Helly Theorems and Generalized Linear Programming

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Abstract

Recent combinatorial algorithms for linear programming also solve certain non-linear problems. We call these Generalized Linear Programming, or GLP, problems. One way in which convexity has been generalized by mathematicians is through a collection of results called the Helly theorems. We show that the every GLP problem implies a Helly theorem, and we give two paradigms for constructing a GLP problem from a Helly theorem. We give many applications, including linear expected time algorithms for finding line transversals and hyperplane fitting in convex metrics. These include GLP problems with the surprising property that the constraints are non-convex or even disconnected. We show that some Helly theorems cannot be turned into GLP problems.

1 Introduction

It has frequently been noted that recent linear programming algorithms can be applied to various non-linear problems as well. The randomized expected linear time algorithms for fixed dimensional problems of [C90], [S90] were applied to specific examples, including convex programming. The deterministic fixed dimensional algorithms of [D84] and [M83] were generalized to specific non-linear problems with rather more effort [D92], [M89]. A major advance was the sub-exponential algorithm of [MSW92]. They defined an abstract framework defining the combinatorial structure of a problem solved by their algorithm, and showed a number of non-linear problems which it solved. This same abstract framework applies to [C90] as well, which, with an additional assumption, can be derandomized to give deterministic linear time algorithms for fixed dimensional problems [CM93]. We may begin to wonder, then, what is the relationship of this abstract framework to the vast body of previous work in combinatorial geometry and mathematical programming? Does it merely restate some known characterization of "easy" non-linear problems, such as convex programming? We will call the class of problems described by the abstract framework *Generalized Linear Programming*, or GLP, problems.

In this paper we forge a connection to one relevant area of combinatorial geometry, the class of results know as Helly theorems. The archetype is

Helly's Theorem *Let K be a family of at least $d+1$ convex sets in E^d, and assume K is finite or that every member of K is compact. Then if every $d+1$ members of K have a point in common, there is a point common to all the members of K.*

This is one of the fundamental properties of convexity. There are many similar theorems with the same logical structure, for objects other than convex sets, for properties other than intersection, or for special cases in which $d+1$ is replaced by some other constant k. These are generically called Helly theorems. Combinatorial geometers collect Helly theorems, in much the same way that computer scientists collect NP-complete problems [DGK63], [GPW93], [E93].

The algorithmic implications of Helly theorems has been a question of some interest. For Helly's theorem itself, an algorithm given by [AH91] finds a point in the intersection of n convex sets, if one exists, with $O(n^{d+1})$ calls to a subroutine which finds some point in the intersection of $d+1$ convex sets. The GLP algorithms find the *minimum* point in the intersection, with respect to some convex objective function (this is the convex programming problem), using only expected $O(n)$ calls to a stronger subroutine which also finds the minimum. To apply GLP in an analogous way to other Helly theorems, we need to come up with an objective function.

Section 2 formally defines these notions. In section 3 we begin by showing that there is a Helly theorem about the constraint family of every fixed dimensional GLP problem; that is, the class of problems for which there are Helly theorems includes the fixed dimensional GLP problems. We can use this to produce new Helly theorems from known GLP problems. For example, we state a Helly theorem for line transversals of boxes in E^3.

In the following sections, we give two paradigms for constructing a GLP objective function for a family of constraints which has a Helly theorem, using some

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additional geometric assumptions about the constraints. We use these paradigms to show that the following are fixed dimensional GLP problems, giving expected linear time algorithms:

1. Finding a line transversal of translates of a convex (but possibly complicated) object in the plane,
2. Finding a line transversal in E^d for some special classes of objects,
3. Finding the closest hyperplane to a family of points under the *weighted L^∞ metric*, in which every coefficient of every point is equipped with a weight,
4. Finding the closest line to a family of points under the weighted L^∞ metric,
5. Finding the closest hyperplane to a family of points under a convex polygonal metric,
6. Finding, for a convex object C of constant complexity and a family K of convex objects of constant complexity, the largest homothet of C contained in the intersection of K, or the smallest homothet of C containing K, or the smallest homothet of C intersecting every member of K,
7. Finding a point in the intersection of a family of sets, each of which is the union of two convex sets, given that the intersection never has more than two connected components.

In the first paradigm, we introduce a scale parameter into the constraints, and use this parameter as the objective function. For instance, to find a line transversal for a family of translates we scale them down to points, and then gradually grow them until we can fit a line through them. In section 5 we show that this paradigm is also useful for formulating certain non-linear optimization problems as GLP. In the second paradigm, we use a nested family of constraints as an objective function. Our examples include problems in which the constraints are non-convex and even disconnected; this shows that the class of GLP problems strictly contains the class of convex programming problems.

Finally, in section 7, we show that some additional geometric assumption on the structure of the constraint family is necessary, by exhibiting a set system with a Helly theorem which does not become a fixed dimensional GLP problem under any objective function.

Besides the applications presented here, we expect this work will be useful in identifying and solving new GLP problems. It is often difficult and sometimes impossible to reduce a GLP problem to linear or convex form; these results give some alternative approaches to getting an efficient algorithm. In addition, it makes it easier to implement programs for these problems, since a single implementation of a GLP algorithm can be equipped with specialized subroutines to solve any one of them.

2 Definitions

Let C be a family of objects, and $\mathcal{P}$ a predicate on subsets of C. A *Helly-type theorem* for the pair $(C, \mathcal{P})$ is something of the form:

There is a constant k such that for all finite $H \subseteq C$, H has property $\mathcal{P}$ if and only if every $B \subseteq H$ with $|B| \leq k$ has property $\mathcal{P}$

The constant k is called the *Helly number* of $(H, \mathcal{P})$. Helly's theorem proper is that convex sets in R^d have Helly number $d+1$ with respect to the intersection property, but there are many other sets systems which have Helly theorems [DGK63]. We call a pair $(C, \mathcal{P})$ a *Helly system* if it has a Helly theorem.

Now we review the abstract framework for generalized linear programming from [MSW92]. A *generalized linear programming* (or GLP) problem is a family H of *constraints* and an objective function w from subfamilies of H to some totally ordered set S. The pair (H, w) must obey the following conditions:

1. **Monotonicity:** For all $F \subseteq G \subseteq H$: $w(F) \leq w(G)$
2. **Locality:** For all $F \subseteq G \subseteq H$ such that $w(F) = w(G)$ and for each $h \in H$:
$w(F+h) > w(F)$ if and only if $w(G+h) > w(G)$

The set S must contain a special maximal element Ω; for $G \subseteq H$, if $w(G) = \Omega$, we say G is *infeasible*; otherwise we call G *feasible*. A *basis* for $G \subseteq H$ is a minimal subfamily $B \subseteq G$ such that $w(B) = w(G)$. Here minimal is to be taken in the sense that for every $h \in B$, $w(B-h) < w(B)$. The *combinatorial dimension* d of a GLP problem is the maximum size of any basis for any feasible subfamily G; an infeasible subfamily G may have a basis of size $d+1$. A GLP problem is *fixed dimensional* if d is constant.

A GLP algorithm takes a GLP problem (H, w) and returns a basis B for H. Matoušek, Sharir and Welzl [MSW92] give a randomized GLP algorithm which uses two primitive operations. A *basis computation* takes a family G of at most $d+1$ constraints and finds a basis for G. A *violation test* takes a basis B and a constraint h, and returns true if B is a basis of $B+h$. Let t_v be the time required for a violation test and t_b be the time required for a basis computation. Their algorithm runs in expected time linear in n and subexponential in d, assuming that both t_v and t_b are polynomial in d.

In our applications, d will always be a constant much smaller than n. With the dependence on d hidden in the big-Oh notation, the algorithm takes expected

$O(t_v n + t_b \lg n)$ time. Thus to get an algorithm which runs in expected linear time, t_v must be $O(1)$, although we can afford to spend up to $O(n/\lg n)$ time per basis computation.

3 Helly Theorems from GLP Problems

We begin by showing that there is a Helly theorem about the constraint family of every GLP problem.

Theorem 3.1 *Let (H, w) be a GLP problem with combinatorial dimension k.*
H has the property $w(H) \leq m$ if and only if every $B \subseteq H$ with $|B| \leq k+1$ has the property $w(B) \leq m$

Proof: Let $w(H) \leq m$. By the monotonicity condition, every $B \subseteq H$ must have $w(B) \leq w(H) \leq m$. Going in the other direction, H must contain some basis B with $w(B) = w(H)$, with $|B| \leq k+1$. So if every B with $|B| \leq k+1$ has $w(B) \leq m$, then $w(H) = w(B) \leq m$.
□

It is interesting that this proof does not use the locality condition of the abstract GLP framework. Notice that when m is the special symbol Ω, this means that every infeasible family of constraints contains an infeasible family of size $\leq k+1$. We can use this theorem to prove new Helly theorems.

Application 3.1 *Line transversal of boxes in E^3.*

A *line transversal* of a family of objects is a line which intersects every object. Let a *positive* line transversal be one directed into the positive orthant of E^d. The argument in [A92] implies that finding a positive line transversal of a family of axis-aligned boxes in E^3 is a GLP problem with combinatorial dimension 4. This gives the following Helly theorem: there is a positive line transversal of a family of axis-aligned boxes in E^3 if and only if there is a positive line transversal for every subfamily B of boxes such that $|B| \leq 5$ [1]

We can also show that certain problems are *not* GLP using Theorem 3.1. This is useful in the same way as a lower bound, in that it rules out a particular line of attack.

Application 3.2 *Line transversal of convex sets in E^3.*

In [AGPW], they give an example of a linearly ordered family A of n disjoint convex compact sets in E^3 which has no line transversal, although every n-2-element subfamily of A has a line transversal consistent with the ordering. This tells us that we cannot hope to apply GLP in this situation, even given a linear ordering.

[1]Nimrod Megiddo has since reduced this problem to linear programming, but the Helly theorem follows from the fact that it is GLP.

4 First Paradigm

Now we turn to the question of constructing a GLP problem, for a Helly system $(C, \mathcal{P})$, to determine, for any $H \subseteq C$, whether H has property $\mathcal{P}$.

We begin by assuming a rather general geometric context. Let X be a set, and C a family of subsets of X. For $G \subseteq C$, we write $\bigcap G$ for $\{x \in X \mid x \in h, \forall h \in G\}$. Let $\cap$, with no argument, be the property that $\bigcap G \neq \emptyset$. We say that a family of sets which has property $\cap$ *intersects*. Assume that $(C, \cap)$ is a Helly system, with some fixed Helly number k. We say that $(C, \cap)$ is *embedded* in X. For any $H \subseteq C$, we will construct an algorithm which determines whether H has property $\cap$ by actually exhibiting some $m \in X$ such that $m \in \bigcap H$, if one exists.

There are many geometric Helly theorems that fall into this context. For instance the theorem in application 3.1 can be restated: Let A be the family of all axis-aligned boxes in E^3, and for any $a \in A$, let $c(a)$ be the set of positive lines which intersect a. Let C be the family $\{c(a) \mid a \in A\}$. Then the system $(C, \cap)$ has Helly number 5.

We can define an objective function on subfamilies $G \subseteq H$ using some well-behaved ordering on the points $x \in X$, just as we do in linear programming. Recall that S is some totally ordered set.

Observation 4.1 *Let C be a family of subsets of X, and let $w' : X \rightarrow S$ be any function such that $w(G) = min\{w'(m) \mid m \in \bigcap G\}$ exists for any $G \subseteq H$ with $\bigcap G \neq \emptyset$. Define $w(G) = \Omega$ when $\bigcap G = \emptyset$. Then w satisfies the monotonicity condition of the abstract GLP framework.*

It is also true that if the function w' has a unique value on every element of x, then (H, w) also obeys the locality condition. However, such an (H, w) will *not* necessarily have bases of fixed size.

We now define some additional structure on a Helly system which produces a natural objective function w. Let $\mathcal{I}$ be an interval on the real line $\mathcal{R}$. Define a *nested family* $\overline{h}$ to be $\{h_\lambda \mid \lambda \in \mathcal{I}\}$, where $h_\lambda \subset X$ for each λ, and $h_i \subset h_j$ for $i < j$. Now consider a collection $\overline{H}$ of nested families $\overline{h}$, all indexed by the same parameter λ. For $\overline{G} \subseteq \overline{H}$, we write G_λ as shorthand for $\{h_\lambda : \overline{h} \in \overline{G}\}$. For a fixed value of λ, we say that $\overline{G}$ *intersects at* λ if $\bigcap G_\lambda \neq \emptyset$. If $\overline{H}$ has the property that $(H_\lambda, \cap)$ is a Helly system of dimension k, for every λ, we say that $(\overline{H}, \cap)$ is a *parameterized Helly system* with Helly number k. Notice that if $\overline{G} \subseteq \overline{H}$ does not intersect at some value λ_2, then $\overline{G}$ also fails to intersect at all $\lambda_1 < \lambda_2$, and if

$\overline{G} \subseteq \overline{H}$ intersects at λ_1, then $\overline{G}$ also intersects at all $\lambda_2 > \lambda_1$.

A parameterized Helly system has a natural objective function w. For $\overline{G} \subseteq \overline{H}$, let $w(\overline{G})$ be the minimum value λ^* such that $\overline{G}$ intersects at λ^*, or Ω if $\overline{G}$ does not intersect at any value of λ. The only remaining difficulty is that $\bigcap G_{\lambda^*}$ might consist of more than one point, which is insufficient to establish the locality condition. In the event, however, that $\bigcap G_{\lambda^*}$ consists of a unique point for every $\overline{G} \subseteq \overline{H}$, we get a GLP problem of dimension k.

Theorem 4.1 *If $(\overline{H}, \cap)$ is a parameterized Helly system with Helly number k such that, for all $\overline{G} \subseteq \overline{H}$,*

1. *$l^* = w(G)$ exists, and*
2. *$|\bigcap G_{\lambda^*}| = 1$ when $\lambda^* \neq \Omega$,*

then $(\overline{H}, w)$ is a GLP problem of combinatorial dimension k.

Proof: We can interpret the elements of $\overline{H}$ as subsets of the space $\mathcal{I} \times X$, so that a point $(i, x) \in \mathcal{I} \times X$ is in $\overline{h}$ if $x \in h_i$. Observe that the $\overline{h}$ are closed subsets of $\mathcal{I} \times X$, and the projection into $\mathcal{I}$ is a function w' on the points of $\mathcal{I} \times X$. Since we assume that $l^* = w(G)$ always exists, observation 4.1 tells us that the problem (H, w) obeys the monotonicity condition.

We also assume that for any $\overline{G} \subseteq \overline{H}$, $\bigcap G_{\lambda^*}$ consists of a single point $m \in X$. $(\overline{H}, w)$ therefore also obeys the locality condition, since for $\overline{F} \subseteq \overline{G}$ with $w(\overline{F}) = w(\overline{G})$ and any additional constraint $\overline{h}$, $w(\overline{G} + \overline{h}) > w(\overline{G})$ only when $m \notin \overline{h}$, so that $w(\overline{F} + \overline{h}) > w(\overline{F})$ as well.

To prove that (H, w) has combinatorial dimension k, we have to show that the size of any basis is $\leq k$. Consider any $\overline{G} \subseteq \overline{H}$ and a basis $\overline{B}$ for $\overline{G}$. The definition of a basis says that for any $\overline{h} \in \overline{B}$, $w(\overline{B} - \overline{h}) < w(\overline{B})$. Let $\lambda^{max} = max\{w(\overline{B} - \overline{h}) \mid \overline{h} \in \overline{B}\}$. The basis $\overline{B}$ does not intersect at λ^{max}, but for any $\overline{h} \in \overline{B}$, $w(\overline{B} - \overline{h}) \leq \lambda^{max}$, which means that $\overline{B} - \overline{h}$ intersects at λ^{max}. Since $(H_{\lambda^{max}}, \cap)$ is a Helly system with Helly number k, $\overline{B}$ must contain some subfamily $\overline{A}$ with $|\overline{A}| \leq k$, such that $\overline{A}$ does not intersect at λ^{max}. Every $\overline{h} \in \overline{B}$ must be in $\overline{A}$, since otherwise it would be the case that $\overline{A} \subseteq (\overline{B} - \overline{h})$ for some $\overline{h}$. This must be false, because $\overline{A}$ does not intersect at λ^{max} while every $(\overline{B} - \overline{h})$ does. Therefore $\overline{B} = \overline{A}$ and $|\overline{B}| \leq k$.
□

Application 4.1 *Line transversal of translates in the plane.*

Let T be a family of disjoint translates of a single convex object O in E^2. Tverberg [T89] showed that if every family $B \subseteq H$ with $|B| \leq 5$ admits a line transversal, then H also admits a line transversal. Egyed and Wenger [EW89] gave a deterministic linear time algorithm to find a line transversal. Showing that the problem can be formulated as GLP gives a simpler, although randomized, linear time algorithm.

We assume that the family of translates is in general position (we will define general position in a moment); if not, we use a standard perturbation argument. The set X is the set of lines in the plane. We abuse notation so that t refers both to a translate $t \in T$ and to the set of lines intersecting t. So a subfamily $G \subseteq T$ intersects when there is a line which intersects every translate in G. We pick a distinguished point q in the interior of the object O. Consider the family $\overline{t}$ of homothets formed by scaling translate t by a factor of λ keeping the point in t corresponding to q fixed in the plane. Every line which intersects the homothet $\lambda_1 t$ also intersects $\lambda_2 t$ for any $\lambda_2 > \lambda_1$. So each $\overline{t}$ is a nested family of lines. For a family $G \subseteq T$, let $\lambda G = \{\lambda t \mid t \in G\}$. If we let λ range over $[0, 1]$, then the t_λ are always disjoint, every $(\lambda T, \cap)$ is a Helly system with Helly number 5, and $(\overline{T}, \cap)$ is a parameterized Helly system.

The natural objective function $w(\overline{G})$ is the minimum λ such that G_λ intersects. In the case where $\overline{G} \subseteq \overline{T}$ consists of a single translate, we define $w(\overline{G}) = 0$. Notice that for certain degenerate placements of the translates (see figure 1) it is possible for there to be two or even three distinct line transversals at $l^* = w(\overline{G})$.

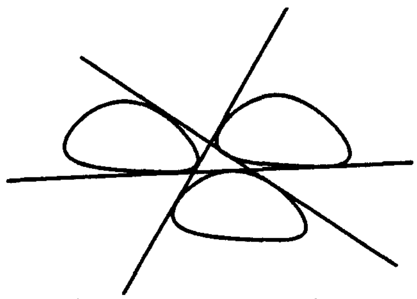

Figure 1: Degenerate input

The general position assumption is that the line transversal at l^* is always unique.

$(\overline{T}, w)$ is a GLP problem with combinatorial dimension 5. Either the GLP algorithm finds a line transversal at some value of $\lambda \leq 1$, or no line transversal of the input exists.

When O is a polygon with a constant number of sides, neither this algorithm nor Egyed and Wenger's is very interesting, since we can find a line transversal via a constant number of fixed dimensional linear programming problems. Either algorithm is intended for more complicated polygons, in which the number of sides depends on n, or for non-polygonal objects.

Recall that the algorithm described in section 2 runs in $O(t_v n + t_b \lg n)$ time, where t_v is the time required for a violation test and t_b is the time required for a basis computation. In this application, a violation test determines whether the current minimum line m intersects a new homothet λt. For any m, there is a diameteral pair of points on O such that m intersects a homothet λt if and only if m passes between the corresponding points on λt. We find such a diameteral pair during the basis computation whenever we find a new minimum line m, so t_v is $O(1)$. The running time is then limited by t_b; when the complexity of O is such that t_b is $O(n/\lg n)$, we get an expected linear time algorithm.

Notice that here the dimension of the space X of lines in the plane is 2. If there were some affine structure on X such that the constraints T were convex subsets of X, then the Helly number of the system $(T, \cap)$ would be 3. But examples show that the bound of 5 is in fact tight, which means that this is a GLP which is *not* a convex program. This is also a natural example of a GLP problem in which the minimal object does not "touch" every constraint in the basis.

This paradigm may be profitably applied to many other Helly theorems.

Application 4.2 *Homothets spanning convex sets.*

We use theorem 2.1 from [DGK63],

Theorem (Vincensini and Klee) *Let K be a finite family of at least $d+1$ convex sets in E^d, and let C be a convex set in R^d. Then there is some translate of C which [intersects/ is contained in/contains] all members of K if and only if there is such a translate for every every $d+1$ members of K.*

We apply the paradigm by either growing or shrinking the convex body C, to get an algorithm which takes as input a finite family K of at least $d+1$ convex sets in E^d and a convex set C and returns either the smallest homothet of C which contains $\bigcup K$, the largest homothet of C contained in $\bigcap K$, or the smallest homothet of C which intersects every member of K. These problems can be seen a a generalization of Megiddo's ball spanning balls problem [M89]. The combinatorial dimension in each case is $d+1$, and the running time again depends on the complexity of the objects. When C and all the elements of K are of constant complexity we get an expected linear time algorithm. In other cases, preprocessing can often be used to reduce the obvious running times; see, eg. [KM91] for a development of this idea in a different context.

Application 4.3 *Special cases of line transversals in E^d.*

In general, finding line transversals is significantly more difficult in dimension $d > 2$ than it is in the plane, but there are a few special cases in which Helly theorems help us get a linear time algorithm. Theorems 5.6 and 5.7 in [DGK63], due to Grünbaum, concern, respectively, a family of $d-1$ dimensional polytopes, all of which lie in a family of parallel hyperplanes, and a family of spheres such that the distance between any two is greater than the sum of their diameters. In both these cases, if there is a line through every $2d-1$ objects then there is a line through all of them. Again, the first paradigm can be applied to give a linear time algorithm to find a line transversal. [2]

5 GLP Problems from other GLP Problems

Notice that the line transversal algorithm for translates finds a line which minimizes the maximum distance from the family of fixed points, under the metric whose unit ball is the object O. This mini-max property is useful in and of itself. We can apply the first paradigm to known GLP problems much as we would apply parametric search, to find the minimum value of λ at which the problem is feasible. While parametric search usually adds an additional logarithmic factor to the running time, the expected time here remains linear.

Application 5.1 *Weighted L^∞ linear interpolation.*

The input to this problem is a family of n points in R^d, with an axis-aligned rectangle T_p centered at each point p. Note that each T_p may have different dimensions. The distance from a hyperplane h to p is the smallest nonnegative real value λ such that T_p intersects h when scaled by λ. We call this the *weighted L^∞ metric.* The linear interpolation problem is to find the h which minimizes the maximum distance to any point.

This problem arises when we want to fit a hyperplane to a family of points, and each coefficient of each point is given a weight, producing box-shaped error regions. This occurs, for example, when the coefficients are calculated and error is bounded using interval arithmetic, or when complicated error regions are approximated by bounding boxes. The general-dimensional version of the problem has been considered in [R89], [D91], and in [PR92], where it is shown to be NP-hard. Showing that the problem can be formulated as GLP gives a expected linear time algorithm for the fixed dimensional case.

Define a *positive* hyperplane to be one which is oriented so that its normal vector is directed into the positive orthant of E^d. There is a diameteral pair of vertices v^+, v^- on each box such that, at any fixed value of λ, a positive hyperplane m goes through the

[2] The first of these problems can in fact be formulated as a linear program.

box if and only if v^+ and v^- lie in its positive and negative halfspaces, respectively. Finding a positive hyperplane transversal of the boxes at a fixed value of λ is thus the geometric dual form of a d-dimensional linear program with $2n$ constraints. If a fixed positive hyperplane m goes through a box at λ_1, it also does so at any $\lambda_2 > \lambda_1$. So for each constraint point v in the linear program, the set of hyperplanes for which v lies in the correct halfspace form a nested family parameterized by λ. Finding the minimum value λ^* which admits a positive hyperplane transversal of the boxes is a GLP problem of combinatorial dimension $d+1$, again using a perturbation argument to ensure a unique minimum hyperplane. In the special case in which the boxes all have the same dimensions, the problem can be formulated as a linear program. In general, however, each point moves along a unique trajectory as λ varies, and the constraints cannot be linearized.

For a given family of boxes, we define a separate GLP problem for each orthant of E^d. The solution to the whole problem will be hyperplane which achieves the minimum λ of any of the 2^d GLP problems.

Application 5.2 *Linear interpolation with a polyhedral metric.*

Consider the problem of finding a hyperplane transversal of a family of polytopes whose facets are drawn from a set U which is the union of a constant number of families of parallel hyperplanes. Avis and Dorskis [AD92] show a similar reduction of this problem to a fixed number of linear programming problems. Applying the first paradigm gives an expected linear time algorithm for fitting a hyperplane to a point family under any metric whose unit ball is a polytope with a constant number of sides, or, more generally, in which each point has an error metric whose unit ball is a polytope with facets drawn from U.

Application 5.3 *Line fitting in the weighted L^∞ metric.*

Megiddo has shown [M91] that the problem of finding a line transversal for a family of axis-aligned boxes in E^d can be formulated as a collection of linear programs in dimension $2d-2$. We can again apply the first paradigm to find the closest line to a family of points under the weighted L^∞ metric defined above.

Observe that, in the previous examples, at any λ (not necessarily one such that $\lambda = w(G)$ for some G), we can find some point in $\bigcap G_\lambda$ by linear programming. This suggests a general way to remove the unique minimum assumption from the first paradigm.

Theorem 5.1 *Let $(\overline{H}, \cap)$ be a parameterized Helly system with parameter λ and natural objective function w_0. If there is a function w_1 such that every (H_λ, w_1) is a GLP problem of combinatorial dimension $\leq k$, then there is a function w such that $(\overline{H}, w)$ is a GLP problem of dimension $\leq 2k+1$.*

The proof of this theorem, which is similar to the proof of theorem 4.1, is omitted. The idea is that w is the parametric objective function (w_0, w_1), where w_0 is the most significant parameter and w_1 is used as a tie-breaker.

The following example shows that unfortunately this upper bound of $2k+1$ on the combinatorial dimension is the best possible in such a general context.

Theorem 5.2 *There is a GLP problem, of the form described in theorem 5.1, in which every (H_λ, w_1) is a GLP problem of combinatorial dimension $\leq k$, and $(\overline{H}, w)$ is a GLP problem of combinatorial dimension $2k+1$.*

Proof: Consider an optimization problem in which the constraints $h \in H$ are sets of the form

$$\left\{ x \in E^k, \lambda \in R \mid \bar{a}\bar{x} \geq \begin{array}{c} c: \text{ if } \lambda < b \\ -\infty: \text{ if } \lambda \geq b \end{array} \right\}$$

Here $\bar{a}$ is a constant vector, and b and c are constants. The most significant objective w_0 is to minimize λ, and the tie-breaking function w_1 is to minimize x.

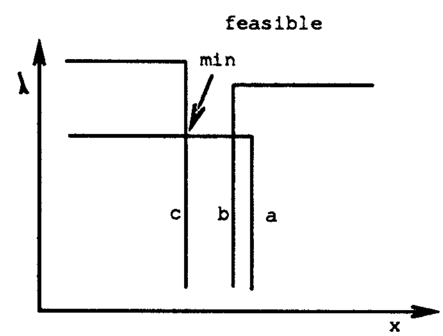

Figure 2: Size of basis is three

Observe that in figure 2, constraints a and b determine the minimum value of λ, while constraint c determines the minimum value of x. In general a basis for (H, w) may contain of a basis of an infeasible k-dimensional linear program, determining λ, and another, disjoint, basis of a feasible k-dimensional linear program determining x, so that its total size may be as large as $2k+1$.
□

6 Second Paradigm

While the first paradigm is useful for many problems, it is *not* the one used by linear programming with a parametric objective function. Can that objective function be applied to other Helly theorems? Hoffman [H79] gave a paradigm for constructing a parameterized objective function for a general Helly system, relating the Helly number to something he called the *binding constraint number*. Here we relate it to the combinatorial dimension.

Assume that we have a set X and a family C of subsets of X, such that C contains a nested family $\overline{P}$. For example, the family of convex subsets of E^d contains the nested families of parallel halfspaces. For any $x \in X$, let the *parameter* of x defined by $\overline{P}$ be the index of the smallest set in $\overline{P}$ which contains x, or the special symbol Ω if there is no such set. Thus one nested family defines a partial order on the elements of X.

Several nested families act like a coordinate system. Assume that C contains d nested families $\overline{P}_i, \ldots, \overline{P}_d$. Each $x \in X$ is thus equipped with a string of parameters $(p_1, \ldots, p_d)$, although more than one x might share the same parameter string. We say that X is *parameterized* by $\overline{P}_i, \ldots, \overline{P}_d$.

Theorem 6.1 *Let $(C, \cap)$ be a Helly system with Helly number k, embedded in X, where X is parameterized by $\overline{P}_1, \ldots \overline{P}_d \subseteq C$. Define $w(G)$ as the lexicographic minimum of the parameters of all $x \in \bigcap G$, or the special symbol Ω if $\bigcap G = \emptyset$. Assume that, for all $G \subseteq H$,*
1. *$w(G)$ exists, and*
2. *if $\bigcap G \neq \emptyset$, there is always a unique point $x \in \bigcap G$ with parameter string $w(G)$.*

Then for any $H \subseteq C$, (H, w) is a GLP problem with combinatorial dimension $\leq (k-1)d$.

Proof: The constraints H are subsets of X, so observation 4.1 again implies that (H, w) obeys the monotonicity condition. Since there is exactly one point m with parameters $w(G)$, the argument in the proof of theorem 4.1 shows that (H, w) meets the locality condition.

Consider any $G \subseteq H$, and any basis B of G. If G does not intersect then $|B| \leq k$, and $w(G) = w(B) = \Omega$.

So assume that G intersects. We consider each of the parameters of $w(B)$ separately. Let p_1 be the most significant, p_2 the next most significant, and so on. For a parameter p_i and a family $G \subset C$, let $p_i(G)$ be the ith parameter of $w(G)$.

For each parameter p_i, let $B_i = \{h \in B \mid p_i(B - h) < p_i(B) \text{ and } p_j(B-h) = p_j(B), \forall j < i\}$. That is, B_i is the family of constraints whose removal from B causes $w(B)$ to decrease in the ith most significant parameter. Let $p_i^* = max\{p_i(B - h) \mid h \in B_i\}$. The value p_i^* is the index of some set in $P^* \in C$, a member of the nested family $\overline{P}_i$. For every $h \in B_i$, $B_i - h$ intersects some member P^h of $\overline{P}_i$, such that $P^h \subseteq P^*$, and hence $B_i - h$ intersects P^*. But we know from the definition of B_i that $\bigcap B_i$ fails to intersect P^*. Therefore the family $B_i \cup \{P^*\}$ must contain some family A of size $\leq k$ such that A fails to intersect. It has to be the case that $P^* \in A$, because otherwise $A \subseteq B_i$, which is impossible because we are assuming that B_i intersects and A does not. Also every $h \in B_i$ is also in $\in A$, since otherwise $A \subseteq (B_i - h) \cup \{P^*\}$, which again is impossible because $(B_i - h) \cup \{P^*\}$ intersects and A does not. So $|B_i| = |A| - 1 \leq k - 1$.

For every i the number of elements in B_i is $\leq k-1$, and there are d parameters, so for the whole basis $|B| \leq d(k-1)$.
□

The archetypal example of a Helly system of this sort is the convex sets in E^d, with the halfspaces defined by the coefficient hyperplanes as the nested families. But this problem has combinatorial dimension d, not d^2 as theorem 6.1 would suggest. We get a better bound on the combinatorial dimension with the following little

Theorem 6.2 *If, for every $H \subseteq C$, there exists some single nested family Q which imposes the same total order on the set of minima $M = \{m \mid m = w(G), G \subseteq H\}$ as the parametric function w, then the combinatorial dimension of (H, w) is $k - 1$.*

Proof: Let w_q be the single parameter function associated with Q. Since w_q determines the same total order on M as w, it meets conditions 1 and 2. By the same argument used in the last proof, every basis B of any $G \subseteq H$ under w_q has size $\leq k - 1$. But the bases are the same under the two functions; so (H, w) also has combinatorial dimension $k - 1$.
□

For problems in E^d involving a finite number of convex objects and using a parameterized linear objective function determined by a lexicographic ordering on the coefficients, there is always some delicately tilted family of nested halfspaces which imposes the same ordering on M as the parameterized linear function. One such family of halfspaces is the one with normal vector $\epsilon, \epsilon^2, \ldots, \epsilon^d$, for some $\epsilon < 1$. There will always be some ϵ small enough, because the family M of constraints is finite. This explains why using a parameterized objective function does not increase the combinatorial dimension for convex programming.

For some Helly systems, the second paradigm can be used to construct a GLP problem but there is no such family Q. For instance, the family of axis aligned boxes in E^d has Helly number 2, but the combinatorial dimension of the resulting GLP problem is d.

We can use this paradigm to define a GLP problem in which the constraints are not even connected, let alone convex. This is interesting in that it implies we can sometimes use a single GLP problem to solve mathematical programming problems in which the feasible region becomes disconnected, so long as the number of connected components remains fixed.

Application 6.1 *Pairs of convex sets.*

A family I of sets is *intersectional* if, for every $H \subseteq I$, $\bigcap H \in I$. Let C be the set of all convex sets in R^d, defined so as to include the empty set. Let $Z = \{(C_1 \cup C_2) \mid C_1, C_2 \in C\}$. The whole family Z is not intersectional. But consider some subfamily Z' which *is* intersectional. Then $(Z', \cap)$ is a Helly system with Helly number $2(d+1)$ [GM61]. As a concrete example, let each $h \in Z'$ be a pair of spheres of radius 1, separated by a distance of 1 (kind of like dumbbells).

Notice that we can adjoin the set S of spheres centered at the origin to any intersectional family of pairs Z'. Each sphere can be considered a pair with the empty set, and set $Z' \cup S$ remains intersectional since the intersection of a sphere with any pair of convex sets will produce ≤ 2 convex sets. Let S_r be the sphere of radius r centered at the origin. For any $G \subseteq Z'$, let $w(G)$ be the smallest r such that $\bigcap G \cap S_r \neq \emptyset$. Notice that for any G, if $r = w(G)$, $|\bigcap G \cap S_r| = 1$. So (Z', w) is a GLP problem, where w is the one-parameter function defined by S, with combinatorial dimension $2d-1$.

7 Non-GLP Helly systems

So far we have given paradigms for constructing a GLP objective function for a Helly system. These paradigms required additional geometric assumptions on the constraint families, beyond having a fixed Helly number. We now show that some such additional assumption is necessary, by exhibiting a set system with a fixed Helly number which *cannot* be turned into a fixed dimensional GLP problem.

Theorem 7.1 *There is a family H of $2n$ sets with Helly number 2 such that for any valid GLP objective function w the combinatorial dimension of (H, w) is n.*

Proof: Let the universe X consist of the 2^n points at the vertices of an n dimensional hypercube, and let the constraint family H be the $2n$ subsets each if which lies in a facet of the hypercube. Notice that if a subfamily $G \subseteq H$ includes any pair of opposite facets, then G fails to intersect, and otherwise G does intersect. So the Helly number of (X, H) is 2.

Any valid objective function w must assign $w(G) = \Omega$ to the infeasible families G which contain a pair of opposite facets. Meanwhile any feasible G which does not contain a pair of opposite facets will have $w(G) = s \in S$, with $s < \Omega$. Let $s^* = \max\{s \in S | s < \Omega \text{ and } s = w(G) \text{ for some } G \subseteq H\}$, and consider some G with $w(G) = s^*$. If $|G| < n$, then there exists some pair (h^+, h^-) of facets, such that G contains neither h^+ nor h^-. This means that $G + h^+$ is also feasible. By the monotonicity condition, so $w(G+h^+) \geq w(G)$; and since $w(G)$ is maximal, we can conclude that $w(G + h^+) = w(G) = s^*$. This argument shows that there must be a subfamily G of size n with $w(G) = s^*$.

Now we show that there is no basis B for such a subfamily G such that $B \neq G$. Assume, for the purpose of contradiction, and without loss of generality, that there is some element $h^+ \in G$ such that $h^+ \notin B$. $B + h^-$ is still feasible, so $w(B + h^-) = w(B) = s^*$. But $w(G + h^-) = \Omega$. Since $B \subset G$ and $w(B) = w(G)$, this means that w is not a valid objective function because it fails to satisfy the locality condition. So any valid objective function w must have $B = G$, and (H, w) must have combinatorial dimension n.
□

This theorem says that the class of problems whose constraint sets have a fixed Helly number is strictly greater than the class of fixed dimensional GLP problems.

8 Concluding Remarks

The two paradigms should be useful in producing computational versions of other interesting Helly theorems such as those using spherical convexity and those concerning separating surfaces.

There are many theorems similar to Helly theorems, such as Gallai-type theorems and Hadwiger-type theorems, and Helly-type theorems in which the fact that all subfamilies of size $\leq k$ have some property p implies that the whole family has some other property q. It would be interesting to find algorithmic applications of these.

Generalizing the many results about linear programming to other Helly systems may give interesting geometric results.

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