The Regular Problem and Green Equivalences for Special Monoids

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ABSTRACT

For the monoid presented by a finite special Church-Rosser Thue system, whether it is a regular semigroup is decidable in polynomial time. The number of each kind of Green equivalence classes is either one or infinite and it is computable in polynomial time.

1. Introduction

During recent years, string rewriting systems have received a lot of attention [8]. Many interesting results about formal languages defined by congruences of rewriting systems of certain types were obtained, and a number of decision problems were deeply investigated[2, 9]. Of particular interest are those systems that are complete(Noetherian and confluent). If a system T is complete, then each congruence class (mod T) contains a unique irreducible word, that is to say, T defines unique normal forms for their congruence classes via the process of reduction. Each reduction sequence starting with a word w can be extended to reach the irreducible word in the congruence class $[w]_T$ in finitely many steps. Obviously, for a finite complete system T, the word problem is effectively decidable[3]. Furthermore, other decision problems that are undecidable in general also become decidable when they are restricted to finite complete systems[2, 9].

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Finite systems, in which all the rules are length-reducing, are always Noetherian. Hence, a system of this form is complete if and only if it is confluent. A finite Thue system is said to be Church-Rosser if it is length-reducing and confluent. This property has been shown to be a powerful tool in providing decidability results for monoids. Many such results have been established. One of them is about the decidability of linear sentences for finite monadic Church-Rosser Thue systems[1]. However, many decision problems, such as the regular problem and the existence of non-trivial idempotents, can not be solved by this method. In this paper, we begin with Green equivalences and use some of their properties to solve several such problems, mainly the regular problem.

Obviously, a semigroup is regular if and only if all $\mathcal{D}$-classes on this semigroup are regular. this is why we first discuss the quantitative properties of Green equivalences. On the other hand, up till now, the authors have never found any papers that discussed the quantitative properties of the monoids presented by finite Thue systems, this paper is a try at this. We are only interested in finite special and Church-Rosser Thue systems for the reason that the process of rewriting modulo a finite special Thue system is particularly simple, since it only amounts to the deletion of subwords, and the systems of this type is very interesting[7]. We call the monoids defined by such thue systems special monoids.

In section 2, we give some basic definitions and results about Thue systems, Green equivalences and regular semigroup; In section 3, we prove that in the monoid M_T presented by a finite special Church-Rosser Thue system T, each $\mathcal{R}$-class, ($\mathcal{L}$-class, $\mathcal{J}$-class) is described completely by an element of M_T and R_e (L_e, $R_e \cap L_e$). The number of each kind of these five equivalence classes is one or infinite and it is computable in polynomial time. Mainly, we prove that it is decidable in polynomial time whether or not M_T is regular. In the last section, we have a simple discussion.

2. Preliminaries

In this section, the definitions of Green equivalences and regular semigroups are given, and some basic properties of reductions in finite special Church-Rosser Thue systems are described. The results presented in the following section will be based on these properties. It is assumed that the reader is familiar with the basic results of automata as covered in a text such as the book [6] by Hopcroft and Ullman and the theory of Church-Rosser Thue systems. Therefore, we repeat only those definitions and results on which our investigations are directly based. For more details and background, see the survey paper [2].

Let Σ be a finite alphabet, then Σ^* denotes the free monoid with identity e generated by Σ. As usual, the **length** of a word $w \in \Sigma^*$ is defined by $\mid w \mid$: $\mid e \mid = 0$; $\mid a \mid = 1$, $a \in \Sigma$; $\mid wa \mid = \mid w \mid +1$, $a \in \Sigma$, $w \in \Sigma^*$. The **concatenation** of two word u and v is written as uv. For any two subsets A, B of Σ^*, define $AB = \{ab \in \Sigma^* \mid a \in A, b \in B\}$. Let w, w_1 be two words. If there exist words w_0 and w_2 such that $w = w_0 w_1 w_2$, then we call w_1 a **subword** of w; if $w_0 = e (w_2 = e)$, then w_1 is called a **prefix(suffix)** of w, furthermore, if we also have $w_1 \neq e$, then w_1 is called a **proper prefix(suffix)** of w.

A **Thue system T** on Σ is a subset of Σ^*. An element (l, r) of T is called a **(rewrite) rule**. The **Thue congruence** $\Leftrightarrow_T^*$ generated by T is the reflexive, symmetric and transitive closure of the **sigle-step reduction** $\Rightarrow_T$, which is defined as follows:

$u \Rightarrow_T v$ if and only if there exist $x, y \in \Sigma^*$, and $(l, r) \in T$ such that $u = xly, v = xry$.

The reduction $\Rightarrow_T^*$ induced by T is the reflexive transitive closure of $\Rightarrow_T$. If $u \Leftrightarrow_T^* v$, then u and v are called **congruent** (*modT*). By $[u]_T$, we denote the congruence class $[u]_T = \{v \in \Sigma^* \mid u \Leftrightarrow_T^* v\}$ of u (*modT*). If $u \Rightarrow_T^* v$, then u is an **ancestor** of v and v is a **descendant** of u (*modT*). For $u \in \Sigma^*$, $\Delta_T^*(u) = \{v \in \Sigma^* \mid u \Rightarrow_T^* v\}$ is the set of the descendants of u and for $R \subseteq \Sigma^*$, $\Delta_T^*(R) = \bigcup_{u \in R} \Delta_T^*(u)$. If $\Delta_T^*(u) = u$, then, u is called **irreducible**; otherwise, it is called **reducible**. $Irr(T)$ is the set of all irreducible words. For a Thue system T, let

$$Left(T) = \{l \in \Sigma^* \mid \exists r \in \Sigma^* : (l, r) \in T\},$$

$$Right(T) = \{r \in \Sigma^* \mid \exists l \in \Sigma^* : (l, r) \in T\}.$$

i.e. $Left(T)$ and $Right(T)$ are the sets of all left-hand sides of rules in T and that of all right-hand sides of rules in T respectively.

Theorem 2.1[1] *Let T be a Thue system on the alphabet Σ. If T is finite, then $Irr(T)$ is a regular set, and from T, we can construct a deterministic finite state acceptor M for $Irr(T)$ in polynomial time.*

QED.

A Thue system T is called **length-reducing**, if $\mid l \mid > \mid r \mid$ holds for all rules $(l, r) \in T$, and it is **monadic (special)** if it is length-reducing and $Right(T) \subseteq \Sigma \cup \{e\}$ ($Right(T) = \{e\}$). It is easy to see that special Thue systems are monadic.

Let T be a length-reducing Thue system on the alphabet Σ. The system T is said to be **Church-Rosser**, if for all $u, v \in \Sigma^*$, $u \Leftrightarrow_T^* v$ implies there exists a $w \in \Sigma^*$, such that $u \Rightarrow_T^* w$, and $v \Rightarrow_T^* w$. If a Thue system T is Church-Rosser, then, every congruence class has a unique irreducible word, Thus, for a finite Church-Rosser Thue system, its word problem is decidable in linear time[3]. For other properties of Church-Rosser Thue systems, see papers [3] and [4].

Suppose that T is a Thue system on alphabet Σ, the set $M_T = \{[u]_T \mid u \in \Sigma^*\}$ of all congruence classes forms a monoid under the operation $[u]_T \circ [v]_T = [uv]_T$ with identity $[e]_T$. Accordingly, the ordered pair (Σ, T) is called a **monoid-presentation** of M_T. In the following sections, we consider Green equivalences on M_T.

Let M be a monoid, for any two elements $a, b \in M$, $(a, b) \in \mathcal{R}$ means that a and b generate the same principal right ideal of M; Dually, we define

$(a,b) \in \mathcal{L}$ if a and b generate the same principal left ideal of M; $(a,b) \in \mathcal{J}$ if a and b generate the same two-side principal ideal; The intersection of the relations $\mathcal{R}$ and $\mathcal{L}$ is denoted by $\mathcal{H}$ and their join by $\mathcal{D}$. Clearly, these five relations are equivalence relations, we call them **Green equivalences**. $m \in M$ is call a regular element, if there exists $x \in M$, such that $mxm = m$; If all the elements of M are regular, we call M a **regular semigroup**. In addition, $m \in M$ is called an **idempotent**, if $m^2 = m$. It is easy to see that idempotents are regular, and if there is one regular element in a $\mathcal{D}$-class, then all the elements of this $\mathcal{D}$-class are regular, i.e. this $\mathcal{D}$-class is a regular one. Also, we know, in a regular $\mathcal{D}$-class, each $\mathcal{R}$-class and $\mathcal{L}$-class contains at least one idempotent(see [5]).

In the following, the $\mathcal{R}$-class($\mathcal{L}$-class, $\mathcal{J}$-class, $\mathcal{H}$-class, and $\mathcal{D}$-class) of M_T containing the element a of $Irr(T)$ will be denoted by $R_a(L_a, J_a, H_a$, and $D_a)$.

Theorem 2.2 *Suppose that T is a Church-Rosser Thue system on the alphabet Σ, $a, b \in Irr(T)$.*

(1). $(a,b) \in \mathcal{R}$ iff there exist $x, y \in Irr(T)$ such that $ax \Rightarrow_T^* b$, $by \Rightarrow_T^* a$;

(2). $(a,b) \in \mathcal{L}$ iff there exist $x, y \in Irr(T)$ such that $xa \Rightarrow_T^* b$, $yb \Rightarrow_T^* a$;

(3). $(a,b) \in \mathcal{J}$ iff there exist $x_1, x_2, y_1, y_2 \in Irr(T)$ such that $x_1 a y_1 \Rightarrow_T^* b$, $x_2 b y_2 \Rightarrow_T^* a$.

QED.

For $w \in Irr(T)$, let $Pref(w)$ and $Suff(w)$ be the sets of all prefixes and suffixes of w respectively.

Theorem 2.3 *Suppose that the Thue system T on the alphabet Σ is finite special and Church-Rosser, then R_e and L_e are all regular sets, and we can construct two finite state acceptors for them in polynomial time respectively.*

Proof. Let

$$pl_T = \{l_1 \in \Sigma^* \mid \exists l \in Left(T) : l_1 \in Pref(l)\};$$

$$sr_T = \{l_2 \in \Sigma^* \mid \exists l \in Left(T) : l_2 \in Suff(l)\},$$

then,

$$R_e = pl_T^* \cap Irr(T); L_e = sr_T^* \cap Irr(T).$$

Since pl_T and sr_T are finite sets, we get the results as required.

QED.

Theorem 2.4 Let T be a finite special Church-Rosser Thue system on the alphabet Σ. $X = \{x \in Irr(T) \mid Suff(x) \cap (R_e \setminus \{e\}) = \emptyset\}$, and $Y = \{y \in Irr(T) \mid Pref(y) \cap (L_e \setminus \{e\}) = \emptyset\}$. Then,

(1). X and Y are regular sets, two finite state acceptors for them can be constructed in polynomial time respectively;

(2). $X = X^*, Y = Y^*$.

Proof. (1). Let $M = (Q, \Sigma, \delta, q_0.F)$ be a deterministic finite state acceptor for $Irr(T)$. For every ordered pair (p,q) of states of M, let $L_{p,q}$ be the set of all the words w, such that on input w, M goes from state p to state q. Clearly, $L_{p,q}$ is a regular set and a finite state acceptor for it can be constructed in polynomial time. Now, we can easily prove

$$X = Irr(T) \setminus \bigcup_{q \in Q, q_F \in F} L_{q_0,q}\left(L_{q,q_F} \cap (R_e \setminus \{e\})\right);$$

$$Y = Irr(T) \setminus \bigcup_{q \in Q, q_F \in F} \left(L_{q_0,q} \cap (L_e \setminus \{e\})\right) L_{q,q_F}.$$

From Theorem 2.1, Theorem 2.3 and the fact that Q, F are finite sets, we know X and Y are regular sets , that a finite state acceptor for X(or Y) can be constructed in polynomial time.

(2). For $x_1, x_2 \in X$, if $x_1 x_2 \notin X$, there exist $x_{12} \in Suff(x_1)$ such that

$$x_{12} x_2 \in R_e \setminus \{e\}.$$

This implies $x_{12} \in R_e$, then,

$$x_{12} = e, x_2 \in R_e \setminus \{e\}$$

and this is impossible.

Similarly, we can prove the results about Y.

QED.

3. Main results

In this section, we first discuss the structure of each kind of Green equivalence classes, and then, as an application, we show that the regular problem is decidable in polynomial time. Of course, the Thue system T we considered on the alphabet Σ is finite special and Church-Rosser.

3.1 The structure of Green equivalences

Since $\mathcal{H}$ is the intersection of $\mathcal{R}$ and $\mathcal{L}$, and $\mathcal{D}$ the join of them, we only consider the structure of $\mathcal{R}, \mathcal{L}$ and J now.

Theorem 3.1 *Let T be a finite special Church-Rosser Thue system on the alphabet Σ, $a \in Irr(T)$.*

(1). If $a \notin X$, i.e. there exists $a_2 \in Suff(a)$ such that $a_2 \in R_e \setminus \{e\}$. Let $a = a_1a_2 (a_1 \in \Sigma^*)$, then, $R_a = R_{a_1}$; otherwise, $R_a = aR_e$;

(2). If $a \notin Y$, i.e. there exists $a_1 \in Pref(a)$ such that $a_1 \in L_e \setminus \{e\}$. Let $a = a_1a_2 (a_2 \in \Sigma^*)$, then, $L_a = L_{a_2}$; otherwise, $L_a = L_e a$;

(3). If $a \notin X$ (or $a \notin Y$), i.e. there exists $a_2 \in Suff(a)$ (or $a_1 \in Pref(a)$), such that $a_2 \in R_e \setminus \{e\}$ (or $a_1 \in L_e \setminus \{e\}$). Let $a = a_0a_2$ (or $a = a_1a_0$). $(a_0 \in \Sigma^*)$, then, $J_a = J_{a_0}$; otherwise, if $a \notin Sub(Left(T))$, $J_a = L_e a R_e$; otherwise, $J_a = J_e$. Where Sub(Left(T)) is the set of all subwords of the words of Left(T).

Proof. (1). Let

$$x \in Irr(T), a_2x \Rightarrow_T^* e,$$

then

$$ax \Rightarrow_T^* a_1, (a, a_1) \in \mathcal{R}.$$

For each $b \in R_a$, there exist $y, z \in Irr(T)$ such that

$$bz \Rightarrow_T^* a, ay \Rightarrow_T^* b.$$

Since

$$Suff(a) \cap (R_e \setminus \{e\}) = \emptyset,$$

for each $a_2 \in Suff(a), y_1 \in Pref(y)$,

$$a_2y_1 \notin Left(T).$$

Hence,

$$ay \in Irr(T), \quad aR_e \subseteq Irr(T)$$

and from the Church-Rosser property of T we know

$$b = ay.$$

Thus,

$$ayz \Rightarrow_T^* a.$$

Since

$$Suff(a) \cap (R_e \setminus \{e\}) = \emptyset,$$

there are $y_2 \in Suff(y), z_1 \in Pref(z)$ such that

$$y_2z_1 \Rightarrow_T^* e, ay_1z_2 = a,$$

where $y_1, z_2 \in \Sigma^*, y = y_1y_2, z = z_1z_2$. Then,

$$y_1 = z_2 = e, yz \Rightarrow_T^* e, R_a \subseteq aR_e.$$

Obviously,

$$aR_e \subseteq R_a.$$

(2). The proof is similar to that of (1).

(3). Since

$$\mathcal{R} \subseteq J, \mathcal{L} \subseteq J,$$

from the first results of (1) and (2), we know that if

$$a \notin X \text{ or } a \notin Y,$$

then

$$J_a = J_{a_0},$$

where $a_0 \in Irr(T)$, $a = a_0a_2$ (or $a = a_1a_0$), $a_2 \in Suff(a)$ ($a_1 \in Pref(a)$), and $a_2 \in R_e \setminus \{e\}$ ($a_1 \in L_e \setminus \{e\}$).

Suppose

$$a \in X, a \in Y.$$

If $a \in Sub(Left(T))$, there are $x, y \in Irr(T)$,

$$xay \Rightarrow_T^* e.$$

Hence,

$$a \in J_e.$$

If $a \notin Sub(Left(T))$, for every $b \in J_a$, there are $x_1, x_2, y_1, y_2 \in Irr(T)$,

$$x_1ay_1 \Rightarrow_T^* b \text{ and } x_2by_2 \Rightarrow_T^* a.$$

If x_1ay_1 is reducible, according to the properties of $Suff(a)$ and $Pref(a)$, we know that there exist $x_{12} \in Suff(x_1)$, $y_{11} \in Pref(y_1)$, $y_{11} \neq e \neq x_{12}$ such that

$$x_{12}ay_{11} \in Left(T).$$

This is contradiction with $a \notin Sub(Left(T))$. Hence,

$$x_1ay_1 \in Irr(T), b = x_1ay_1$$

and this fact also implies

$$L_eaR_e \subseteq Irr(T).$$

Obviously,

$$x_2x_1ay_1y_2 \Rightarrow_T^* a.$$

Thus, there exist

$$x_{22} \in Suff(x_2), x_{11} \in Pref(x_1),$$

$$y_{12} \in Suff(y_1), y_{21} \in Pref(y_2),$$

$$x_{12}, x_{21}, y_{11}, y_{22} \in \Sigma^*,$$

$$x_1 = x_{11}x_{12}, x_2 = x_{21}x_{22},$$

$$y_1 = y_{11}y_{12}, y_2 = y_{21}y_{22}$$

and

$$x_{22}x_{11} \Rightarrow_T^* e,$$

$$y_{12}y_{21} \Rightarrow_T^* e,$$

$$x_{21}x_{12}ay_{11}y_{22} = a.$$

Hence,

$$x_{12} = y_{11} = e, x_1 = x_{11}, y_1 = y_{12}, x_1 \in L_e, y_1 \in R_e.$$

This implies

$$b \in L_e a R_e.$$

Hence

$$J_a \subseteq L_e a R_e.$$

Conversely, $L_e a R_e \subseteq J_a$ is obvious.

QED.

3.2 Properties of Green equivalences

Corollary 3.2 *Suppose that T is a finite special Church-Rosser Thue system on the alphabet Σ. For $a \in Irr(T)$, R_a, L_a, J_a, H_a and D_a are all regular sets, and the finite state acceptors for them can be constructed in polynomial time respectively.*

Proof. Choose

$$a_1, a_2, a_3, a_4 \in \Sigma^*,$$

such that

$$a = a_1a_2 = a_3a_4, a_2 \in R_e, a_3 \in L_e$$

and

$$a_1 \in X, a_4 \in Y.$$

Then,

$$R_a = R_{a_1} = a_1R_e, L_a = L_{a_4} = L_e a_4,$$

we know that R_a and L_a are regular sets, two finite state acceptors for them can be constructed in polynomial time.

Since

$$H_a = R_a \cap L_a,$$

we obtain the result about H_a.

Let

$$a = a_1a_0a_2(a_0, a_1, a_2 \in \Sigma^*), a_1 \in L_e, a_2 \in R_e$$

and

$$a_0 \in X, a_0 \in Y.$$

we have

$$J_a = J_{a_0}.$$

If $a_0 \notin Sub(Left(T))$, then,

$$J_a = L_e a_0 R_e.$$

In this case, J_a is a regular set and we can construct a finite state acceptor for it in polynomial time; If $a_0 \in Sub(Left(T))$, then $J_a = J_e$. Now, we show the results about J_e. In fact,

$$J_e = L_e Sub(Left(T)) R_e \cap Irr(T)$$

and the fact that $Sub(Left(T))$ is finite implies the results as required.

Now, we prove that the following set A_a :

$$\{b_1 \in Irr(T) \mid \exists b \in L_a, \exists b_2 \in R_e, b = b_1b_2, b_1 \in X\}$$

is a regular set and a finite state acceptor for it can be constructed in polynomial time.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite state acceptor for L_a , $L_{p,q}$ be the same as that of Theorem 2.4. By B_q and C, we denote $L_{q_0,q} \cap X$ and $L_a \setminus \Sigma^*(R_e \setminus \{e\})$ respectively. Then,

$$A_a = \bigcup_{q \in Q, q_F \in F, L_{q,q_F} \cap (R_e \setminus \{e\}) \neq \emptyset} B_q \cup C.$$

For each $b_1 \in A_a$, $b = b_1b_2$, where $b \in L_a, b_2 \in R_e$, and $b_1 \in X$. If $b_2 = e$, then

$$b \in C;$$

If $b_2 \neq e$, there exist $q \in Q, q_F \in F$, such that

$$b_1 \in L_{q_0,q}, b_2 \in L_{q,q_F}.$$

Hence,

$$b_1 \in B_q \subseteq \bigcup_{q \in Q, q_F \in F, L_{q,q_F} \cap (R_e \setminus \{e\}) \neq \emptyset} B_q.$$

Conversely, for $c_1 \in \bigcup_{q \in Q, q_F \in F} B_q$, there are $q \in Q, q_F \in F, c_2 \in \Sigma^*$, such that

$$c_1 \in B_q, c_2 \in L_{q,q_F} \cap (R_e \setminus \{e\}).$$

Let $c = c_1c_2$, then

$$c \in L_a, c_1 \in A_a$$

and

$$\bigcup_{q\in Q, q_F\in F, L_{q,q_F}\cap(R_e\setminus\{e\})\neq\emptyset} B_q \subseteq A_a.$$

It is clear that

$$C \subseteq A_a.$$

Using Theorem 3.1(1), we have

$$D_a = \cup_{b\in L_a} R_b = \cup_{b\in A_a} R_b = \cup_{b\in A_a} bR_e = A_aR_e.$$

Thus we get the results about D_a.

QED.

Let $|\mathcal{R}|, |\mathcal{L}|, |\mathcal{J}|, |\mathcal{H}|$ and $|\mathcal{D}|$ represent the number of $\mathcal{R}$-classes, $\mathcal{L}$-classes, $\mathcal{J}$-classes, $\mathcal{H}$-classes and $\mathcal{D}$-classes in M_T respectively. For a set $S \subseteq \Sigma^*, |S|$ represents the number of the words of S.

Theorem 3.3 *Let T be a finite special Church-Rosser Thue system on the alphabet Σ, then we have*

(1). $|\mathcal{R}| = |X|$;
(2). $|\mathcal{L}| = |Y|$;
(3). $|\mathcal{J}| = |(X\cap Y)\setminus Sub(Left(T))| + 1$;
(4). $|(X\cap Y)\setminus Sub(Left(T))| \leq |\mathcal{D}| \leq |X\cap Y|$.

Proof. (1). For $a \in Irr(T)$, let

$$a = a_1a_2,$$

where

$$a_2 \in R_e, a_1 \in \Sigma^*, a_1 \in X.$$

Then,

$$R_a = R_{a_1}, \text{ i.e. } a_1 \in R_a.$$

This implies that there is at least one word in R_a which belongs to X.

Let

$$a_1, a_2 \in X \text{ and } a_1, a_2 \in R_a,$$

if

$$a_1R_e = R_{a_1} = R_{a_2} = a_2R_e,$$

there exists $c_0 \in R_e$, such that

$$a_1 = a_2c_0.$$

Since

$$a_1 \in X,$$

$$c_0 = e, a_1 = a_2.$$

So there is a 1-1 correspondence between X and the set of all $\mathcal{R}$-classes in M_T.

(2). Similar to (1).

(3). Let

$$a \in Irr(T),$$

if $J_a \neq J_e$, let

$$a = a_1a_0a_2,$$

where

$$a_1 \in L_e, a_2 \in R_e, a_0 \in \Sigma^*$$

and

$$a_0 \in X\cap Y.$$

Then,

$$J_a = J_{a_0}.$$

Thus, for each $\mathcal{J}$-class J_a which is not J_e, there is at least one word in J_a that belongs to $(X\cap Y)\setminus Sub(Left(T))$. For $b_1, b_2 \in (X\cap Y)\setminus Sub(Left(T))$, we know

$$J_{b_1} = L_eb_1R_e, J_{b_2} = L_eb_2R_e.$$

If

$$J_{b_1} = J_{b_2},$$

there exist $c \in L_e, d \in R_e$ such that

$$b_1 = cb_2d,$$

thus,

$$c = d = e, b_1 = b_2.$$

Then, if J_a is not J_e, there is a unique word in J_a belonging to $(X\cap Y)\setminus Sub(Left(T))$. Obviously, if $J_e = Irr(T)$, then

$$(X\cap Y)\setminus Sub(Left(T)) = \emptyset.$$

So we have proved there is a 1-1 correspondence between $(X\cap Y)\setminus Sub(Left(T))$ and the set of all $\mathcal{J}$-classes of M_T except J_e.

(4). Clearly,

$$|\mathcal{D}| \geq |\mathcal{J}| = |(X\cap Y)\setminus Sub(Left(T))| + 1.$$

For $a \in Irr(T)$, let

$$a = a_1a_2,$$

where

$$a_2 \in R_e, a_1 \in X,$$

then

$$R_{a_1} = R_a \subseteq D_a, a_1 \in D_a.$$

Let

$$a_1 = a_{11}a_{12}, a_{11} \in L_e, a_{12} \in Y,$$

then

$$L_{a_{12}} = L_{a_1} \subseteq D_{a_1} = D_a.$$

So we have

$$a_{12} \in D_a, a_{12} \in X \cap Y.$$

This implies in each $\mathcal{D}$-class , there is at least one word belonging to $X \cap Y$. Hence,

$$| \mathcal{D} | \leq | X \cap Y | .$$

QED.

In section 2, we have shown $X = X^*, Y = Y^*$. So $| X |$ ($| Y |$,$| X \cap Y |$) $= 1$ or ∞. Obviously, $| \mathcal{H} | = \infty$ iff $| \mathcal{R} | = \infty$ or $| \mathcal{L} | = \infty$. By theorem 3.3 and Theorem 2.4(1), we have

Theorem 3.4 *Suppose that T is a finite special Church-Rosser Thue system on the alphabet Σ, the number of each kind of Green equivalence classes is one or infinite and it is decidable in polynomial time.*

QED.

Actually, there are two ways to decide the cardinality of $\mathcal{R}, \mathcal{L}$ and $\mathcal{J}$-classes. One is to compute the number of the words of X, Y and $(X \cap Y) \setminus Sub(Left(T))$; The other is first to decide whether $Irr(T) = R_e, L_e$ or J_e, and if this is the case, there is only one $\mathcal{R}, \mathcal{L}$ or $\mathcal{J}$-class, otherwise, it is infinite. In addition, the second way is applicable for $\mathcal{H}$-classes.

3.3 Applications

Now, we investigate the regular problem for M_T using above results, where T is finite special and Church-Rosser.

We know that in each $\mathcal{D}$-class there is at least one word of $X \cap Y$. However, all the words of $(X \cap Y) \setminus \{e\}$ are not regular elements of M_T. So there is only one regular $\mathcal{D}$-class D_e in the monoid presented by a finite special Church-Rosser Thue system. Thus, the monoid of this form is regular if and only if it contains only one $\mathcal{D}$-class. Then, we have

Theorem 3.5 *It is decidable in polynomial time whether or not a monoid presented by a finite special Church-Rosser Thue system is a regular semigroup.*

QED.

In addition, in the monoid of this type, there are non-trivial regular elements if and only if $D_e \neq \{e\}$. So it is decidable in polynomial time whether this monoid contains non-trivial regular elements. And also whether it contains non-trivial idempotents is decidable in polynomial time.

In fact, if $D_e \neq R_e$, there is at least one non-idempotent; Otherwise, let $a \in D_e$ be an idempotent, i.e. $a^2 \Rightarrow_T^* a$. Since $a \in R_e$, there exists $x \in Irr(T)$ such that $ax \Rightarrow_T^* e$, then $a^2x \Rightarrow_T^* a$. On the other hand, we also have $a^2x \Rightarrow_T^* ax \Rightarrow_T^* e$. Thus, $a = e$. So in this case, there are no non-trivial idempotents.

Theorem 3.6 *There is no zero element in the non-trivial monoid presented by a finite special Church-Rosser Thue system.*

Proof. Suppose o is the zero element of the monoid. Since o is a regular element of this monoid, we have $D_e = D_o = \{o\}$. This implies that the monoid is trivial.

QED.

obviously, by manipulating finite state acceptors, we can design computer programs to decide whether a given special monoid has above properties and the number of each kind of Green equivalences.

Now we give an example to illustrate our results.

Example 3.7 *Let $\Sigma = \{a, b\}$, $T = \{(ab, e)\}$. Obviously, M_T is bicyclic semigroup, and*

$$Irr(T) = \{b^n a^m : n, m \geq 0\},$$

$$X = \{b^n : n \geq 0\},$$

$$Y = \{a^m : m \geq 0\}.$$

Then

$$X \cap Y = \{e\}.$$

Therefore, there are infinite $\mathcal{R}, \mathcal{L}$ and $\mathcal{H}$-classes, one $\mathcal{J}$-class, and one $\mathcal{D}$-class. And M_T is a regular semigroup.

4. Discussion

We are now investigating whether our results can be generalized to other classes of monoids. If the Thue system T is more general than special ones, for example, if T is finite monadic and Church-Rosser, we can easily give examples in which Theorem 3.1 does not hold. That is to say, a word of $Irr(T)$ and $R_e(L_e, J_e)$ cannot describe a $\mathcal{R}(\mathcal{L}, \mathcal{J})$-class completely. So in this case, the structure of each kind of Green equivalence classes is more complicated.

Up till now, we only know that R_e and L_e are all regular sets for the monoids presented by such Thue systems, and two finite state acceptors for them can be constructed in polynomial time.

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