On the Assignment Complexity of Uniform Trees

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Abstract – This paper discusses the assignment complexity of the uniform tree, which is made up of identical cells realizing a function f. The assignment complexity of a tree is defined as the cardinal number of the minimum complete assignment set of the tree. When a complete assignment set is applied to the primary input lines of the tree, every internal f cell in the tree can be excited by all possible input combinations. The assignment problem is a basic problem in the VLSI system design, test and optimization. The relation between the property of f and the assignment complexity of the uniform tree is analyzed. It is shown that the assignment complexity of a balanced uniform tree with n primary input lines is either $O(1)$ or $\Omega((\lg n)^\alpha)$ ($\alpha \in (0,1]$). In the first case, the cardinal number of the minimum complete assignment set for a tree is constant and independent of the size and structure of the tree. In the second case, the assignment complexity depends on the number of the primary input lines of the tree. If a balanced uniform tree is based on a commutative function, then it is either $\Theta(1)$ or $\Theta(\lg n)$ assignable.

1 Introduction

The test is an unavoidable procedure for the VLSI system manufacture and maintenance. With the rapid development of the VLSI technology, the circuit density is increasing dramatically, and the test of VLSI circuits is becoming more and more difficult and expensive. The study for the efficient test methods is of growing importance. This paper studies the theoretical aspect of the test problem associated with tree-like structure circuits.

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Tree-like structure circuits are called trees, which are basic components in many VLSI circuits, especially in circuits for parallel and fast computations. Therefore, the study of the test complexity of trees is very useful for the design, optimization and test of VLSI systems.

The test complexity of trees based on primitive gates of type AND, OR, NAND and NOR has been extensively studied in [6, 7]. Uniform trees consisting of more complex identical nodes computing an associative or commutative function have been studied in [1,2,3,4,9,10,12]. The assignment complexity of uniform trees based on commutative functions of two variables is discussed in [11]. In this paper we further extend the above theory by studying the assignment complexity of uniform trees based on functions of several variables.

The test problem can be divided into two subproblems, namely the test pattern assignment and the diagnosis signal propagation. A complete assignment set for a circuit with n primary input lines is a set of n-component input patterns. When it is applied to the primary input lines of the circuit, every internal cell in the circuit can be excited by all possible input combinations. The construction of a complete assignment set is the first step towards the generation of a complete test set for a circuit. For I_{DDQ} test technique [5], a complete assignment set is just a complete test set for the circuit. For other test techniques the propagation of the diagnosis signal to the primary output line of the circuit must be considered additionally. In this paper we pay our attention to the *assignment* problem. The theoretical framework developed here is also useful to solve the *propagation* problem.

The assignment complexity of a tree is defined as the cardinal number of the minimum complete assignment set of the tree and is measured as a function of the number of the primary input lines in the tree. This paper consists of six sections. In the next section we give a formal definition of the assignment complexity of uniform trees and make some conventions. The third section is on the sufficient and necessary condition of $O(1)$ assignable uniform trees. The fourth section ex-

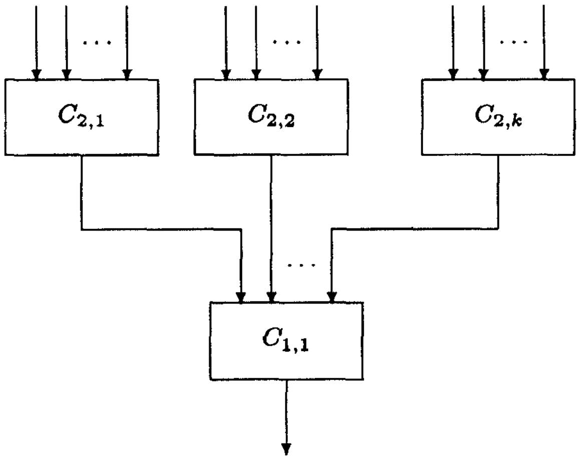

Fig. 1: A balanced tree

plores the jump of the assignment complexity from $O(1)$ to $\Omega((\lg n)^{\alpha})$. In the fifth section, we convert the assignment problem into the algebraic problem. The sixth section shows that a balanced uniform tree based on a commutative function is either $\Theta(1)$ or $\Theta(\lg n)$ assignable.

2 Assignment Complexity of Uniform Trees

Let M be a set of m symbols, and $f : M^k \longrightarrow M$ a surjective function. Without loss of generality we assume $M = \{1, 2, ..., m\}$. We use the symbol f to represent a function as well as a cell implementing the function. A uniform f-tree is made up of identical cells implementing the function f. The set of all f-trees is denoted by T_f. $T_f^{(n)}$ is used to denote a balanced uniform f-tree with n primary input lines. Fig. 1 shows a balanced tree. If every cell $C_{i,j}$ realizes the same function $f : M^k \longrightarrow M$, then it is a uniform tree.

We assign every line and cell in $T_f^{(n)}$ a unique level. The levels is arranged in ascending order from the primary output line to the primary input lines of $T_f^{(n)}$. The primary output line is assigned level 0. An f cell and all its input lines are assigned level $k+1$, if its output line is in level k. A tree is said to be of k-level, if it has k levels.

For the sake of convenience, we make some conventions. Throughout this paper, $\{a, a, a\}$ and $\{a, a\}$ are recognized as two different *multiple* sets. The cardinal number of the former is three, and that of the latter is two. A multiple set can be changed into a conventional set by using operator $\top$. For example, $\top\{a, a, a\} = \top\{a, a\} = \{a\}$ and $\top\{b, c, b\} = \{b, c\}$. For a multiple set A, $\#A$ represents the number of the elements in A. For example, $\#\{a, a, a\} = 3$.

Let

$$(\vec{I_1}, \vec{I_2}, \ldots, \vec{I_k}) := \begin{pmatrix} I_{11}, I_{12}, ..., I_{1k} \\ I_{21}, I_{22}, ..., I_{2k} \\ \vdots \\ I_{t1}, I_{t2}, ..., I_{tk} \end{pmatrix}$$

for $\vec{I_j} = (I_{1j}, I_{2j}, ..., I_{tj})^T, j \in [1, k], \ I_{ij} \in M$.

Based on function f we define a vector function $\tilde{f}$ as follows:

$$\tilde{f}(\vec{I_1}, \vec{I_2}, \ldots, \vec{I_k}) := \begin{pmatrix} f(I_{11}, I_{12}, ..., I_{1k}) \\ f(I_{21}, I_{22}, ..., I_{2k}) \\ \vdots \\ f(I_{t1}, I_{t2}, ..., I_{tk}) \end{pmatrix}$$

It is easy to see that applying t k-component patterns to an f cell is equal to assigning k t-dimension vectors to the k input lines of the f cell.

Let $\mathcal{D}_t = \{(x_1, ..., x_t)^T \mid x_i \in M\}$, namely, the set of all t-dimension vectors ($t \in \mathbf{N}$). Given k vectors in $\mathcal{D}_t$, using operator ∇ one can construct a set of t k-component patterns, and (1) is the formal definition of this operator.

$$\nabla(\vec{I_1}, \cdots, \vec{I_k}) := \{(I_{i1}, \cdots, I_{ik}) \mid i \in [1, t]\}, \ \ \vec{I_j} \in \mathcal{D}_t \quad (1)$$

Example 2.1: Function f_1 is defined as follows.

f_1	0	1
0	1	1
1	1	0

Let

$$\vec{I_0} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \ \vec{I_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ \vec{I_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Then

$$\tilde{f}\left(\vec{I_1}, \vec{I_2}\right) = \begin{pmatrix} f(1,1) \\ f(1,1) \\ f(1,0) \\ f(0,1) \\ f(0,0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \vec{I_0}$$

and

$$\nabla\left(\vec{I_1}, \vec{I_2}\right) = \{(1,1), (1,1), (1,0), (0,1), (0,0)\}.$$

Assume that a $T_f^{(n)}$ consists of cells $C_{1,1}, C_{2,1}, C_{2,2}, ..., C_{k,l}, ...,$ and cell $C_{i,j}$ is the jth cell in the ith level

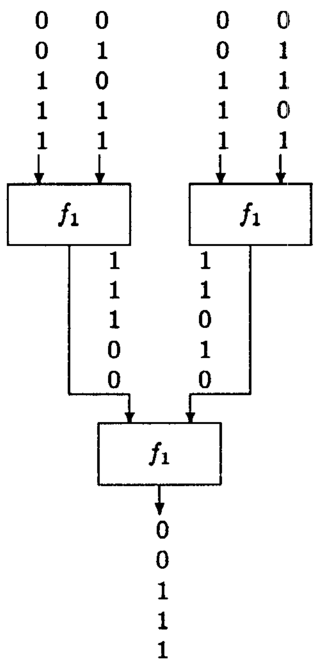

Fig. 2: Complete assignment for $T_{f_1}^{(4)}$

of $T_f^{(n)}$. Let A be a set of n-component patterns, and $\#A = t$. When all of the patterns in A are applied to the primary input lines of $T_f^{(n)}$, a t-component vector is delivered to every line in $T_f^{(n)}$. To apply all of the t patterns to the n primary input lines of $T_f^{(n)}$ is the same as to apply n t-component vectors to the n primary input lines. We use $\vec{I}_l(A,i,j)$ to denote the corresponding vector applied to the lth input line of an f cell $C_{i,j}$, and $\vec{I}_0(A,i,j)$ to denote the vector delivered to the output line of the cell. Then $\vec{I}_0(A,i,j) = \tilde{f}(\vec{I}_1(A,i,j), \cdots, \vec{I}_k(A,i,j))$.

In order to classify the assignment complexity of uniform trees we give a formal definition of the complete assignment and the assignment complexity.

Definition 1 $(\vec{I}_1(A,i,j), \ldots, \vec{I}_k(A,i,j))$ *is a complete assignment of cell* $C_{i,j}$ *if and only if*

$$M^k \subset \nabla(\vec{I}_1(A,i,j), \cdots, \vec{I}_k(A,i,j)).$$

A is a complete assignment set of $T_f^{(n)}$ *if and only if* $(\vec{I}_1(A,i,j), \cdots, \vec{I}_k(A,i,j))$ *is a complete assignment for every cell* $C_{i,j}$ *in* $T_f^{(n)}$.

Fig. 2 shows a complete assignment set to $T_{f_1}^{(4)}$. By assigning patterns $(1,1,1,1)$, $(1,1,1,0)$, $(1,0,1,1)$, $(0,1,0,1)$ and $(0,0,0,0)$ to the four primary input lines of $T_{f_1}^{(4)}$, one can guarantee that each of $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$ can be applied to every cell in $T_{f_1}^{(4)}$. Thus we can state that the five patterns comprise a complete assignment set for $T_{f_1}^{(4)}$.

It is obvious that f has to be surjective to M, otherwise, it is impossible to construct a complete assignment set for a tree system $T_f^{(n)}$ $(n > 2)$.

Definition 2 *The assignment complexity of* $T_f^{(n)}$ *is defined by the mapping* $AC_f : T_f \longrightarrow \mathbf{N}$.

$$AC_f(T_f^{(n)}) = \min\left\{\#A \;\middle|\; \begin{array}{c} A \text{ is a complete} \\ \text{assignment set for } T_f^{(n)} \end{array}\right\}$$

In case $AC_f(T_f^{(n)}) = \Theta(1)$, we say $T_f^{(n)}$ to be $\Theta(1)$ assignable. In a tree, all cells at the same level can be assigned simultaneously since their input lines are independent of each other. Furthermore, all cells at the same level can be excited completely by using m^k patterns. A straightforward conclusion is that all cells in $T_f^{(n)}$ can be excited completely by using $m^k\lceil\lg n\rceil$ patterns since $T_f^{(n)}$ has at most $\lceil\lg n\rceil$ levels. Thus we have the following observation.

Observation 1 *For an arbitrary surjective function* $f : M^k \longrightarrow M$

$$\begin{aligned} AC_f(T_f^{(n)}) &\leq m^k\lceil\lg n\rceil \\ &= O(\lg n) \end{aligned} \tag{2}$$

3 $\Theta(1)$ Assignable Trees

In this section we discuss the criteria of $\Theta(1)$ assignable uniform trees.

Lemma 1 $T_f^{(n)}$ *is* $\Theta(1)$ *assignable if there are a* $t \in \mathbf{N}$ *and a set* $W \subset \mathcal{D}_t$ *so that every* $\vec{I}_0 \in W$ *can be generated by using* k *vectors* $\vec{I}_1, \cdots, \vec{I}_k$ *which belong to* W *and comprise a complete assignment to an* f *cell. Put it formally, there are a* $t \in \mathbf{N}$ *and a set* $W \subset \mathcal{D}_t$ *so that*

$$\forall \vec{I}_0 \in W \exists \vec{I}_1, ..., \vec{I}_k \in W \left\{ \begin{array}{c} M^k \subset \nabla(\vec{I}_1, ..., \vec{I}_k) \\ \tilde{f}(\vec{I}_1, ..., \vec{I}_k) = \vec{I}_0 \end{array} \right\} \tag{3}$$

Proof: We prove that for every N-level $T_f^{(n)}$ we can construct a complete assignment set $A^{(N)}$ by assigning to every primary input line a vector in $W \in \mathcal{D}_t$. Then $\#A^{(N)}$ is equal to the constant t. This can be proven through induction on the number of the level of the tree.

In case $N = 1$, the tree has only one cell. We choose arbitrarily an $\vec{I}_0 \in W$, then determine k vectors $\vec{I}_1, \cdots, \vec{I}_k \in W$ so that $M^k \subset \nabla(\vec{I}_1, \cdots, \vec{I}_k)$ and

$\vec{f}(\vec{I_1}, \cdots, \vec{I_k}) = \vec{I_0}$. It is clear that $\nabla(\vec{I_1}, \cdots, \vec{I_k})$ is a complete assignment to a tree with only one cell.

Assume that for $N = i$ one can construct a complete assignment set $A^{(i)}$ for an i-level $T_f^{(n)}$ by assigning to every primary input line of $T_f^{(n)}$ a vector in W, and the vector assigned to the jth primary input line is $\vec{I}_{j,0} \in W$. Suppose $T_f^{(kn)}$ is of $(i+1)$ levels and is constructed by connecting every primary input line in $T_f^{(n)}$ to the output line of an f cell. According to the assumption, there are $\vec{I}_{j,1}, \cdots, \vec{I}_{j,k} \in W$ so that

$$M^k \subset \nabla(\vec{I}_{j,1}, \cdots, \vec{I}_{j,k}) \ \wedge \ \vec{f}(\vec{I}_{j,1}, \cdots, \vec{I}_{j,k}) = \vec{I}_{j,0}.$$

Hence $(\vec{I}_{j,1}, \cdots, \vec{I}_{j,k})$ is a complete assignment to an f cell. When $\vec{I}_{j,1}, \cdots \vec{I}_{j,k}$ are applied to the k input lines of the cell directly linked to the jth input line in the level i, the vector offered to this input line is just $\vec{I}_{j,0}$. Thus we can construct a complete assignment to every cell in level $(i+1)$ by assigning to every primary input line in $T_f^{(kn)}$ a vector in W, and all of the vectors delivered to the lines in level i comprise $A^{(i)}$, which is a complete assignment set to $T_f^{(n)}$ as assumed. All of the vectors assigned to the lines in level $(i+1)$ comprise the $A^{(i+1)}$ which is a complete assignment set to $T_f^{(kn)}$. Thus we can conclude that $\#A^{(i)} = \#A^{(i+1)}$, and $T_f^{(n)}$ is $\Theta(1)$ assignable.

Q.E.D.

For $\vec{I_1}, \cdots, \vec{I_k} \in \mathcal{D}_t$, we regard $\left(\vec{I_1}, \cdots, \vec{I_k}\right)$ as a $t \times k$ matrix, and $\vec{I} \in \mathcal{D}_t$ as a $1 \times k$ matrix.

Definition 3 *Matrices* $\left(\vec{I_1}, \cdots, \vec{I_k}\right)$ *and* $\left(\vec{I}_{i_1}, \cdots, \vec{I}_{i_k}\right)$ *are said to be* similar *to each other, denoted by*

$$\left(\vec{I_1}, \cdots, \vec{I_k}\right) \sim \left(\vec{I}_{i_1}, \cdots, \vec{I}_{i_k}\right),$$

if and only if the former can be changed to the latter by using row exchanges.

For example,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

It is easy to see that for three arbitrary matrices A_i, A_j, A_l, the following three statements hold.

1. $A_i \sim A_i$;
2. $A_i \sim A_j \Longrightarrow A_j \sim A_i$;
3. $A_i \sim A_j \wedge A_j \sim A_l \Longrightarrow A_i \sim A_l$.

Hence $\sim$ is an equivalence relation.

In order to simplify the expression we will use $\mathbf{P}_f(\vec{I_0}, \vec{I_1}, \cdots, \vec{I_k})$ to stand for

$$M^k \subset \nabla(\vec{I_1}, \cdots, \vec{I_k}) \ \wedge \ \vec{f}(\vec{I_1}, \cdots, \vec{I_k}) \sim \vec{I_0}.$$

Corollary 1 $T_f^{(n)}$ *is* $\Theta(1)$ *assignable if there are a* $t \in \mathbf{N}$ *and a set* $W' \subset \mathcal{D}_t$ *so that for every* $\vec{I_0} \in W'$ *there are* $\vec{I_1}, \cdots, \vec{I_k} \in W'$, *and they comprise a complete assignment and can be transferred into a vector* similar *to* $\vec{I_0}$. *Put it formally, there are a* $t \in \mathbf{N}$ *and a set* $W' \subset \mathcal{D}_t$ *so that*

$$\forall \vec{I_0} \in W' \exists \vec{I_1}, \cdots, \vec{I_k} \in W' \left\{ \mathbf{P}_f(\vec{I_0}, \vec{I_1}, \cdots, \vec{I_k}) \right\} \qquad (4)$$

Proof: Given a set $W' \subset \mathcal{D}_t$ $(t \in \mathbf{N})$, we can always induce a set W so that

$$\forall \vec{I} \in \mathcal{D}_t \left\{ \exists \vec{I'} \in W' \left\{ \vec{I'} \sim \vec{I} \right\} \Longrightarrow \vec{I} \in W \right\}.$$

The set W includes every vector which is *similar* to a vector in W'. It is obvious that W satisfies (3) if W' fulfills (4).

Q.E.D

As mentioned, applying t n-component patterns to the n primary input lines of $T_f^{(n)}$ is equal to applying n t-dimension vectors to the n primary input lines respectively.

Apply a complete assignment A to $T_f^{(n)}$. Let W be the set of the corresponding vectors applied to the primary input lines and the vectors delivered to other lines in other levels of $T_f^{(n)}$. Set W can be partitioned into a number of equivalence classes based on the equivalence relation $\sim$. It is not hard to see that the larger the number of the equivalence classes in W, the greater the dimension t of the vectors in W. The dimension t is just the cardinality of A. In the following we explore the relation between the cardinality of A and the number of the equivalence classes in W.

Given a complete assignment A to an N-level $T_f^{(n)}$, we construct a set in the following way:

$$W_s(A) := \top \left\{ \vec{I_l}(A, i, j) | i \in [1, s], j \in [1, k^{i-1}], l \in [0, k] \right\}$$

for every $s \in [1, N]$.

$W_s(A)$ includes all vectors delivered to a line in level i $(i \in [1, s])$ and the vector delivered to the primary output line. Partition $W_s(A)$ into equivalence classes according to the equivalence relation $\sim$, and let $\#W_s(A)/_\sim$ denote the number of equivalence classes in $W_s(A)$. Observation 2 is obvious.

Observation 2 *Assume A to be a complete assignment set for an N-level $T_f^{(n)}$. Then*

1. $\forall s \in [2,N]\{W_{s-1}(A) \subseteq W_s(A) \subseteq \mathcal{D}_t\}$;
2. $\forall s \in [2,N]\{1 \leq \#W_{s-1}(A)/_\sim \leq \#W_s(A)/_\sim\}$.

Lemma 2 *Assume A to be a complete assignment set for an N-level $T_f^{(n)}$. Then $T_f^{(n)}$ is $\Theta(1)$ assignable if*

$$\#W_1(A)/_\sim = 1$$

or

$$\exists s \in [2,N]\{\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim\} .$$

Proof: Assume A to be a complete assignment set for an N-level $T_f^{(n)}$. In case $\#W_1(A)/_\sim = 1$, $W_1(A)$ includes only one equivalence class, and every two vectors in $W_1(A)$ are *similar* to each other. Suppose $W_1(A) = \{\vec{I_0}, \vec{I_1}, \cdots, \vec{I_k}\}$, and $\vec{I_1}, \cdots, \vec{I_k}$ are the corresponding vectors applied to the k input lines of the f cell in the first level, and $\vec{I_0}$ is the vector delivered to the output line. $\#W_1(A)_\sim = 1$ means that

$$\forall l \in [1,k]\left\{\vec{I_l} \sim \vec{I_0}\right\}$$

and

$$M^k \subset \nabla(\vec{I_1}, \cdots, \vec{I_k}) \wedge \vec{f}(\vec{I_1}, \cdots, \vec{I_k}) = \vec{I_0}.$$

This implies that

$$\forall \vec{I_0^l} \in W_1(A) \exists \vec{I_1^l}, \cdots, \vec{I_k^l} \in W_1(A)\left\{\mathbf{P}_f(\vec{I_0^l}, \vec{I_1^l}, \cdots, \vec{I_k^l})\right\}.$$

Thus $T_f^{(n)}$ is $\Theta(1)$ assignable according to Corollary 1.

Suppose $\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim$ for an $s \in [2,N]$. As mentioned

$$\forall s[2,N]\{W_{s-1}(A) \subseteq W_s(A)\} .$$

This means that $W_s(A)$ and $W_{s-1}(A)$ have the same number of equivalence classes. It is not hard to see that

$$\forall \vec{I_0} \in W_s(A) \exists \vec{I_1}, \cdots, \vec{I_k} \in W_s(A)\left\{\mathbf{P}_f(\vec{I_0}, \vec{I_1}, \cdots, \vec{I_k})\right\}.$$

Based on Corollary 1, $T_f^{(n)}$ is $\Theta(1)$ assignable.

Q.E.D.

Lemma 3 *Assume A to be a complete assignment set for an N-level $T_f^{(n)}$ $(N > 1)$, then*

$$\forall s \in [2,N]\{W_{s-1}(A) \subsetneq W_s(A)\} \tag{5}$$

if $T_f^{(n)}$ is not $\Theta(1)$ assignable.

Proof: Suppose $T_f^{(n)}$ is not $\Theta(1)$ assignable, and A is a complete assignment set of an N-level $T_f^{(n)}$. As mentioned, $W_{s-1}(A) \subset W_s(A)$ for all $s \in [2,N]$. If $W_s(A) = W_{s-1}(A)$ for an $s \in [2,N]$, then

$$\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim.$$

According to Lemma 2, $T_f^{(n)}$ is $\Theta(1)$ assignable. This contradicts the assumption directly.

Q.E.D.

Lemma 4 *For every complete assignment set A for an N-level $T_f^{(n)}$*

$$\forall s \in [2,N]\{\#W_s(A)/_\sim > s\} \tag{6}$$

if $T_f^{(n)}$ is not $\Theta(1)$ assignable.

Proof: Suppose $T_f^{(n)}$ is not $\Theta(1)$ assignable. According to Lemma 2,

$$\#W_1(A)/_\sim > 1$$

and

$$\forall s \in [2,N]\{\#W_s(A)/_\sim > \#W_{s-1}(A)/_\sim\} .$$

Therefore, $\#W_s(A)/_\sim > s$.

Q.E.D.

In order to prove Theorem 1 and 2, we define $P(M,t)$ as a set of vectors in the following form.

$$(\underbrace{1, \cdots, 1}_{t_1}, \underbrace{2, \cdots, 2}_{t_2}, ..., \underbrace{m, \cdots, m}_{t_m})^T, \sum_{1 \leq i \leq m} t_i = t \tag{7}$$

It is easy to see that every t-component vector in $\mathcal{D}_t$ is *similar* to a vector in $P(M,t)$.

Observation 3 *For every complete assignment set A $(\#A = t)$ to an N-level $T_f^{(n)}$*

$$\forall \vec{I'} \in W_N(A) \exists \vec{I} \in P(M,t)\left\{\vec{I'} \sim \vec{I}\right\}$$

and

$$\#W_N(A)/_\sim \leq \#P(M,t).$$

Theorem 1 *$T_f^{(n)}$ is $\Theta(1)$ assignable if and only if there is a $t \in \mathbf{N}$ and exists a $W \subset \mathcal{D}_t$ so that*

$$\forall \vec{I_0} \in W \exists \vec{I_1}, \cdots, \vec{I_k} \in W\left\{\mathbf{P}_f(\vec{I_0}, \vec{I_1}, \cdots, \vec{I_k})\right\}.$$

Proof: The *if* part follows from Corollary 1 directly. Assume $T_f^{(n)}$ to be $\Theta(1)$ assignable. Then there is a constant $t \in \mathbf{N}$, and one can construct a complete assignment set A of t patterns for an arbitrary $T_f^{(n)}$. Suppose $\lg n = N$ and $N > \#P(M,t)$. Since

$$\forall s \in [1,N]\{1 \leq \#W_s(A)/_\sim \leq \#P(M,t) < N\},$$

there must be such an $s \in [2,N]$ that

$$\#W_s(A)/_\sim = \#W_{s-1}(A)/_\sim.$$

Thus we can state that there is such an $s \in [2,N]$ that

$$\forall \vec{I_0} \in W_s(A) \exists \vec{I_1}, \cdots, \vec{I_k} \in W_s(A) \left\{ \mathbf{P}_f(\vec{I_0}, \vec{I_1}, \cdots, \vec{I_k}) \right\}.$$

We have the Theorem.

Q.E.D

4 Jump from $\Theta(1)$ to $\Omega((\lg n)^{\frac{1}{m-1}})$

In this section we show that the assignment complexity of a $T_f^{(n)}$ is either $\Theta(1)$ or $\Omega((\lg n)^{\frac{1}{m-1}})$, In other words, there is a jump from $\Theta(1)$ to $\Omega((\lg n)^{\frac{1}{m-1}})$.

Coming up next we explore the upper boundary of $\#P(M,t)$.

Lemma 5 *For $\#M = m$,*

$$\#P(M,t) = \binom{t+m-1}{m-1} \tag{8}$$

Proof: Let $p(\#M,t)$ denote $\#P(M,t)$. We prove this lemma by induction on m, which is the cardinal number of M.

For $m = 1$, $p(1,t) = \binom{t}{0}$ for every $t \in \mathbf{N}$. Suppose $p(m,t) = \binom{t+m-1}{m-1}$ for $m \leq i$ and every $t \in \mathbf{N}$. For $m = i+1$, based on the inductive assumption in the last step.

$$\begin{aligned} p(m,t) &= p(i+1,t) \\ &= \sum_{0 \leq j \leq t} p(i, t-j) \\ &= \sum_{0 \leq j \leq t} \binom{t-j+i-1}{i-1} \\ &= \sum_{i-1 \leq j \leq t+i-1} \binom{j}{i-1} \\ &= \binom{t+i}{i} \\ &= \binom{t+m-1}{m-1}. \end{aligned}$$

Q.E.D.

Theorem 2 *$T_f^{(n)}$ is either $\Theta(1)$ or $\Omega((\lg n)^{\frac{1}{m-1}})$ assignable.*

Proof: Suppose $T_f^{(n)}$ is not $\Theta(1)$ assignable. It suffices to show that $\#A = \Omega((\lg n)^{\frac{1}{m-1}})$ for every complete assignment set A of $T_f^{(n)}$.

Assume $T_f^{(n)}$ to be of N levels. According to Lemma 4 and Observation 3.

$$N < \#W_N(A)/_\sim \leq \#P(M,t) \tag{9}$$

for every complete assignment set A. Based on Lemma 5,

$$N < \binom{t+m-1}{m-1}.$$

Then $t \geq N^{\frac{1}{m-1}} - m$ for $m > 1$. We know $N = \lceil \lg n \rceil$. Thus we can conclude that

$$\begin{aligned} t &= \Omega(N^{\frac{1}{m-1}}) \\ &= \Omega((\lg n)^{\frac{1}{m-1}}). \end{aligned}$$

Q.E.D.

The parameter m in Theorem 2 is the cardinality of M. For $M = \{0,1\}$, the parameter m is 2. The following lemma is immediate from Observation 1 and Theorem 2.

Corollary 2 *Assume f to be a surjective function from $\{0,1\}^k$ to $\{0,1\}$. Then $T_f^{(n)}$ is either $\Theta(1)$ or $\Theta(\lg n)$ assignable.*

5 Problem Conversion

Theorem 1 gives a criterion of judging $\Theta(1)$ assignable uniform trees. And Theorem 2 explores the structure of assignment complexity. In this section, we give a new criterion for deciding the assignment complexity and convert the assignment problem of balanced uniform trees into the algebraic problem for exploring its aspects further.

We use $\vec{0}$ to denote the all-zero vector and $\vec{1}$ the all-one vector. Assume that L is a matrix, and $\vec{x}$, $\vec{b}$, $\vec{y}$, and $\vec{c}$ are vectors. When notations like

$$L\vec{x} = \vec{b}, \quad \vec{y}L = \vec{c}$$

are used, we implicitly assume that the compatibility of sizes and forms of L, $\vec{x}$, $\vec{y}$, and $\vec{c}$. If L is an $m \times n$ matrix, then $\vec{x}$ is a column vector with n components, $\vec{b}$ is a column vector with m components, $\vec{y}$ is a row vector of dimension m, and $\vec{c}$ is a row vector of dimension n.

For $\vec{x} = (x_1, \cdots, x_n)$ and $\vec{y} = (y_1, \cdots, y_n)$, we define

$$\begin{aligned} \vec{x} \geq \vec{y} &\iff \forall i \in [1,n]\, \{x_i \geq y_i\} \\ \vec{x} > \vec{y} &\iff \vec{x} \geq \vec{y} \wedge \exists i \in [1,n]\, \{x_i > y_i\} \end{aligned}$$

Assume that $s = \#M^k$ and $P_j = (p_j^{(1)}, ..., p_j^{(k)})$ denotes the jth element of M^k. Let Π_l ($l \in [1,k]$) be a projection of the lth component of P_j. For instance, $\Pi_l(P_j) = p_j^{(l)}$. Parameters $b_{ij}^{(l)}$ for $i \in [1,m]$, $j \in [1,s]$, and $l \in [0,k]$ are defined as following.

$$\begin{aligned} b_{ij}^{(0)} &= \begin{cases} 1 & : \ f(P_j) = i \\ 0 & : \ \text{otherwise} \end{cases} \\ b_{ij}^{(l)} &= \begin{cases} 1 & : \ \Pi_l(P_j) = i \\ 0 & : \ \text{otherwise} \end{cases} . \end{aligned}$$

By using the above parameters we construct $k+1$ matrices $B^{(l)} = \left(b_{ij}^{(l)}\right)_{m \times s}$ ($l \in [0,k]$). It is obvious that every column of these matrices has only one nonzero element. Given an $i \in M$, M^k includes m^{k-1} elements P_j satisfying $\Pi_l(P_j) = i$. Hence, every row of $B^{(l)}$ ($l \in [1,k]$) has m^{k-1} nonzero components.

Before giving the new criterion for deciding the assignment complexity of uniform trees, we define two mappings

$$\begin{aligned} G &: \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{k} \longrightarrow \underbrace{\mathbf{N}_0 \times \cdots \times \mathbf{N}_0}_{s} \\ \mathcal{G} &: \underbrace{\mathbf{N}_0 \times \cdots \times \mathbf{N}_0}_{s} \longrightarrow \underbrace{\mathcal{D} \times \cdots \times \mathcal{D}}_{k} \end{aligned}$$

For $\vec{I}_i \in \mathcal{D}_t (i \in [1,k])$, $G(\vec{I}_1, \cdots, \vec{I}_k) = (x_1, \cdots, x_s)^T$ if

$$(\vec{I}_1, \cdots, \vec{I}_k) \sim (\underbrace{P_1, \cdots, P_1}_{x_1}, \cdots, \underbrace{P_s, \cdots, P_s}_{x_s})^T.$$

Given $x_i \in \mathbf{N}_0 (i = 1, ..., s)$,

$$\mathcal{G}(x_1, \cdots x_s) = (\underbrace{P_1, \cdots, P_1}_{x_1}, \cdots, \underbrace{P_s, \cdots, P_s}_{x_s})^T.$$

Assume $\vec{I}_0 = \vec{f}(\vec{I}_1, \cdots, \vec{I}_k)$ for $(\vec{I}_1, \cdots \vec{I}_k) \in \mathcal{D}_t^k$. We call $B^{(l)} G(\vec{I}_1, \cdots, \vec{I}_k)$ *characteristic vector* of $\vec{I}_l$ ($l \in [0,k]$), and use $Ch(\vec{I}_l)$ to denote it. Vector $\vec{I}_l$ belongs to $\mathcal{D}_t$, and its characteristic vector $Ch(\vec{I}_l)$ belongs to $\mathbf{N}_0^m$. We have the following observation.

Observation 4 *Given* $(\vec{I}_1, \cdots, \vec{I}_k) \in \mathcal{D}_t^k$. *If*

$$\begin{aligned} Ch(\vec{I}_l) &= B^{(l)} G(\vec{I}_1, \cdots, \vec{I}_k) \\ &= \left(c_1^{(l)}, \cdots, c_m^{(l)}\right)^T \end{aligned}$$

for all $l \in [0,k]$, *then*

$$\forall l \in [0,k] \left\{ \vec{I}_l \sim (\underbrace{1, \cdots, 1}_{c_1^{(l)}}, \cdots, \underbrace{m, \cdots, m}_{c_m^{(l)}})^{\mathrm{T}} \right\} \qquad (10)$$

Theorem 3 $T_f^{(n)}$ *is* $\Theta(1)$ *assignable if and only if there is a finite set* $X = \{\vec{X}_i \mid \vec{X}_i \in \mathbf{N}^s\}$ *so that*

$$\forall l \in [1,k]\, \{S_l(X) \subset S_0(X)\} \qquad (11)$$

where

$$S_l(X) \ := \ \{B^{(l)} \vec{X}_i \mid \vec{X}_i \in X\}, \ \ l \in [0,k] \qquad (12)$$

Proof: We prove the *only if* part at first. Assume $T_f^{(n)}$ to be $\Theta(1)$ assignable. According to Theorem 1, there are a $t \in \mathbf{N}$ and a set $W \subset \mathcal{D}_t$ so that

$$\forall \vec{I}_0 \in W \exists \vec{I}_1, \cdots, \vec{I}_k \in W \left\{ \mathbf{P}_f(\vec{I}_0, \vec{I}_1, \cdots, \vec{I}_k) \right\}.$$

We construct such a complete assignment set for the $T_f^{(n)}$ that every $\vec{I}_0 \in W$ is the output vector of a cell in $T_f^{(n)}$, and the input vector of another cell as well. Let X be the smallest set of vectors that includes every $G(\vec{I}_1, \cdots, \vec{I}_k)$ if $(\vec{I}_1, \cdots, \vec{I}_k)$ is the complete assignment for an f cell in $T_f^{(n)}$. Then $S_l(X) \subset S_0(X)$ for every $l \in [1,k]$.

Now we prove the *if* part. Suppose there is a finite set $X \subset \mathbf{N}^s$ and $S_l(X) \subset S_0(X)$ for every $l \in [1,k]$. Let $W' = \left\{ \vec{f}(\mathcal{G}(\vec{X}_i)) \mid \vec{X}_i \in X \right\}$. It is easy to show that for every $\vec{I}_0 \in W'$ there are $\vec{I}_1, \cdots, \vec{I}_k \in W'$ and exist $\vec{I}'_1, \cdots, \vec{I}'_k \in \mathcal{D}_t$ so that

$$\forall l \in [1,k] \{\vec{I}_l \sim \vec{I}'_l\} \ \wedge \ \mathbf{P}_f(\vec{I}_0, \vec{I}'_1, \cdots, \vec{I}'_k)$$

Let W be the smallest set so that

$$\forall \vec{I} \in \mathcal{D}_t \left\{ \exists \vec{I}' \in W' \left\{ \vec{I}' \sim \vec{I} \right\} \Longrightarrow \vec{I} \in W \right\}.$$

Then

$$\forall \vec{I}_0 \in W \exists \vec{I}_1, \cdots, \vec{I}_k \in W \left\{ \mathbf{P}_f(\vec{I}_0, \vec{I}_1, \cdots, \vec{I}_k) \right\}.$$

Based on Theorem 1, $T_f^{(n)}$ is $\Theta(1)$ testable.

Q.E.D.

Corollary 3 $T_f^{(n)}$ *is not* $\Theta(1)$ *assignable if*

$$B^{(0)} \vec{y} = B^{(l)} \vec{y}, \quad \vec{y} \geq \vec{1} \qquad (13)$$

has no feasible solution for an $l \in [1,k]$.

Proof: Assume $T_f^{(n)}$ to be $\Theta(1)$ assignable. According to Theorem 3 there is such a set $X \subset \mathbf{N}^s$ that (11) holds. We show that if (14) holds for an $l \in [1,k]$, then (13) has a feasible solution for the given l.

$$S_l(X) \quad \subset \quad S_0(X) \tag{14}$$

In case $\#X = 1$ and $X = \{\vec{y}\}$, $\vec{y}$ is just a feasible solution of (13). Assume that in case $\#X = N$, (13) has a feasible solution if (14) holds for the given $l \in [1,k]$. For $\#X = N+1$, there are three cases to be considered.

Case 1, $\#S_l(X) = \#S_0(X) = \#X$.

Case 2, $\#S_l(X) = \#S_0(X) < \#X$.

Case 3, $\#S_l(X) < \#S_0(X)$.

For the first case, $\#S_l(X) = \#S_0(X) = \#X$ and $S_l(X) = S_0(X)$, then $\vec{y} = \sum_{\vec{X}_i \in X} \vec{X}_i$ is a solution of (13).

For the second case, there must be $\vec{X}_i, \vec{X}_j \in X$ so that $\vec{X}_i \neq \vec{X}_j$ and $B^{(0)}\vec{X}_i = B^{(0)}\vec{X}_j$. Thus $X \setminus \vec{X}_i$ satisfies (14) also, and its cardinal number is N. This implies that (13) has a feasible solution.

For the third case, X must include such an $\vec{X}_i$ that $B^{(0)}\vec{X}_i \notin S_l(X)$. This indicates that $X \setminus \vec{X}_i$ satisfies (14) for the given l, and its cardinal number is N.

Q.E.D

In the rest of this section, we present two basic theorems in linear programming. They will be used in the next section.

Theorem 4 (Farkas' Lemma) *Assume A to be an $s \times t$ matrix.*

$$A\vec{x} = \vec{b}, \qquad \forall j \in [1,t]\{x_j \geq 0\} \tag{15}$$

has feasible solutions if and only if

$$\forall \vec{y} \in \mathbf{R}^s \left\{ \vec{y}A \geq \vec{0} \Longrightarrow \vec{y}\vec{b} \geq 0 \right\} \tag{16}$$

The proof of **Farkas' Lemma** can be found almost in every linear programming book.

Theorem 5 *If*

$$A\vec{x} = \vec{0}, \qquad \forall j \in [1,t]\{x_j \geq 1\} \tag{17}$$

has a feasible solution, then it has feasible integer solutions, provided that the terms of the constraint matrix A are all integers.

Proof: Assume that A is an $s \times t$ integer matrix and its rank is r. For $r \leq s$, we can determine an $r \times t$ matrix A' including r independent rows. Then (17) and (18).

$$A'\vec{x} = \vec{0}, \qquad \forall j \in [1,t]\{x_j \geq 1\} \tag{18}$$

have the same solution space.

It is obvious that (18) has a feasible solution if and only if

$$A'\vec{x} = -A'\vec{1}, \qquad \forall j \in [1,t]\{x_j \geq 0\} \tag{19}$$

has a feasible solution.

Suppose that $B = (b_{ij})_{r \times r}$ is a nonsingular submatrix of A'. Without loss of generality, assume that B includes the first r columns of A'. Thus

$$x_i = \begin{cases} \text{the } i\text{th component of } -B^{-1}A'\vec{1} & : \ i \leq r \\ 0 & : \ i > r \end{cases}$$

define a basic solution of (19). It is clear that so defined basic solution is a rational solution.

It has been proven that at least one of its basic solutions is feasible if (19) has a feasible solution [8]. It implies that (19) has a feasible rational solution if and only if it has a feasible solution. Based on the relationship between (19) and (18), $\vec{x}$ is a feasible solution of (19) if and only if $\vec{x} + \vec{1}$ is a feasible solution of (18).

Given a feasible rational solution of (18), we can always construct a feasible integer solution since (18) is a homogeneous linear equation system.

Q.E.D

6 Commutative Trees

The $k+1$ matrices $B^{(l)}$ ($l \in [0,k]$) defined in section 5 are determined completely by the function definition of f. In this section we use B to denote matrix

$$\begin{bmatrix} B^{(0)} - B^{(1)} \\ B^{(0)} - B^{(2)} \\ \vdots \\ B^{(0)} - B^{(k)} \end{bmatrix}$$

For commutative function f we have the following result.

Theorem 6 *Assume surjective function $f : M^k \longrightarrow M$ to be commutative. Then $T_f^{(n)}$ is $\Theta(1)$ assignable if and only if*

$$B\vec{y} = -B\vec{1}, \qquad \vec{y} \geq \vec{0} \tag{20}$$

has a feasible solution.

Proof: We prove the *if* part at first. Suppose (20) has a feasible solution. This means that

$$B\vec{y} = \vec{0}, \qquad \vec{y} \geq \vec{1} \tag{21}$$

has a feasible solution. Furthermore, it has a feasible integer solution according to Theorem 5. Suppose $\vec{y} \in \mathbf{N}^s$ is a feasible integer solution of (21). Let $X = \{\vec{y}\}$. Then

$S_l(X) \subset S_0(X)$ for all $l \in [1,k]$. According to Theorem 3, $T_f^{(n)}$ is $\Theta(1)$ assignable.

Now we turn to the proof of the *only if* part. Based on Farkas' Lemma, (20) has a feasible solution if and only if

$$\forall \vec{z} \in \mathbf{R}^{km} \left\{ \vec{z}B \geq \vec{\mathbf{0}} \Longrightarrow -\vec{z}B\vec{\mathbf{1}} \geq 0 \right\} \tag{22}$$

Suppose (20) has no feasible solution. This means that

$$\exists \vec{z} \in \mathbf{R}^{km} \left\{ \vec{z}B > \vec{\mathbf{0}} \right\} \tag{23}$$

Thus we can choose a $\vec{z}$ so that for every $\vec{y} \geq \vec{\mathbf{1}}$

$$\vec{z}B\vec{y} > k \tag{24}$$

This implies that for an arbitrary complete assignment $(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k)$ to an f cell

$$\vec{z}BG(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) > k \tag{25}$$

since $G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) \geq \vec{\mathbf{1}}$ for the complete assignment $(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k)$.

Based on Observation 4

$$\begin{aligned}
Ch(\vec{I}_1) &= B^{(1)}G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) \\
&= B^{(2)}G(\vec{I}_k, \vec{I}_1, \cdots, \vec{I}_{k-1}) \\
&= \cdots = B^{(k)}G(\vec{I}_2, \vec{I}_3, \cdots, \vec{I}_1) \\
Ch(\vec{I}_k) &= B^{(1)}G(\vec{I}_k, \vec{I}_1, \cdots, \vec{I}_{k-1}) \\
&= B^{(2)}G(\vec{I}_{k-1}, \vec{I}_k, \cdots, \vec{I}_{k-2}) \\
&= \cdots = B^{(k)}G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) \\
&\vdots \\
Ch(\vec{I}_2) &= B^{(1)}G(\vec{I}_2, \vec{I}_3, \cdots, \vec{I}_1) \\
&= B^{(2)}G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) \\
&= \cdots = B^{(k)}G(\vec{I}_3, \vec{I}_4, \cdots, \vec{I}_2).
\end{aligned}$$

This indicates that

$$\vec{z} \begin{bmatrix} B^{(1)} \\ B^{(2)} \\ \vdots \\ B^{(k)} \end{bmatrix} \left\{ \begin{array}{c} G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) + G(\vec{I}_k, \vec{I}_1, \cdots, \vec{I}_{k-1}) \\ + \cdots + G(\vec{I}_2, \vec{I}_3, \cdots, \vec{I}_1) \end{array} \right\}$$

equals

$$\vec{z} \begin{bmatrix} B^{(1)} + B^{(2)} + \cdots + B^{(k)} \\ B^{(1)} + B^{(2)} + \cdots + B^{(k)} \\ \vdots \\ B^{(1)} + B^{(2)} + \cdots + B^{(k)} \end{bmatrix} G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k).$$

Assume $\vec{I}_0 = \vec{f}(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k)$. For commutative function f

$$\begin{aligned}
\vec{I}_0 &= \vec{f}(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) \\
&= \vec{f}(\vec{I}_k, \vec{I}_1, \cdots, \vec{I}_{k-1}) \\
&= \cdots = \vec{f}(\vec{I}_2, \vec{I}_3, \cdots, \vec{I}_1),
\end{aligned}$$

hence

$$\begin{aligned}
Ch(\vec{I}_0) &= B^{(0)}G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) \\
&= B^{(0)}G(\vec{I}_k, \vec{I}_1, \cdots, \vec{I}_{k-1}) \\
&= \cdots = B^{(0)}G(\vec{I}_2, \vec{I}_3, \cdots, \vec{I}_1).
\end{aligned}$$

Then

$$\vec{z} \begin{bmatrix} B^{(0)} \\ B^{(0)} \\ \vdots \\ B^{(0)} \end{bmatrix} kG(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k)$$

is not smaller than

$$\vec{z} \begin{bmatrix} B^{(1)} + B^{(2)} + \cdots + B^{(k)} \\ B^{(1)} + B^{(2)} + \cdots + B^{(k)} \\ \vdots \\ B^{(1)} + B^{(2)} + \cdots + B^{(k)} \end{bmatrix} G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) + k^2,$$

according to (25). Thus we can state that if $T_f^{(n)}$ is not $\Theta(1)$ assignable, then there is $\vec{z} \in \mathbf{R}^m$ so that

$$k\vec{z}B^{(0)}G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k)$$

is not smaller than

$$\vec{z}\left(B^{(1)} + B^{(2)} + \cdots + B^{(k)}\right) G(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k) + k$$

for every complete assignment $(\vec{I}_1, \vec{I}_2, \cdots, \vec{I}_k)$ to an f cell. In other words, if $T_f^{(n)}$ is not $\Theta(1)$ assignable then there is such a $\vec{z} \in \mathbf{R}^m$ that for every complete assignment $(\vec{I}_1, \cdots, \vec{I}_k)$ to an f cell there exists an $i \in [1,k]$

$$\vec{z}B^{(0)}G(\vec{I}_1, \cdots, \vec{I}_k) \geq \vec{z}B^{(i)}G(\vec{I}_1, \cdots, \vec{I}_k) + 1 \tag{26}$$

Suppose $T_f^{(n)}$ has k^N primary input lines. We determine a path, called *downhill path*, from the primary output line to a primary input line using the following procedure.

1. Choose the cell with the primary output line as the first cell on the downhill path, and let $(\vec{I}_{1,1}, \cdots, \vec{I}_{1,k})$ denote the complete assignment set to this cell. Then

$$\vec{z}B^{(0)}G(\vec{I}_{1,1}, \cdots, \vec{I}_{1,k}) \geq \vec{z}B^{(0)}G(\vec{I}_{1,1}, \cdots, \vec{I}_{1,k}) + 1 - 1.$$

2. Let $(\vec{I}_{l,1}, \cdots, \vec{I}_{l,k})$ denote the complete assignment to the lth cell on the downhill path, and suppose

$$\vec{z}B^{(0)}G(\vec{I}_{1,1}, \cdots, \vec{I}_{1,k}) \geq \vec{z}B^{(0)}G(\vec{I}_{l,1}, \cdots, \vec{I}_{l,k}) + l - 1.$$

3. According to (26) there is such an i that

$$\vec{z}B^{(0)}G(\vec{I}_{l,1},\cdots,\vec{I}_{l,k}) \geq \vec{z}B^{(i)}G(\vec{I}_{l,1},\cdots,\vec{I}_{l,k}) + 1.$$

We choose the cell linked directly to the ith input line of the lth cell as the $(l+1)$th cell on the downhill path. $(\vec{I}_{l+1,1},\cdots,\vec{I}_{l+1,k})$ is the complete assignment to this cell. We can state that

$$\begin{aligned}\vec{z}B^{(0)}G(\vec{I}_{1,1},\cdots,\vec{I}_{1,k}) &\geq \vec{z}B^{(0)}G(\vec{I}_{l,1},\cdots,\vec{I}_{l,k}) + l - 1\\ &\geq \vec{z}B^{(i)}G(\vec{I}_{l,1},\cdots,\vec{I}_{l,k}) + l\\ &= \vec{z}B^{(0)}G(\vec{I}_{l+1,1},\cdots,\vec{I}_{l+1,k}) + l.\end{aligned}$$

In this way, we can finally determine the Nth cell on the downhill path. Suppose $(\vec{I}_{N,1},\cdots,\vec{I}_{N,k})$ is the complete assignment set to this cell. Based on the above calculation

$$\vec{z}B^{(0)}G(\vec{I}_{1,1},\cdots,\vec{I}_{1,k}) \geq \vec{z}B^{(0)}G(\vec{I}_{N,1},\cdots,\vec{I}_{N,k}) + N - 1.$$

Let $|\vec{y}|$ denote the sum of the absolute values of the components in $\vec{y}$. Then $|G(\vec{I}_{1,1},\cdots,\vec{I}_{1,k})|$ is the cardinal number of the complete assignment set to $T_f^{(N)}$.

$$\begin{aligned}\left|G(\vec{I}_{1,1},\cdots,\vec{I}_{1,k})\right| &\geq \frac{N}{|\vec{z}B^{(0)}|}\\ &= \Omega(N).\end{aligned}$$

We know that $N = \lceil \lg n \rceil$, and every $T_f^{(n)}$ is $O(\lg n)$ assignable. Therefore, $T_f^{(n)}$ is $\Theta(\lg n)$ assignable.

Q.E.D.

The following Corollary is immediate from the above theorem

Corollary 4 *Assume $f : M^k \longrightarrow M$ to be commutative, then $T_f^{(n)}$ is either $\Theta(1)$ or $\Theta(\lg n)$ assignable.*

Assume f to be commutative and $T_f^{(n)}$ $O(1)$ assignable. The problem of searching for the minimum complete assignment set for $T_f^{(n)}$ is related to solving the following integer programming.

$$\begin{aligned} &\min \sum_{i=1}^{s} y_i \\ B\vec{y} &= -B\vec{\mathbf{1}} \\ \vec{y} &\geq \vec{\mathbf{0}} \end{aligned} \tag{27}$$

The scene changes if f is not commutative. The problems of deciding the assignment complexity of $T_f^{(n)}$ and searching for the minimum complete assignment set for an $O(1)$ assignable $T_f^{(n)}$ are still open.

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References

[1] J. A. Abraham and D. D. Gajski. Design of testable structure defined by simple loops. *IEEE Trans. on Computers*, C-30, 1981.

[2] B. Becker, and J. Hartmann. Optimal-time multipliers and C-testability. *Proceedings of the 2nd Annual Symposium on Parallel Algorithms and Architectures.* pp146-154, 1990.

[3] B. Becker and U. Sparmann, Computations over Finite Monoids and their Test Complexity, *Theoretical Computer Science,* pp225-250, 1991.

[4] D. Bhattacharya and J. P. Hayes. Fast and easily testable implementation of arithmetic functions. *Proceedings of the 16th International Symposium on Fault Tolerant Computing Systems,* July 1986.

[5] R. Fritzemeier, J. Soden, R. K. Treece, and C. Hawkins. Increased CMOS IC stuck-at fault coverage with reduced I_{DDQ} test sets. *Proceedings of ITC.* pp427-435, 1990.

[6] J. P. Hayes, On realizations of boolean functions requiring a minimal or near-minimal number of tests, *IEEE Trans. on Computers* C-20, pp1506-1513, 1971.

[7] G. Markowsky. A straightforward technique for producing minimal multiple fault test sets for fanout-free combinational circuits. Technical Report RC 6222, IBM T. J. Watson Research Center, Yorktown Heights, 1976.

[8] Christos H. Papadimitriou and K, Steiglitz, **Combinatorial Optimization**– Algorithms and Complexity, Prentice-Hall, Inc., 1982.

[9] S.C. Seth, and K.L. Kodandapani. Diagnosis of faults in linear tree networks. *IEEE Trans. on Computers*, C-26(1), pp29-33, Jan. 1977.

[10] H. Wu, "On Tests of Uniform Tree Circuits", Proceedings of Second Joint International Conference on Vector and Parallel Processing, pp527-538,1992. (Lecture Notes in Computer Science 634).

[11] H. Wu, U. Sparmann, "On the Assignment Complexity of VLSI Tree Systems", Proceeding of The Seventh International Symposium on Computer and Information Sciences, pp97-103, 1992.

[12] H. Wu, "On the Test Complexity of Tree VLSI Systems", Technical Report, SFB 124-B1, 08/1992, FB-Informatik, Universität des Saarlandes, Germany.