On algebraic solutions of first order Riccati equation

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Abstract

In this paper we prove the following theorem. If the Riccati equation $w' + w^2 = R(x)$, $R \in Q(x)$, has algebraic solutions then one can find a minimal polynomial defining such solutions whose coefficients are in a quadratic extension of the field Q.

1 Introduction

Kovacic [1] has given an algorithm for finding Liouvillian solutions of the second-order linear homogeneous differential equations

$$au'' + bu' + cu = 0 \tag{1}$$

with $a, b, c \in C(x)$, the set of complex rational functions of one complex variable x. Equation (1) can always, without altering its Liouvillian character, be written in the reduced form

$$y'' = Ry, \quad R \in C(x). \tag{2}$$

Using the classification of the differential Galois groups of equation (2) Kovacic has proved that there are precisely four cases that can occur.

Case 1. Equation (2) has a solution of the form $e^{\int w}$ where $w \in C(x)$.

Case 2. Equation (2) has a solution of the form $e^{\int w}$ where w is algebraic over $C(x)$ of deg. 2, and case 1 does not hold.
Case 3. All solutions of equation (2) are algebraic over $C(x)$ and cases 1 and 2 do not hold.
Case 4. Equation (2) has no Liouvillian solutions.

On account of this theorem the problem of integrating of equation (2) in Liouvillian functions is reduced to finding a minimal polynomial $P(w, x) \in C[w, x]$ for the function $w = y'/y$ which is algebraic over $C(x)$ and satisfies the Riccati equation

$$w' + w^2 = R. \tag{3}$$

The complete decision procedure for finding $P(w, x)$ given by Kovacic [1] and implemented in a number of the computer algebra systems [2-4] is based on the knowledge of the even-order poles of R. This implies a complete splitting of even-order square free factors of the denominator of R and requires also that computations should be carried out in an algebraic extension of the constant field Q generated by the coefficients of R. In this paper we prove that the minimal polynomial $P(w, x)$ (if it exists) can be alwyas chosen so that its coefficients are in a quadratic extension of Q. This enables to construct an algorithm for finding such polynomial which does not require computing the poles of R at all.

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2 Proof of the theorem

Theorem. Let Q be a field of characteristic zero generated by the coefficients of R. If equation (3) has algebraic solutions then there exists a minimal polynomial $P(w, x)$ for such solutions whose coefficients are in $Q(\sqrt{q})$, $q \in Q$.

Proof. If equation (3) has an algebraic solution w then the following system of equations holds

$$P(w, x) = 0, \quad P_w(R - w^2) + P_x = 0. \tag{4}$$

For fixed $deg_w P$ and $deg_x P$ equations (4) imply a system of polynomial equations over Q for the unknown coefficients of $P(w, x)$. Computing the Groebner basis for these equations we immediately prove the theorem for the case when equation (3) has not more than 2 algebraic solutions.

Assume that for fixed $deg_w P$ and $deg_x P$ equation (3) has more than 2 algebraic solutions and show that in this case one can find the appropriate $P(w, x)$ with coefficients in $Q(\sqrt{q})$. We shall give the proofs separately for the cases 1,2,3 listed above and use the following lemma.

Lemma. If equation (3) has 3 different solutions w_1, w_2, w_3 in some field F then $R = -(1/2)\cdot\{\sigma, x\}$ where $\{\sigma, x\} \equiv (\sigma''/\sigma')' - (1/2)\cdot(\sigma''/\sigma')^2$ is a Schwarzian derivative and $\sigma \in F$.

Proof of the lemma. The proposition of the lemma can be easily verified for $\sigma = e^{\int (w_1 - w_2)dx}$ taking into account that $e^{\int w_1}, e^{\int w_2}$ are linear independent solutions of equation (2) and $e^{\int w_3} = c_1 \cdot e^{\int w_1} + c_2 \cdot e^{\int w_2}$ where $c_1 \cdot c_2 \neq 0, \quad c_1, c_2 \in C$. □

Proof of the theorem for the case 1. Assume that $R \in Q(x)$ and equation (3) has more than 2 solutions in $C(x)$. Then according to the Lemma $R = -(1/2)\cdot\{\sigma, x\}$ where $\sigma \in C(x)$. In this case we can write a solution of (3) involving 2 arbitrary constants:

$$w = -(1/2)\cdot\theta + \pi'/\pi, \tag{5}$$

$$\theta = \tau'/\tau, \quad \tau = u'v - v'u,$$

$$\pi = \alpha\cdot u + \beta\cdot v, \quad \alpha, \beta \in C,$$

$$u = num(\sigma), \quad v = den(\sigma),$$

$$gcd(u, v) = 1, \quad u, v, \tau, \pi \in C[x].$$

Choose α, β so that $gcd(\pi', \pi) = 1$ (it is always possible because $gcd(u, v) = 1$) and expand τ to a product of the square-free factors $\tau = \tau_1\tau_2^2...\tau_n^n$. Then substituting (5) into (3) and collecting the terms involving the second-order poles we find that the proper rational function

$$R^* \equiv -(1/4)\cdot\sum_{i=1}^{n} i(i+2)\tau_i'/\tau_i$$

is equal (up to an additive arbitrary constant) to the rational part of the integral $\int R dx$. Hence $R^* \in Q(x)$. Since $i \neq j$ implies $i(i+2) \neq j(j+2), \quad \tau_i \in Q[x]$ for $i = 1, 2, ..., n$. Hence $\theta \in Q(x)$. Substituting (5) into (3) we find that the coefficients of the polynomial π can be determined from the equation

$$\pi'' + 2\theta\cdot\pi' + (\theta' + \theta^2 - R)\cdot\pi = 0$$

using linear algebra. Hence π may be chosen in $Q[x]$ and $w \equiv -(1/2)\cdot\theta + \pi'/\pi \in Q(x)$. This proves the theorem for the case 1. □

Proof of the theorem for the case 2. Assume that $R \in Q(x)$, equation (3) has more than two solutions algebraic over $C(x)$ of degree 2 and case 1 does not hold. Then at least two different minimal polynomials $P(w, x)$ exist, $deg_w P = 2$, which define 4 different solutions of (3): $w_{1,2} = u \pm \sqrt{v}, \quad w_{3,4} = g \pm \sqrt{h}, \quad u, v, g, h \in C(x)$. The well-known relation for different 4 solutions of the Riccati equation

$$\frac{w_3 - w_1}{w_4 - w_1} : \frac{w_3 - w_2}{w_4 - w_2} = const$$

implies $\sqrt{vh} \in C(x)$. Consequently

$$\sigma \equiv e^{\int (w_1 - w_2)dx} \equiv const\cdot\frac{w_2 - w_3}{w_3 - w_1} = a + \sqrt{b},$$

$$a, b \in C(x).$$

By the other hand, $\sigma'/\sigma \equiv w_1 - w_2 = 2\sqrt{v}$. Hence $(\sigma'/\sigma)^2 \in C(x)$, which is possible iff

$$(2a'b - ab')\cdot(2aa' - b') = 0$$

If $2a'b - ab' = 0$ then $b = const\cdot a$ and $\sigma \in C(x)$. According to the lemma $R = -(1/2)\cdot\{\sigma, x\} \in Q(x)$ and (3) is solvable in $Q(x)$. This contradicts our assumption that case 1 does not hold. Another possibility is $2aa' - b' = 0$ which implies $b = a^2 + \alpha, \quad \alpha \in C$. Hence $\sigma = a + (a^2 + \alpha)^{1/2}$ and $R = -(1/2)\cdot\{\sigma, x\}$. Taking into account the invariance of the Schwarzian derivative under the fractional - linear transformations we find a solution in $C(x)$

$$w = -(1/2)\cdot(\vartheta''/\vartheta') \in C(x), \quad \vartheta = \frac{\sigma - i\sqrt{\alpha}}{\sigma + i\sqrt{\alpha}}.$$

Thus case 1 holds that contradicts our assumption. The theorem is proved for the case 2. □

Proof of the theorem for the case 3. Assume that $R \in Q(x)$ and case 3 holds. In this case the differential Galois group G of equation (2) is a finite algebraic subgroup of $SL(2)$ and is conjugate either to the tetrahedral ($ord\, G = 24$), octahedral ($ord\, G = 48$) or icosahedral group ($ord\, G = 120$) [1]. Let G be a conjugate to the tetrahedral group which is generated by the matrices $((\xi, 0), (0, \xi^{-1})), \ ((\phi, \phi), (2\phi, -\phi))$ where $\phi = (2\xi - 1)/3$ and $\xi^2 - \xi + 1 = 0$. We fix a fundamental system of solutions (y_1, y_2) of equation (2) so that G coincides with the tetrahedral group. Then $deg_{C(x)} w_1 = deg_{C(x)} w_2 = 4$ where $w_1 = y_1'/y_1, \ w_2 = y_2'/y_2$ (see [1]). Let (z_1, z_2) be another fundamental system of solutions of equation (2) and let H be a subgroup of G that fixes z_1'/z_1. Since G is finite, H is cyclic and $deg_{C(x)} z_1'/z_1 = deg_{C(x)} z_2'/z_2 = [G : H]$. There are four 6-order cyclic subgroups of G which are generated respectively by the following matrices

$$((\xi, 0),\ (0, \xi^{-1})),$$

$$((\xi + 1,\ 1 - 2\xi),\ (2 - 4\xi,\ 2 - \xi)),$$

$$((\xi + 1,\ \xi + 1),\ (2\xi - 4,\ 2 - \xi)),$$

$$((\xi + 1,\ \xi - 2),\ (2\xi + 2,\ 2 - \xi)).$$

and there are respectively 4 families of linear independent systems of solutions of equation (2) such that their logarithmic derivatives are algebraic over $C(x)$ of degree 4. Thus there are precisely 8 solutions of equation (3) algebraic over $C(x)$ of degree 4. It means that only 2 different minimal polynomials $P(w,x)$ exist such that $deg_w P = 4$. This proves the theorem for the "tetrahedral subcase" of the case 3.
To prove the theorem for the other subcases it is sufficient to check that the octahedral group has 6 cyclic subgroups of order 8 and the icosahedral group has 12 cyclic subgroups of order 10. This proves the main theorem. □

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