Generating Quadratic Bilevel Programming Test Problems

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This paper describes a technique for generating sparse or dense quadratic bilevel programming problems with a selectable number of known global and local solutions. The technique described here does not require the solution of any subproblems. In addition, since most techniques for solving these problems begin by solving the corresponding relaxed quadratic program, the global solutions are constructed to be different than the global solution of this relaxed problem in a selectable number of upper- and lower-level variables. Finally, the problems that are generated satisfy the requirements imposed by all of the solution techniques known to the authors.

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1. INTRODUCTION

Bilevel programming has become an important field of mathematical programming [Dirickx and Jennegren 1979; Kolstad 1985; Mesanovic et al. 1970]. Applications of these problem are numerous [Bard 1983; Ben-Ayed et al. 1988; Fortuny-Amat and McCarl 1981], and a significant range of techniques have been proposed for solving these programs [Bi et al. 1991; Edmunds and Bard 1991; Gauvin and Savard 1989; Kolstad 1985]. From the

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computational point of view, test problems play an important role, helping to test and improve codes and allowing for the comparison of different solution techniques. Although several papers concerning the generation of nonlinear programming test problems have been published (see, e.g., Floudas and Pardalos [1990], Kalantari and Rosen [1986], and Lenard and Minkoff [1984]), none have addressed the need for standardized quadratic bilevel test problems. We hope the technique presented in this paper fills this void.

The paper is divided as follows: Sections 2 and 3 describe, respectively, the general quadratic bilevel program (QBP) and a corresponding separable parametric QBP. Sections 4 and 5 demonstrate that the solution of this parametric QBP is straightforward, since it only involves the solution of a number of simple two-variable one-parameter QBPs. Section 5 also derives the important properties of the separable parametric QBP. In Section 6 we discuss extensions and modifications to the separable parametric problem $QBP(\rho)$, and in Section 7 we introduce a transformation to make the problems more general. In Section 8 we illustrate our technique with an example. Finally, in Section 9 we demonstrate that the generated problems satisfy the requirements of three different solution techniques, and report our conclusions.

2. THE QUADRATIC PROBLEM

Define problem $QBP(C, c, S, s, A_x, A_y, b)$ as

$$\min_{x,y} Q(x,y) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy}^T & C_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \kappa$$

(the upper-level problem), where $y = y(x)$ solves (the lower-level problem)

$$\min_{y} q(x,y) = \frac{1}{2}\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} S_{xx} & S_{xy} \\ S_{xy}^T & S_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} s_x \\ s_y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix},$$

subject to

$$A_x x + A_y y \le b,$$

with

$$C = \begin{bmatrix} C_{xx} & C_{xy} \\ C_{xy}^T & C_{yy} \end{bmatrix}, \quad c = \begin{bmatrix} c_x \\ c_y \end{bmatrix}, \quad S = \begin{bmatrix} S_{xx} & S_{xy} \\ S_{xy}^T & S_{yy} \end{bmatrix}, \quad s = \begin{bmatrix} s_x \\ s_y \end{bmatrix},$$

$$c_x, s_x, x \in \boldsymbol{R}^{nx}, \qquad c_y, s_y, y \in \boldsymbol{R}^{ny},$$

$$C_{xx}, S_{xx} \in \boldsymbol{R}^{nx \times nx}, \qquad C_{yy}, S_{yy} \in \boldsymbol{R}^{ny \times ny}, \qquad C_{xy}, S_{xy} \in \boldsymbol{R}^{nx \times ny},$$

$$A_x \in \boldsymbol{R}^{\beta \times nx}, \qquad A_y \in \boldsymbol{R}^{\beta \times ny}, \qquad b \in \boldsymbol{R}^{\beta}, \qquad \text{and} \qquad \kappa \in \boldsymbol{R}.$$

In addition to this problem, define the corresponding *relaxed* quadratic program $QP(C, c, A_x, A_y, b)$ as

$$\min_{x,y} \{Q(x, y) : (x, y) \in \Omega\},$$

where $\Omega = \{(x, y): A_x x + A_y y \le b\}$.

3. A SIMPLE SEPARABLE PARAMETRIC QBP

Our technique for generating QBPs involves randomly transforming the parametric QBP that results when the following substitutions are made in the original problem definitions:

$$C = I_n, \qquad c_x = -1_{nx}, \qquad c_y = 0_{ny}, \qquad s = 0_n \qquad S_{xx} = \mathbf{0}_{nx}, \qquad S_{yy} = I_{ny},$$

$$(S_{xy})_{ij} = \begin{cases} -1 & 1 \le i = j \le m, \\ 0 & \text{otherwise}, \end{cases}$$

$$A_x = \begin{bmatrix} P_x \\ P_x \\ -P_x \end{bmatrix}, \qquad A_y = \begin{bmatrix} -P_y \\ P_y \\ -P_y \end{bmatrix}, \qquad b = \begin{bmatrix} 1_m \\ \rho \\ -1_m \end{bmatrix}, \qquad \text{and} \qquad \kappa = \frac{nx}{2},$$

where

$$n = nx + ny \qquad \text{and} \qquad m = \min\{nx, ny\},$$

I_γ is the order-γ identity matrix and $\mathbf{0}_{nx}$ is the order-nx zero matrix,

1_γ is the ones-vector of length γ and 0_γ is the zeros-vector of length γ,

$$P_x \in \boldsymbol{R}^{m \times nx} \qquad \text{and} \qquad P_y \in \boldsymbol{R}^{m \times ny} \qquad \text{satisfy} \quad P_{ij} = \begin{cases} 1 & 1 \le i = j \le m, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$\rho \in \boldsymbol{R}^m \qquad \text{with} \quad \rho_i \ge 1 \qquad \text{for} \quad i = 1, \ldots, m.$$

With these substitutions we obtain the following parametric QBP, denoted $QBP(\rho)$:

$$\begin{aligned} \min_{x,y} Q(x, y) &= \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -1_{nx} \\ 0_{ny} \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} + \frac{nx}{2} \\ &= \frac{1}{2} \left\| \begin{matrix} x - 1_{nx} \\ y \end{matrix} \right\|_2^2 \\ &= \frac{1}{2} \left\{ \sum_{i=1}^{m} \left((x_i - 1)^2 + y_i^2 \right) + \sum_{m < i \le nx} (x_i - 1)^2 + \sum_{m < i \le ny} y_i^2 \right\}, \end{aligned}$$

subject to $y = y(x)$, solving

$$\min_{y} q(x, y) = \frac{1}{2} y^T S_{yy} y + y^T S_{xy} x$$
$$= \sum_{i=1}^{m} \left(\frac{1}{2} y_i^2 - y_i x_i \right) + \frac{1}{2} \sum_{m < i \leq ny} y_i^2,$$

subject to

$$x_i - y_i \leq 1, \quad i = 1, \ldots, m,$$
$$1 \leq x_i + y_i \leq \rho_i, \quad i = 1, \ldots, m,$$

with $\rho_i \geq 1$ for $i = 1, \ldots, m$.

4. SOLUTION OF THE TWO-VARIABLE QBP

Let $(x^G(\rho), y^G(\rho))$ be a global solution of problem $QBP(\rho)$. To obtain $(x_i^G(\rho), y_i^G(\rho))$, for $i = 1, \ldots, m$, we exploit the fact that problem $QBP(\rho)$ is separable in these pairs of variables. Thus, to obtain $(x_k^G(\rho), y_k^G(\rho))$, $k \in \{1, \ldots, m\}$, we consider the following two-variable one-parameter QBP, denoted $QBP(\rho_k)$:

$$\min_{x_k, y_k} Q_k(x_k, y_k) = \frac{1}{2}\left\{(x_k - 1)^2 + y_k^2\right\},$$

where $y_k = y(x_k)$ solves

$$\min_{y_k} q_k(x_k, y_k) = \frac{1}{2} y_k^2 - x_k y_k,$$

subject to

$$x_k - y_k \leq 1,$$
$$1 \leq x_k + y_k \leq \rho_k,$$

with $\rho_k \geq 1$.

There are four cases to consider:

(1) Case 1 (Figure 1), where $\rho_k = 1$;
(2) Case 2 (Figure 2), where $1 < \rho_k < 2$;
(3) Case 3 (Figure 3), where $\rho_k = 2$; and
(4) Case 4 (Figure 4), where $\rho_k > 2$.

In each of these four cases, the feasible region $\Omega(\rho_k)$, for (x_k, y_k), is the (unbounded) region bounded above by $x_k + y_k \leq \rho_k$, bounded below by $x_k + y_k \geq 1$, and bounded on the right by $x_k - y_k \leq 1$.

The set of all feasible points of problem $QBP(\rho_k)$ is called the *induced region* (see Edmunds and Bard [1991] for a complete mathematical description).

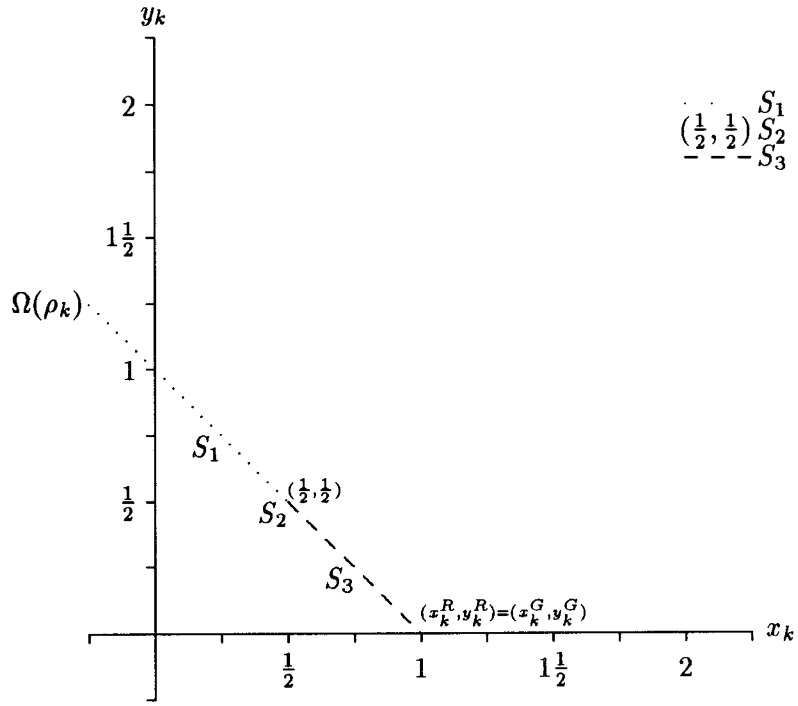

Fig. 1. Case 1, where $\rho_k = 1$.

PROPOSITION 1. *If throughout, the induced region, denoted S, in all four cases consists of the union of the three sets*

$$S_1 = \{(x_k, y_k) \in \Omega(\rho_k): y_k - x_k \geq 0 \quad \textit{and} \quad x_k + y_k = 1\},$$
$$S_2 = \{(x_k, y_k) \in \Omega(\rho_k): y_k - x_k = 0\},$$
$$S_3 = \{(x_k, y_k) \in \Omega(\rho_k): y_k - x_k \leq 0 \quad \textit{and} \quad x_k + y_k = \rho_k\},$$

which describe three line segments in $\Omega(\rho_k)$.

PROOF. Suppose $(x_k, y_k) \in \Omega(\rho_k)$ solves problem $QBP(\rho_k)$. The Karush–Kuhn–Tucker conditions of the corresponding lower-level problem therefore imply that there exists $\lambda_1, \lambda_2, \lambda_3 \in \boldsymbol{R}$ such that

$$\begin{aligned} y_k - x_k &= \lambda_1 - \lambda_2 + \lambda_3, \\ \lambda_1(x_k + y_k - 1) &= 0, \\ \lambda_2(-x_k - y_k + \rho_k) &= 0, \\ \lambda_3(-x_k + y_k + 1) &= 0, \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0. \end{aligned}$$

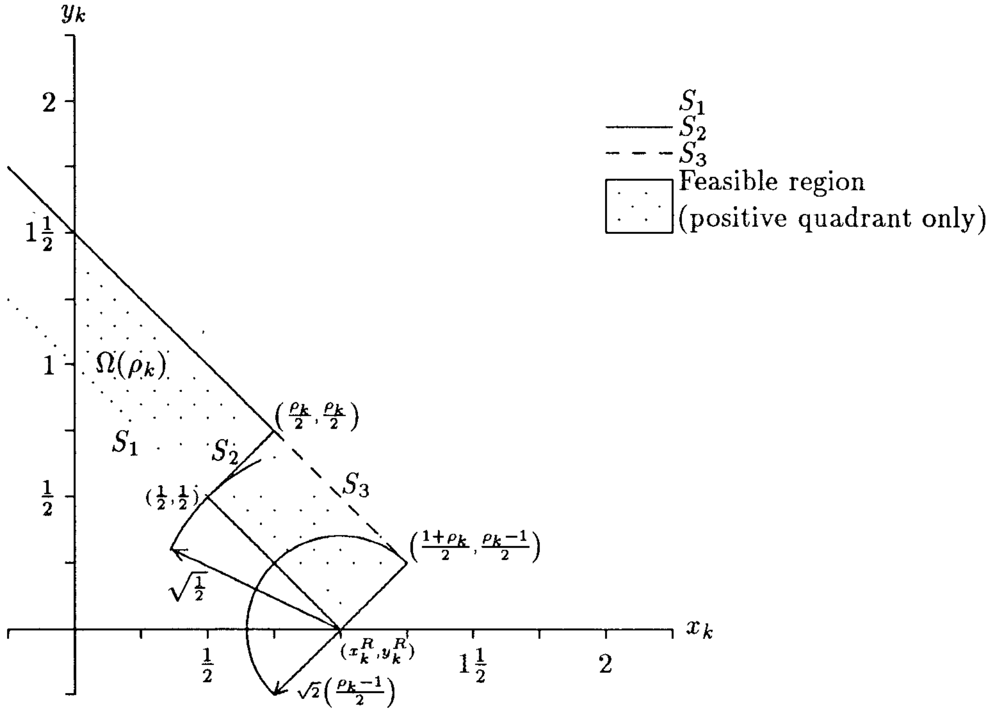

Fig. 2. Case 2, where $1 < \rho_k < 2$.

Thus, for $(x_k, y_k) \in \Omega(\rho_k)$ with $\rho_k > 1$, we have four possibilities:

(1) $(x_k, y_k) \in Int(\Omega(\rho_k))$. In this case, all constraints are inactive, which implies that $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Consequently, $y_k - x_k = 0$. This describes the interior of S_2.

(2) (x_k, y_k) satisfies $x_k + y_k = 1$ with $x_k \neq 1$. In this case, $\lambda_2 = \lambda_3 = 0$. Consequently, $y_k - x_k = \lambda_1 \geq 0$. This describes all of S_1 and the bottom endpoint of S_2.

(3) (x_k, y_k) satisfies $x_k + y_k = \rho_k$ with $x_k \neq (1 + \rho_k)/2$. In this case, $\lambda_1 = \lambda_3 = 0$. Consequently, $y_k - x_k = -\lambda_2 \leq 0$. This describes the top endpoint of S_2 and all of S_3 except its right endpoint.

(4) (x_k, y_k) satisfies $x_k - y_k = 1$. In this case, there are three subcases:
(a) $x_k \neq 1$ and $x_k \neq (1 + \rho_k)/2$. In this case, $\lambda_1 = \lambda_2 = 0$. Consequently, $y_k - x_k = \lambda_3 \geq 0$, which yields a contradiction.
(b) $x_k = 1$. In this case, $\lambda_2 = 0$. Consequently, $y_k - x_k = \lambda_1 + \lambda_3 \geq 0$, which yields a contradiction.
(c) $x_k = (1 + \rho_k)/2$. In this case, $\lambda_1 = 0$. Consequently, $y_k - x_k = -\lambda_2 + \lambda_3$. Choosing $\lambda_2 > \lambda_3$ avoids a contradiction, implies that $y_k - x_k < 0$, and yields the right endpoint of S_3. □

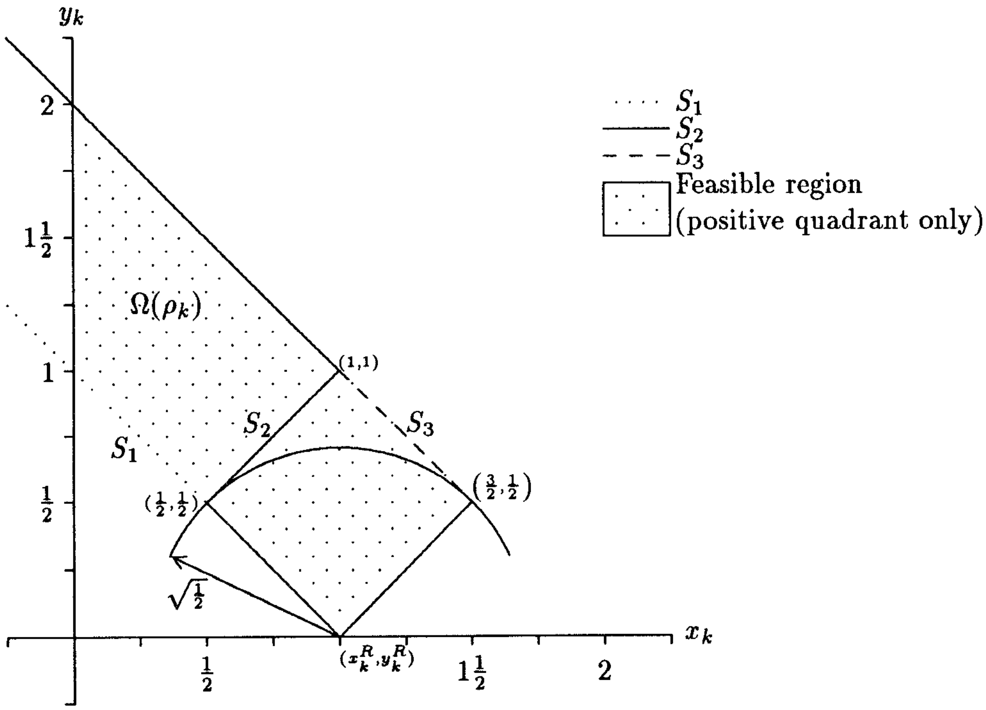

Fig. 3. Case 3, where $\rho_k = 2$.

If $\rho_k = 1$, the induced region is the half-line $\{(x_k, y_k): x_k + y_k = 1$ and $x_k - y_k \leq 0\}$.

We now examine problem $QBP(\rho_k)$ for each of the four cases for ρ_k:

Case 1: $\rho_k = 1$. In this case, the set S_2 is the single point $(1/2, 1/2)$, and the union of the sets S_1 and S_3 describe the half-line $\{(x_k, y_k): x_k + y_k = 1$ and $x_k - y_k \leq 1\}$. For this case, depicted in Figure 1, $(x_k^G, y_k^G) = (x_k^R, y_k^R) = (1, 0)$, where (x_k^R, y_k^R) corresponds to the minimizer of the relaxed problem, yielding $Q_k(x_k^G, y_k^G) = 0$.

Case 2: $1 < \rho_k < 2$. In this case, depicted in Figure 2, the set S_1 describes the half-line

$$\{(x_k, y_k): x_k + y_k = 1 \quad \text{and} \quad x_k - y_k \leq 0\}$$

(i.e., the points on the line $x_k + y_k = 1$ to the left of, and including, the point $(1/2, 1/2)$). The set S_3 describes the line segment

$$\{(x_k, y_k): x_k + y_k = \rho_k \quad \text{and} \quad 0 \leq x_k - y_k \leq 1\}$$

(i.e., the points on the line segment joining the point $(\rho_k/2, \rho_k/2)$ to the point $((1 + \rho_k)/2, (\rho_k - 1)/2))$. The set S_2 describes the points on the line segment joining the point $(1/2, 1/2)$ to the point $(\rho_k/2, \rho_k/2)$.

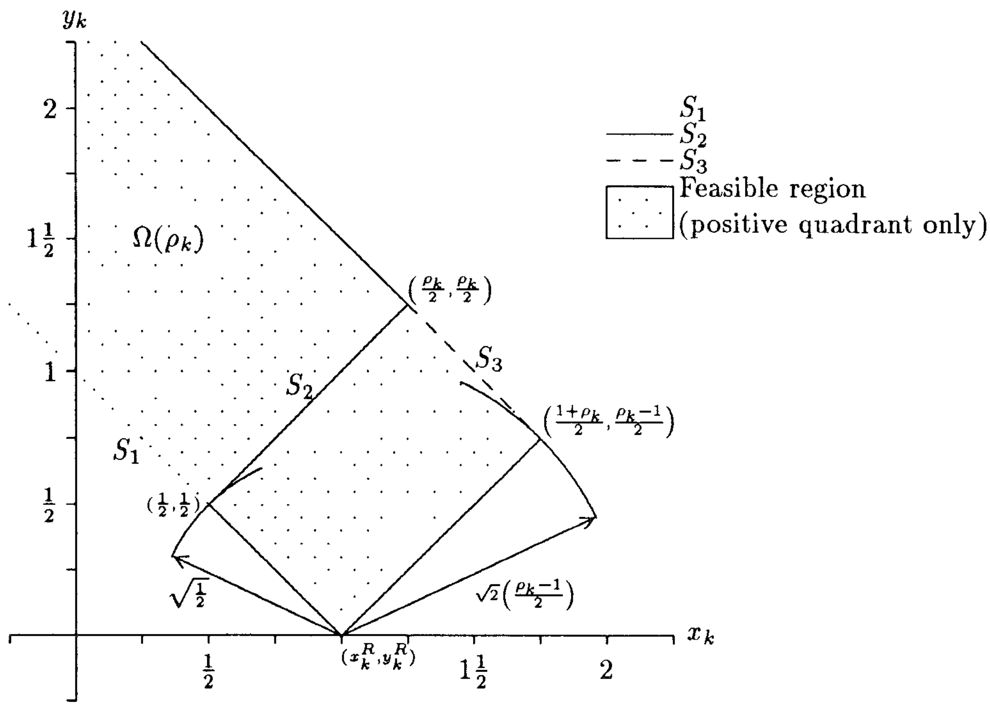

Fig. 4. Case 4, where $\rho_k > 2$.

The circle of radius $r_k = (\rho_k - 1)/\sqrt{2}$, centered at the point $(x_k^R, y_k^R) = (1, 0)$, contains all of the points (x_k, y_k) for which $Q_k(x_k^G, y_k^G) \le r_k^2/2 = ((\rho_k - 1)/2)^2$. Since the intersection of this circle with S includes only the point $((1 + \rho_k)/2, (\rho_k - 1)/2)$, we have $(x_k^G, y_k^G) = ((1 + \rho_k)/2, (\rho_k - 1)/2)$, with $Q_k(x_k^G, y_k^G) = ((\rho_k - 1)/2)^2$. In addition, every point in $S_1 \cup S_2$, except their unique intersection point $(1/2, 1/2)$, lies outside the circle of radius $r_k = \sqrt{1/2}$, centered at $(1, 0)$. Consequently, the point $(x_k^L, y_k^L) = (1/2, 1/2)$, with $Q_k(x_k^L, y_k^L) = r_k^2/2 = 1/4$, is a local minimizer of problem $QBP(\rho_k)$ when $1 < \rho_k < 2$.

Case 3: $\rho_k = 2$. In this case, depicted in Figure 3, the set S_1 is the same as in Case 2, S_2 describes the line segment joining the point $(1/2, 1/2)$ to the point $(1, 1)$, and S_3 describes the line segment joining the point $(1, 1)$ to the point $(3/2, 1/2)$. The points $(1/2, 1/2)$ and $(3/2, 1/2)$ are the only two points in S within the circle of radius $r_k = \sqrt{1/2}$, centered at $(1, 0)$. Consequently, both of these points are (strict) global minimizers of problem $QBP(\rho_k)$, when $\rho_k = 2$, with $Q_k(1/2, 1/2) = Q_k(3/2, 1/2) = 1/4$.

Case 4: $\rho_k > 2$. Figure 4 depicts the last possible case. Here the set S_1 is the same as in Cases 2 and 3, S_2 describes the line segment joining point $(1/2, 1/2)$ to the point $(\rho_k/2, \rho_k/2)$, and S_3 describes the points on the line segment joining the point $(\rho_k/2, \rho_k/2)$ to the point $((1 + \rho_k)/2, (\rho_k - 1)/2)$.

The circle of radius $r_k = \sqrt{1/2}$, centered at $(x_k^R, y_k^R) = (1, 0)$, contains all of the points for which $Q_k(x_k, y_k) \le r_k^2/2 = 1/4$. Since the intersection of this circle with S includes only the point $(1/2, 1/2)$, we have $(x_k^G, y_k^G) = (1/2, 1/2)$, with $Q_k(x_k^G, y_k^G) = 1/4$.

In addition, every point in S_3, except the one endpoint $((1 + \rho_k)/2, (\rho_k - 1)/2)$, lies outside the circle of radius $r_k = (\rho_k - 1)/\sqrt{2}$, centered at the point $(x_k^R, y_k^R) = (1, 0)$. Consequently, the point $(x_k^L, y_k^L) = ((1 + \rho_k)/2, (\rho_k - 1)/2)$, with $Q_k(x_k^L, y_k^L) = r_k^2/2 = ((\rho_k - 1)/2)^2$, is a local minimizer of problem $QBP(\rho_k)$ when $\rho_k > 2$.

5. SOLUTION AND PROPERTIES OF PROBLEM $QBP(\rho)$

Let $(x^R(\rho), y^R(\rho))$ be the unique global solution of the relaxed (convex) quadratic program $QP(\rho)$ that corresponds to $QBP(\rho)$. Since $(1_{nx}, 0_{ny})$ is feasible (i.e., $(1_{nx}, 0_{ny}) \in \Omega$) and $Q(x, y) \ge 0 = Q(1_{nx}, 0_{ny})$, we have $x^R(\rho) = 1_{nx}$ and $y^R(\rho) = 0_{ny}$. In addition, $x_i^G(\rho) = 1$ for every $m < i \le nx$, and $y_i^G(\rho) = 0$ for every $m < i \le ny$.

Since $QBP(\rho)$ is separable in each of the m two-variable pairs (x_i, y_i), $i = 1, \dots, m$, we see (as a consequence of the results presented in the previous section) that $(x_i^G(\rho), y_i^G(\rho))$ can easily be obtained by examining the value of ρ_i and setting $(x_i^G(\rho), y_i^G(\rho)) = (x_i^G(\rho_i), y_i^G(\rho_i))$. In addition, the features of problem $QBP(\rho)$ can be controlled by adjusting the magnitude of the m parameters ρ_1 through ρ_m. In order to describe this more precisely, define the four sets, M_1 through M_4 (corresponding to Cases 1 through 4, respectively) as

$$
\begin{aligned}
M_1 &= \{k \in \{1, \dots, m\} : \rho_k = 1\},\\
M_2 &= \{k \in \{1, \dots, m\} : 1 < \rho_k < 2\},\\
M_3 &= \{k \in \{1, \dots, m\} : \rho_k = 2\},\\
M_4 &= \{k \in \{1, \dots, m\} : \rho_k > 2\},
\end{aligned}
$$

and let m_1, m_2, m_3, and m_4 equal the cardinality of the sets M_1 through M_4, respectively.

PROPERTY 1. *Problem $QBP(\rho)$ has 2^{m_3} global solutions (i.e., when $m_3 = 0$, the global solution is unique) with value $Q_G = \sum_{k \in M_2}((\rho_k - 1)/2)^2 + (m_3 + m_4)/4$.*

The reason for this is that, for each $k \in M_3$, there are two global solutions of problem $QBP(\rho_k)$ in the variables (x_k, y_k). The result follows by considering all combinations of these global solutions in the variables (x_k, y_k) for $k \in M_3$.

PROPERTY 2. *Problem $QBP(\rho)$ has additional local solutions (i.e., not global) only when $m_2 + m_4 > 0$. In this case, the number of additional local minima to problem $QBP(\rho)$ equals $2^{m_2 + m_3 + m_4} - 2^{m_3}$.*

The reason for this is that, for each $k \in M_2 \cup M_3 \cup M_4$, problem $QBP(\rho_k)$ has two local solutions in the variables (x_k, y_k). By considering all combinations of these local solutions in the variables (x_k, y_k), for $k \in M_2 \cup M_3 \cup M_4$, we obtain $2^{m_2+m_3+m_4}$ local solutions for problem $QBP(\rho)$. However, by Property 1, 2^{m_3} of these local solutions must be global solutions of problem $QBP(\rho)$.

PROPERTY 3. *The global solution of problem $QBP(\rho)$ differs from the relaxed solution of the corresponding quadratic program $QP(\rho)$ in both upper-and lower-level variables as long as $m_1 < m$ (i.e., $m_2 + m_3 + m_4 \neq 0$).*

This result follows since, for $k \in M_2 \cup M_3 \cup M_4$, we have $(x_k^G(\rho), y_k^G(\rho)) = (x_k^G(\rho_k), y_k^G(\rho_k)) \neq (x_k^R(\rho_k), y_k^R(\rho_k))$.

PROPERTY 4. *The gradients, in the lower-level variables y, of the active constraints at the global solution of problem $QBP(\rho)$ are linearly independent if $m_4 = m$ and linearly dependent if $m_1 + m_2 > 0$. If $m_1 + m_2 = 0$, but $m_3 > 0$, then exactly one of the (nonunique) global solutions of problem $QBP(\rho)$ will have linearly independent gradients, and all other global solutions will have dependent gradients.*

The proof follows directly upon observing that one of the two local minimizers, namely, $(1/2, 1/2)$, is on the boundary of only one constraint, whereas the other, $((1 + \rho_k)/2, (\rho_k - 1)/2)$, is on the boundary of two constraints. Consequently, the linear independence of the gradients of the active constraints of problem $QBP(\rho)$ is due to its separability and to the linear independence of the gradients of the active constraint in each two-variable parametric problem $QBP(\rho_k)$.

PROPERTY 5. *The complementarity conditions associated with the lower-level problem of problem $QBP(\rho)$ are satisfied strictly at the (unique) global solution of problem $QBP(\rho)$ whenever $m_4 = m$ (i.e., the corresponding Lagrange multipliers are strictly positive in this case). These complementarity conditions are not satisfied strictly whenever $m_1 + m_2 > 0$. If $m_3 > 0$, then exactly one of the (nonunique) global minimizers satisfies these conditions strictly.*

This property follows directly from Property 4.

6. EXTENSIONS AND MODIFICATIONS

Problem $QBP(\rho)$ can be extended and modified in a number of ways. For instance, constraints in x and y could be added to the upper-level problem to bound Ω. We describe three other changes here.

6.1 Reducing the Number of Inequality Constraints

The lower-level problem associated with problem $QBP(\rho)$ has ny variables $(y(i), i = 1, \ldots, ny)$ and $3m$ constraints

$$\begin{aligned} x_i - y_i &\leq 1, \quad i = 1, \ldots, m, \\ 1 \leq x_i + y_i &\leq \rho_i, \quad i = 1, \ldots, m. \end{aligned}$$

The number of constraints can be reduced, without changing the properties of problem $QBP(\rho)$, by redefining the sets M_1 through M_4 (and their corresponding cardinalities m_1 through m_4) such that

$$M_1 = \{k \in \{1, \dots, \overline{m}\}: \rho_k = 1\},$$
$$M_2 = \{k \in \{1, \dots, \overline{m}\}: 1 < \rho_k < 2\},$$
$$M_3 = \{k \in \{1, \dots, \overline{m}\}: \rho_k = 2\},$$
$$M_4 = \{k \in \{1, \dots, \overline{m}\}: \rho_k > 2\},$$

where $\overline{m} \le m = \min\{nx, ny\}$, and by replacing the original constraints with

$$\begin{aligned} 1 \le x_i + y_i, &\quad i = 1, \dots, \overline{m}, \\ x_i + y_i \le \rho_i, &\quad i \in M_2 \cup M_3 \cup M_4, \\ x_i - y_i \le 1, &\quad i \in M_2 \cup M_3 \cup M_4 \end{aligned}$$

(which induce corresponding changes to the definitions of A_x, A_y, and b).

If $\overline{m} < m$, then the variables $(x_k, y_k), k = \overline{m} + 1, \dots, m$, are unconstrained. This means that (x_k^G, y_k^G) must be the point in S_2 closest to $(x_k^R, y_k^R) = (1, 0)$. Thus, $(x_k^G, y_k^G) = (1/2, 1/2)$ when $k \in \{\overline{m} + 1, \dots, m\}$.

This modification yields $3\overline{m} - 2m_1$ constraints in ny variables, which corresponds to a reduction of $2m_1 + 3(m - \overline{m})$ lower-level constraints. Alternatively, the sets $\Omega(\rho_k)$ could be defined by the strip $\{(x_k, y_k): 1 \le x_k + y_k \le \rho_k\}$, $k \in \{1, \dots, \overline{m}\}$, or by the quarter space defined by $\{(x_k, y_k): x_k - y_k \le 1$ and $x_k + y_k \le \rho_k\}$, $k \in \{1, \dots, \overline{m}\}$.

6.2 Adding Equality Constraints

Without changing either the relaxed, local, or global solutions corresponding to problem $QBP(\rho)$, four types of equality constraints can be added to the problem:

(1) $x_i - x_j = 0$, where $i \ne j$ and $i, j \in \{m + 1, \dots, nx\}$, when $nx > ny = m$.
(2) $y_i - y_j = 0$, where $i \ne j$ and $i, j \in \{m + 1, \dots, ny\}$, when $ny > nx = m$.
(3) $x_i - y_j = 1$, where $i, j \in M_1$, when $m_1 > 0$.
(4) $x_i - y_i = 0$, where $i, j \in \{\overline{m} + 1, \dots, m\}$, and $\overline{m}$ and M_1 through M_4 are redefined as in the previous section.

To guarantee the linear independence of the gradients (with respect to the lower-level variables y) of the constraints at optimality, the following conditions must hold:

(1) For type (2) constraints, the variables y_i and y_j of a particular type (2) constraint should appear in that constraint only; and
(2) for both type (3) and type (4) constraints, the variable y_j of a particular constraint in this set should appear in only that specific constraint of this set.

6.3 Adding a Linear Term to the Lower-Level Objective

A linear term can easily be added to the lower-level objective function by introducing the change of variables

$$x_i \leftarrow x_i - \sigma_i, \qquad i = 1, \dots, m,$$

throughout the formulation. This, in effect, is equivalent to setting

$$(s_y)_i = \sigma_i, \qquad i = 1, \dots, m,$$

in the statement of problem *QBP*.

7. THE TRANSFORMATION

Define the order-n matrix $M = HDH$, where

$$H = \begin{bmatrix} H_x & 0 \\ 0 & H_y \end{bmatrix}$$

is a block-diagonal matrix with H_x and H_y constructed as random Householder matrices using

$$H_x = I_{nx} - 2v_x v_x^T, \quad \text{with} \quad v_x^T v_x = 1, \quad \text{and} \quad v_x \in \boldsymbol{R}^{nx} \quad \text{sparse},$$

$$H_y = I_{ny} - 2v_y v_y^T, \quad \text{with} \quad v_y^T v_y = 1, \quad \text{and} \quad v_y \in \boldsymbol{R}^{ny} \quad \text{sparse},$$

and D is a positive definite diagonal matrix with 2-norm condition number $\kappa_2(D) = 10^{\delta}$.

In addition, define the augmented matrix $A = [A_x \quad A_y]$, and let $W = M^{-1} = HD^{-1}H$.

PROPOSITION 2. *Problem $QBP(C, c, S, s, A_x, A_y, b)$ (and consequently problem $QBP(\rho)$) in the variables x and y is equivalent to problem $QBP(MCM, Mc, MSM, Ms, AM, b)$ in the variables $\bar{x}$ and $\bar{y}$ under the nonsingular transformation*

$$\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = W \begin{bmatrix} x \\ y \end{bmatrix}.$$

PROOF. For

$$\begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix},$$

problem $QBP(C, c, S, s, A_x, A_y, b)$ becomes

$$\min_{\bar{x}, \bar{y}} Q(\bar{x}, \bar{y}) = \frac{1}{2} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^T (MCM) \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + \begin{bmatrix} c_x \\ c_y \end{bmatrix}^T M \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + \kappa,$$

subject to $\bar{y} = \bar{y}(\bar{x})$, solving (the lower-level problem)

$$\min_{\bar{y}} q(\bar{y}) = \frac{1}{2} \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix}^T (MSM) \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} + \begin{bmatrix} s_x \\ s_y \end{bmatrix}^T M \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix},$$

subject to

$$[AM]\begin{bmatrix}\bar{x}\\ \bar{y}\end{bmatrix} \le b,$$

which is problem $QBP(MCM, Mc, MSM, Ms, AM, b)$ in the variables $\bar{x}$ and $\bar{y}$. □

Thus,

$$\begin{bmatrix}\bar{x}^G\\ \bar{y}^G\end{bmatrix} = W\begin{bmatrix}x^G\\ y^G\end{bmatrix}$$

is a global solution to the transformed problem. This one-to-one correspondence holds for all minima of problem QBP.

Notes

(1) The sparsity of v_x and v_y controls the sparsity of M (and, consequently, the sparsity of the data).

(2) The 2-norm condition of D controls the 2-norm condition of M (and, consequently, affects the condition of the problem).

(3) The matrix MCM is positive definite in (x, y), and the submatrix $(MSM)_{yy}$ is positive definite in y.

8. EXAMPLE

The following simple example illustrates some of the ideas that have been presented: Suppose the following values of the control parameters were used:

$$nx = 4, ny = 2 \quad \text{and} \quad m = \overline{m} = 2,$$
$$m_1 = m_3 = 0, \quad \text{and} \quad m_2 = m_4 = 1,$$

and

$$\rho_1 = 1.5 \quad \text{and} \quad \rho_2 = 3.$$

This corresponds to the following untransformed bilevel problem:

$$\min_{x,y} Q(x,y) = \frac{1}{2}\left\{\sum_{i=1}^{2}\left((x_i - 1)^2 + y_i^2\right) + \sum_{i=3}^{4}(x_i - 1)^2\right\},$$

subject to $y = y(x)$, solving

$$\min_{y} q(x,y) = \sum_{i=1}^{2}\left(\frac{1}{2}y_i^2 - y_i x_i\right),$$

subject to

$$\begin{array}{rcccrcr}
x_1 & & - & y_1 & & \le & 1,\\
 & x_2 & - & & y_2 & \le & 1,\\
x_1 & & + & y_1 & & \le & 1.5,\\
 & x_2 & + & & y_2 & \le & 3,\\
-x_1 & & - & y_1 & & \le & -1,\\
 & -x_2 & - & & y_2 & \le & -1,
\end{array}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \boldsymbol{R}^4 \qquad \text{and} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \boldsymbol{R}^2.$$

Now suppose the following data were used in the transformation:

$$v_x^T = [0.9 \quad 0.3 \quad 0.3 \quad 0.1], \qquad v_y^T = [0.8 \quad 0.6], \qquad \text{and}$$
$$D = \operatorname{diag}(10, 10, 20, 20, 10, 10).$$

This would yield the following quadratic bilevel programming problem:

$$\min_{x,\,y} Q(x, y)$$
$$= \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 197.2 & 32.4 & -129.6 & -43.2 & 0 & 0 \\ 32.4 & 110.8 & -43.2 & -14.4 & 0 & 0 \\ -129.6 & -43.2 & 302.8 & -32.4 & 0 & 0 \\ -43.2 & -14.4 & -32.4 & 389.2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \end{bmatrix}$$
$$+ \begin{bmatrix} -8.56 \\ -9.52 \\ -9.92 \\ -16.64 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \end{bmatrix} + 2$$

(the upper-level problem), subject to $y = y(x)$, solving (the lower-level problem)

$$\min_{y} q(x, y)$$
$$= \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 0 & -132.4 & -10.8 \\ 0 & 0 & 0 & 0 & -10.8 & -103.6 \\ 0 & 0 & 0 & 0 & 43.2 & 14.4 \\ 0 & 0 & 0 & 0 & 14.4 & 4.8 \\ -132.4 & -10.8 & 43.2 & 14.4 & 100.0 & 0 \\ -10.8 & -103.6 & 14.4 & 4.8 & 0 & 100.0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \end{bmatrix},$$

subject to

$$
\begin{array}{rrrrrrl}
13.24x_1 & +1.08x_2 & -4.32x_3 & -1.44x_4 & -10y_1 & & \leq 1, \\
1.08x_1 & +10.36x_2 & -1.44x_3 & -0.48x_4 & & -10y_2 & \leq 1, \\
13.24x_1 & +1.08x_2 & -4.32x_3 & -1.44x_4 & +10y_1 & & \leq 1.5, \\
1.08x_1 & +10.36x_2 & -1.44x_3 & -0.48x_4 & & +10y_2 & \leq 3, \\
-13.24x_1 & -1.08x_2 & +4.32x_3 & +1.44x_4 & -10y_1 & & \leq -1, \\
-1.08x_1 & -10.36x_2 & +1.44x_3 & +0.48x_4 & & -10y_2 & \leq -1,
\end{array}
$$

where

$$
x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \boldsymbol{R}^4 \quad \text{and} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \in \boldsymbol{R}^2.
$$

Since $m_3 = 0$ and $m_2 + m_4 = 2$, this problem has a unique global minimum with value $Q_G = ((1.5 - 1)/2)^2 + (m_3 + m_4)/4 = 0.3125$, as well as an additional $2^{m_2+m_3+m_4} - 2^{m_3} = 3$ local solutions.

9. REMARKS AND CONCLUSIONS

Test problems are only useful when they can be used to test solution techniques. The quadratic bilevel test problems generated by the proposed technique satisfy this criterion. We confirm this statement by considering three existing solution techniques.

Branch and Bound and Cutting Algorithms [Edmunds and Bard 1991]. The requirements of these methods are met since, for

$$
\Psi(x) = \{y : \exists x \quad \text{with} \quad (x, y) \in \Omega\},
$$

we have that

(1) all functions of the lower-level problem are twice continuously differentiable in y for all $y \in \Psi(x)$;
(2) the lower-level objective function is strictly convex in y for all $y \in \Psi(x)$;
(3) for each $x, \Psi(x)$ is a compact and convex set; and
(4) the upper-level objective function is continuous and convex in x and y.

Steepest Descent Methods [Gauvin and Savard 1989]. The requirements imposed by these techniques are met since the lower-level objective function is strictly convex in y (thus guaranteeing the uniqueness of the optimal solution of the lower-level problem for all x) and since there are no upper-level constraints.

Exact Penalty Function Approaches [Bi et al. 1991]. The requirements of these methods (for $\rho > 1$) are met since

(1) the upper-level objective function is twice continuously differentiable and the lower-level objective function and the constraint functions are convex in y, for fixed x, and three times continuously differentiable;
(2) the interior of Ω is nonempty and its closure is Ω;
(3) for each $(x, y(x))$ in the induced region of the untransformed problem, the Hessian of the Lagrangian function associated with the lower-level problem in y, denoted $\nabla_{yy} L$, satisfies $z^T \nabla_{yy} Lz = \|z\|^2$ for all directions $z \in \boldsymbol{R}^{ny}$; and
(4) Ω is a compact set; the closure property always holds, and to get a bounded set, without changing any other property, it is sufficient to introduce the constraint $x_k \geq 0$ in each problem $QBP(\rho_k)$.

The technique proposed in this paper exhibits a number of favorable properties, not the least of which is that the test problems can be generated without any significant computational effort. Besides having control over the number and type of minimizers, the sparsity and condition of the problems can also be affected. In addition, the number of upper- and lower-level variables in which the global solutions differ from the corresponding relaxed solution is controllable.

A FORTRAN 77 code that implements the technique described in this paper can be obtained by sending an email request to *phcalamai@dial.uwaterloo.ca*. In addition to the technique described in this paper, the authors are currently working on a similar method for the linear and linear-quadratic bilevel programming problems.

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