transition player)
	N.N.	N-M+40
$Verify_{q',i',i,\sigma'}$	β^p for all $p \in Q \setminus q'$	M (state player, transition player)
for each $q' \in Q$, $0 \le i \le t'$,	β^j for all $j \neq i'$	M (position player, transition player)
$i' \in \{i-1, i, i+1\},\$	β_i^{σ} for all $\sigma \in \Gamma \setminus \sigma'$	M (cell _i , transition player)
and $\sigma' \in \Gamma$	$igg eta^0_{V,q',i',i,\sigma'}$	M (Control _{V,q',i',i,σ'} , transition player)
	$\alpha_D^{0.1}$	1 (transition player, $Control_D$)
	N.N.	$N - (Q + t' + \Gamma - 1)M + 60$
Done	triggerClock	80 (transition player, clock player)
	$\mid eta_D^0 \mid$	$M(\text{Control}_D, \text{ transition player})$
	$\alpha^1_{W,q',i',i,\sigma'}$ for all q',i',i,σ'	1 (transition player, $Control_{W,q',i',i,\sigma'}$)
	$\alpha_{V,q',i',i,\sigma'}^{1}$ for all q',i',i,σ'	1 (transition player, $Control_{V,q',i',i,\sigma'}$)
	N.N.	N-M+20
Halt	β^q for all $q \in Q \setminus q_h$	M (state player, transition player)
	N.N.	N-M

Figure 6: Strategies of the transition players. Markets denoted by N.N. are used by the transition players only. Let $N = |Q|(t+1)6|\Gamma|M$

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B Details of the proof of Theorem 7

Player	strategy	partitions with utility 2
cell_i	change	$ \text{tape-change} \neq 0 \text{ and } \text{position}^1 = i$
	cell ⁰	$ \text{new-tape} \neq 0 \text{ and } \text{new-sym}^0 > 0$
	cell ¹	$ \text{new-tape} \neq 0 \text{ and } \text{new-sym}^1 > 0$
	$ e^{cell^b} $	$ \text{new-tape} \neq 0 \text{ and } \text{new-sym}^b > 0$
$tape_i$	$tape^0$	$ \text{init} \neq 0 \text{ and } \text{cell}^0 > \text{tape}^0 $
	$tape^1$	$ \text{init} \neq 0 \text{ and } \text{cell}^1 > \text{tape}^1 $
	$tape^b$	$ \text{init} \neq 0 \text{ and } \text{cell}^b > \text{tape}^b $
symbol	symbol ⁰	$ \text{eval-tape} \neq 0 \text{ and } \text{cell}^0 - \text{tape}^0 < 0$
	$symbol^1$	$ \text{eval-tape} \neq 0 \text{ and } \text{cell}^1 - \text{tape}^1 < 0$
	symbol^b	$ \text{eval-tape} \neq 0 \text{ and } \text{cell}^b - \text{tape}^b < 0$
new-sym	new-sym ⁰	$ \text{new-symbol} \neq 0 \text{ and if } 0 \text{ is } new \ symbol$
	new-sym ¹	$ \text{new-symbol} \neq 0 \text{ and if } 1 \text{ is } new \ symbol$
	new-sym^b	$ \text{new-symbol} \neq 0 \text{ and if } b \text{ is } new \ symbol$
new-pos	new-pos ¹	$ \text{new-pos} \neq 0 \text{ and } \text{new-pos}^1 < new position$
	new-pos^0	$ \text{new-pos} \neq 0 \text{ and } \text{new-pos}^1 > new position$
new-state	new-state ¹	$ \text{new-state} \neq 0 \text{ and } \text{new-state}^1 > new \ state$
	new-state ⁰	$ \text{new-state} \neq 0 \text{ and } \text{new-state}^1 < new \ state$
position	position ¹	$ \text{new-pos}2 \neq 0 \text{ and } \text{position}^1 < \text{new-pos}^1 $
	position ⁰	$ \text{new-pos}2 \neq 0 \text{ and } \text{position}^1 > \text{new-pos}^1 $
state	state ¹	$ \text{new-state2} \neq 0 \text{ and } \text{state}^1 < \text{new-state}^1 $
	state ⁰	$ \text{new-state2} \neq 0 \text{ and } \text{state}^1 > \text{new-state}^1 $
halt	$ state^1 = m$	

Figure 7: The strategy partition combinations are listed that induce utility 2. Note that the *new* symbol, new position, and new state can be coded as a function of $|symbol^0|$, $|symbol^1|$, $|symbol^b|$, and $|state^1|$.

strategy	partitions with utility 2
tape-change	$ Xinit > 0$ and $ cell^0 = tape^0 $ and $ cell^1 = tape^1 $
	and $ \text{cell}^b = \text{tape}^b $
eval-tape	Xtape-change > 0 and $ cell-change = 1$
new-sym	$ Xeval-tape > 0$ and $ cell^0 + symbol^0 = tape^0 $ and $ cell^1 + symbol^1 = tape^1 $
	and $ \text{cell}^b + \text{symbol}^b = \text{tape}^b $
new-sym2	$ \text{Xnew-sym} > 0 \text{ and } \text{new-sym}^{\sigma'} = 1 \text{ for } \sigma' = new \ symbol$
new-pos	Xnew-sym2 > 0 and $ change = 0$
new-pos2	$ Xnew-pos > 0$ and $ new-pos^1 = new position$
new-state	$ Xnew-pos2 > 0$ and d $ position^1 = new-pos^1 $
new-state2	$ Xnew-state > 0$ and $ new-state^1 = new state$
init	$ Xnew-state2 > 0 \text{ and } state^1 = new-state^1 $
stop	$ position^1 = m$

Figure 8: The strategy/partition combinations of the first control player are listed that induce utility of 2.

strategy	partitions with utility 2
Xinit	$ \mathrm{init} > 0$
Xtape-change	tape-change > 0
Xeval-tape	eval-tape > 0
Xnew-sym	new-sym > 0
Xnew-sym2	new-sym2 > 0
Xnew-pos	new-pos > 0
Xnew-pos2	new-pos2 > 0
Xnew-state	new-state > 0
Xnew-state2	new-state2 > 0

Figure 9: The strategy/partition combinations of the second control player that induce utility of 2.

	Configuration players	tape	symbol	new-sym	new-pos	new-state	control1	control2
П	$(\sigma_1,\ldots,\sigma_{i-1},\sigma_i,\sigma_{i+1}\ldots\sigma_{t'}),q,i$	*	*	*	*	*	init	*
2	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma_i, \sigma_{i+1} \ldots \sigma_{t'}), q, i$	p_0, p_1, p_b	*	*	*	*	$\frac{\text{init}}{}$	Xinit
က	$\Big \; ig(\sigma_1, \dots, \sigma_{i-1}, \sigma_i, \sigma_{i+1} \dots \sigma_{t'}), q, i$	p_0, p_1, p_b	*	*	*	*	tape-change	Xinit
4	$\mid (\sigma_1, \dots, \sigma_{i-1}, \overline{\operatorname{change}}, \sigma_{i+1} \dots \sigma_{t'}), q, i$	p_0, p_1, p_b	*	*	*	*	tape-change	Xtape-change
വ	$\mid (\sigma_1, \dots, \sigma_{i-1}, \text{change}, \sigma_{i+1} \dots \sigma_{t'}), q, i$	p_0, p_1, p_b	*	*	*	*	eval-tape	Xtape-change
9	$\mid (\sigma_1, \dots, \sigma_{i-1}, \text{change}, \sigma_{i+1} \dots \sigma_{t'}), q, i \mid$	p_0, p_1, p_b	σ_i	*	*	*	eval-tape	Xeval-tape
7	$\mid (\sigma_1, \dots, \sigma_{i-1}, \operatorname{change}, \sigma_{i+1} \dots \sigma_{t'}), q, i$	p_0, p_1, p_b	σ_i	*	*	*	new-sym	Xeval-tape
∞	$\mid (\sigma_1, \dots, \sigma_{i-1}, \operatorname{change}, \sigma_{i+1} \dots \sigma_{t'}), q, i$	p_0, p_1, p_b	σ_i	σ'	*	*	new-sym	Xnew-sym
6	$\mid (\sigma_1, \dots, \sigma_{i-1}, \operatorname{change}, \sigma_{i+1} \dots \sigma_{t'}), q, i$	p_0, p_1, p_b	σ_i	φ'	*	*	$\overline{\mathrm{new-sym2}}$	Xnew-sym
10	$\left \; (\sigma_1, \ldots, \sigma_{i-1}, \overline{\sigma', \sigma_{i+1}} \ldots \sigma_{t'}), q, i \right $	p_0, p_1, p_b	σ_i	σ'	*	*	new-sym2	$\overline{\mathrm{Xnew\text{-}sym2}}$
11	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma', \sigma_{i+1} \ldots \sigma_{t'}), q, i$	p_0, p_1, p_b	σ_i	σ'	*	*	new-pos	Xnew-sym
12	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma', \sigma_{i+1} \ldots \sigma_{t'}), q, i$	p_0, p_1, p_b	σ_i	σ'	i'	*	new-pos	Xnew-pos
13	$\mid (\sigma_1, \dots, \sigma_{i-1}, \sigma', \sigma_{i+1} \dots \sigma_{t'}), q, ar{i} \mid$	p_0,p_1,p_b	σ_i	ω'	i'	*	new-pos2	Xnew-pos
14	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma', \sigma_{i+1} \ldots \sigma_{t'}), q, i'$	p_0,p_1,p_b	σ_i	ω'	i'	*	new-pos2	Xnew-pos2
15	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma', \sigma_{i+1} \ldots \sigma_{t'}), q, i'$	p_0,p_1,p_b	σ_i	ω'	i'	*	new-state	Xnew-pos2
16	$\mid (\sigma_1, \dots, \sigma_{i-1}, \sigma', \sigma_{i+1} \dots \sigma_{t'}), q, i'$	p_0, p_1, p_b	σ_i	σ'	i'	d'	new-state	Xnew-state
17	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma', \sigma_{i+1} \ldots \sigma_{t'}), q, i'$	p_0,p_1,p_b	σ_i	ω'	i'	d'	new-state2	Xnew-state
18	$\mid (\sigma_1, \dots, \sigma_{i-1}, \sigma', \sigma_{i+1} \dots \sigma_{t'}), q', i'$	p_0, p_1, p_b	σ_i	σ'	i'	d'	$\frac{\text{new-state2}}{\text{new-state}}$	Xnew-state2
19	$\mid (\sigma_1, \ldots, \sigma_{i-1}, \sigma', \sigma_{i+1} \ldots \sigma_{t'}), q', i'$	p_0, p_1, p_b	σ_i	σ'	i'	d'	init	Xnew-state2

position, new-pos, new-state players, we give the number of players on state¹, position¹, new-pos¹, and new-state¹, respectively. The strategy profile of the tape players is described by a vector $p \in \{0, \dots, t'\}^3$ that denotes the number of players on tape⁰, tape¹, and $tape^b$, respectively. For the players symbol and new-sym, σ denotes strategy symbol σ and new-sym σ , respectively. The round starts profile of the cell players is given as a vector $\sigma \in \{0,1,b,change\}^{t'}$ where σ_i denotes strategy cell σ_i for player cell_i. For the state, with each player cell_i on $\sigma_i \in \{0,1,b\}$, q state players on state¹, i position players on position¹ and the first control player on init. Table 1: This figure shows the sequence of strategy profiles during one round. A strategy profile is described es follows. The strategy The underlined strategies indicate the players that have an incentive to deviate from their current strategies.