# A Strong Direct Product Theorem for Disjointness 

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#### Abstract

A strong direct product theorem says that if we want to compute $k$ independent instances of a function, using less than $k$ times the resources needed for one instance, then the overall success probability will be exponentially small in $k$. We establish such a theorem for the randomized communication complexity of the Disjointness problem, i.e., with communication const • $k n$ the success probability of solving $k$ instances can only be exponentially small in $k$. We show that this bound even holds in an $A M$ communication protocol with limited ambiguity.

The main result implies a new lower bound for Disjointness in a restricted 3-player NOF protocol, and optimal communication-space tradeoffs for Boolean matrix product.

Our main result follows from a solution to the dual of a linear programming problem, whose feasibility comes from a so-called Intersection Sampling Lemma that generalizes a result by Razborov Raz92.

We also discuss a new lower bound technique for randomized communication complexity called the generalized rectangle bound that we use in our proof.


## 1 Introduction

### 1.1 Direct product theorems

One of the fundamental questions that can be asked in any model of computation is how well computing several instances of the same problem can be composed. Are significant savings possible when computing the same function on $k$ independent inputs? How do the resources needed for computing $k$ independent instances of $f$ scale with the resources needed for one instance and with $k$ ? "Resources" may refer to any complexity measure. Similarly we need to define what we mean by "computing $f$ ".

In this paper we consider randomized communication complexity. A protocol between players Alice and Bob is given $k$ inputs $\left(x_{1}, y_{1}\right), \ldots,\left(x_{k}, y_{k}\right)$, and has to output the vector of $k$ answers $f\left(x_{1}, y_{1}\right), \ldots, f\left(x_{k}, y_{k}\right)$. The issue is how the protocol can optimally distribute its resources among the $k$ instances it needs to compute. We focus on the relation between the total amount of communication and the best-achievable success probability $\sigma$ (in the worst-case).

If every protocol with communication $c$ must have some constant error probability when computing just one instance of $f$, then for computing $k$ instances with communication $c$ we expect a constant error to occur on each instance and an exponentially small success probability for the

[^0]$k$-vector as a whole. If this is really the case for all functions, we say that a weak direct product theorem holds.

However, even if we allow our protocol to use communication $k c$, on average it still has only communication $c$ available per instance. So even in this more generous case we might expect to incur constant error per instance and overall an exponentially small success probability (unless a protocol could somehow correlate its computation on several instances for all possible choices of inputs). If such a statement is true we call it a strong direct product theorem (SDPT).

Strong direct product theorems are usually hard to prove and sometimes not even true. In particular Shaltiel [Sha01 exhibits a general setup in which strong direct product theorems cannot be expected. His main argument is that in the distributional complexity setting one can construct functions for which there is a "hard core" of some size $\epsilon$ that cannot be ignored when allowing only error probability $\epsilon / 3$ (making computing one instance hard), yet given $k$ instances only roughly an $\epsilon k$ of them will be in the hard core, and we can re-allocate most of our resources to those, and easily solve the other instances (the function is defined to be almost trivial outside the hard core), altogether using around $\epsilon k$ times the resources for one instance while having small overall error.

An incomplete list of examples of "positive" results about DPT's are Nisan et al. 's NRS94] strong direct product theorem for "decision forests", Parnafes et al.'s PRW97 direct product theorem for "forests" of communication protocols, Shaltiel's strong direct product theorems for "fair" decision trees and for the discrepancy bound for communication complexity under the uniform distribution [Sha01, Lee et al.'s analogous result for arbitrary distributions LSS08], Viola and Wigderson's extension to the multiparty case [VW08], Ambainis et al.'s SDPT for the quantum query complexity of symmetric functions [ASW09], Jain et al.'s SDPT for subdistribution bounds in communication complexity JKN08, Ben-Aroya et al.'s SDPT for the quantum one-way communication complexity of the Index function [BRW08] and several more. In a similar vein are "XOR"-lemmas like Yao's Yao82. "Direct Sum" results which state that $k$ times the resources are needed without the success probability deterioration are also important in communication complexity, see KN97.

In this paper we focus on the Disjointness problem in communication complexity. Suppose Alice has an $n$-bit input $x$ and Bob has an $n$-bit input $y$. These $x$ and $y$ represent sets, and $\operatorname{DISJ}_{n}(x, y)=1$ iff those sets are disjoint. Note that $\mathrm{DISJ}_{n}$ is the negation of $\operatorname{NDISJ}_{n}=\mathrm{OR}_{n}(x \wedge y)$, where $x \wedge y$ is the $n$-bit string obtained by bitwise AND-ing $x$ and $y$. In many ways, NDISJ $_{n}$ plays a central role in communication complexity. In particular, it is "NP complete" BFS86] in the communication complexity world. The communication complexity of $\operatorname{NDISJ}_{n}$ has been well studied: e.g. it takes $\Theta(n)$ bits of communication classically [KS92, Raz92] and $\Theta(\sqrt{n})$ quantumly AA03, Raz03].

For the case where Alice and Bob want to compute $k$ instances of Disjointness, we establish a strong direct product theorem in Section 3,

## SDPT for randomized communication complexity:

Every randomized protocol that computes NDISJ $n_{n}^{(k)}$ using $T \leq \beta k n$ bits of communication has worst-case success probability $\sigma=2^{-\Omega(k)}$.
Previously, Klauck et al. KSW07] proved that the same success probability bound holds when the communication is $\beta k \sqrt{n}$ (but even in the quantum case). The same result was obtained by Beame et al. [BPSW06], who actually give an SDPT for the rectangle/corruption bound under product distributions (under such distributions DISJ $_{n}$ has complexity $\sqrt{n}$ ). Klauck [K04] also showed a weak DPT for the rectangle/corruption bound under all distributions, which implies that with communication $\beta n$ the success probability goes down exponentially in $k$.

Our approach is as follows. First we massage the problem in a very similar manner as in [KSW07]. This leads to the problem of finding $k$ elements in the intersection of two $N$ bit strings. Since these can easily be verified, we can assume that the protocol either gives up or produces correct outputs. We are interested in the tradeoff between success probability and communication.

The next step is to formulate a linear program that corresponds to a relaxation of an integer program that expresses a convex combination of partitions of the communication matrix with the desired acceptance probabilities. Similar programs have been considered before by Lovasz [L90] and by Karchmer et al. [KKN95], but have been rarely used to bound randomized communication complexity. The program expresses, that we can detect inputs with intersection $k$ with "high" probability, while not accepting inputs with smaller intersection at all, and, trivially but importantly, accepting the remaining inputs with probability at most 1 . This extra constraint expresses the fact that we do not talk about covers of the communication matrix, but partitions. Unsurprisingly we prove the lower bound by exhibiting a solution to the dual.

To prove feasibility of the solution we provide a new lemma that we call the intersection sampling lemma. This lemma is a generalization of Razborov's main lemma from Raz92 and follows from it by a rather simple induction argument. The lemma states that any rectangle that is large among the disjoint $x, y$ is also large for inputs that have intersection size $k$ for every $k$ under a suitable distribution (losing a $2^{k}$ factor).

### 1.2 Applications

### 1.2.1 Communication-Space Tradeoffs

Our main result has some applications to other problems. First, we consider communication-space tradeoffs. Research on communication-space tradeoffs has been initiated by Lam et al. [LTT92] in a restricted setting, and by Beame et al. BTY94 in a general model of space-bounded communication complexity. In the setting of communication-space tradeoffs, players Alice and Bob are modeled as space bounded circuits, and we are interested in the communication cost when given particular space bounds.

We study the problems of Boolean matrix-vector product and Boolean matrix product. In the first problem there are an $N \times N$ matrix $A$ (input to Alice) and a vector $b$ of dimension $N$ (input to Bob), the goal is to compute the vector $c=A b$, where $c_{i}=\vee_{j=1}^{n}\left(A[i, j] \wedge b_{j}\right)$. In the problem of matrix multiplication two input matrices have to be multiplied with the analogous Boolean product.

Time-space tradeoffs for Boolean matrix-vector multiplication have been analyzed in an average case scenario by Abrahamson [Abr90], whose results give a worst case lower bound of $T S=\Omega\left(N^{3 / 2}\right)$ for classical algorithms. He conjectured that a worst case lower bound of $T S=\Omega\left(N^{2}\right)$ holds, which was later confirmed by KSW07.

Beame et al. gave tight lower bounds for the communication-space tradeoffs for the matrixvector product and matrix product over GF(2), but stated the complexity of Boolean matrix-vector multiplication as an open problem. Klauck [K04] generalized these results to the quantum case, but also showed the following classical lower bounds for the Boolean product and randomized protocols: for matrix-vector product $C S^{2}=\Omega\left(N^{2}\right)$, and for matrix-matrix product $C S^{2}=\Omega\left(N^{3}\right)$. Using our direct product result we are now able to show that any randomized protocol for matrix-vector product satisfies $C S=\Omega\left(N^{2}\right)$, for matrix-matrix product $C S=\Omega\left(N^{3}\right)$. These bounds match the trivial upper bounds.

### 1.2.2 Multiparty Communication

Consider the Nondisjointness problem in the 3 player number-on the forehead setting, i.e., Alice sees inputs $y, z$ Bob sees $x, z$ and Charlie sees $x, y$. They have to decide whether there is an index $i$ such that $x_{i}=y_{i}=z_{i}=1$. Lee and Shraibman [LS09] show that the randomized complexity of this problem is $\Omega\left(n^{1 / 4}\right)$. Prior to that result larger bounds were shown for models in which the interaction between the players is restricted. In particular, in the model with one-way communication, Viola and Wigderson show a $\Omega(\sqrt{n})$ lower bound VW07, and in the model, where Charlie sends a single message, followed by an arbitrary protocol between Alice and Bob, Beame et al. [BPSW06] show an $\Omega\left(n^{1 / 3}\right)$ lower bound, which was later simplified by de Wolf BRW08. Using our main theorem we can show that the latter type of protocol actually needs communication $\Omega(\sqrt{n})$.

### 1.3 The Generalized Rectangle Bound

Our main result is proved by giving a solution to the dual of a linear program. While this program is tailor made for the problem at hand, this is a general approach described e.g. in [L90]. In Section 5 we investigate the LP given by Lovasz and show that its objective function equals the rectangle/corruption bound (see [BPSW06, K03]).

Adding another, seemingly trivial constraint gives us a more powerful lower bound method (via the dual), which goes beyond the power of the rectangle bound by using that protocols partition the inputs into rectangles instead of covering them. The lower bound method is similar to the rectangle bound, but allows the use of negative weights for a small fraction of the 1-inputs. In this it is very similar to the way the generalized discrepancy method [S08, K07] relates to the standard discrepancy bound (both the latter methods are lower bounds on quantum communication). The new method has the potential to overcome limitations of the rectangle bound (e.g., in the situation when the nondeterministic communication complexity is small).

We can also pinpoint the power of the generalized rectangle bound more closely by showing that it actually lower bounds unambiguous AM-protocols. Indeed our main result also holds for AM-protocols with ambiguity $2^{\epsilon k}$. Note that NDISJ $_{n}$ has very efficient nondeterministic protocols (and so does its $k$-fold), so the lower bound really comes from the partition constraints. In particular we show that any unambiguous AM-protocol for $\mathrm{NDISJ}_{n}$ needs linear communication (while nondeterministic protocols need communication $O(\log n)$. This shows that the generalized rectangle bound goes beyond the standard one.

## 2 Preliminaries

In this section we give some definitions of some of the models of communication we study. We refer to KN97 for more background in communication complexity. Note that this section has nothing to offer to readers interested only in our main result.

### 2.1 Some Definitions on Communication Complexity

The randomized protocols we consider are all public coin protocols. Success probability of a protocol is defined to be the probability over the coins to compute the correct output for a worst case input.

A nondeterministic protocol for a Boolean function $f$ is a cover of the communication matrix of $f$ with 1-chromatic rectangles, its cost is the logarithm of the number of rectangles used. Alternatively,
a nondeterministic protocol can be viewed as a proof system, in which a prover sends a proof to Alice, after which Alice and Bob verify the proof. In a valid protocol for all 1-inputs there exists a proof that is accepted, and for no 0 -input any proof is accepted. The cost is the amount of communication between Alice and Bob. A nondeterministic protocol with ambiguity $t$ is a nondeterministic protocol in which each 1 -input has no more than $t$ different proofs. For $t=1$ protocols are called unambiguous.

Karchmer et al. KNSW94 have shown that nondeterministic protocols with ambiguity $t$ have complexity at least $\Omega(\sqrt{D(f)} / t)$. Also the rank lower bound holds for unambiguous protocols.

In the case that we want to compute a $k$-tuple of Boolean functions by a nondeterministic protocol, we assume that the prover wants to convince Alice and Bob of the fact that $f\left(x_{i}, y_{i}\right)=1$ for as many $i$ as possible. Such a protocol is correct, if for all $x_{1}, y_{1}, \ldots, x_{k}, y_{k}$ such that $f\left(x_{i}, x_{i}\right)=1$ for all $i \in I \subseteq\{1, \ldots, k\}$ there is a proof such that Alice and Bob agree on output $o_{1}, \ldots, o_{k}$ with $o_{i}=1 \Longleftrightarrow i \in I$, while for no $i \notin I$ there exists a proof such that $o_{i}=1$ will be an output. Note that in this definition we never require the prover to convince Alice and Bob of the fact that $f\left(x_{i}, y_{i}\right)=0$ for any position $i$, so this is a genuine one-sided nondeterminism for many-output problems.

In other words every nondeterministic protocol with ambiguity $t$ is a collection of at most $2^{c}$ rectangles each labeled by an output sequence such that for each input $x_{1}, y_{1}, \ldots, x_{k}, y_{k}$ and each rectangle $R$ with output $o_{1}, \ldots, o_{k}$ containing that input: $o_{i} \leq f\left(x_{i}, y_{i}\right)$ for all $i$, and there exists a rectangle containing the input where $o_{i}=f\left(x_{i}, y_{i}\right)$ for all $i$. Furthermore each input is contained in at most $t$ such rectangles. The communication cost is then $c$.

An Arthur-Merlin communication protocol (first suggested in [BFS86]) with ambiguity $t$ and communication $c$ is a convex combination of a set of nondeterministic protocols $P_{i}$, each occurring with probability $p_{i}$. Each nondeterministic protocol is a collection of at most $2^{c}$ rectangles each labeled by an output sequence and each input is contained in at most $t$ such rectangles per $P_{i}$. We require that for each input $x_{1}, y_{1}, \ldots, x_{k}, y_{k}$ with probability at least $1-\epsilon$ the protocol $P_{i}$ has $x_{1}, y_{1}, \ldots, x_{k}, y_{k}$ in some rectangle labeled $f\left(x_{1}, y_{1}\right), \ldots, f\left(x_{k}, y_{k}\right)$, whereas with probability at $\operatorname{most} \epsilon$ a $P_{i}$ contains the input in a rectangle labeled with $o_{j}=1$ while $f\left(x_{j}, y_{j}\right)=0$ for some $i$. An AM-protocol with ambiguity 1 is called unambiguous (note that for different values of the public coin different proofs are allowed for the same input).

### 2.2 Communicating Circuits

In the standard model of communication complexity Alice and Bob are computationally unbounded entities, but we are also interested in what happens if they have bounded memory, i.e., they work with a bounded amount of storage. To this end we model Alice and Bob as communicating circuits. In short, these circuits place no restrictions on local gates, but require the number of bits stored locally to be bounded. Communication is the number of wires crossing between Alice and Bob's part of the circuit.

A pair of communicating circuits is actually a single circuit partitioned into two parts. The allowed operations are local computations and access to the inputs. Alice's part of the circuit may read single bits from her input, and Bob's part of the circuit may do so for his input. Otherwise arbitrary gates (of any fan-in) on the locally available bits can be used.

The communication $C$ between the two parties is simply the number of wires carrying bits that cross between the two parts of the circuit. A pair of communicating circuits uses space $S$, if the whole circuit works on $S$ bits.

In the problems we consider, the number of outputs is much larger than the memory of the players. Therefore we use the following output convention. The player who computes the value of an output sends this value to the other player at a predetermined point in the protocol, who is then allowed to "forget" the output. Outputs have to be made in some specified order in the circuit, i.e., we expect the $i$ th output to be made at a specific gate.

## 3 The Direct Product Theorem

In this section we formally state and prove our main result.

### 3.1 Massaging the Problem

In this section we bring the $k$-fold $\operatorname{NDISJ}_{n}$ problem into anther form that will be easier to handle. More precisely, we will consider the following three problems. We freely associate strings $x \in\{0,1\}^{n}$ with the sets they are the characteristic vectors of.

Definition 1 1. $\operatorname{NDISJ}_{n}^{(k)}$ is the problem, given $k$ pairs of strings $x_{i}, y_{i}$ of length $n$ each, to compute the $k$-tuple of function values of $\mathrm{NDISJ}_{n}$ on these.
2. $\operatorname{Search}_{n}^{(k)}$ is the problem, given $k$ pairs of strings $x_{i}, y_{i}$ of length $n$ each, to find indices $j_{1}, \ldots, j_{k}$, such that $x_{i}$ and $y_{i}$ intersect in $j_{i}$. If $x_{i}$ and $y_{i}$ are disjoint, output 0 for position $i$.
3. $\operatorname{Search}_{\binom{N}{k}}$ is the problem, given two strings $x, y$ of length $N$, to find $k$ indices $j_{1}, \ldots, j_{k}$, such that $x$ and $y$ intersect in all $j_{i}$. If $|x \cap y|<k$ output 0 .

We will prove that problem 3) is hard in the following subsections and state the result now.
Lemma 1 (Main) There are constants $0<\alpha, \beta, \gamma \leq 1$ such that any randomized protocol with communication $\beta N$ for the problem Search $\binom{N}{k}$ with $k \leq \gamma N$ has success probability at most $2^{-\alpha k}$.

We now establish that the first two problems are also at least as hard as 3) by reductions very similar to the analogous ones in KSW07.

Theorem 2 (SDPT for Search) There are constants $0<\alpha^{\prime}, \beta \leq 1$ such that any randomized protocol with communication $\beta k n$ for the problem $\operatorname{Search}_{n}^{(k)}$ has success probability at most $2^{-\alpha^{\prime} k}$.

Proof. We show that a protocol for $\operatorname{Search}_{n}^{(k)}$ can be used to solve $\operatorname{Search}_{\left({ }_{\alpha k / 4}^{N}\right)}$. Set $N=k n$, fix a protocol $P$ for $\operatorname{Search}_{n}^{(k)}$ with success probability $\sigma$. Now consider the following protocol that acts on $N$-bit inputs $x, y$ :

1. Apply a random permutation $\pi$ to $x$ and to $y$.
2. Run $P$ on $\pi(x), \pi(y)$.
3. If $P$ makes at least $\alpha k / 4$ outputs $\neq 0$, then output any $\alpha k / 4$ of them (undoing $\pi$ before).

## 4. Otherwise output 0 .

This uses the same communication as before. Note that the above protocol will work correctly and solve $\operatorname{Search}_{(\alpha k / 4}^{N}$, whenever $P$ works, and when at least $\alpha k / 4$ positions $i$ with $x_{i}=y_{i}=1$ end up in different blocks after the permutation, so they can be produced by $P$ (assuming that $\alpha k / 4$ such positions exist).

The probability of at least $\alpha k / 4$ positions $i$ with $x_{i}=y_{i}=1$ being in different blocks (assuming that so many exist) is at least

$$
\frac{N}{N} \cdot \frac{N-n}{N-1} \cdots \frac{N-\alpha(k / 4) n+1}{N-\alpha(k / 4)+1} \geq\left(\frac{N-\alpha(k / 4) n}{N}\right)^{\alpha k / 4} \geq(1-\alpha / 4)^{\alpha k / 4}
$$

So the success probability is at least $\sigma \cdot(1-\alpha / 4)^{\alpha k / 4}$ which defines our $\alpha^{\prime}$ via

$$
2^{-\alpha \cdot(\alpha k / 4)} \geq 2^{-\alpha^{\prime} k} \cdot(1-\alpha / 4)^{\alpha k / 4} .
$$

This gives an $\alpha^{\prime}>0$, since $(1-\alpha / 4)^{\alpha / 4} \geq 2^{-\alpha^{2} / 8}$ for $0 \leq \alpha<1$.
The above argument only works, if $\alpha k / 4 \leq \gamma N \Longleftrightarrow n \geq \alpha /(4 \gamma)$. Since these are constants the opposite case can be covered by assuming $n=O(1)$, i.e., we now have to show that solving many size $O(1)$ instances is hard. So when the communication is less than $\epsilon k=\Theta(k n)$, it can easily be shown via an information theoretic argument, that it is impossible to solve Search ${ }_{n}^{(k)}$ with better success than $2^{-\Omega(k)}$ : under the uniform distribution the players don't communicate enough to agree on a set of $k$ outputs of sufficient entropy.

Theorem 3 (SDPT for NDISJ $_{n}$ ) There exist constants $0<\alpha^{\prime \prime}, \beta^{\prime \prime} \leq 1$ such that every randomized protocol for $\operatorname{NDISJ}_{n}^{(k)}$ with $\beta^{\prime \prime} k n$ communication has success probability $\sigma \leq 2^{-\alpha^{\prime \prime} k}$.

Proof. A protocol $P$ for NDISJ $_{n}^{(k)}$ with success probability $\sigma$ and communication $C \leq \beta^{\prime \prime} k n$ can be used to build a protocol $P^{\prime}$ for $\operatorname{Search}_{n}^{(k)}$ with slightly worse success probability:

1. Run $P$ on the original inputs and remember which blocks are accepted.
2. Run simultaneously (at most $k$ ) binary searches on the accepted blocks. Iterate this $s=$ $2 \log \left(1 / \beta^{\prime \prime}\right)$ times. Each iteration is computed by running $P$ on the parts of the blocks that are known to contain a position $j$ with $x_{i}(j)=y_{i}(j)=1$, halving the remaining instance size each time.
3. Run the trivial protocol on each of the remaining parts of the instances to look for an intersection there (each remaining part has size $n / 2^{s}$ ).

This new protocol $P^{\prime}$ uses $(s+1) C+k n / 2^{s}=\mathrm{O}\left(\beta^{\prime \prime} \log \left(1 / \beta^{\prime \prime}\right) k n\right)$ communication. With probability at least $\sigma^{s+1}, P$ succeeds in all iterations, in which case $P^{\prime}$ solves $\operatorname{Search}_{n}^{(k)}$.

So setting $\beta^{\prime \prime}$ such that $\beta \geq \mathrm{O}\left(\beta^{\prime \prime} \log \left(1 / \beta^{\prime \prime}\right)\right)$ and $\alpha^{\prime \prime}=\alpha^{\prime} /(s+1)$ we get the desired reduction.

### 3.2 The Linear Program

In this section we provide a linear program, whose value gives a lower bound on the communication complexity of solving $\operatorname{Search}_{\binom{N}{k}}$ with success probability $\sigma$. This will be our tool to establish Lemma 1 .

So consider any protocol for Search $\binom{N}{k}$ with success probability $\sigma$. We can assume that the protocol either rejects, or outputs $i_{1}, \ldots, i_{k}$. In the latter case we require that the inputs $x, y$ do actually intersect on those positions, or the other way around, that wrong outputs of this form have probability 0 . This we can assume, because Alice and Bob can simply check an output, before making it "official". The communication overhead is just two bits to agree on the output being correct. Furthermore, at an additional cost of $k \log N$ bits of communication we can make sure that every message sequence has a fixed particular set of outputs that Alice and Bob agree on, i.e., for any rectangle that corresponds to a leaf of the communication tree for any value of the random coins there are $k$ different positions $i_{1}, \ldots, i_{k}$ such that the inputs $x, y$ intersect on them, or the protocol rejects.

We can change such a protocol to a protocol with binary output in which inputs with intersection size $k$ are accepted with probability $\geq \sigma$, whereas all inputs with intersection size smaller than $k$ are accepted with probability 0 . Furthermore on all inputs acceptance happens with probability at most 1. This latter trivial constraint is important in our proof. The linear program is now as follows. We have variables $w_{R}$ for all rectangles $R \subseteq\{0,1\}^{n} \times\{0,1\}^{n}$.

$$
\begin{align*}
& \min \sum_{R} w_{R}  \tag{1}\\
& \text { for all } x, y \text { with }|x \cap y|<k: \sum_{R: x, y \in R} w_{R}=0  \tag{2}\\
& \text { for all } x, y \text { with }|x \cap y|=k: \sum_{R: x, y \in R} w_{R} \geq \sigma  \tag{3}\\
& \text { for all } x, y \text { with }|x \cap y|>k: \sum_{R: x, y \in R} w_{R} \leq 1  \tag{4}\\
& w_{R} \geq 0 \tag{5}
\end{align*}
$$

It is obvious that any randomized protocol with communication $c$ and success probability $\sigma$ for the problem of accepting $k$ intersection input and rejecting smaller intersection inputs perfectly can be used to create a solution to this program with cost $2^{c}$ : A randomized protocol is a convex combination of deterministic protocols $P_{1}, \ldots, P_{m}$ with probabilities $p_{1}, \ldots, p_{m}$, and each deterministic protocol corresponds to a partition of the inputs into rectangles. We restrict our attention to the rectangles on which protocols accept. The weight $w_{R}$ of a rectangle $R$ is the sum of the $p_{i}$ over all $P_{i}$ in which $R$ occurs as accepting rectangle. Then for all input $x, y$ the value $\sum_{R: x, y \in R} w_{R}$ is simply the acceptance probability of the protocol.

Recall that we required not only that the protocol accepts inputs $x, y$ that intersect in exactly $k$ positions with some probability $\geq \sigma$, but that also for each accepting message sequence (i.e., each accepting rectangle $R$ ) there is a set of positions $I \subseteq[n],|I|=k$ such that all inputs $x, y \in R$ we have $I \subseteq x \cap y$. Denote by $\mathcal{R}_{v}$ the set of all rectangles $R$ for which there is an $I \subseteq[n],|I|=k$ such that all $x, y \in R$ satisfy $I \subseteq x \cap y$. We can hence restrict the rectangles $R$ to come from $\mathcal{R}_{v}$ in our LP. This also makes the constraints (2) superfluous.

We now take the dual of the program (with restricted rectangle set) and show a lower bound by exhibiting a feasible solution of high cost.

$$
\begin{align*}
& \max \sum_{x, y} \sigma \phi_{x, y}+\psi_{x, y}  \tag{6}\\
& \phi_{x, y} \geq 0  \tag{7}\\
& \psi_{x, y} \leq 0  \tag{8}\\
& \text { if }|x \cap y| \neq k \text { then } \phi_{x, y}=0  \tag{9}\\
& \text { for all } R \in \mathcal{R}_{v}: \sum_{x, y \in R} \phi_{x, y}+\psi_{x, y} \leq 1 \tag{10}
\end{align*}
$$

The program asks us to put weights on the inputs, where intersection $k$ inputs should receive positive weights, and some other inputs negative weights, such that all rectangles in $\mathcal{R}_{v}$ are either small or contain at least as many negative weights as positive weights (we will discuss this approach further in Section 5), but we can only afford overall negative weight which is smaller by a factor exponential in $k$ compared to the overall positive weight. Negative weights make it easier to satisfy the rectangle constraints (10), but deteriorate the cost function.

Intuitively the LP formulation states that it is hard to cover the inputs with intersection size $k$ while keeping the partition constraints (4) satisfied. Note that the primal without (4), but keeping (2) has a very simple solution of $\operatorname{cost} \exp (k \log n)$, even for $\sigma=1$. The problem with that solution is that it corresponds to a nondeterministic protocol, but not to a randomized one.

### 3.3 The Solution

Having found a dual program which will allow us to prove a lower bound, we start by defining distributions on inputs with different intersection sizes in a similar way to [Raz92.

Definition 2 For $I=\left\{i_{1}, \ldots, i_{k}\right\} \subseteq[n]$ with $|I|=k$ denote by $S_{I, n}$ the set of inputs $x, y$ such that $x \cap y=\left\{i_{1}, \ldots, i_{k}\right\}$. Furthermore let $T_{k, n}=\cup_{I:|I|=k} S_{I, n}$ denote the set of all inputs with intersection size $k$.
$\mu_{k, n, m}$ is a distribution on $\{0,1\}^{n} \times\{0,1\}^{n}$. All $x, y \notin T_{k, n}$ have probability 0. Inputs in $T_{k, n}$ that also satisfy $|x|=|y|=m$ are chosen uniformly, i.e., with probability

$$
\frac{1}{\binom{n}{m}\binom{m}{k}\binom{n-m}{m-k}} .
$$

## Lemma 4

$$
\begin{gathered}
\mu_{2 k, n+k, m+k}(x, y)=\frac{\binom{n}{k}}{\binom{n+k}{2 k}} \cdot \mu_{k, n, m}\left(x^{\prime}, y^{\prime}\right), \\
\mu_{k, n+k, m+k}(x, y)=\frac{1}{\binom{n+k}{k}} \cdot \mu_{0, n, m}\left(x^{\prime}, y^{\prime}\right), \\
\mu_{k, n, m}(x, y)=\frac{1}{\binom{n}{k}} \cdot \mu_{0, n-k, m-k}\left(x^{\prime}, y^{\prime}\right),
\end{gathered}
$$

$$
\mu_{k+1, n, m}(x, y)=\frac{n-k}{\binom{n}{k+1}} \cdot \mu_{1, n-k, m-k}\left(x^{\prime}, y^{\prime}\right)
$$

where $x^{\prime} y^{\prime}$ are inputs resulting from $x, y$, when $k$ intersecting positions are removed.
The solution to the dual program is based on the following intuition. Since the problem is symmetric, we should assign weights uniformly for all inputs with a given intersection size. Naturally we put a good amount of weight on the intersection $k$ inputs, and these are the only inputs with positive weights. We do not need to put negative weights on inputs with smaller intersections, since we already restricted the set of viable rectangles to those with intersection size at least $k$. All we need to do is to find a set of inputs to assign negative weights to (the overall weight we can distribute to those is exponentially smaller in $k$ ) so that all the rectangle constraints are satisfied. It turns out the $2 k$ intersection inputs work just fine. This is because the $2 k$ intersection inputs end up in many more rectangles than their weight suggests compared to the $k$ intersection inputs.

So we define a solution as follows (the input length is set to $n+k$ ):

- The positive weight inputs are in $T_{k, n+k}$. Their weight is defined as $\phi_{x, y}=2^{\beta n} \mu_{k, n+k, m+k}(x, y)$.
- The negative weight inputs are in $T_{2 k, n+k}$. Their weight is $\psi_{x, y}=-2^{\beta n} 2^{-\alpha k} \mu_{2 k, n+k, m+k}(x, y)$.
- For all other inputs $x, y: \phi_{x, y}=\psi_{x, y}=0$.
$\beta, \alpha>0$ are some constants that we choose later. We can right away compute the value of this solution, before checking its feasibility. If we set $\sigma=2^{-\alpha k+1}$, then the value is
$\sum_{x, y} \sigma \phi_{x, y}+\psi_{x, y}=\sum_{x, y \in T_{k, n+k}} \sigma 2^{\beta n} \mu_{k, n+k, m+k}(x, y)-\sum_{x, y \in T_{2 k, n+k}} 2^{\beta n} 2^{-\alpha k} \mu_{2 k, n+k, m+k}(x, y)=2^{\beta n} 2^{-\alpha k}$,
since both $\mu$ 's are distributions. So for $\alpha k$ being less than $(\beta / 2) n$ we get a linear lower bound on the communication, and we will require $k \leq \gamma n / 2$ for some $\gamma \leq \beta$ and set $\alpha=1 / 2$.

The "sign" constraints $(7,8,9)$ are obviously satisfied, so the only thing we need to check are the rectangle constraints (10). The following lemma is the main ingredient of the proof.

Lemma 5 (Intersection Sampling Lemma) There is a constant $\gamma>0$, such that for each rectangle $R=A \times B \subseteq\{0,1\}^{n} \times\{0,1\}^{n}$ with $\mu_{0, n, m}(R) \geq 2^{-\gamma n}$ and all $k \leq \gamma n / 2$ we have $\mu_{k, n, m}(R) \geq \mu_{0, n, m}(R) / 2^{k+1}$.

This lemma is a generalization of Razborov's main lemma in Raz92, which is essentially the same statement for $k=1$. We shall give the proof in the next section, however, now it's time to show that our solution to the dual program is feasible.

So let us check the rectangle constraints. If $R$ is a rectangle first suppose that $\mu_{k, n+k, m+k}(R) \leq$ $2^{-\beta n}$. In this case $\sum_{x, y \in R} \phi_{x, y} \leq \sum_{x, y \in R \cap T_{k, n+k}} 2^{\beta n} \mu_{k, n+k, m+k}(x, y) \leq 1$.

Hence we need only worry about large rectangles in $\mathcal{R}_{v}$. All inputs in $R$ intersect on some positions $I=\left\{i_{1}, \ldots, i_{k}\right\}$. If we remove those positions from the universe $\{1, \ldots, n+k\}$ ( $I$ is actually unique for all rectangles that contain inputs with positive weights at all) we can consider $R$ as a rectangle $R^{\prime}$ in $\{0,1\}^{n} \times\{0,1\}^{n}$. Clearly $\mu_{0, n, m}\left(R^{\prime}\right) \geq \mu_{k, n+k, m+k}(R)$, since all inputs in $R \cap T_{k, n+k}$ have a corresponding input in $R^{\prime} \cap T_{0, n}$, and for each $x, y: \mu_{0, n, m}(x, y)=\mu_{k, n+k, m+k} \cdot\binom{n+k}{k}$.

So the intersection sampling lemma is applicable to $R^{\prime}$ as long as we set $\beta=\gamma$ and $k \leq \gamma n / 2$. The lemma tells us that $\mu_{k, n, m}\left(R^{\prime}\right) \geq \mu_{0, n, m}\left(R^{\prime}\right) / 2^{k+1}$.

Consequently,

$$
\begin{align*}
\mu_{2 k, n+k, m+k}(R) & =\mu_{k, n, m}\left(R^{\prime}\right) \cdot \frac{\binom{n}{k}}{\binom{n+k}{2 k}}  \tag{11}\\
& \geq \mu_{0, n, m}\left(R^{\prime}\right) \cdot \frac{\binom{n}{k}}{\binom{n+k}{2 k} 2^{k+1}}  \tag{12}\\
& =\mu_{k, n+k, m+k}(R) \cdot \frac{\binom{n}{k}\binom{n+k}{k}}{\binom{n+k}{2 k} 2^{k+1}}  \tag{13}\\
& \geq \mu_{k, n+k, m+k}(R) \cdot \Omega\left(2^{k} / \sqrt{k}\right), \tag{14}
\end{align*}
$$

where (11) and (13) follow from Lemma 4, (12) from Lemma 5, and (14) using Sterling approximation.

So, surprisingly, the intersection sampling lemma lets us conclude that $R$ contains a lot more weight on $2 k$ intersections than on $k$ intersections. Of course this is really a consequence of the fact that we forced the original protocol to be correct in its outputs, and hence the fact that rectangles we consider have one set of $k$ positions that all their inputs intersect in.

So

$$
\sum_{x, y \in R} \phi_{x, y}+\psi_{x, y}=\sum_{x, y \in R \cap T_{k, n+k}} 2^{\beta n} \mu_{k, n+k, m+k}(x, y)-\sum_{x, y \in R \cap T_{2 k, n}} 2^{\beta n} \mu_{2 k, n+k, m+k}(x, y) 2^{-\alpha k} \leq 0 .
$$

The rectangle constraints are satisfied and our program is indeed feasible. We have the constants $\beta=\gamma$, and $\sigma=2^{-\alpha k+1}$, and $\alpha=1 / 2$ as well as $k \leq \gamma n / 2$. Overall our solution to the dual proves, that each protocol with communication $\beta n$ cannot solve the $S E A R C H_{\binom{n+k}{k}}$ with success better than $\sigma$, as long as $k \leq \gamma n / 2$. By adjusting constants this proves Lemma 1 .

### 3.4 The Intersection Sampling Lemma

In this section we prove Lemma 5 which we have used to establish the feasibility of the solution to the linear program exhibited in the previous section.

The base of the induction proof will be provided by Razborov's main lemma from Raz92] restated as follows:

Fact 6 There is a constant $\delta>0$, such that for all $m \in\{n / 4-\delta n, \ldots, n / 4\}$ and for each rectangle $R \subseteq\{0,1\}^{n} \times\{0,1\}^{n}$ with $\mu_{0, n, m}(R) \geq 2^{-\delta n}$ we have $\mu_{1, n, m}(R) \geq \mu_{0, n, m}(R) /(3 / 2)$.

The factor $3 / 2$ corresponds to error $2 / 5$ in the original statement, but it can be seen easily, that any error $1 / 2-\epsilon$ can be achieved in Razborov's proof by reducing the size of the rectangles considered suitably (i.e., by lowering the communication bound $\delta n$ considered). Also Razborov fixes $m=n / 4$, but slightly smaller sets can be accommodated in the proof ${ }^{1}$

We prove the following statement by induction.

[^1]Lemma 7 There is a constant $\gamma>0$, such that for $m=n / 4$ and each rectangle $R=A \times B \subseteq$ $\{0,1\}^{n} \times\{0,1\}^{n}$ with $\mu_{0, n, m}(R) \geq 2^{-\gamma n}$ and all $k \leq \gamma n / 2$ we have $\mu_{k, n, m}(R) \geq \mu_{0, n, m}(R) / 2^{k}-k$. $2^{-\delta(n-k+1)}$.

Setting $\gamma=\delta / 3$ (as well as $k \leq \gamma n / 2$ ) implies Lemma 5 as stated in the previous subsection.
Clearly the base of the induction over $k$ is true by Fact 6 . So consider any rectangle $R$, such that $\mu_{0, n, m}(R) \geq 2^{-\gamma n}$ and $k \leq \gamma n / 2$.

For all $I:|I|=k$ let's denote by $R_{I}$ the rectangle that is the intersection of $R$ with the rectangle that fixes $x_{i}=y_{i}=1$ for all $i \in I$. Now $R \cap\left\{T_{k, n} \cup \cdots \cup T_{n, n}\right\}=\cup_{I:|I|=k} R_{I}$. Furthermore each input $x, y \in T_{k+1, n} \cap R$ lies in exactly $k+1$ rectangle $R_{I}$, while all inputs in $x, y \in T_{k, n} \cap R$ lie in exactly one $R_{I}$. Hence

$$
\mu_{k+1, n, m}(R)=\sum_{I:|I|=k} \mu_{k+1, n, m}\left(R_{I}\right) /(k+1) .
$$

Again we can reinterpret the $R_{I}$ as rectangles $R_{I}^{\prime}$ in $\{0,1\}^{n-k} \times\{0,1\}^{n-k}$, because each $R_{I}$ has all its inputs intersecting on the set $I$, so we only consider what happens on the other positions.

Note that $\mu_{0, n-k, m-k}\left(R_{I}^{\prime}\right)=\mu_{k, n, m}\left(R_{I}\right) \cdot\binom{n}{k}$, so we can conclude that $\mu_{0, n-k, m-k}\left(R_{I}^{\prime}\right)$ is large whenever $\mu_{k, n, m}\left(R_{I}\right)$ is.

Let $\mathcal{I}=\left\{I:|I|=k \wedge \mu_{0, n-k, m-k}\left(R_{I}^{\prime}\right) \leq 2^{-\delta(n-k)}\right\}$. Then

$$
\begin{equation*}
\sum_{I \in \mathcal{I}} \mu_{k, n, m}\left(R_{I}\right) \leq \sum_{I \in \mathcal{I}} \mu_{0, n-k, m-k}\left(R_{I}^{\prime}\right) /\binom{n}{k} \leq 2^{-\delta(n-k)} \tag{15}
\end{equation*}
$$

Now

$$
\begin{align*}
\mu_{k+1, n, m}(R) & =\sum_{I:|I|=k} \mu_{k+1, n, m}\left(R_{I}\right) /(k+1)  \tag{16}\\
& =\sum_{I:|I|=k} \mu_{1, n-k, m-k}\left(R_{I}^{\prime}\right) \cdot \frac{n-k}{\binom{n}{k+1} \cdot(k+1)}  \tag{17}\\
& \geq \sum_{I:|I|=k \wedge I \notin \mathcal{I}} \mu_{1, n-k, m-k}\left(R_{I}^{\prime}\right) \cdot \frac{n-k}{\binom{n}{k+1} \cdot(k+1)}  \tag{18}\\
& \geq \sum_{I:|I|=k \wedge I \notin \mathcal{I}} \mu_{0, n-k, m-k}\left(R_{I}^{\prime}\right) \cdot \frac{n-k}{\binom{n}{k+1} \cdot(k+1) \cdot 3 / 2}  \tag{19}\\
& \geq \sum_{I:|I|=k \wedge I \notin \mathcal{I}} \mu_{k, n, m}\left(R_{I}\right) \cdot \frac{(n-k)\binom{n}{k}}{\binom{n}{k+1} \cdot(k+1) \cdot 2}  \tag{20}\\
& =\sum_{I:|I|=k \wedge I \notin \mathcal{I}} \mu_{k, n, m}\left(R_{I}\right) \cdot \frac{1}{2}  \tag{21}\\
& \geq \sum_{I:|I|=k} \mu_{k, n, m}\left(R_{I}\right) \cdot \frac{1}{2}-2^{-\delta(n-k)}  \tag{22}\\
& \geq \sum_{I:|I|=k} \mu_{0, n, m}\left(R_{I}\right) \cdot \frac{1}{2^{k+1}}-(k+1) 2^{-\delta(n-k)} . \tag{23}
\end{align*}
$$

(17), (20) are via Lemma 4, (19) uses Fact 6, (22) is from (15), and (23) uses the induction hypothesis.

## 4 Applications

### 4.1 Communication-Space Tradeoffs for Boolean Matrix Products

In this section use the strong direct product result for the communication complexity of Disjointness Theorem 3 to prove tight communication-space tradeoffs.

Theorem 8 Every bounded-error protocol in which Alice and Bob have bounded space $S$ and that computes the Boolean matrix-vector product, satisfies $C S=\Omega\left(N^{2}\right)$.

Theorem 9 Every bounded-error protocol in which Alice and Bob have bounded space $S$ and that computes the Boolean matrix product, satisfies $C S=\Omega\left(N^{3}\right)$.

Proof of Theorem 8. Alice receives a matrix $A$, and Bob a vector $b$ as inputs. Given a circuit that multiplies these with communication $C$ and space $S$ and success probability $1 / 2$, we proceed to slice it. A slice of the circuit is a set of consecutive gates in the circuit containing a limited amount of communication. In communicating circuits the communication corresponds to wires carrying bits that cross between Alice's and Bob's part of the circuit. Hence we may cut the circuit after $\beta N$ bits have been communicated and so on. Overall there are $C / \beta N$ such circuit slices. Each starts with an initial state computed by the previous part of the circuit. This state on at most $S$ bits may be replaced by the uniform distribution on $S$ bits. The effect is that the success probability decreases to $(1 / 2) \cdot 1 / 2^{S}$.

We want to employ the direct product theorem for the communication complexity of NDISJ $_{N / k}$ (for some $k$ ) to show that a protocol with the given communication has success probability at most exponentially small in the number of outputs it produces, and so a slice can produce at most $\mathrm{O}(S)$ outputs. Combining these bounds with the fact that $N$ outputs have to be produced gives the tradeoff: $C / \beta N \cdot O(S) \geq N$.

To use the direct product theorem we restrict the inputs in the following way: Suppose a protocol makes $k$ outputs. We partition the vector $b$ into $k$ blocks of size $N / k$, and each block is assigned to one of the $k$ rows of $A$ for which an output is made. This row is made to contain zeroes outside of the positions belonging to its block, and hence we arrive at a problem where Nondisjointness has to be computed on $k$ instances of size $N / k$. With communication $\beta N$, the success probability must be exponentially small in $k$ due to Theorem 3. Hence $k=\mathrm{O}(S)$ is an upper bound on the number of outputs produced per slice.

Proof of Theorem 6. The proof uses the same slicing approach as in the other tradeoff result. Note that we can assume that $S=\mathrm{o}(N)$, since otherwise the bound is trivial. Each slice contains communication $\beta N$, and as before a direct product result showing that $k$ outputs can be computed only with success probability exponentially small in $k$ leads to the conclusion that a slice can only compute $\mathrm{O}(S)$ outputs. Therefore $(C / \beta N) \cdot \mathrm{O}(S) \geq N^{2}$, and we are done.

Consider a protocol with $\beta N$ bits of communication. We partition the universe $\{1, \ldots, N\}$ of the Disjointness problems to be computed into $k$ mutually disjoint subsets $U(i, j)$ of size $N / k$, each associated to an output $(i, j)$, which in turn corresponds to a row/column pair $A[i], B[j]$ in the
input matrices $A$ and $B$. Assume that there are $\ell$ outputs $\left(i, j_{1}\right), \ldots,\left(i, j_{\ell}\right)$ involving $A[i]$. Each output is associated to a subset of the universe $U\left(i, j_{t}\right)$, and we set $A[i]$ to zero on all positions that are not in one of these subsets. Then we proceed analogously with the columns of $B$.

If the protocol computes on these restricted inputs, it has to solve $k$ instances of Disjointness of size $n=N / k$ each, since $A[i]$ and $B[j]$ contain a single block of size $N / k$ in which both are not set to 0 if and only if $(i, j)$ is one of the $k$ outputs. Hence Theorem 3 is applicable.

### 4.2 Multiparty

Theorem 10 In the model where Charlie sends one message, followed by an arbitrary interaction between Alice and Bob, the 3-party Disjointness problem has randomized complexity $\Omega(\sqrt{n})$.

Proof. This proof idea is due to Ronald de Wolf BRW08]. Let $P$ be a protocol for the 3-party NDISJ $_{n}$ problem with $\epsilon \sqrt{n}$ communication and error $1 / 3$.

We partition $\{1, \ldots, n\}$ into $\sqrt{n}$ blocks of size $\sqrt{n}$. Charlie's input $z$ is restricted to contain 1's in one particular block, and 0 's elsewhere. So in effect $z$ chooses one of $\sqrt{n}$ instances of NDISJ $_{\sqrt{n}}$ problems given to Alice and Bob. Since Charlie's message does not depend on $z$, Alice and Bob may reuse it in $\sqrt{n}$ runs of $P$ in order to determine, for all of the $\sqrt{n}$ possible $z$, the value of all of their NDISJ $_{\sqrt{n}}$ problems with overall communication $\epsilon n$. For each block the error probability is at $\operatorname{most} \epsilon$. The expected number of blocks where the protocol fails is at most $2 \epsilon \sqrt{n}$ with probability at least $1 / 2$ by the Markov inequality. So for every input $x, y$ to Alice and Bob there is a message from Charlie which will make them give the wrong answer for at most $2 \epsilon \sqrt{n}$ blocks with probability at least $1 / 2$.

We may now simply replace Charlie's message by a uniformly random string which will deteriorate the probability of having at least $(1-2 \epsilon) \sqrt{n}$ blocks correct to $2^{-\epsilon \sqrt{n}} \cdot 1 / 2$. We have found a 2 player protocol with communication $\epsilon n$ and the mentioned success probability to compute $(1-2 \epsilon) \sqrt{n}$ instances of Nondisjointness correctly. In BRW08] a general argument is given that relates the success probabilty in this situation to the standard situation of an SDPT (in which success means all the outputs are correct). For small enough $\epsilon$ this contradicts our main result.

## 5 The Linear Programming Bound, the Rectangle Bound, and Limited Ambiguity

In this section we take a closer look at the method behind our main result. Lower bound methods usually strip away some aspects of the computation involved. In communication complexity almost all lower bound methods strip away the specific tree structure of protocols and instead consider partitions of the inputs into rectangles (it is known that bounds for partitions are polynomially tight at least for total Boolean functions). Our method is no different. However, most other general methods are only considering the properties of single rectangles, while our approach brings in the fact that they have to form a partition.

A major tool to prove randomized communication complexity bounds is the rectangle/corruption bound (see K03, BPSW06]). It was originally introduced by Yao [Y83], and its most prominent but by no means only use is in Razborov's Disjointness bound [Raz92]. We will show that the method is in fact the solution to a dual LP, and then proceed to define a stronger lower bound
method. The rectangle/corruption bound states that all rectangles in the communication matrix are either small or have large error (under some hard distribution).

Let us define the one-sided rectangle/corruption bound. $f:\{0,1\}^{n} \times\{0,1\}^{n} \rightarrow\{0,1\}$ is a function. Let $\mu_{0}$ be a probability distribution on $f^{-1}(0)$ and $\mu_{1}$ be a probability distribution on $f^{-1}(1)$. Let $1 \geq s \geq 0$ be the maximum number such that for all rectangles $R$ with $\mu_{1}(R) \geq s$ it is true that $\mu_{0}(R) \geq \mu_{1}(R) \cdot \epsilon$. This means that under the distribution $1 / 2 \cdot\left(\mu_{0}+\mu_{1}\right)$ all large 1 -rectangles have error at least $\epsilon /(1+\epsilon)$. The one-sided rectangle bound is bound $(f, 1, \epsilon)=-\log s$.

In L90] Lovasz describes the following LP to bound randomized communication complexity.

$$
\begin{align*}
& \min \sum_{R} w_{R}  \tag{24}\\
& \text { for all } x, y \text { with } f(x, y)=1: \sum_{R: x, y \in R} w_{R} \geq 1-\epsilon  \tag{25}\\
& \text { for all } x, y \text { with } f(x, y)=0: \sum_{R: x, y \in R} w_{R} \leq \epsilon  \tag{26}\\
& w_{R} \geq 0 \tag{27}
\end{align*}
$$

He takes the dual.

$$
\begin{align*}
& \max \sum_{x, y: f(x, y)=1}(1-\epsilon) \phi_{x, y}+\sum_{x, y: f(x, y)=0} \epsilon \phi_{x, y}  \tag{28}\\
& \text { for all } x, y \text { with } f(x, y)=1: \phi_{x, y} \geq 0  \tag{29}\\
& \text { for all } x, y \text { with } f(x, y)=0: \phi_{x, y} \leq 0  \tag{30}\\
& \text { for all } R: \sum_{x, y \in R} \phi_{x, y} \leq 1 \tag{31}
\end{align*}
$$

Note that here $R$ ranges over all rectangles in the communication matrix. One can now prove a lower bound by exhibiting a solution to the dual. Let $\phi(x, y)$ describe such a solution.

We now show that the optimum of this program is characterized by the one-sided rectangle bound.

First assume that bound $(f, 1, \delta)=-\log s$. Then all rectangles larger than $s$ have large error under a distribution $\mu=1 / 2 \cdot\left(\mu_{0}+\mu_{1}\right)$. We create a solution to the dual as follows. We set $\phi(x, y)=\mu_{1}(x, y) / s$ if $f(x, y)=1$. For $f(x, y)=0$ we set $\phi(x, y)=-\mu_{0}(x, y) /(\delta s)$.

Now clearly the sign constraints (29-10) are satisfied, and for every rectangle smaller than $s$ under $\mu_{0}$ the rectangle constraints (31) are trivially true. For larger rectangles we have

$$
\sum_{x, y \in R} \phi_{x, y}=\sum_{x, y \in R, f(x, y)=1} \mu_{1}(x, y) / s-\sum_{x, y \in R, f(x, y)=0} \mu_{0}(x, y) /(\delta s) \leq 0 .
$$

The cost function is

$$
\sum_{x, y: f(x, y)=1}(1-\epsilon) \mu_{1}(x, y) / s-\epsilon \sum_{x, y: f(x, y)=0} \mu_{0}(x, y) /(\delta s) \geq 1 /(3 s)
$$

for $\epsilon \leq \delta / 3$ and $\delta \leq 1$.

Now conversely assume that $\phi$ defines a solution to the dual LP, with cost $c$.
We define two distributions, $\mu_{1}$ is on the 1 -inputs of $f$, and is simply given by

$$
\mu_{1}(x, y)=\phi(x, y) / \sum_{x, y: f(x, y)=1} \phi(x, y) .
$$

Note that $t_{1}=\sum_{x, y: f(x, y)=1} \phi(x, y) \geq c /(1-\epsilon) . \mu_{0}$ is on the 0 -inputs of $f$, and is given by

$$
\mu_{0}(x, y)=\phi(x, y) / \sum_{x, y: f(x, y)=0} \phi(x, y) .
$$

Note that $t_{0}=-\sum_{x, y: f(x, y)=0} \phi(x, y) \leq \sum_{x, y: f(x, y)=1}(1-\epsilon) \phi(x, y) / \epsilon$, since $c \geq 0$.
Consider any rectangle $R$. Now

$$
\mu_{0}(R) \cdot t_{0}=-\sum_{x, y \in R: f(x, y)=0} \phi(x, y) \geq \sum_{x, y \in R: f(x, y)=1} \phi(x, y)-1=t_{1}\left(\mu_{1}(R)-1 / t_{1}\right)
$$

Since $t_{1} / t_{0} \geq \epsilon /(1-\epsilon)$ we have $\mu_{0}(R) \geq(\epsilon /(1-\epsilon)) \mu_{1}(R)-\epsilon / c$. This means that all rectangles with $\mu_{1}(R) \geq 2 / c$ must satisfy $\mu_{0}(R) \geq(\epsilon / 2) \mu_{1}(R)$ and hence bound $(f, 1, \epsilon / 2) \geq \log (c / 2)$.

This completes our proof that the rectangle/corruption bound is equivalent to Lovasz's LP.
Instead of proceeding like Lovasz, who relaxes (31) (and obtains the operator norm bound, which can be exponentially smaller), we note the absence of the "trivial" constraint

$$
\begin{equation*}
\text { for all } x, y \text { with } f(x, y)=1: \sum_{R: x, y \in R} w_{R} \leq 1 . \tag{32}
\end{equation*}
$$

We propose the primal Lovasz LP augmented with (32) as a new lower bound on randomized communication. Consider the dual of the augmented program.

$$
\begin{align*}
& \max \sum_{x, y: f(x, y)=1}(1-\epsilon) \phi_{x, y}+\sum_{x, y: f(x, y)=0} \epsilon \phi_{x, y}+\sum_{x, y: f(x, y)=1} \psi_{x, y}  \tag{33}\\
& \text { for all } x, y \text { with } f(x, y)=1: \phi_{x, y} \geq 0 ; \psi_{x, y} \leq 0  \tag{34}\\
& \text { for all } x, y \text { with } f(x, y)=0: \phi_{x, y} \leq 0, ; \psi_{x, y}=0  \tag{35}\\
& \text { for all } R: \sum_{x, y \in R} \phi_{x, y}+\psi_{x, y} \leq 1 \tag{36}
\end{align*}
$$

In this program we are still choosing a hard distribution and a threshold, but we are also allowed to create negative weights for a small portion of 1-input. This facilitates satisfying (36) but deteriorates the cost function quickly. We call the resulting bound the generalized rectangle bound due to the similarity of this approach to the generalized discrepancy bound [S08, K03]. In fact it is easy to see that the discrepancy bound and the generalized discrepancy bound can also be defined in a similar LP fashion, by allowing positive and negative weights for rectangles, and the difference between the two again boils down to on the presence of a constraint of the type (32).

Now is the generalized rectangle bound really stronger than the standard one? Let us find out the strongest type of communication protocol that we can still lower bound. Consider unambiguous

AM-protocols (i.e., convex combinations of partitions with bounded error, see Section 2. Combinatorially inclined people might prefer the name randomized partition number). It is easy to see that the LP with constraint (32) lower bounds such protocols.

We can now give a partial answer to our question by showing a linear lower bound for the Nondisjointness problem and unambiguous AM-protocols. Note that since $N\left(\right.$ NDISJ $\left._{n}\right)=O(\log n)$, also $A M\left(\mathrm{NDISJ}_{n}\right)=O(\log n)$. Only the extra partition constraint (32) leads to the lower bound. While this does not prove in principle that the generalized rectangle bound beats the rectangle bound (we could just work with $\mathrm{DISJ}_{n}$ and use the rectangle bound), it demonstrates the difference between the bounds nicely. Ultimately we would like to show a linear lower bound for the TRIBES function (i.e., a fan-in $\sqrt{n}$ AND of NDISJ $\sqrt{n}$ problems, for which the rectangle bound can only provide $O(\sqrt{n})$ bounds on both the function and its complement). Such a bound holds, but has so far only been established using information theoretic techniques JKS03.

So let us prove that the generalized rectangle bound is large for NDISJ ${ }_{n}$. Again we can restrict the set of rectangles to $\mathcal{R}_{v}=\{R: \exists i: x, y \in R \Rightarrow i \in x \cap y\}$. We define a solution to the dual as follows:

- Inputs in $T_{0, n}$ have weight $\phi_{x, y}=-\infty$.
- Inputs in $T_{1, n}$ have weight $\phi_{x, y}=2^{\beta n} \mu_{1, n}(x, y)$.
- Inputs in $T_{2, n}$ have weight $\psi_{x, y}=-2^{\beta n} \mu_{2, n}(x, y) \cdot 3 / 4$.
- All other inputs have weight 0 .

All other inputs have weight 0 .
The cost is $2^{\beta n}((1-\epsilon)-3 / 4)$. The sign constraints are satisfied. Now consider a rectangle $R \in \mathcal{R}_{v}$. Let $R^{\prime}$ denote the rectangle in which we ignore its intersection position $i$. Then

$$
\begin{align*}
\mu_{2, n, m}(R) & =\mu_{1, n-1, m-1}\left(R^{\prime}\right) \cdot(n-1) /\binom{n}{2}  \tag{37}\\
& \geq \mu_{0, n-1, m-1}\left(R^{\prime}\right) \cdot(n-1) /\left(1.5\binom{n}{2}\right)  \tag{38}\\
& =\mu_{1, n, m}(R) \cdot(n-1) n /\left(1.5\binom{n}{2}\right)  \tag{39}\\
& \geq \mu_{1, n, m}(R) \cdot 4 / 3 \tag{40}
\end{align*}
$$

We use that $\mu_{0, n-1}\left(R^{\prime}\right) \geq n \cdot \mu_{1, n}(R) \geq n 2^{-} \beta n \geq 2^{\delta(n-1)}$ and of course Fact 6 in (38).
Then

$$
\sum_{x, y \in R: f(x, y)=1} \phi_{x, y}+\sum_{x, y \in R: f(x, y)=0} \phi_{x, y} \leq 2^{\beta n} \mu_{1, n, m}(x, y)(1-3 / 4 \cdot 4 / 3)=0
$$

This shows that any unambiguos AM-protocol for $\mathrm{NDISJ}_{n}$ needs communication $\Omega(n)$. It is known KNSW94] that nondeterministic protocols with ambiguity $t$ need communication $\sqrt{D(f)} / t$ for the deterministic complexity $D$, and the rank lower bound is also known to hold for unambiguous nondeterministic protocols. However, these methods do not allow errors, and the first bound cannot achieve linear lower bounds at all (and the approximate rank cannot give better bounds than $\sqrt{n}$ either since it lower bounds quantum protocols ( $(\underline{\mathrm{Raz} 03}])$.

We can also generalize our main result Theorem 3 in a similar fashion:

Theorem 11 Assume an AM-protocol with ambiguity $2^{\epsilon k}$ computes $k$ instances of NDISJ $J_{n}$. Then the success probability of the protocol (over the random bits) is at most $2^{\Omega(-k)}$ unless the communication is at least $\beta k n$.

The proof of this statement is practically identical to the proof of Theorem 3. Going through the reductions in section 3 we see that they need only a constant repetition of the original protocol, and a polynomial increase in the ambiguity.

When we come to the search problem and the linear programming formulation note that we have to replace the right hand side of constraint (4) by $\leq 2^{\epsilon k}$. In the dual this increases the gap between $\phi$ 's and $\psi$ 's in the objective function, but that gap is already exponentially large in $k$, so nothing in the construction really changes.

Note that this bound is quite tight, since with ambiguity $2^{c k}$ we can reduce the communication to any fraction of $k n$, and with ambiguity $n^{k}$ the communication collapses to $O(k \log n)$ even without any error.

## 6 Conclusions

In this paper we have proved a strong direct product theorem for Disjointness. One ingredient of the proof is a new lower bound technique for randomized communication complexity, that relates to the standard rectangle/corruption bound like the generalized discrepancy bound relates to the discrepancy bound. Instead of lower bounding the function itself we consider the rectangle bound for some function which is close, or, similarly, we allow hardness to be proved under a distribution that allows negative weights. This is a way to express the partition constraint inherent in communication, that is not fully exploited by both the rectangle and discrepancy bounds. So in particular we are able to show that the Nondisjointness problem has no efficient unambiguous AM-protocols, whereas nondeterministic protocols are very efficient (and large 1-rectangles without error exist). We believe our lower bound method may prove to be general enough to tackle problems like the TRIBES function, for which the standard rectangle bound fails.

A long standing open problem is whether the rectangle bound is polynomially tight for randomized communication complexity (it is well known to be quadratically tight in the 0 -error case). A new approach might be to show tightness of the generalized rectangle bound instead (which might in fact be tight within a constant factor).

Last but not least, every lower bound method that goes beyond the level of arguing about single rectangles is welcome, since progress in several areas of complexity seems to be stalled at the rectangle level of argumentation. In particular we believe that the question of separating the polynomial hierarchy in communication complexity deserves more attention.

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[^1]:    ${ }^{1}$ The proof needs to be adjusted in several ways. First of all, instead of mixing the distributions on intersection size 1 and 0 in the proportions $1 / 4$ and $3 / 4$ we need to mix them uniformly. Secondly, the constant $1 / 3$ in the definition of $x$-bad can be replaced with a constant close to 1 , and consequently the numbers in Claims 3 and 4 need to be adjusted. A bit more troublesome is allowing $m$ to be slightly smaller than $n / 4$, since this makes Fact 2 false, although it remains approximately true, tilting all other estimates by $1+\delta$ factors.

