Inclusion Problems in Parallel Learning and Games (Extended Abstract)

Martin Kummer
Institut für Logik, Komplexität
und Deduktionssysteme
D-76128 Universität Karlsruhe, Germany
kummer@ira.uka.de

Frank Stephan *
Institut für Logik, Komplexität
und Deduktionssysteme
D-76128 Universität Karlsruhe, Germany
fstephan@ira.uka.de

1 Introduction

In a recent paper Kinber, Smith, Velauthapillai, and Wiehagen [8, 9] introduced a notion of parallel learning to model the learning of a collection of concepts all chosen from a single set. More precisely, they call a collection S of functions (m,n)-*learnable* iff there is a learning machine which for any n-tuple of pairwise distinct functions from S learns at least m of the n functions correctly from examples of their behavior after seeing some finite amount of input. One of the basic questions in this area is the "inclusion problem", i.e., the question for which m, n, m', n', every (m,n)-learnable class is also (m',n')-learnable. This question turns out to be combinatorially difficult and in [8, 9] it could be solved only for some few instances.

In this paper we propose a general approach for attacking this problem. The idea is to associate with each m, n, m', n' in a uniform way a finite 2-player game such that the second player has a winning strategy in this game iff every (m,n)-learnable class is (m',n')-learnable. In this way we take off the recursion theoretic disguise of the problem and isolate its combinatorial core. This works out nicely for the *popperian version* of parallel learning (where all guesses have to be total) and we get a complete characterization of the corresponding inclusion problem. For the general case we get a strong sufficient condition for noninclusions which gives us an explicit solution of the "equality problem", i.e., the question for which m, n, m', n', (m,n)-learnable and (m',n')-learnable coincide.

In the popperian case we are also able to explicitly characterize the "strength" of each particular noninclusion by the complexity of an oracle which is needed to overcome it. There are four different types of noninclusions.

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1.1 Notation and Definitons

The set of all natural numbers is denoted by ω. The set of all finite sequences of natural numbers is ω^*. $\sigma \star \tau$ is the concatenation of σ and τ, for $\sigma, \tau \in \omega^*$. Sometimes we simply write 131 for $1 \star 3 \star 1$, etc. We write $\sigma \preceq \tau$ if σ is an initial segment of τ. The set ω^* can be identified with an infinite tree whose nodes are ordered by $\preceq$. The root of this tree is the empty sequence λ. The functions $f : \omega \to \omega$ can be identified with an infinite branches of ω^*. The initial segment of f of length t is denoted by $f \restriction t$, i.e., $f \restriction t$ is the finite function with domain $\{0, \ldots, t-1\}$ which agrees with f on its domain. The recursion theoretic notation is standard and follows the books [14, 16]. Let REC be the set of all total recursive functions.

We recall the definitions of some well-known inference criteria, see [15] for further background. An inductive inference machine (IIM) M is a total recursive function with domain ω^* and range $\omega \cup \{?\}$. M *finitely infers* $f \in REC$ if there exists $t \in \omega$ such that $M(f \restriction s) = ?$, for all $s < t$ and $M(f \restriction t) = e$ where e is an index of f, i.e., $\varphi_e = f$. In this case we also write $M(f) = e$. We say that M diverges on input f if $M(f \restriction t) = ?$, for all $t \in \omega$. M finitely infers $S \subseteq REC$ iff M finitely infers all $f \in S$. Intuitively, after reading a certain finite initial segment of $f \in S$, M knows an index of f. $FIN = \{S \subseteq REC : (\exists M)[M \text{ finitely infers } S]\}$.

An IIM M is called *popperian* if every number in the range of M is an index of a total recursive function (see [2, Definition 2.16]). $PFIN$ is the class of all $S \subseteq REC$ which can be finitely inferred by a popperian IIM.

Below we consider a slight generalization of IIM's which take as input initial segments of n functions in parallel and output n-tuples of programs.

1.2 Basic Definitions and Facts for Parallel Learning

Now we turn to the notion of inference which is central for our paper.

Definition 1.1 [8] Let S be a set of recursive functions, $1 \leq m \leq n$. S is *finitely* (m,n)-*learnable* iff there is an inductive inference machine M that takes as input any pairwise distinct functions $f_1, \ldots, f_n \in S$ and computes an n-tuple of indices $e_1, \ldots, e_n$ such that at least m of them are correct, i.e., satisfy $f_i = \varphi_{e_i}$. Formally,

$$(\forall \text{ distinct } f_1, \ldots, f_n \in S)$$
$$(\exists t, e_1, \ldots, e_n)(\forall s < t)$$
$$[M(f_1 \upharpoonright s, \ldots, f_n \upharpoonright s) = ?,$$
$$M(f_1 \upharpoonright t, \ldots, f_n \upharpoonright t) = \langle e_1, \ldots, e_n \rangle, \text{ and}$$
$$|\{i : \varphi_{e_i} = f_i\}| \geq m].$$

Let $(m, n)FIN$ be the class of all S that are finitely (m, n)-learnable. Furthermore let $(m, n)PFIN$ be the class of all S that are finitely (m, n)-learnable via some popperian IIM M.

The following is a generalization of the basic fact that classes which contain an accumulation point are not finitely learnable.

Fact 1.2 [8, Theorem 11] *The following inclusions hold for FIN:*

- $(m+1, n+1)FIN \subseteq (m, n)FIN \subseteq (m, n+1)FIN$ *and*
- $(m, n)FIN \cap (h, k)FIN \subseteq (m+h, n+k)FIN$.

The same inclusions also hold for PFIN:

- $(m+1, n+1)PFIN \subseteq (m, n)PFIN \subseteq (m, n+1)PFIN$,
- $(m, n)PFIN \cap (h, k)PFIN \subseteq (m+h, n+k)PFIN$.

Trivially, $(n, n)FIN = FIN$, $(n, n)PFIN = PFIN$.

Fact 1.3 [8, Lemma 6] *If* $n-m > k-h$ *then the following noninclusions hold:*

- $(m, n)FIN \not\subseteq (h, k)FIN$ *and*
- $(m, n)PFIN \not\subseteq (h, k)PFIN$.

Corollary 1.4 *Let* $n \geq k$. *Then*

$$\begin{aligned} & (m, n)FIN \subseteq (h, k)FIN \\ \Leftrightarrow \quad & n - m \leq k - h \\ \Leftrightarrow \quad & (m, n)PFIN \subseteq (h, k)PFIN. \end{aligned}$$

2 A Game-Theoretical Characterization of the Inclusion Problem for Parallel Learning

In this section we provide game-theoretical characterizations for the inclusion problems for $PFIN$ and FIN. The idea of using games to study recursion theoretic questions was also used in investigations of the lattice of r.e. sets by Dëgtev [4] and Lachlan [12].

2.1 A Characterization of the Inclusion Problem for PFIN

Definition 2.1 A finite two person game $\mathcal{G}$ is a 5-tuple (G_1, G_2, W, s_0, t_0) where $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ are finite directed acyclic graphs, $W \subseteq V_1 \times V_2$, and $(s_0, t_0) \in (V_1 \times V_2) - W$.

We say that node v is *adjacent* to w in a directed graph G iff there is a directed path in G from v to w (the path may be empty, i.e., v is adjacent to v; we say that w is *properly* adjacent to v if w is adjacent to v and $v \neq w$).

The game (G_1, G_2, W, s_0, t_0) is played in rounds as follows. There are two players: Anke and Boris. At the beginning Anke has a marker at node $s_0 \in V_1$ and Boris has a marker at node $t_0 \in V_2$. A *position* is just an element of $V_1 \times V_2$. So the starting position is (s_0, t_0). In each round both players move their marker to some adjacent node. Boris moves first. All previous moves are known to both players. The position after Boris' move must belong to W. Anke is not allowed to perform empty moves. The first player who is unable to move according to these rules loses the game. By the restriction on the moves of Anke, it is clear that the game ends after at most $|V_1|$ rounds. Since the game is finite, one of the players has a winning strategy.

We will now describe for $1 \leq m \leq n \leq k$ and $1 \leq h \leq k$ a finite game $G(m, n; h, k)$ for which we prove that Boris has a winning strategy iff $(m, n)PFIN \subseteq (h, k)PFIN$. This characterizes the inclusion problem for $PFIN$. Since the game is finite, one can effectively decide which player has a winning strategy. Thus the inclusion problem for $PFIN$ is decidable.

For the sake of readability we formulate our game not quite according to Definition 2.1 but in a more intuitive way.

Definition 2.2 Let $1 \leq m \leq n \leq k$ and $1 \leq h \leq k$. The game $G(m, n; h, k)$ is played as follows: There are two players Anke and Boris. Each of them is equipped with several movable markers: For every n-element set $D \subseteq \{1, \ldots, k\}$ and every $j \in D$ Anke has a marker $\mu_{D,j}$. Boris has k markers $\nu_1, \ldots, \nu_k$.

The markers are moved on the infinite "board" ω^*. At the beginning each $\mu_{D,j}$ is placed on node j and each ν_j is placed on node λ. In each move a player is allowed to shift her (his) markers downwards in the tree to adjacent nodes. Boris moves first.

Anke's moves have to be of the following type ("node-splittings"): She selects a node σ which contains at least two of her markers and distributes all of her markers from σ onto the successor nodes $\sigma \star 1, \ldots, \sigma \star a$, for some $a \geq 2$, such that each of these nodes receives at least one marker.

Boris chooses for each of his markers ν_j an adjacent node $\sigma_j \succeq j$, containing at least one marker of Anke, and moves ν_j to node σ_j. Furthermore at any time Boris may move one of his markers from any node σ to 0 (and stay there forever); this is only for technical reasons, we need it to model a silly move of Boris.

Note that after each move of Boris any two markers either belong to incomparable nodes or they belong to the same node.

We say that the markers are in an A-configuration via $L \subseteq \omega^*$ iff

- Every node in L contains a marker of Anke and for each $j = 1, \ldots, k$ there is at most one node $\sigma \succeq j$ in L;

- For every D, at least m of Anke's markers $\mu_{D,1},\ldots,\mu_{D,n}$ are on nodes in L;
- Less than h of Boris' markers ν_j are on nodes in L.

The other configurations of the game are called B-configurations. Boris wins the game iff after each of his moves the markers are in a B-configuration.

Intuitively, Boris is trying to establish with each of his moves a B-configuration, while Anke tries to eventually establish an A-configuration which cannot be transformed in a B-configuration by any of Boris' moves.

At the beginning Anke's p markers are distributed on k nodes. Every move of Anke increases the number of nodes which contain at least one of her markers; so after Anke has moved j times at least $k+j$ different nodes contain one of her markers. Thus Anke cannot make more than $p-k$ moves and the game ends after at most $1+p-k$ rounds. Therefore we do not really need an infinite board, the finite tree $\bigcup_{s\leq p}\{0,\ldots,p\}^s$ would be enough.

Now it is easy to see that we can reformulate the game $G(m,n;h,k)$ as a finite game according to Definition 2.1: W corresponds to the set of all B-configurations, etc.

Theorem 2.3 *Let $k\geq n$.*
Then $(m,n)PFIN\subseteq(h,k)PFIN$
iff Boris has a winning strategy in $G(m,n;h,k)$.

Proof: $(\Rightarrow)$: This is shown by contraposition. Assume that Boris has no winning strategy. Since the game is finite, Anke has a winning strategy. Furthermore, we may assume that if Anke plays according to her winning strategy then after each of her moves she reaches an A-configuration. We show that this winning strategy is the basic building block to construct a class $S\in(m,n)PFIN-(h,k)PFIN$ by diagonalization.

Let $\{M_i\}_{i\in\omega}$ be a recursive listing of all inductive inference machines. We define S inductively and add for every i a set of k functions which is not $(h,k)PFIN$-inferred by M_i. This diagonalizes every $(h,k)PFIN$-algorithm. It should be noted that S is defined nonuniformly. This basic idea is due to [8, 9] who used it in their proofs that $(b-1,b)FIN\not\subseteq BC$ and $(1,2)FIN\not\subseteq(2,3)FIN$.

To ensure that $S\in(m,n)PFIN$ we construct a uniformly recursive family of total functions $\{F_{i,e,D,j}: i,e\in\omega\wedge D\subseteq\{1,\ldots,k\}\wedge|D|=n\wedge j\in D\}$ and a further (nonuniform) family $\{f_{i,e,j}: i,e\in\omega\wedge 1\leq j\leq k\}$ of total recursive functions with the following properties:

(I) $f_{i,e,j}(0)=\langle i,e,j\rangle$ and $(\forall^\infty x)[f_{i,e,j}(x)=0]$;

(II) For all $D\subseteq\{1,\ldots,k\}$, $|D|=n$, there are m distinct indices $j_1,\ldots,j_m\in D$ such that $f_{e,i,j_1}=F_{e,i,D,j_1}$, $f_{e,i,j_2}=F_{e,i,D,j_2}$, $\ldots$, $f_{e,i,j_m}=F_{e,i,D,j_m}$;

(III) M_i does not $(h,k)PFIN$-infer $f_{i,e,1},\ldots,f_{i,e,k}$.

S is the ascending union of finite sets S_i: Let $S_0=\emptyset$. In each step i there exists by (I) a constant e_i such that $f(x)=0$ for all $f\in S_i$ and all $x\geq e_i$ since S_i is finite. Let $S_{i+1}=S_i\cup\{f_{i,e_i,1},\ldots,f_{i,e_i,k}\}$.

$S\in(m,n)PFIN$: Consider any n pairwise different functions $g_1,\ldots,g_n\in S$. The inference algorithm first reads $g_1(0),\ldots,g_n(0)$ which gives the corresponding values $\langle i_1,e_{i_1},j_1\rangle,\ldots,\langle i_n,e_{i_n},j_n\rangle$ for all functions. Let e be the maximum of these e_i. Then the algorithm reads the initial segments of length e of every function. W.l.o.g., let $g_1,\ldots,g_u$ with maximal first component $i=i_1,\ldots,i_u$. The remaining functions have $i_j<i$ and, by (I), they satisfy the equation $g_j=(g_j\restriction e_i)\star 0^\omega$. Thus we can compute their indices which gives us $n-u$ correct components.

Since $j_1,\ldots,j_u$ are pairwise different there is a set $E\subseteq\{1,\ldots,k\}$, $|E|=n-u$ such that $D=\{j_1,\ldots,j_u\}\cap E$ is an n-elements set. Then we output in the first u components the indices of $F_{e,i,D,j_1},\ldots,F_{e,i,D,j_u}$. By (II), at least $m-(n-u)$ of them are correct. So we get a total of m correct components, as required.

$S\notin(h,k)PFIN$: In stage i of the construction of S, functions $f_{i,e_i,1},\ldots,f_{i,e_i,k}$ are added to S which are not inferred by M_i. So there is no IIM M such that $S\in(h,k)PFIN$ via M.

For the following fix i,e. The diagonalization implements the game $G(m,n;h,k)$ in a recursion theoretic way and uses the winning strategy of Anke to satisfy (I), (II) and (III). There is a translation between the nodes σ of the board and their current value $\tau_s(\sigma)$. This is either undefined or is a finite function defined on an initial segment. If $\tau_s(\sigma)$ is defined then $\tau_s(\sigma)\preceq\tau_{s+1}(\sigma)$. Furthermore let $\mu_{D,j,s}$ denote the position of marker $\mu_{D,j}$ at stage s, then

$$(*)\quad F_{i,e,D,j}=\lim_s\tau_s(\mu_{D,j,s})=\tau(\lim_s\mu_{D,j,s}).$$

The position $\nu_{j,s}$ of the marker ν_j are constructed from the functions ψ_j which denote the functions guessed by M_i:

$$\psi_j(x)=\begin{cases}\varphi_{e_j}(x) & \text{if } M_i(\langle i,e,1\rangle\star0^\omega,\ldots,\langle i,e,k\rangle\star0^\omega)\\ & \quad=(e_1,\ldots,e_k)\\ \uparrow & \text{otherwise.}\end{cases}$$

W.l.o.g. we assume that $\psi_{j,s}(x)$ is undefined if $x\geq s$ or $M_i(\langle i,e,1\rangle\star0^{s+2},\ldots,\langle i,e,k\rangle\star0^{s+2})$ is undefined. In every stage s the diagonalization procedure does the following:

- Check whether Boris has moved;
- If yes, check whether the game is in an B-configuration;
- If both conditions are satisfied, implement Anke's next move;
- Extend τ in order to make all functions total.

Anke's markers are at each stage on the *leaves* of the tree spanned by the positions of all markers. The other nodes are called interior nodes. If σ becomes an *interior* *node* in stage s then $\tau_{s'}(\sigma) = \tau_s(\sigma)$ for all $s' \geq s$. Only the τ-values of the leaves may change. Furthermore, if $\sigma \preceq \sigma'$ and $\tau_s(\sigma), \tau_s(\sigma')$ are defined then $\tau_s(\sigma) \preceq \tau_{s'}(\sigma')$. Now we present the algorithm in detail:

(1) Initialize the algorithm.
Let $\tau_0(\lambda) = \lambda$ and $\tau_0(j) = \langle i, e, j\rangle$ for $j = 1, \ldots, k$. Furthermore place Anke's markers $\mu_{D,j}$ on the node j and Boris' markers in the root λ. Let $s = 0$.

(2) Reconstruct the positions of Boris' markers.
Note that $dom(\psi_{j,s}) \subseteq \{0, \ldots, s\}$ and $dom(\tau_s(\sigma)) = \{0, \ldots, s\}$ for every leaf σ. For every marker ν_j define its position $\nu_{j,s}$ as follows:

$$\nu_{j,s} = \begin{cases} \sigma & \text{if there is } \sigma \in dom(\tau_s) \text{ such that } \sigma \text{ is the shortest string with } \psi_{j,s} \preceq \tau_s(\sigma); \\ 0 & \text{otherwise (no } \tau_s(\sigma) \text{ extends } \psi_{j,s}). \end{cases}$$

The 0 in the second case stands for markers no longer to be considered in the game; it means that for each leaf σ there is $x \leq s$ such that $\psi_{j,s}(x) \downarrow \neq \tau_s(\sigma)(x)$ and thus $\psi_j \neq F_{i,e,j,D}$ for all j and D. In particular if a marker ν_j once moved onto 0, it remains there forever, i.e., $\nu_{j,t} = 0$ for all $t \geq s$.

(3) Check whether Boris has completed his move.
A move of Boris is complete only if all his markers are in the leaves and if the game is in a B-configuration. If this has not already been achieved, goto step (5); otherwise continue at step (4).

(4) Implement Anke's move according to the winning strategy.
Since the game is in a B-configuration, Anke selects according to the winning strategy a leaf σ and distributes the markers from the node σ onto the nodes $\sigma \star 1, \ldots, \sigma \star a$ $(a \geq 2)$.

(5) Extend τ on the leaves.
If σ is an interior node then let

$$\tau_{s+1}(\sigma) = \tau_s(\sigma).$$

If σ is an old leaf (i.e. $\tau_s(\sigma)\downarrow$) then let

$$\tau_{s+1}(\sigma) = \tau_s(\sigma) \star 0.$$

If $\sigma = \eta \star b$ is a new leaf from step (4) then let

$$\tau_{s+1}(\sigma) = \tau_s(\eta) \star b \star 0.$$

Otherwise $\tau_{s+1}(\sigma)$ remains undefined.
Let $s = s + 1$ and goto step (2).

Note that $\mu_{D,j,s}$ is always placed on a leaf. By induction on s and the update-rule for τ_s, it follows that $|\tau_s(\sigma)| \geq s$ for all leaves σ. Therefore the functions $F_{i,e,D,j} = \lim_s \tau_s(\mu_{D,j,s})$ are total. If σ, η are incomparable nodes and $\tau_s(\sigma), \tau_s(\eta)$ are both defined, then they are also incomparable.

Anke moves only finitely often. After her last move she reaches an A-configuration. Choose a set of nodes L witnessing the A-configuration. Boris cannot reach a B-configuration (otherwise Anke would need at least one further move to win the game). Therefore Boris will never complete his last move. On the other hand, there are only finitely many stages where he moves his markers. Let s be sufficiently large such that after stage s no marker is moved and consider the final configuration in stage s.

(a) If one of Boris' markers ν_j is not in a leaf or in node 0, then the corresponding ψ_j is not total, i.e., M_i is not a $PFIN$-machine. In this case we let $f_{e,i,j} = \lim_t \tau_t(\eta)$ if $\eta \in L \wedge \eta \succeq j$. If there is no $\eta \in L$ with $\eta \succeq j$, we let $f_{e,i,j} = \langle e, i, j\rangle 0^\omega$. Then it is easy to see that (I), (II), (III) are satisfied.

(b) If in the final configuration every v_j is on a leaf or on node 0, then we are in an A-configuration (since Anke has won the game), say via L'. We let $f_{e,i,j} = \lim_t \tau_t(\eta)$ if $\eta \in L' \wedge \eta \succeq j$. If there is no $\eta \in L'$ with $\eta \succeq j$, then we choose $f_{e,i,j} \succeq \langle e, i, j\rangle$ such that $f_{e,i,j}$ is almost always zero and different from ψ_j.

As above (I), (II) are satisfied. Suppose for a contradiction that M_i $(h,k)PFIN$-infers $f_{e,i,1}, \ldots, f_{e,i,k}$. Then at least h of the equations $f_{e,i,j} = \psi_j$ must hold. If there is no $\eta \in L'$ with $\eta \succeq j$ then clearly $f_{e,i,j} \neq \psi_j$. Thus there are at least h nodes $\eta \in L'$ such that $\psi_j = \lim_t \tau_t(\eta)$. However, since the τ-values of incomparable nodes are incomparable, it follows that $\psi_j = \lim_t \tau_t(\eta)$ holds only if the final position of v_j is η. Thus in the final configuration at least h of Boris' markers are on nodes in L'. This contradicts the hypothesis that the final configuration is an A-configuration via L'. Hence (III) holds.

$(\Leftarrow)$: Assume that Boris has a winning strategy in $G(m,n;h,k)$ and $S \in (m,n)PFIN$ via M. We describe an $(h,k)PFIN$-machine N which infers S.

Given k pairwise different functions $f_1, \ldots, f_k$, N simulates $M(f_{i_1}, \ldots, f_{i_n})$ for every n-element set $D = \{i_1 < \cdots < i_n\} \subseteq \{1, \ldots, k\}$. N waits until M converges for each such D, say with output $e_{D,i_1}, \ldots, e_{D,i_n}$. By hypothesis, all of these programs compute total functions. Let $F_{D,j}$ denote the function computed by $e_{D,j}$.

Then N outputs programs which compute the functions $g_1, \ldots, g_k$ defined as follows: We consider the $F_{D,j}$'s, translate them into configurations of the game, move the markers according to the winning strategy, and translate the positions $\nu_{i,s}$ of ν_i back into g_i: $g_{i,s} = \tau_s(\nu_{i,s})$.

(1) Initialization.
Place the markers $\mu_{D,j}$ and ν_j on node j. Let $\tau_0(j) = \lambda$, $s = 0$, $x = 0$ and goto step (2).

(2) Check whether Anke has moved.
Select a leaf η such that $x = |\tau_s(\eta)|$ is minimal among the lengths $|\tau_s(\sigma)|$ of all leaves σ. For every marker $\mu_{D,j}$ placed on η calculate $F_{D,j}(x)$. Since the guesses $F_{D,j}$ are always total functions, these calculations terminate. Let $y_1, \ldots, y_a$ be the values. If $a > 1$ then we discovered a move of Anke and goto step (4). Otherwise, Anke did not move, and we goto step (3).

(3) Adjust τ while waiting for Anke's move.
Since Anke did not move, the game remains in a B-configuration and the only activity is to update τ:

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\sigma) * y_1 & \text{if } \sigma = \eta; \\ \tau_s(\sigma) & \text{otherwise } (\sigma \neq \eta). \end{cases}$$

Let $s = s + 1$ and goto step (2).

(4) Implement Anke's move.
The computations of the $F_{D,j}(x)$ with $\mu_{D,j}$ placed on the leaf η give several different values $y_1, \ldots, y_a$. Now τ is adjusted on the new leaves $\eta * b$ ($b = 1, \ldots, a$) as follows:

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\eta) * y_b & \text{if } \sigma = \eta * b; \\ \tau_s(\sigma) & \text{otherwise.} \end{cases}$$

All markers $\mu_{D,j}$ with $F_{D,j}(x) \succeq \tau_s(\eta) * y_b$ move from η to $\eta * b$. Goto step (5).

(5) Implement Boris' move.
If Boris has no marker on η then he does not move. Otherwise some marker ν_i remained on η while all markers of Anke moved to some leaf. Then Boris moves this marker according to his winning strategy from η to a new leaf $\eta * b$. Now the game is again in a B-configuration. Let $s = s + 1$ and goto step (2).

Anke makes only finitely many moves. Therefore the game ends in a B-configuration and for all leaves σ of this final configuration, $\tau_s(\sigma)$ is extended infinitely often. Since every ν_i eventually moves onto such a leaf, all $g_i = \lim_s \tau_s(\nu_{i,s})$ are total. Thus N is a $PFIN$-machine.

Now suppose that $f_1, \ldots, f_k \in S$. Let $L = \{\sigma : (\exists j)[f_j = \lim_s \tau_s(\sigma)]\}$. Since the f_j are total functions, the nodes $\sigma \in L$ must be leaves of the final configuration. Since for every n-element set D, m of the functions $F_{D,j}$ coincide with f_j, m of the markers $\mu_{D,j}$ are placed on nodes in L. Thus h of the markers ν_j must be placed on nodes in L since L otherwise the final configuration would be an A-configuration via L. Therefore $g_j = f_j$ for these $v_j \in L$, so N infers at least h of the $f_1, \ldots, f_k$. Thus $S \in (h,k)PFIN$. ∎

2.2 Noninclusions for FIN

In this section we define a slight modification of the game $G(m,n;h,k)$. This modification used to give a sufficient condition for the noninclusion $(m,n)FIN \not\subseteq (h,k)FIN$.

Definition 2.4 The game $G'(m,n;h,k)$ is a variant of the game $G(m,n;h,k)$. The players receive the same markers. Anke has for every n-element set $D \subseteq \{1,...,k\}$ and each $j \in D$ a marker $\mu_{D,j}$, Boris has the markers $\nu_1, \ldots, \nu_k$. Anke's markers $\mu_{D,j}$ are initially placed on the node j, Boris' markers on the root λ. As in the game G the markers move on the tree ω^* from nodes σ to adjacent nodes $\eta \succeq \sigma$. From now on the words leaf, interior node and successor refer to the subtree generated by the current positions of Anke's markers.

The definition of an A-configuration via a set L is the same as in the game G, but the implicit requirement that L consists of leaves must be made explicit since Anke's markers may remain on interior nodes:

- Every node in L is a leaf (and therefore contains a marker of Anke).
- For each $j = 1, \ldots, k$ there is at most one node $\sigma \succeq j$ in L.
- For every D, at least m of Anke's markers $\mu_{D,1}, \ldots, \mu_{D,n}$ are on nodes in L.
- Less than h of Boris' markers ν_j are on nodes in L.

The rules to move the markers are less restrictive:

- Anke moves her markers from nodes σ to any adjacent node $\eta \succeq \sigma$.
- Boris moves his markers from σ to $\eta \succeq \sigma$ or to 0, where η is inside the subtree generated by Anke's markers and markers on 0 do never leave this node.
- After Anke's move the game is in an A-configuration, after Boris' move it is in a B-configuration.

Boris moves first, then the players move alternately. Boris wins the game if he always moves into a B-configuration; otherwise the game comes to an end in an A-configuration and Anke wins the game.

Theorem 2.5
If Anke has a recursive winning strategy for the game $G'(m,n;h,k)$ then $(m,n)FIN \not\subseteq (h,k)FIN$.

Proof: The diagonalization works as in Theorem 2.3. In general it is the same except that the $F_{i,e,D,j}$ may be partial, the conditions (I), (II) and (III) are the same, also their verification after the algorithm to implement the winning strategy is the same. Again $F_{i,e,D,j} = \lim_s \tau_s(\mu_{D,j,s})$ and $\psi_j \preceq \tau_s(\nu_{j,s})$ if $\nu_{j,s} \neq 0$. The algorithm has to be partially adapted:

(1) Initialize the algorithm.
Place the markers $\mu_{D,j}$ and ν_j on node j. Let $\tau_0(j) = \lambda$, $s = 0$, $x = 0$ and goto step (2).

(2) Reconstruct the positions of Boris' markers.
Let $\nu_{j,s}$ be the shortest string $\sigma \in dom(\tau_s)$ such that $\psi_j \preceq \tau_s(\sigma) * 0^s$; if such a string does not exist let $\nu_{j,s} = 0$.

(3) Check whether Boris has completed his move.
If the game is in a B-configuration then Boris completed his move and the algorithm continues at step (5) otherwise goto step (4).

(4) Extend τ on the leaves while waiting for Boris' move.

$$\tau_{s+1}(\sigma) = \begin{cases} \tau_s(\sigma) * 0 & \text{if } \sigma \text{ is a leaf;} \\ \tau_s(\sigma) & \text{if } \sigma \text{ is an interior node;} \\ \uparrow & \text{otherwise.} \end{cases}$$

Let $s = s + 1$ and goto step (2).

(5) Implement Anke's move according to her winning strategy.
Since the game is in a B-configuration, Anke moves the markers according to her winning strategy from nodes σ onto nodes $\sigma' \in \sigma \star \{1, 2, \ldots\}^*$. The game is in an A-configuration again. Goto step (6).

(6) Update τ for stage $s+1$ after Boris and Anke have moved.
Let "old tree" refer to the tree generated by Anke's marker positions before step (5) and let "new tree" refer to that of the marker positions after step (5). Every node σ in the new tree can be split into an old part η which is the longest initial segment of σ belonging to the old tree and a new part η' defined by the equation $\sigma = \eta \star \eta'$. If σ already belongs to the old tree then $\eta' = \lambda$ otherwise $\eta' \in \{1, 2, \ldots\}^+$.
If σ is on the new tree, but was not an interior node on the old tree, then let

$$\tau_{s+1}(\sigma) = \tau_s(\eta) \star 0^s \star \eta'.$$

If σ was an interior node on the old tree, then let

$$\tau_{s+1}(\sigma) = \tau_s(\sigma).$$

Otherwise, i.e. when σ is not on the new tree, the value $\tau_{s+1}(\sigma)$ remains undefined.
Let $s = s + 1$ and goto step (2).

Since Anke follows in step (5) a recursive winning strategy, this strategy can be coded into the programs of the $F_{i,e,D,j}$. Further by the winning strategy, she moves only finitely often. After Anke's last move, Boris has only finitely many possibilities to shift his markers but he will not reach a B-configuration. So the game ends in a final A-configuration at some stage s witnessed via some set L of leaves. Now the $f_{i,e,j}$ are defined via L as in Theorem 2.3 and the further verification of the local step is the same. Note that the $f_{i,e,j}$ are total since $\lim_t \tau_t(\eta)$ is total iff η is a leaf at stage s. Those ψ_j, which belong to markers ν_j remaining on an interior node at stage s, are partial. ▮

A further modification is the game G'' which is a version between G and G'. The only difference between G'' and G is that Boris - as in the game G' - is not required to move all markers onto leaves while Anke's moves have to fulfil the same requirements as in the game G. Also the definition of A-configuration and B-configuration is the same as in game G. A small modification of the proof of Theorem 2.3 gives that $(m,n)PFIN \subseteq (h,k)FIN$ iff Boris has a winning strategy for the game $G''(m,n;h,k)$.

The game $G'(m,n;h,k)$ does not characterize the inclusion-problem for FIN. Nevertheless this can be done with a more complicated game using the methods of [12, 13]. However, it might not be worth the effort, since by now we cannot guarantee that there are any nontrivial inclusions for FIN besides those that follow from Fact 1.2. If this is indeed the case then one would have an easy explicit description of the inclusion structure and no games would be needed.

Open Problem: Are there any inclusions for FIN besides those generated by Fact 1.2?

There are certain partial results on the way to this conjecture. Proposition 3.5 shows that $(n,n+1)FIN \not\subseteq (n+1,n+2)FIN$. Furthermore Corollary 3.10 establishes the conjecture for $m = 1$: $(1,n)FIN \subseteq (h,k)FIN$ iff $k \geq hn$. For $m = 2$ we can show as a first result that $(2,n)FIN \subseteq (3,k)FIN$ iff $k \geq 2n-1$. But already the questions whether

- $(2,n)FIN \subseteq (5,k)FIN \Leftrightarrow k \geq 3n-1$ and
- $(3,n)FIN \subseteq (4,k)FIN \Leftrightarrow k \geq 2n-2$

are open.

3 Explicit Results on the Inclusion Problem for Parallel Learning and Popperian Parallel Learning

The next results are application of the game-theoretic characterization of the inclusion-relation.

3.1 On Popperian Parallel Learning

Proposition 3.1 $(2,3)PFIN \not\subseteq (3,4)PFIN$.
$(3,4)PFIN \not\subseteq (4,5)PFIN$.

Proof: Both noninclusions follow from suitable winning strategies for the first player in the corresponding game $G(m,n;h,k)$; we show only the first noninclusion.

The winning strategy of Anke starts with creating three new leaves 11, 12, 13 below 1. Then Boris places his marker w.l.o.g. onto the leaf 11. Now Anke creates three new leaves below 2; Boris answers by moving to an node $2x$. The following diagram illustrates the situation, the first four rows show the positions of the four classes of Anke's markers $\{\mu_{D,j} : j \in D\}$ for $D = \{2,3,4\}, \{1,3,4\}, \{1,2,4\}$, and $\{1,2,3\}$. The last row shows the positions of Boris' markers $\nu_1, \ldots, \nu_4$.

-	21	3	4
11	-	3	4
12	22	-	4
13	23	3	-
11	$2x$	3	4

If $2x = 22$ then the game is in an A-configuration via $\{12, 23, 3, 4\}$. Otherwise the game is in an A-configuration via $\{13, 22, 3, 4\}$; thus Boris lost the game.

In this strategy Anke's second move depends on the first move of Boris. One can check that there is no winning strategy for Anke which is independent of Boris' moves. ▮

Theorem 3.2 *$(n,n+1)PFIN = (n+1,n+2)PFIN$ for all $n \geq 4$.*

Open Problem: Find an explicit characterization of the equality problem for $PFIN$, i.e., of the set

$$\{(m,n;h,k) : (m,n)PFIN = (h,k)PFIN\}.$$

3.2 On Parallel Learning

Since the condition of Theorem 2.5 is not a characterization as in Theorem 2.3, the following Proposition 3.3 must be proved in a direct way.

Proposition 3.3 *If $(m,n)FIN \not\subseteq (h,k)FIN$ then $(m,n+1)FIN \not\subseteq (h,k+1)FIN$.*

Proof: Let $S' \in (m,n)FIN - (h,k)FIN$, w.l.o.g. $f(0) = 0$ for all $f \in S$. A set $S = S' \cup \{g_i : i \in \omega\} \in (m,n+1)FIN - (h,k+1)FIN$ is constructed via a sequence g_i; the functions g_i are of the form $e_i 0^{a_i} b_i 0^{\omega}$ where $e_0 = 1$, $e_{i+1} = e_i + a_i + 2$ and $a_i, b_i \in \omega$.

Already these construction-requirements guarantee that $S \in (m,n+1)FIN$: Given $n+1$ functions ordered by the first value ($f_1(0) \leq f_2(0) \leq \ldots \leq f_{n+1}(0)$) there is an u such that $0 = f_u(0) < f_{u+1}(0)$, w.l.o.g. $u \leq n$. The indices of $f_{u+1}, \ldots, f_n$ can be calculated from their initial segments of length $f_{n+1}(0)$. So one obtains $n-u$ correct indices. Since $f_1, \ldots, f_u \in S'$ and, by Fact 1.3, $S' \in (m-(n-u), n-(n-u))FIN$, u indices can be calculated such that $m-(n-u)$ of them are correct. In total there are n indices for $f_1, \ldots, f_n$ of which m are correct. Suggesting for f_{n+1} some default index, $S \in (m,n+1)FIN$.

One diagonalizes against the i-th $(h,k+1)PFIN$-machine M_i while defining g_i. Since $S' \notin (h,k)FIN$, there are $f_1, \ldots, f_k \in S'$ such that

- either M_i does not converge on input $f_1, ..., f_k, e_i 0^{\omega}$. Then let $g_i = e_i 0^{\omega}$, $a_i = 0$, $b_i = 0$ and $e_{i+1} = e_i + 2$.
- or M_i converges after reading a_i arguments to $k+1$ indices such that $k+1-h$ of the indices for $f_1, \ldots, f_k$ are incorrect.
  Then select b_i such that the index for $g_i = e_i 0^{a_i} b_i 0^{\omega}$ is also incorrect and let $e_{i+1} = e_i + a_i + 2$.

In both cases g_i is selected as a witness against the M_i (together with $f_1, \ldots, f_k$). So $S \notin (h,k+1)FIN$. ▮

In the following we show explicit noninclusions by providing winning strategies for Anke in $G'(m,n;h,k)$.

Proposition 3.4 $(2,3)FIN \not\subseteq (3,4)FIN$.

Proof: The first move of Anke's winning strategy for the game $G'(2,3;3,4)$ creates an A-configuration via the set $\{11, 21, 3, 4\}$ in order to force Boris to move at least one marker, w.l.o.g. Boris moves his first marker:

–	2	3	4
1	–	3	4
11	2	–	4
1	21	3	–
11	2	3	4

Now Anke moves the markers, which remained on 1, to the node 12 and those, which remained on 2, to the node 21. Now the set $\{12, 21, 4\}$ witnesses an A-configuration, since it contains two markers of each row, but it does not contain three of Boris' markers.

–	21	3	4
12	–	3	4
11	21	–	4
12	21	3	–
11	21	3	4

Boris can move his second marker to 21, but he cannot move his marker from 11 to 12. So $\{12, 21, 4\}$ contains at most two of Boris' markers and thus he has lost the game. ▮

Proposition 3.5 $(n,n+1)FIN \not\subseteq (n+1,n+2)FIN$.

Propositions 3.3 and 3.5 confirm the following conjecture of Kinber and Wiehagen [9]:

For all m and n with $1 \leq m < n$:
$(m,n)FIN \not\subseteq (m+1,n+1)FIN$.

They already indicated in [9, p. 15] that their conjecture implies that there are no nontrivial equalities between the FIN-classes:

Theorem 3.6 *$(m,n)FIN = (h,k)FIN$ iff $m = h \wedge n = k$ or $m = n \wedge h = k$.*

Proof: The if-direction is trivial. For the converse assume that $(m,n)FIN = (h,k)FIN$, and say $n \leq k$. By Fact 1.3 it follows that $n - m = k - h$, By Fact 1.2 $(h,k)FIN \subseteq (h-1,k-1)FIN \subseteq \cdots \subseteq (h-b,k-b)FIN$, for every $b < h$. Therefore, if $k > n$ then $(h,k)FIN \subseteq (m+1,n+1)FIN$ (let $b = k-n-1$). But as we noted above, $(m,n)FIN \not\subseteq (m+1,n+1)FIN$ unless $m = n$. So it follows that $m = n \wedge h = k$ or $n = k \wedge m = h$. ▮

Together with the facts $(3,4)PFIN \not\subseteq (4,5)PFIN$, $(4,5)PFIN = (5,6)PFIN$, $(3,4)PFIN \subseteq (4,5)FIN$, Proposition 3.5 shows that all three inclusion-problems differ in general. (The fourth type of inclusion (FIN versus $PFIN$) is not considered since it never holds: $(\forall n)[FIN \not\subseteq (1,n)PFIN]$. This is witnessed by the family $S = \{f \in REC : \varphi_{f(0)} = f\}$.)

Corollary 3.7 *The following three inclusion structures do not coincide in general:*
- $(m,n)PFIN \subseteq (h,k)PFIN$;
- $(m,n)PFIN \subseteq (h,k)FIN$;
- $(m,n)FIN \subseteq (h,k)FIN$.

3.3 Admissible Sets

In his investigation of the inclusion problem for frequency computation, Dëgtev [4] introduced the notion of (m,n)-admissible sets. They also appear implicitly in Kinber's thesis [7]. We show that they are also of use in the study of parallel learning, since they give us further explicit noninclusions.

Definition 3.8 Let $s \geq n \geq m \geq 1$. A finite set $V \subseteq \omega^s$ is called (m,n)-*admissible* iff for every n numbers x_i ($1 \leq x_1 < \cdots < x_n \leq s$) there exists a vector $(b_1, \ldots, b_n) \in \omega^n$ such that for every $v \in V$:

$$|\{i : v[x_i] = b_i\}| \geq m.$$

In other words, there is a function $f : \{1, ..., s\}^n \to \omega^n$ such that for all pairwise distinct $x_1, \ldots, x_n \in \{1, ..., s\}$, $|\{i : v[x_i] = (f(x_1, \ldots, x_n))_i\}| \geq m$.

Theorem 3.9 *If V is (m,n)-admissible, but not (h,k)-admissible, then $(m,n)FIN \not\subseteq (h,k)PFIN$, in particular, $(m,n)FIN \not\subseteq (h,k)FIN$ and $(m,n)PFIN \not\subseteq (h,k)PFIN$.*

Proof: If $k < n$, then an (m,n)-admissible set V which is not (h,k)-admissible exists only for $n - m > k - h$ and so Theorem 3.9 reduces to Fact 1.3.

Let $n \leq k$ and let $V \subseteq \{1, \ldots, q\}^k$ be (m,n)-admissible but not (h,k)-admissible. By the remark following Theorem 2.5 it suffices to show that Anke has a winning strategy in the game $G''(m,n;h,k)$.

In the first move, Anke places her markers on the leaves according to an (m,n)-operator for V; i.e., if the (m,n)-operator for $D = \{i_1, \ldots, i_n\}$ gives $(b_1, \ldots, b_n)$ then each marker μ_{D,i_j} is placed on the leaf $i_j \star b_j$. Thus for every $v \in V$ the associated set $L_v = \{i \star v[i] : 1 \leq i \leq k\}$ witnesses the A-configuration.

Assume that Boris could move into a B-configuration by placing his markers on nodes $1 \star c_1, 2 \star c_2, \ldots, k \star c_k$. Then for each v, h markers are in the set L_v and h components of $(c_1, c_2, \ldots, c_k)$ agree with the corresponding components of v. Thus V would be (h,k)-admissible via $(c_1, c_2, \ldots, c_k)$, a contradiction. Thus whatever Boris does, the game remains in an A-configuration and Anke wins the game. ∎

The set $\{1^k, 2^k, \ldots, n^k\}$ is an example for an $(1,n)$-admissible set which is not (h,k)-admissible for any h,k with $h/k > 1/n$. So this set provides following noninclusion:

Corollary 3.10 *If $1/n < h/k$ then*
$(1,n)FIN \not\subseteq (h,k)FIN \wedge (1,n)PFIN \not\subseteq (h,k)PFIN$.

Further noninclusions can be derived from the following fact:

Fact 3.11 [10, Lemma 9.5] *If one of the following conditions hold then there is an (m,n)-admissible set V which is not (h,k)-admissible:*
(a) $n - 2m > k - 2h \geq 0$;
(b) $n = 2m + 1$, $k = 2h + 1$ *and* $k > n$;
(c) *There is an $(m,n-1)$-admissible set W which is not $(h,k-1)$-admissible.*

Proof: (a) Let $V = \{0,1\}^{n-2m} \times \{0\}^{k+m}$.

(b) Let V contain 0^k, 1^k, all vectors 0^i10^{k-i-1} for $i = 0, ..., k-1$, $10^{k-2}1$ and 0^i110^{k-i-2} for $i = 0, ..., k-2$. Note that V is the closure of $\{0^k, 10^{k-1}, 110^{k-2}, 1^k\}$ under "rotational shifts".

(c) Let $V = W \times \{0,1\}$.
See [10] for the verification that the sets V have the required properties. ∎

Fact 3.12 [7, Theorem 1.6] *Every $(n,n+1)$-admissible set is $(n+1,n+2)$-admissible for $n \geq 2$.*
Therefore the inclusion-relations of FIN and $PFIN$ both differ from the admissibility-criterion.

Proof: It is sufficient to show this for subsets $V \subseteq \omega^{n+2}$. Le V be $(n, n+1)$-admissible and let $0^{n+2} \in V$. If V is not $(n+1, n+2)$-admissible via 0^{n+2} then some vector has two nonzero components, say $1^20^n \in V$. Since V is $(1,2)$-admissible, there are a and b such that $v[0] = a$ and $v[1] = b$ for every $v \in V$, say $a = 1$ and $b = 0$. Now V is $(n+1, n+2)$-admissible via 10^{n+1}: Otherwise there would exist some $v \in V$ differing on two components from 10^{n+1}, e.g., $v[i] = 1$ and $v[j] = 1$ and $i > j > 0$. Since either $v[0] = 1$ or $v[1] = 0$, $j > 1$. Thus the projection onto the coordinates $0, i, j$ contains the set $\{100, 011, 000\}$ which is not $(2,3)$-admissible.
$(2,3)FIN \not\subseteq (3,4)FIN$ and $(2,3)PFIN \not\subseteq (3,4)PFIN$ while every $(2,3)$-admissible set is $(3,4)$-admissible, so the second statement follows. ∎

Nevertheless results on admissible sets allow a further partial result for the $PFIN$-equality-problem:

Proposition 3.13 *If $m/n < 2/3$ and $(h,k) \neq (m,n)$ then there exists either an (m,n)-admissible set which is not (h,k)-admissible or an (h,k)-admissible set which is not (m,n)-admissible.*
If $m/n < 2/3$ and $(h,k) \neq (m,n)$ then $(m,n)PFIN \neq (h,k)PFIN$.

4 Oracles for Finitary Games

In Definition 2.1 we have introduced the notion of a finite game $\mathcal{G} = (G_1, G_2, W, s_0, t_0)$ in order to characterize the inclusion problem for $PFIN$. Our next goal is to determine when $(m,n)PFIN \subseteq (h,k)PFIN[A]$. Here $(h,k)PFIN[A]$ is the class of all $S \in REC$ which are $(h,k)PFIN$-inferable by an algorithm which has access to oracle $A \subseteq \omega$. To this end we have to investigate the "off-line" version of $G(m,n;h,k)$. But this is only a special case of a more general approach which works for arbitrary finite games $\mathcal{G}$, and which may be of use in similar situations and for other inference criteria. In this section we study the general approach and in the next section we discuss the application to $PFIN$.

Definition 4.1 In the *off-line* version of G, Anke announces at the beginning the list of her moves $(v_1, ..., v_k)$ in rounds $1, \ldots, k$. Here, v_{i+1} must be properly adjacent to v_i for $i = 0, \ldots, k$ (where $v_0 = s_0$) and v_k must not have outcoming edges. Boris wins iff there is a list of counter moves $(w_0, \ldots, w_k)$ such that Boris wins the original game if both players play according to their move lists, i.e., Boris moves from t_0 to w_0, Anke from s_0 to v_1, Boris from w_0 to w_1, ... until Anke moves from v_{k-1} to v_k and Boris wins the game by his last move from w_{k-1} to w_k. Formally the w_i have to satisfy that w_0 is adjacent to t_0, w_{i+1} is adjacent to w_i for $i = 0, \ldots, k-1$ and $(v_i, w_i) \in W$ for $i = 0, \ldots, k$.

In the *infinite* version of $\mathcal{G}$ both players are allowed to perform empty moves and we drop the condition that the position after each move of Boris belongs to W. There are ω many rounds. Since G_1, G_2 are finite and acyclic it follows that at almost all rounds the marker of Anke [Boris] is at some fixed node s_1 [t_1]. Boris wins the game iff $(s_1, t_1) \in W$.

It is easy to see that any winning strategy for the finite version can be translated into a winning strategy for the infinite version.

We are interested in computability questions for the *off-line version of the infinite game.* Suppose we are given an index i for the list of moves of Anke in the infinite game. Can we compute uniformly in i a list of counter moves for Boris such that he wins the corresponding infinite game? We want to characterize the oracles A such that this computation can be done recursive in A. Let *comp*($\mathcal{G}$) denote the class of all such oracles A.

Let PA denote the class of all degrees containing a complete and consistent extension of Peano Arithmetic. See [14, pp. 510-515] for background information. Let $DNR_k = \{g : \omega \to \{0, \ldots, k-1\} \mid (\forall i)[g(i) \neq \varphi_i(i)]\}$. Jockusch [6, Proposition 2] showed that PA coincides with the degrees of functions in DNR_k for all $k \geq 2$.

Theorem 4.2 *There are exactly four possible cases:*
(1) *If Boris has a winning strategy for $\mathcal{G}$ then comp($\mathcal{G}$) $= \{A : A \subseteq \omega\}$.*
(2) *If Boris has a winning strategy for the off-line version of $\mathcal{G}$ but not for $\mathcal{G}$, then comp($\mathcal{G}$) $= \{A : dg_T(A) \in PA\}$.*
(3) *If Anke has a winning strategy for the off-line version and for every s adjacent to s_0 there is t adjacent to t_0 with $(s,t) \in W$, then comp($\mathcal{G}$) $= \{A : A \geq_T K\}$.*
(4) *If there is s adjacent to s_0 such that $(s,t) \notin W$ for all t adjacent to t_0, then comp($\mathcal{G}$) $= \emptyset$.*

Proof: (1) and (4) are obvious.
(2) Assume that Boris has a winning strategy for the (finite) off-line version of $\mathcal{G}$. Every list of counter moves $(w_0, \ldots, w_k)$ induces in a uniform way a list of counter moves for the infinite off-line version as we now explain.

Suppose we are given an index i for the list of moves for Anke. W.l.o.g. assume that $\varphi_i(0) = s_0$. We define h_i, the induced list of counter moves, as follows:

Let $h_i(0) = w_0$. If $h_i(n) = w_m$, $(\varphi_i(n+1), h_i(n)) \notin W$, and $m < k$, then let $h_i(n+1) = w_{m+1}$, else let $h_i(n+1) = h_i(n)$.

We say that $w = (w_0, \ldots, w_k)$ loses against i in step n if n is minimal such that $(\varphi_i(n), h_i(n)) \notin W$. In that case we write $l(w,i) = n$. If n does not exist then $l(w,i) = \infty$. Note that the graph of $l(-,-)$ is uniformly recursive.

If $l(w,i) = \infty$ then in particular the induced h_i wins against φ_i in the infinite version of the game. It easily follows from the hypothesis that for every infinite list of moves φ_i of Anke, there exists w with $l(w,i) = \infty$.

Since the off-line version of the finite game has at most $k = |V_1|$ rounds, we may assume that all lists of counter moves have length k. Let L be the finite set of all these lists.

Now suppose that we are given an index i of the list of moves of Anke in the infinite game. We show that if $dg_T(A) \in PA$ then we can A-recursively compute a finite list w which does not lose against i in any step. By the remarks above, this completes the proof.

By the hypothesis we know that a suitable w is contained in L. So it suffices to provide an A-recursive reduction procedure which reduces L to a one-element set that still contains a suitable w.

Construction:
As long as $|L| > 1$ choose different lists $u, w \in L$ and compute an index e of the following constant function f.

$$f(x) = \begin{cases} 0, & \text{if } l(u,i) < \infty \ \wedge \ l(u,i) \leq l(w,i); \\ 1, & \text{if } l(w,i) < l(u,i); \\ \uparrow, & \text{otherwise.} \end{cases}$$

Since $dg_T(A) \in PA$ we can A-recursively *exclude* either $\varphi_e(e) = 0$ or $\varphi_e(e) = 1$. In the first case we let $L = L - \{w\}$, else we let $L = L - \{u\}$. Then we repeat the procedure. □

Note that if the list which we remove does not lose against i at any step, then the list that we keep in L has the same property. Thus at each step L contains a suitable list, i.e., the reduction procedure is correct. This completes the proof of the first part.

For the other direction, assume that Anke has a winning strategy for the off-line version of $\mathcal{G}$. In our case this is just a function $p : V_1 \times V_2 \to V_1$ such that Anke wins if she plays $p(v_1, v_2)$ in every position (v_1, v_2) where it is her turn to move.

(∗) We may assume w.l.o.g. that $(p(v_1, v_2), v_2) \notin W$ in every position $(v_1, v_2) \in W$ which is reachable when Anke plays according to her winning strategy.

Let $a = |V_1|$, $V_2 = \{w_0, \ldots, w_{k-1}\}$. Suppose that $A \in$ *comp*($\mathcal{G}$). We shall show that there is an A-recursive function in DNR_k. As was mentioned above this implies $dg_T(A) \in PA$.

To this end we define inductively for every sequence of a numbers $\sigma = (z_1, \ldots, z_a)$ a move list $g = g_\sigma$ for Anke in the infinite off-line version of $\mathcal{G}$ as follows:

Construction:
Initialization: Let $n = 0$; $g(0) = s_0$; $v = s_0$; $w = t_0$. Goto step 1.
Step j: Let $C_j = \{i : w_i \text{ adjacent to } w, (v, w_i) \in W\}$. While $\varphi_{z_j,n}(z_j) \notin C_j$ let $g(n+1) = g(n)$, $n = n+1$. (Now $\varphi_{z_j,n}(z_j) \in C_j$.) Let $w = w_i$ for $i = \varphi_{z_j,n}(z_j)$, let $g(n+1) = p(v, w)$, $v = p(v, w)$, $n = n+1$, and goto step $j+1$. □

Note that g is the sequence of moves according to the winning strategy of Anke against a potential Boris who chooses his move in round j as follows: He waits until $\varphi_{z_j}(z_j)$ is defined, say equal to i. Then he moves to w_i (if this is correct and produces a position in W).

Thus any A-recursive counter strategy that wins against g must be different from this potential strategy. We complete the proof by showing that if one can A-recursively compute different counter strategies for all such g, then $dg_T(A) \in PA$.

By the hypothesis, there exists in an uniform way an A-recursive infinite list f_σ of counter moves for Boris with $(s_1, t_1) \in W$ for $s_1 = \lim_n g_\sigma(n)$ and $t_1 = \lim_n f_\sigma(n)$. We may assume w.l.o.g.:

$$(**) \quad [g_\sigma(n+1) = g_\sigma(n) \ \wedge \ (g_\sigma(n), f_\sigma(n)) \in W] \Rightarrow f_\sigma(n+1) = f_\sigma(n).$$

Let $n_j(\sigma)$ denote the j-th number n (in increasing order) such that $g_\sigma(n+1) \neq g_\sigma(n)$, if it exists. For every $\sigma = (z_1, \ldots, z_a)$ and every i $(1 \leq i \leq a)$ we define a predicate $P(i, \sigma)$ as follows:

$$P(i,\sigma) \ \Leftrightarrow \ (\forall j,\ 1 \leq j < i)[n_j(\sigma)\downarrow \ \wedge \ f_\sigma(n_j(\sigma)) = w_m \text{ for } m = \varphi_{z_j}(z_j)].$$

Note that trivially $P(1, \sigma) \equiv \mathit{true}$. Also note that $P(i, \sigma)$ is r.e. in A. Intuitively, if $P(i, \sigma)$ holds then g_σ has correctly predicted the behaviour of f_σ up to round i.

If g_σ would correctly predict the game up to round a then $(g(n_{a-1}+1), f(n_{a-1}))$ would be a final position in $\mathcal{G}$ such that $(g(n_{a-1}+1), w) \notin W$ for any node w adjacent to $f(n_{a-1})$. Furthermore $g(n) = g(n_{a-1}+1)$ for all $n > n_{a-1}$. Since $\lim_n f(n)$ is adjacent to $f(n_{a-1})$ we would have $(\lim_n g(n), \lim_n f(n)) \notin W$, contradicting the property of f. Therefore, $P(a, \sigma) \equiv \mathit{false}$. Consider the least i with $1 \leq i < a$ such that:

$$(\exists z_{i+1}, \ldots, z_a)(\forall z_1, \ldots, z_i)[\neg P(i+1, \sigma) \text{ for } \sigma = (z_1, \ldots, z_a)].$$

Note that i exists because $\neg P(a, \sigma) \equiv \mathit{true}$. For the following we fix witnesses $z_{i+1}, \ldots, z_a$. If $i > 1$ then, using the minimality of i, we get

$$\neg(\exists z_i)(\forall z_1, \ldots, z_{i-1})[\neg P(i, \sigma)].$$

Or equivalently,

$$(+) \quad (\forall z_i)(\exists z_1, \ldots, z_{i-1})[P(i, \sigma)].$$

For $i = 1$ this holds trivially since $P(1, \sigma) \equiv \mathit{true}$. Now we can A-recursively compute a function $d \in DNR_k$ as follows:

Construction:
On input z_i we search for $z_1, \ldots, z_{i-1}$ such that $P(i, \sigma)$ holds. The search is effective since $P(-,-)$ is r.e. in A. By (+), the search terminates.

Let $n_{i-1} = n_{i-1}(\sigma), f = f_\sigma, g = g_\sigma$. By the choice of $z_{i+1}, \ldots, z_a$ we know that $P(i+1, \sigma)$ does not hold. This means:

$$(++) \quad \text{If } n_i\downarrow \ \wedge \ (g(n_i), f(n_i)) \in W, \text{ then there is } m \text{ with } f(n_i) = w_m \ \wedge \ m \neq \varphi_{z_i}(z_i).$$

Therefore we search for the least $n' > n_{i-1}$ such that
(a) $n' = n_i$, or
(b) $(g(n'), f(n')) \in W$.
If the search terminates by (a) then we know $\varphi_{z_i}(z_i)$ and define $d(z_i) = \min\{x : x \neq \varphi_{z_i}(z_i)\}$. If the search terminates by (b) then we let $d(z_i) = m$ with $w_m = f(n')$. □

Clearly $d(z_i) < k$. By the property of f we have that $(g(n), f(n)) \in W$ for all sufficiently large n. Thus the search terminates and d is total.

If n_i is undefined and $\varphi_{z_i}(z_i) = m'$ then $(g(n'), w_{m'}) \notin W$ or $w_{m'}$ is not adjacent to $f(n_{i-1})$. Hence in this case $m \neq m'$.

Now suppose that the search terminates by (b) and n_i is defined. Then $n_i > n'$. Since $g(n) = g(n')$ for $n' \leq n \leq n_i$ we get by assumption $(**)$ that $f(n) = f(n')$ for $n' \leq n \leq n_i$. Using (++) we get $d(z_i) = m \neq \varphi_{z_i}(z_i)$.

Thus we have $d \in DNR_k$ and therefore $dg_T(A) \in PA$. This completes the proof of part (2).

(3) Suppose we are given an index i of the move list of Anke. Let s_1 be the final position of the marker of Anke. Then $s_1 = \lim_{n\to\infty} \varphi_i(n)$. Using a K-oracle we can compute s_1 from i. By hypothesis, there exists $t_1 \in V_2$ adjacent to t_0 with $(s_1, t_1) \in W$. So the list of counter moves $(t_1, t_1, \ldots)$ wins for Boris.

Now suppose that Boris can A-recursively compute from every index i of a move list of Anke an A-recursive function f_i which is a winning list of counter moves. Let $(v_1, \ldots, v_k)$ be a winning list of moves for Anke in the off-line version of the finite game.

For any $x_1, \ldots, x_k$ we define a recursive function $g = g_{x_1,\ldots,x_k}$ as follows: $g(0) = s_0$, and $g(n) = v_m$ where $m = |\{i : x_i \in K_n\}|$ for $n > 0$.

Now we can A-recursively enumerate for all $x_1, \ldots, x_k$ a set of at most k strings such that $F_k^K(x_1, \ldots, x_k)$ is among them. By the Nonspeedup Theorem [1] it follows that $K \leq_T A$.

The enumeration procedure works as follows:

Compute an index i of $g = g_{x_1,\ldots,x_k}$. In step n enumerate $(K_n(x_1), \ldots, K_n(x_k))$ if $(g(n), f_i(n)) \in W$.

Since f_i wins against g it follows that $(g(n), f_i(n)) \in W$ for all sufficiently large n, so $F_k^K(x_1, \ldots, x_k)$ is enumerated. Suppose for a contradiction that we enumerate

$k+1$ different strings. Choose n_j minimal such that a string with exactly j 1s is enumerated in step n_j, $j = 0, \ldots, k$. Note that $(g(n_0), \ldots, g(n_k)) = (s_0, v_1, \ldots, v_k)$.

Then the list of counter moves $(f_i(n_0), \ldots, f_i(n_k)$ wins against $l = (v_1, \ldots, v_k)$ in the off-line version of the finite game. This contradicts the hypothesis that l is a winning list of moves. ∎

5 On the Strength of Noninclusions in Parallel Learning

Suppose that $(m,n)PFIN \not\subseteq (h,k)PFIN$, i.e., there exists a set $S \subseteq REC$ which can be inferred by an $(m,n)PFIN$-machine, but not by any $(h,k)PFIN$-machine. What happens if we scale-up the $(h,k)PFIN$-machines and allow them to access an oracle A? Then S might become $(h,k)PFIN$-inferable if A is sufficiently powerful. How powerful must A be? This question is studied for teams of finite learners in [11], by similar methods.

In this section we characterize the oracles A such that every $S \in (m,n)PFIN$ can be A-recursively inferred by an $(h,k)PFIN$-machine. Let $strength(m,n;h,k)$ denote the class of all such A. The strength of the noninclusion $(m,n)PFIN \not\subseteq (h,k)PFIN$ is measured by the class $strength(m,n;h,k)$: the stronger the noninclusion the smaller is $strength(m,n;h,k)$. Applying Theorem 4.2 we show that there are four possibilities for $strength(m,n;h,k)$.

Theorem 5.1 *Let $1 \le m \le n$, $1 \le h \le k$.*

(1) *$strength(m,n;h,k) = \{A : A \subseteq \omega\}$ iff $[n \ge k \wedge n-m \le k-h] \vee [n \le k$ and Boris has a winning strategy in $G(m,n;h,k)]$.*

(2) *$strength(m,n;h,k) = \{A : dg_T(A) \in PA\}$ iff $[n \le k \wedge$ Anke has a winning strategy in $G(m,n;h,k)$, but Boris has a winning strategy in the off-line version of $G(m,n;h,k)]$.*

(3) *$strength(m,n;h,k) = \{A : A \ge_T K\}$ iff $[n \le k \wedge n-m \le k-h \wedge$ Anke has a winning strategy in the off-line version of $G(m,n;h,k)]$.*

(4) *$strength(m,n;h,k) = \emptyset$ iff $n-m > k-h$.*

Proof: Since the right-hand sides of (1)-(4) is a complete case distinction, it suffices to show the if-direction in (1)-(4).

(1) If the condition on the right hand side holds then we have $(m,n)PFIN \subseteq (h,k)PFIN$ by Corollary 1.4 and Theorem 2.3, respectively.

(2) Assume that $n \le k$ and that Boris has a winning strategy for the off-line version of $G(m,n;h,k)$. We show that $strength(m,n;h,k) \supseteq \{A : dg_T(A) \in PA\}$.

Fix any A with $dg_T(A) \in PA$. By Theorem 4.2, Boris has a winning strategy for the infinite off-line version of $G(m,n;h,k)$. Similar as in the proof of Theorem 2.3, ($\Leftarrow$), we can build, for any given $(m,n)PFIN$-machine M that infers a set $S \subseteq REC$, an A-recursive $(h,k)PFIN$-machine N^A which simulates M:

On input $f_1, \ldots, f_k$ we simulate $M(f_{i_1}, \ldots, f_{i_k})$ for every n-element subset $D = \{i_1 < \cdots < i_n\} \subseteq \{1, \ldots, k\}$ until it outputs programs $(e_{D,i_1}, \ldots, e_{D,i_n})$, for every such D. These programs determine in a uniform way an off-line strategy for Anke in $G(m,n;h,k)$. We compute an index i of this strategy. Now we are using the oracle A to compute a finite list l of counter moves for Boris such that l does not lose against i. This is done as in the proof of Theorem 4.2, (2). Only at this point the machine N^A outputs programs for k functions $g_1, \ldots, g_k$. These are equipped with the move-list l which they use in the same way as the winning strategy for Boris was used in the proof of Theorem 2.3, ($\Leftarrow$). By an analogous argument as in this proof it follows that at least h of the g_i's are correct, and all of them are total. $S \in (h,k)PFIN[A]$ via N^A.

Now assume that $n \le k$ and that Anke has a winning strategy in $G(m,n;h,k)$. Fix any oracle A with $dg_T(A) \notin PA$. We show that $A \notin strength(m,n;h,k)$.

By a modification of the proof of Theorem 2.3, ($\Rightarrow$), we can construct a set $S \in (m,n)PFIN - (h,k)PFIN[A]$. The idea to diagonalize a single machine M_i^A is to build a uniformly recursive sequence $F_{\langle i,p\rangle,D,j}$ for $p = 0, 1, \ldots$ The functions $F_{\langle i,p\rangle,D,j}$ are defined according to the moves of Anke which are given by the p-th recursive off-line strategy $strat_p$. Here we refer to the corresponding listing $\{strat_p\}_{p\in\omega}$ of recursive off-line strategies for Anke as they are used in the proof of Theorem 4.2, (2). Note that as in the proof of the Theorem 2.3, ($\Rightarrow$), the action of M_i^A defines an A-recursive counter strategy for each $strat_p$. Recall from the proof of Theorem 4.2, (2), that each $strat_p$ wins against some potential strategy of Boris where the moves in each round are correctly predicted by $strat_p$. We have shown there that if one can A-recursively compute for each $strat_p$ a counter strategy which wins against $strat_p$, then $dg_T(A) \in PA$. The action of M_i^A on the initial segments of the $F_{\langle i,p\rangle,D,j}$'s however defines us an A-recursive counter strategy for Boris.

In order to formally cover the case where the $F_{\langle i,p\rangle,D,j}$'s split before M_i^A has produced its guess, we may introduce the convention that the corresponding positions are B-configurations. I.e., if Boris' markers are in node λ and one of Anke's marker is not in $\{1, \ldots, k\}$ then this a B-configuration. In particular, Boris wins the off-line version of the infinite game if he keeps his markers in λ and finds a stage where Anke moves. However, if Anke would never move then this strategy would not be successful.

Since $A \notin PA$ it follows that there exists p such that the strategy provided by M_i^A loses against $strat_p$. This means that we can define functions $f_1, \ldots, f_k$ which are not $(h,k)PFIN$-inferred by M_i^A, but which are inferred in a uniform way by a recursive $(m,n)PFIN$-algorithm.

As in the proof of Theorem 2.3, ($\Rightarrow$), we define $S \in (m,n)PFIN - (h,k)PFIN[A]$ by pasting together the k-tuples that diagonalize the different M_i^A for $i = 0, 1, \ldots$

(3) Assume that $n \leq k$ and $n - m \leq k - h$. Let an $(m,n)PFIN$-machine M be given which infers a class $S \subseteq REC$. We can build a K-recursive $(h,k)PFIN$-machine N^K which simulates M.

As above, on input $f_1, \ldots, f_k$ we simulate $M(f_{i_1}, \ldots, f_{i_n})$ for every n-element subset $D = \{i_1 < \cdots < i_n\} \subseteq \{1, \ldots, k\}$ until it outputs programs $(e_{D,i_1}, \ldots, e_{D,i_n})$, for every such D. Since the $F_{D,j}$ are total we can K-recursively compute which of them are equal and which are different. Then we find s_0 such that if two of these functions differ then they differ on an argument less than s_0. If there is a function $F_{D,j}$ which agrees with f_j for all arguments less than s_0 then let $g_j = F_{D,j}$, otherwise let $g_j = \lambda x.0$. We output a k-tuple of programs for $(g_1, \ldots, g_k)$.

Clearly, every program which we output computes a total function. We claim that at most $n - m$ of them are incorrect. Suppose for a contradiction that there is a set E of $n - m + 1$ indices j with $f_j \neq g_j$. Choose an n-element set $D \subseteq \{1, \ldots, k\}$ with $E \subseteq D$. For every $j \in E$: if $F_{D,j} = g_j$ then $F_{D,j} \neq f_j$, by the hypothesis on g_j; if $F_{D,j} \neq g_j$ then $F_{D,j} \neq f_j$, since $F_{D,j}$ must already differ from f_j on some argument less than s_0. Thus more than $n - m$ of the $F_{D,j}$ are incorrect, i.e., M does not (m,n)-infer $\{f_i : i \in D\}$, a contradiction. This shows that N^A makes at most $n - m \leq k - h$ errors, i.e., it (h,k)-infers S.

Finally, assume that $n \leq k$, Anke has a winning strategy in the off-line version of $G(h,k;m,n)$, and $(m,n)PFIN \subseteq (h,k)PFIN[A]$. Then $A \geq_T K$. This is shown by combining the proofs of Theorem 2.3 with the proof of Theorem 4.2 in a similar way as in (3) above. We omit the details.

(4) This follows from the observation that the diagonalization in the proof of Fact 1.3 in [8] also works against $(h,k)PFIN$-algorithms which have access to an oracle. ∎

Each of these four cases may occur in a nontrivial way:

(1) $(4,5)PFIN \subseteq (5,6)PFIN$, see Proposition 3.2.

(2) This holds for $(2,3)PFIN$ versus $(3,4)PFIN[A]$, one can check that Boris has a winning strategy for the off-line version of the game $G(2,3;3,4)$ (cf. the proof of Proposition 3.1).

(3) This holds if $n \leq k$, $n - m \leq k - h$, and there is an (m,n)-admissible set which is not (h,k)-admissible: the proof of Theorem 3.9 actually provides an off-line winning strategy for Anke. For example this holds for $PFIN(1,3)$ versus $PFIN(2,5)[A]$.

(4) Obvious.

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