Which Recursive Equilibrium?*

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Abstract

For simple versions of the Cass-Diamond growth model where the fundamental welfare theorems of economics fail, we prove the set of minimal state space recursive equilibrium (RE) is large, with multiple RE existing in multiple subclasses of functions. This situation remains the true even when *uniqueness* results are known, for both OLG and infinity-lived agent versions of the model. This implies the set of Generalized Markov equilibrium (GME) is also large. We also provide explicit iterative procedures from computing particular RE in each subclass, with some procedures globally stable from a topological persective for particular RE when their domains are restricted. All iterative methods are order stable relative to perturbations of deep parameters. Finally, we construct an simple economy where existing correspondence-based methods for computing GME fails, and propose a new method that computes the set of GME as minimal state space

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RE using fixed points of the expanded set of state variables. Our results point to the complicated nature of the RE approximation problem even in simple macro models.

1. Introduction

Since the seminal work of Kydland and Prescott [28], a great deal of focus in macroeconomics and industrial organization as been on the quantitative assessment of dynamic equilibrium models and dynamic games. For example, in macroeconomics the calibration of recursive equilibrium (RE) provides a prototype for this approach. In a calibration method, after defining (and hopefully proving the existence) of a RE, one first partitions the set of deep parameters of the economies into 2 distinct subsets, the first will be used to fit a particular equilibrium of the model to the data, while the second subset that will be perturbed in numerical experiments that are of interest in the calibration.[1] More precisely, in the best case of calibration, after fitting some approximate RE solution to the actual RE to macroeconomic data using the subset of calibration parameters, one simulates changes in the remaining subset of parameters to study provide a quantitative comparison of the simulated equilibrium dynamics under two different parametric configurations. In this sense, in a very precise way, calibration can be thought of as a numerical approach to equilibrium comparative statics/dynamics. Similar sorts of statements can be made about econometric procedures (e.g., GMM or simulated moments procedures) that are used to estimate deep parameters of the economy under study.

Given such approaches to characterizing RE of such models, one obvious question that arises naturally, yet is rarely addressed, for such numerical approaches to economics concerns the *stability* of the set of approximate vs. actual RE under perturbation. More precisely, what do calibration (or estimation) results mean in the presence of in the presence of multiple equilibrium?[2] For example, say one perturbs the counterfactual parameter comparable to the old RE at the initial set

[1] By "deep parameters", we mean parameters that characterize variations in the primitive data governing taste, technology, policy, information, etc. The first subset of parameters will be referred to in this paper as *calibration parameters,* while the second subset will be referred to as the *counterfactual parameters.*

[2] As RE are functions, multiple equilibrium here comes in two forms: (i) multiplicities of RE within a given space of functions, and (ii) multiplicities *across* sets of functions. In this paper, we shall consider both types of multiplicities.

of parameters? How about comparing the approximate RE under the new counterfactual parameters? Additionally, what if RE can exist within *multiple distinct subclasses for functions*? How can we develop numerical approaches to computing RE that allow researcher to know which subclass of RE are actually being approximated? These are but a few of the questions on can ask of numerical approaches to RE in the presence of multiplicities. It bears mentioning, calibration and estimation methods are being applied to very complicated dynamic general equilibrium models, including models with (i) random measures as state variables, (ii) heterogeneous agents, (iii) models where the welfare theorems fail (hence, no Negishi method is known to work), etc.

In this paper, we first show how the set of RE can be in *very simple models.* To show the robust nature of the problem, we display the problems with multiplicities within and across subclasses of function spaces for both OLG models, as well as models with infinitely-lived agents. To keep things simple, we just use a very simple version of the Solow-Cass growth model of production in all cases. For the case of simple OLG models, we just work with very simple model with 2 period-lived, identical cohorts and capital, and consider both the case of classical and nonclassical production.[3] For the infinitely-lived agent case, to make the model nonoptimal, we just add a simple production externality as in Romer [50].[4]

In these two simple settings, we are able to show some very important difficulties associated with the rigorous characterization of RE numerically. The problems hold for both function-based and correspondence-based continuation methods, all of which carryover to more complicated nonempty dynamic economies that are typical in the existing literature in macroeconomics. For the OLG cases, we first consider OLG models with concave production functions in private returns for firms (e.g., often Cobb-Douglas) and power utility. For the case where production processes do not satisfy "capital income monotonicity", we prove existence of RE in *four distinct* subclasses of the space of bounded functions, none of which include continuous functions. Then, under the stronger condition of capital income monotonicity, we first prove (i) uniqueness of continuous RE, and show (ii) this uniqueness result holds relative to the space of bounded increasing functions. We

[3] For nonclassical production, we just consider Romer [50], where there is a difference between private returns to production for each firm, and the social return to production. There is nothing special about this form of "externality" as many other specifications could be used for our arguments.

[4] We will also show how our results for this specification can be developed for the income tax problem considered in Coleman [10][11][12], Greenwood and Huffman [23], Morand and Reffett [40], and Mirman, Morand, and Reffett [35].

then show the uniqueness result fails relative to the space of bounded functions. In all cases, we give explicit successive approximation methods for computing least and greatest RE in all subclasses of functions. The constructive nature of our arguments are critical (as it allows us to show explicitly how iterative methods can arrive at very different limits based upon the initial versions they chooses.

We then tur our focus on the case of state-contingent taxes and infinitely-lived agents. The first case we study the progressive tax case studied first in Coleman [10].[5] For these economies, focusing first on the well-known Euler equation method first proposed in Coleman [10], we first extend the uniqueness result obtained in Coleman [10][12] to much *larger* class of functions, namely, particular spaces of *bounded* functions. We then show using two different fixed point procedures (i) the existence of continuous recursive equilibrium using a related Euler equation method to Colemans where his uniqueness argument fails, and (ii) develope a very simple policy iteration method, whose domain is exactly the set of functions Coleman studies, but for which Coleman's uniqueness approach cannot be used to rule out multiple solutions in this same domain of functions.

Finally, we then consider the question of continuous RE in models often thought not to admit them (e.g., the example presented in Santos ([54], section 3.2)). As the Santo's example has been a central motivation for adopting correspondence-based approaches to characterizing recursive equilibrium (e.g., Miao and Santos [34], Feng et. al. [21], and Peralta-Alva and Santos [47]), reconsidering the question of continuous RE in this class of economies is particularly important. Using a new simple variation of Coleman's policy iteration procedure, we construct multiple subclasses of RE ranging from locally Lipschitz isotone continuous recursive equilibrium (for investment) to bounded (not isotone) RE. In doing this, we also show the Miao-Santos procedure equilibrium correspondence fails to verify RE. Finally, we propose a new correspondence-based approach, based upon "interval iterations" of interval operators valued in function spaces, that integrate the function-based policy-iteration approach of Coleman [10], with the correspondence-based continuation method advocated in Kubler and Schmedders [26] and Miao and Santos [34]. For this interval mapping, we prove the existence of continuous recursive equilibrium for our class of economies, and prove a simple method for computing it approximately. We then develop a formal partial ordering method for making comparative dynamics comparisons of our new interval

[5] Our results also apply to the papers of Greenwood and Huffman [23], Coleman [11][12], Datta, Mirman, and Reffett [14], Morand and Reffett [40], and Mirman, Morand, and Reffett [35].

iteration method versus the existing class of correspondence-based continuation methods.

We can be a bit more formal in describing the problem. Say one writes down a class of economies $E(t)$ parameterized by a vector of "deep" parameters $t = (t_1, t_2) \in T$, where T is some ordered linear topological space (e.g., $\mathbf{R}^n$), with t_1 a subset of parameters that will be taken as given for the calibration step (e.g., parameter values are taken from micro or macro data, others summarize policy, etc), and t_2 will be the set of parameters used to calibrate the model to the actual data. Assume for each $t \in T$ the existence of a nonempty set of RE in some space of functions, say $RE(X)$ where X is a minimal state space for each t (i.e., there exists RE selections $\psi(x,t) \in \Psi(x;t) \subset RE(X)$, where $\Psi(x;t)$ is a nonempty correspondence valued in $RE(X)$ for each $t \in T$).[6] Then, for the calibration step of the modeling, one first collects some relevant data (say $\{Y_t\} = Y$), imposes a loss function $L(\hat{\psi};Y)$ on the class of approximations $\hat{\Psi}(x;\alpha,t_2)$ (where α is just a vector of parameters for the approximation scheme), and uses L to define some goodness of fit measure that compares the fit of the models equilibrium dynamics $\{y_t\} = y$ to the observed data Y, chooses the best parameter vector, say $t_2^*(Y) = t_2^*$ that is the best approximation under the loss function. Then, the resulting approximate solution $\hat{\psi}_1^*$ is now used as an approximation to the actual RE $\psi(x,t^*)$ $\in \Psi(x,t^*)$.[7] Finally, calibrated equilibrium comparative statics (or counterfactuals for the actual model) are then generated from perturbing the initial parameters t_1, recomputing the RE to arrive at a new approximate solution $\hat{\psi}_2^*(x,t_1^{'},t_2^*) \in \hat{\Psi}(x;t_1',t_2^*)$ where t_2^* is now taken as fixed, and $\hat{\psi}_2$ is an approximation solutions to an actual RE selection $\psi_2(x,t_1',t_2^*) \in \Psi(x;t_1',t_2^*)$. One then just compares the models new dynamic properties at new parameter $t_1^{'}$ to the old one t_1.

The question of continuity (i.e., stability) of this numerical equilibrium comparative statics method then immediately arises. For simplicity of notation, let $t_* = (t_1, t_2^*)$ and $t_*^{'} = (t_1^{'}, t_2^*)$. Then, a few immediate questions that come to mind are the following:

[6] Notice, this existence assumption is a very strong.

[7] Notice, here we assume here in this discussion that a *unique* approximate solution under this loss function exists (i.e., the best approximation problem generated by the loss function is strictly convex). Also, recall to design an accurate approximation scheme rigorously, one must know the structural properties of the objects that will be approximated (i.e., the structural properties of $RE(\mathbf{X})$). You cannot approximate an arbitrary function to an reasonable standard of accuracy. Therefore, knowing where one has existence is critical here to knowing how construct an appropriate approximation scheme where error bounds can constructed. Lets also assume all these problem can be resolved (a nontrivial task).

(a) *Is the approximation scheme* $(\hat{\Psi}(x;t), L)$ *stable under perturbation:* i.e., does there exist some topology such that the numerical approximation scheme has desirable continuity properties and satisfies $\hat{\psi}_1^*(x,t) = \hat{\psi}_2^*(x,t) = \hat{\psi}^*(x,t)$ for all t in any neighborhood of t_* that includes $t_*^{'}$?

(b) *Are actual RE* $\Psi(x,t)$ *stable under perturbation in* t*: i.e.,* $\exists$ a RE selection $\psi^*(x,t)$ in any neighbor t_* that includes $t_*^{'}$ where $\psi^*(x,t)$ has desirable continuity properties; and

(c) *Is the approximation* $\hat{\psi}^*(x,t)$ *accurate relative the actual selection* $\psi^*(x,t)$ at t_* and $t_*^{'}$ (e.g., can one conduct error analysis for the approximation $\hat{\psi}^*(x,t)$ under L in $\hat{\Psi}$ relative to the actual RE selection $\psi^*(x,t) \in \Psi(x,t)$?

In an important sense, answering question (b) is critical if one is to have a hope at resolving (a)-(c). That is, if one cannot produce conditions under which particular RE has desirable continuity/stability properties at least under local perturbations in t, achieving numerical stability and accuracy in the sense of (a)-(c) seems hopeless. Further, keep in mind that our problems, RE will not in general be sufficiently smooth to attempt applications of classically implicit function theorems.[8]

In this paper, we propose a different method to attack the stability question in (b) using order topologies (and in particular, we develop order continuous computational methods for least and greastest RE in each subclass of functions where we verify RE exist). In particular, using "lower" and "upper" solutions, we are able to develop simple parameterized iterative procedures that are *order stable* under *ordered* perturbations of t (and even in some cases, have extremal RE being continuous in an appropriate order topology). Then, given we are working in function spaces (and hence know the exact property of the unknown function at hand), resolving (a) and (c) becomes a fairly standard problem in approximation theory.

We should mention, this innovation is important for nonoptimal problems, verifying properties (a)-(c) is a daunting task. In the early work on quantitative macro using numerical methods to compute RE, resolving (a)-(c) was quite simple. That is, numerical implementations for computing RE could be where based upon the results in Prescott and Mehra [45]), where the economies studied were homogeneous agent Pareto optimal economies, so the second welfare theorem held. This situation is particularly convenient, as RE allocations and prices could be

[8]Further, nonsmooth implicit function theorems will be difficult to apply also because of the infinite dimensional nature of the RE fixed point problem, along with the fact that RE in many cases need not be even be continuous functions, let alone locally Lipschitz continuous.

constructed by solving a single functional equation, namely a Bellman equation describing a pure resource allocation problem faced by a social planner on only aggregate states. Then, appealing to arguments of Bewley [7] or Prescott and Lucas [44], the social planner's optimal solutions could be supported as recursive competitive equilibrium (RCE) with the implied sequential equilibrium price system existing in a suitable infinite dimensional space. Further, well-known tools are available for solving the Bellman equations (both from an theoretical and numerical perspective), the resulting methodological approach was very powerful. In particular, under standard strict concavity conditions on preferences, and convexity conditions for technologies, the RE computed from planner's solutions were unique, the the planner's optimal solutions $\psi_p^*(x,t)$ to each t (and associated value function $v_p^*(x;t)$) where *continuous in* t.[9] Then, using standard methods for approximating solutions to dynamic programs (and appealing to duality and the second welfare theorem), the planners solutions can be decentralized under suitable support prices as a competitive (and recursive) equilibrium. [10]

Unfortunately, things have changed a great deal over the last three decades. In most recent work, the setting for the analysis are dynamic economies where the second welfare theorem fails (i.e., so-called "nonoptimal" economies"). For such economies, for even the simplest of cases (e.g., a homogeneous agent economy with one sector production, a state contingent tax and lump-sum transfers), the social planning procedures of Prescott and Mehra [45] are known generally to fail. As numerous interesting problems in dynamic general equilibrium take place in such nonoptimal settings (e.g., models of optimal taxation, monetary economies, heterogeneous agent models with incomplete markets, among others), and researchers want to compute and quantitatively assess elements of the set of RE $\Psi(x,t)$ in meaningful sense consist when conditions (a)-(c), the first question one must ask is what does the set of RE $\Psi(x,t)$ look like in even simple dynamic

[9] To establish the continuity of RE in t, on can simply appeal the Bonsall-Nadler theorem (e.g, Nadler [41] Theorems 1 and 2 and Duemmel [18], Main Theorem, p294) for parameterized contractions, notiing as the modulus for the contraction is constant, the value function is continuous in t (so $v_p^*(x;t)$ is continuous in t),and by strict concavity and Berge's theorem, $\psi_p^*(x;t)$ were continuous in t.

[10] From a theoretical perspective when entertaining questions concerning equilibrium comparative statics, pricing, and/or existence of stationary Markov equilibrium; from a numerical perspective because of the availability of value and policy iteration techniques for obtaining accurate approximation solutions. For motivation from a theoretical perpective, see Prescott and Mehra [45]; for motivation from a numerical perspective, see the survey of Rust [53].

economies? As even for simple models, one is forces to solve a systems of functional equations (e.g., often both parameterize dynamic programs and Euler equations simultaneously), the question of computing RE can become quite complex.

We should finally mention, various methods have been proposed in the current literature to resolve some (or all) these technical issues in the context of various economies. One important class of such methods are known as "monotone continuation methods".[11] To date, two types of monotone continuation approaches have been proposed: (i) continuation methods defined in spaces of correspondences (i.e., so-called *correspondence-based method* first proposed in Kydland and Prescott [27], but extended in numerous papers including Kubler and Schmedders [26], or Miao and Santos [34]), among others), or (ii) continuation methods defined in spaces of functions (i.e., so-called *function-based continuation method,* or so-called *monotone-map methods* first proposed in Coleman [10][12], Greenwood and Huffman [23], Datta, Mirman, and Reffett [14] or Mirman, Morand, and Reffett [35]).

The paper proceeds as follows: in the next section, we study RE in simple two-period OLG versions of the Solow-Cass growth models. In section three, we use function-based continuation methods to study the structure of recursive equilibrium in situation of progressive taxation as in Coleman [10]. Section four then considers the same methods, but for the regressive tax case as in Santos [54]. Section five reconsiders both models, but only with correspondence-based methods. Here, in addition to applying the Miao-Santo's procedure, we also construct a new correspondence-based procedure for constructing recursive equilibrium. Section six, then, makes concluding remarks. In the appendix, will include the proofs, a detailed appendix with all the mathematical terminology used in the paper, and statements of key fixed point theorems used in the proofs.

[11] By a "continuation method", we mean a operator theoretic approach to solving functional equations that constructs a mapping $T : X \rightarrow X$ where X is a collection of functions or correspondences, where an element $x \in X$ is used to parameterized the future structure of equilibrium of the economy, then one computes the implied decision rules, from when $T(x)$ can be computed. The approach does bear, in an abstract sense, a resemblence to a continuation method in numerical analysis, but the details are still somewhat different.

A classic continuation method, in our terminology, is stationary dynamic programming. Here, one parameterized the continuation structure of the optimization problem on a state space is $s \in S$ with $V(s) \in \mathbf{V(S)}$, with $\mathbf{V(S)}$ is a complete metric space of function on $\mathbf{S}$**).** Then, one construct TV in $\mathbf{V(S)}$ in the Bellman equation that implied by $V(s)$. A (unique) fixed point is then constructed to solve the functional equation. Correspondence-based extensions of these methods are also available.

2. The Economies

To keep the issues raised in this paper clear, we focus our attention on recursive equilibrium in simple subclasses of the Solow-Cass-Diamond growth models. Time is discrete and indexed by $t \in T = \{0, 1, 2, ...\}$. We work with production processes are concave in both private and social returns (but in some cases exhibit social increasing returns as in Romer [50]). To avoid measurability issues, there no uncertainty. We consider both OLG models and models where a stand-in household lives infinitely-many periods.

For our OLG models, we assume the economy has a large number of identical agents are born each period who live for two periods. In their first period of life, they are endowed with a unit of time which they supply inelastically to the firm to earn a wage which they consume and/or save. In their second period of life, agents consume their savings. For the simpliest case, we assume preferences are represented time separable utility function and completely standard (e.g., power utility). For the case of infinitely-lived agent models, we then just extend the life spans of the typical agent to be infinite, so there the economiy consists of a continuum of infinitely-lived and identical household/firm agents.

As for preferences, we assume time separable preferences with constant discounting. Utility in any period is derived from consumption and given by a strictly concave and smooth function $u(c)$ that is bounded below (or $u(c) = \ln c$), with discounting summarized by $\beta \in (0, 1)$. For the OLG models, consumption for a household born in any period when "young" is denoted by c_1, and consumption when old is denoted by c_2. Therefore, in this case, the consumption set in any period is simply $X \subset \mathbf{R}_+$ (or $X \subset \mathbf{R}_{++}$ if period utility is given by $u(c) = \ln c$). For the infinitely-lived agent case, we simply extend the lifetime of the household. Then, household's lifetime preferences are defined over sequences indexed by dates and histories $c = (c_t)$ and are given by:[12]

$$U(\mathbf{c}) = \sum_{t=0}^{T} \beta^i u(c_t) \tag{1}$$

where β is the discount rate, where for the OLG case, we just take in (1) that $T = 2$, while for infinitely-lived agent models, we assume $T = \infty$.

We first discuss some

[12] Although the original work of Coleman [10] did not cover the unbounded homogeneous returns case, Morand and Reffett [40] show all his results can be extended to this case.

Assumption 1. The utility function $u : X \to \mathbb{R}_+$ is either $u(c) = \ln c$ or $u(c)$ satisfies:[13]

I. once continuously differentiable;

II. strictly increasing in each of its arguments and jointly concave;

III. $\lim_{c \to 0^+} u(c) = +\infty$

Households are endowed with a unit of time which is supplied inelastically to competitive firms, and enters any given period with an individual level of the capital $k \in \mathbf{K}$, facing an economy in aggregate state $K \in \mathbf{R}_+$ (where K is the per-capita capital stock). Following that tradition of work on monotone map methods for RE in nonoptimal economies (e.g., Greenwood and Huffman [23]. Coleman [12], and Mirman, Morand, and Reffett [35]), we consider reduced-form production functions that admit a specification denoted $F(k, n, K, N)$.[14] We assume F is constant returns to scale and concave in private inputs (k, n) for each level of aggregate inputs (K, N). As firms are identical, and endowed with standard constant returns to scale technology, profits are zero in equilibrium. As firms are also profit maximizing, anticipating equilibrium where $n^* = N^* = 1$, and $k = K$, if production F is sufficiently smooth, we have

$$\begin{aligned} r &= F_1(k, 1, k, 1) \\ w &= F_2(k, 1, k, 1) \end{aligned}$$

We now state our assumptions on F as follows:

Assumption 2. The production function $F(k, n, K, N) : X \times [0, 1] \times X \times [0, 1] \to \mathbb{R}_+$ is:

I. twice continuously differentiable jointly all its arguments;

II. strictly increasing and strictly concave in all its arguments, and supermodular in its first two arguments;

IIIa. such that $r(k, z) = F_1(k, 1, k, 1)$ is antitone in k, and $\lim_{k \to 0} r(k) = +\infty$;

IIIb. such that $w(k, z) = F_2(k, 1, k, 1)$ is isotone in k, and $\lim_{k \to 0^+} w(k) = 0$;

IV. such that there exists a maximal sustainable capital stock $k_{\max}$ (i.e., $\forall k \geq k_{\max}$ such that $F(k, 1, k, 1) \leq k_{\max}$, and with $F(0, 1, 0, 1) = 0$.

These assumptions are completely standard. It is well known that Assumption 2 IV implies that the set of feasible capital stocks can be restricted to be in the

[13] For example, one can just assume $u(c)$ is power utility as is typical in applied work.

[14] A number of well-known economies fit this reduced-form specification. See Greenwood and Huffman [23], for example, for a detailed discussion.

compact interval $X = [0, k_{\max}]$ as long as we place the initial capital stocks in X. This condition, along with (IIIa and IIIb) also place restrictions on the amount of nonconvexity we can allow.

The following two additional assumptions will help establish sharper properties of the RE, the latter being sufficient to exclude economies in which 0 may be the only RE (and will lead to the construction of minimal RE by successive approximations).[15]

Assumption 3 F is such that $\lim_{k \to 0^+} r(k)k = 0$

Finally, when we consider uniqueness of RE for OLG models, we will use the following capital monotonicity assumption:

Assumption 4: F is such that (a) $r(k)k$ is increasing in k, and (b) $r(k)$ is decreasing in k.

Assumptions 3 and 4 are satisfied for example in the standard Cobb-Douglas production case, as well as the economies in Romer [50]). Assumption 4 is not satisfied for general concave production in private inputs (with no externalities). Further, in the case of production externalities as in Romer [50] we could have $F(k, 1, K, 1) = k^{\alpha}K^{\eta}$ for $\eta < 0$, $\alpha \in (0, 1)$, and $|\eta| > \alpha$, which also violates Assumption 4.[16] There are no shocks.

In some cases, to relate our result to the existing literatyre, we create a simple class of equilibrium distortions for the infinitely-lived agent models that involve government taxes all sources of income using a state-contingent marginal tax rate of $\tau(K) \in [0, 1)$, with rebates of these tax revenues returned lump sum in the amount $J = \tau m$ to households. In this paper, we only consider only two cases of taxation under perfect commitment: (i) $\tau(K)$ increasing in K (e.g., "progressive" taxation"), and (ii) $\tau(K)$ is decreasing in K (e.g., "regressive" taxation).[17]

Assumption 5: (i) (Progressive taxation) $\tau : R_+ \to [0, 1)$, is locally Lipschitz, and monotone increasing (i.e., isotone); (ii) (Regressive taxation): $\tau : R_+ \to$

[15] The isotonicity assumption is standard; the continuity assumption may be weakened to upper semicontinuity.

[16] This latter case can be thought of a negative production externalities (e.g., one way of introducing a social cost to capital accumulation like in models of environmental degradation).

[17] The term "progressive" and "regressive" is a bit casual. We can easily rewrite the state variable for this economy to be after-tax income, and then make the tax literally an progressive/regressive income tax. Here, we are using K to proxy for the level of income.

$[0, 1)$, is locally Lipschitz, and monotone increasing (i.e., isotone); (iii) (Lump sum transfers) $J : R_+ \to R_+$ is locally lipschitz continuous.

The class of equilibrium distortions consistent with Assumption 5 have are common in the literature. The local Lipschitz structure for taxes and transfers is a very mild assumption, and have been used in many papers (e.g., Santos [54]).

3. RE in OLG Models

This section addresses the set of RE in OLG versions of our model. We begin by defining the numerous classes of functions where we shall both prove existence of (minimal state space) RE, and provide monotone iterative methods for computing particular elements of the set.[18] All our spaces of functions will be subsets of the space of bounded functions on compact set X. For any bounded function $m_b : X \to \mathbb{R}^+$, define the set $B_{m_b}(X) = \{h : X \to \mathbb{R}_+,\ 0 \leq h \leq m^b\}$ (we shall refer to this set as the set of "bounded functions"). If the upper bound m_I is isotone (i.e., non-decreasing in its arguments), the set $H_{m_I}(X) = \{h : X \to \mathbb{R}_+,$ $0 \leq h \leq m_I\}$. If in addition, m is continuous in k (in the usual topology on $\mathbb{R}$), define the set $H_m^u = \{h : X \to \mathbb{R}_+,\ 0 \leq h \leq m,\ h$ upper semicontinuous in $k \in X\}$ (resp., $H_m^l = \{h : X \to \mathbb{R}_+, 0 \leq h \leq m,\ h$ lower semicontinuous in $k \in X$ $\})$. We have the following proposition

Lemma 1. *The posets $(B_{n_b}, \leq), (H_{m_I}, \leq)$ $(H_m^u, \leq)$ and $(H_m^l, \leq)$ are complete lattices.*

Proof. That $(B_{m_b}, \leq)$ and $(H_{m_I}, \leq)$ are complete lattices is obvious. Let $B \subset H_m^u$, denote $g_\wedge(k) = \inf_{h \in B} h(k)$. Clearly $0 \leq g_\wedge \leq w$, g is isotone, and $g_{\bar{\wedge}}(k)$ is usc (i.e., see Aliprantis and Border [5], Lemma 2.41). Thus $g_\wedge$ is an greatest lower bound of B. Since m is the top element of H_m^u, it is a complete lattice (e.g., Davey and Priestley [16], Theorem 2.31). Dually, for $A \subset H_m^l$, define $g_\vee(k) = \sup_{h \in A} h(k)$. Clearly $0 \leq g_\vee \leq w$, g is isotone, and $g_\vee(k)$ is lsc. Again, as the bottom element is continuous (i.e., $\wedge H_m^l = 0$), H_m^l is a complete lattice. ■

[18] The order theoretic terminology we use in the paper is not standard in the literature. Useful references for such terminology include [16][56][58].

3.1. Computing RE in Each Subclass

We now discuss how to compute RE in distinct subclasses of functions (i.e., RE with distinct structural properties). We first formulate the household's problem. Given a competitive wage $w = m_b$ in the labor to the market, in a candidate RE $h \in B_w$, a typical young agent of any generation must decide what amount y to save for next period consumption when they retire. To make this decision, the agent uses h to compute the expected continuation return on her capital investment, as well as future competitive wages and returns on capital use the firms profit maximization problem with $w(k,z) = F_2(k,1,k,1)$ and $r(k) = F_1(k,1,k,1)$. Let $X^* = X \backslash 0$, and select $k \in X^*$ and $h \in B_w$. Then, a young agent solves:

$$\max_{x \in [0,w(s)]} u(w(k) - x) + u(r(h(s))x) \tag{2}$$

Let $x^*(k, h(k))$ be the optimal solution to this household problem in 2. Let $RE(X) \subset \mathbf{K}^X$, where the exponential space $\mathbf{K}^X$=$\{h : h : X \rightarrow \mathbf{K}\}$ given the topology of pointwise convergence and the pointwise partial order. Then *any* RE can be characterized as follows:

Definition 2. *A Recursive Equilibrium (RE) is any function $h^*(k) \in RE(X)$ and a policy function $x^*(k; h^*(k))$ such that (i) for all $k \in X^*$, $h^*(k) > 0$, we have*

$$x^* = x^*(k; h^*(k)) = h^*(k) \tag{3}$$

with $h^*(s) = 0$, else, and (ii)

$$-u'(w(k) - x^*) + u'(r(h^*(k))x^*)r(h^*(k)) = 0 \tag{4}$$

Notice, in our definition, we restrict our attention to the case of RE that have memory only on the current states of the economy. We consider existence of RE in our four subclasses of $RE(X)$ (namely, $B(X)$, $H(X)$, $H^u(X)$, and $H^l(X)$).To construct such RE in each of these subclasses, we introduce the nonlinear operator A defined implicitly in the HH equilibrium Euler equation follows

Definition 3. *Given any $h \in B$ (resp, H, H^u, H^l), define the operator A as follows: If $h(k) > 0$, then, $A(h(k))$ is the unique solution for x to:*

$$Z(y; k, h) = -u'(w(k) - x) + u'(r(h(k))x)r(x) = 0 \tag{5}$$

and $A(h)(k) = 0$ whenever $h(k) = 0$.[19]

[19] It is easy to verify the existence of a unique solution under Assumption 1.

By inspecting the definition of our operator, as $A(h)(k)$ corresponds with $x^*(k;h)$ for each (k,h), we then have a function h^* is a RE if and only if it is a non-zero fixed point of the operator A, and the issues of existence, characterization, and construction of extremal RE simply follow from the study of the set of nontrivial fixed points of Ah.

We now prove existence of RE existence in each of these subclasses, and provide explicit iterative procedures that convergence within in each subclass (but not in the other subclasses). To do this, we prove three lemmas first. In our first lemma, we show how Ah transforms on four spaces, and is isotone.

Lemma 4. *Under Assumptions 1, 2, (a) A is an isotone self map on $(H_w^u, \leq)$ and (H_w^l). Further, (b) $\exists$ upper solutions m_b (resp, m_I) such that Ah is an isotone self map on $(B_{m_b}, \leq)$ and $(H_{m_I}, \leq)$.*

Proof. (a) Consider $h \in H^u$, $h > 0$. As h is usc and isotone in k, $Z(y,k,h)$ is right continuous at every $k \in X$, increasing in k and strictly decreasing in x under A1 and A2, the unique solution to 5 is $A(h)(k)$, which is usc and isotone in k. Hence, $A(h)(k) \in H_w^u$ for such h. Noting the definition of Ah elsewhere, we have $A(h)(k) \in H_w^u$.

To see Ah is isotone on H_w^u, as Z is also increasing in h, each k, we have Ah isotone on H_w^u whenever $h > 0$. Noting the definition of Ah elsewhere, Ah isotone on H^u. As similar argument shows $Ah \in H_w^l$ and is isotone.

(b) First, we use $Ah \in H_w^u$ to construct both m_b and m_I. Consider $0 < k^* < k^{**} < k_{\max}$. Define the following two functions

$$\begin{aligned} m_b(k) &= 0 \text{ for } k = 0 \\ &= w(k) \text{ for } 0 < k < k^* \\ &= A(w)(k) \text{ for } k^* \leq k \leq k^{**} \\ &= A^2(w)(k) \text{ else} \end{aligned} \tag{6}$$

and

$$\begin{aligned} m_I(k) &= 0 \text{ for } k = 0 \\ &= A^2(w)(k) \text{ for } 0 < k < k^* \\ &= A(w)(k) \text{ for } k^* \leq k \leq k^{**} \\ &= w(k) \text{ else} \end{aligned} \tag{7}$$

By construction, m_b (resp, m_I) is a bounded function, not semicontinuous or increasing (resp, a bounded increasing but not semicontinuous) function. Further, m_b (resp, m_I) are upper solutions in the space $B_{m_b}(X)$ (resp, $H_{m_I}(X)$) as they are clearly not fixed points of Ah in either space (and all the fixed points, if nonempty, my be lower in order than m_b (resp, m_I). Finally, they are each not ordered with the iteration $A(w)$.

Now, consider $h \in B_{m_b}$ (resp, $h \in H_{m_I}$) with $h > 0$. As Z is bounded in k (resp, Z is bounded and increasing in k), and Z is strictly decreasing in x, when $h > 0$, $A(h)(k) \in B_{m_b}$ (resp, $A(h)(k) \in H_{m_I}$. Noting the definition of Ah elsewhere, $A(h)(k) \in B_{m_b}$ (resp, $A(h)(k) \in H_{m_I}$. Further, as Z is increasing in h (resp, increasing in h), Ah is isotone on B_{m_b} (resp., isotone on H_{m_I}). Noting the definition of $A(h)(k)$ elsewhere, Ah is isotone on B_{m_b} (resp, H_{m_I}). ■

We now define two distinct, yet related, notions of order continuity that we shall use in the paper to check conditions under which we can compute RE by successive approximation. Both notions of continuity refer to the interval topology.

Definition 5. *A function $F : (P, \leq) \to (P, \leq)$ is sup (resp, inf) order continuous if for any countable chain $C \subset P$ such that $\vee C$ and $\wedge C$ both exist,*

$$\vee\{F(C)\} = F(\vee C) \text{ (resp, } \wedge \{F(C)\} = F(\wedge C)).$$

The function F is order continuous if it is sup/inf order continuous. The function F is sup (resp, inf) order continuous along an F generated chain from $x_0 \in P$ if for all n

$$\vee F^n(x_0) = F(\vee x_n) \text{ (resp, } \wedge F^n(x_0) = F(\wedge x_n)$$

where the sequence $\{x_j\}_j^n$ is generated recursively as

$$x_{j+1} = F(x_j),\ x_0 \in P \text{ given}$$

We now show under pointwise partial orders, our operator Ah is order continuous along A- generated chains in all of its relevant domains (e.g., W, H, H^u, and H^l).

Lemma 6. *(i) Under Assumptions 1 and 2, (a) the set of fixed points of A in $(B_{m_b}, \leq)$ (resp, $H_{m_I}, \leq$), $(H^u_w, \leq)$ and $(H^l_w, \leq)$ is a non-empty complete lattice, (b) A is order continuous along F generated chains in $(B_{m_b}, \leq)$ (resp, $H_{m_I}, \leq$). It is inf-order continuous along chains in $(H^u_w, \leq)$, and sup-order continuous along chains in $(H^l_w, \leq)$*

Proof. (a) The complete lattice structure of these sets of fixed points follows from Tarski's fixed point theorem, noting each subclass $(B_{m_b}, \leq)$ (resp, $H_{m_I}, \leq)$, $(H^u_w, \leq)$ and $(H^l_w, \leq)$ is a complete lattice by Lemma 1, and Ah is an isotone self map by Lemma 4.

(b) Next, we prove order continuity along increasing F chains by showing that for any increasing sequence $\{g_n\}$ in $(B_{m_b}, \leq)$ or in $(H_{m_I}, \leq)$, we have

$$\sup(\{Ag_n(s)\}) = A(\sup\{g_n(s)\}).$$

For such a sequence and for all $s \in S$, the sequence of real numbers $\{g_n(k)\}$ is increasing and bounded above (by $w(k)$), thus $\lim_{n\to\infty} g_n(k) = \sup\{g_n(k)\}$. For the same reason $\lim_{n\to\infty} Ag_n(k) = \sup\{Ag_n(k)\}$. By definition, for all $n \in \mathbb{N}$, and all $k \in X^*$:

$$-u'(w(k) - Ag_n(k)) + u'(r(g_n(k))Ag_n(k))r(Ag_n(k)) = 0$$

The function u' is continuous (Assumption 1), and r is continuous (Assumption 2), hence taking limits when n goes to infinity, we have:

$$-u'(w(k) - \sup\{Ag_n(k)\}) - u'(r(\sup\{g_n(s)\})\sup\{Ag_n(k)\})r(\sup\{Ag_n(k)) = 0$$

which implies that $A(\sup\{g_n(k)\}) = \sup\{Ag_n(k)\}$. A symmetric argument can easily be made for any decreasing sequence $\{g_n\}$ in $(B_{m_b}, \leq)$ or in $(H_{m_I}, \leq)$. This establishes (b) for $(B_{m_b}, \leq)$ or in $(H_{m_I}, \leq)$. A similar argument can be used for $(H^u_w, \leq)$ and $(H^l_w, \leq)$ to show A is inf (resp, sup) order continuous noting the lower envelope (resp, upper envelope) of a collection of upper semicontinuous (resp, lower semicontinuous) functions is upper semicontinuous (resp, lower semicontinuous). ■

Our final lemma is particularly important for verifying the existence of nontrivial minimal RE. As is clear from the definition of Ah, in all cases of subsets of W, $h^* = 0$ is a trivial fixed point. Therefore, the next lemma find a minimal element of H^l that maps up. Note that we construct this lower bound h_0 to be lsc so that the iterations $\{A^n h_0\}$ will be an increasing sequence of lsc functions, which therefore converges in order to the lsc function $\vee\{A^n h_0\}$.

Lemma 7. *Under assumptions 1, 2, and 3, (a) there exists a function $h_0 \in (H^l_w, \leq)$ such that (i) $\forall k \in X^*$, $Ah_0(k) > h_0(k) > 0$, and (ii) $\forall h \in (0, h_0]$, $Ah > h$ on X^*.*

Proof. See McGovern, Morand, and Reffett [33], Appendix A, noting there are no shocks in our economies, and under Assumptions A1, preferences are additively separable. ■

We are now prepared to prove our first theorem on the existence of RE in the class of bounded functions and isotone bounded function, as well as characterize the structure of the set of RE. In the Theorem, h_0 is the function constructed in Lemma 7. What will be critical in the next few theorems is to notice the key role played in the arguments by the upper and lower solutions relative to fixed point sets of our operator Ah when it maps in different domains. That is, first notice in this theorem below, we study existence of RE in subcomplete order intervals spaces B_w and H_w (where w is taken to be the upper solution that tops the space, and the lower element of the order interval is given by h_0 in Lemma 7. [20] Therefore, what this theorem verifies is two facts: (i) how to very the existence of a complete lattice of of nontrivial RE, as well as compute least and greatest RE for our OLG economies under A1-A3 relative to the space of bounded functions $B_w \cap [h_0, w]$ (resp, bounded isotone functions $H_w \cap [h_0, w]$) with upper solution w, as well as (ii) showing that *both* least and greatest RE in this case actually belong to spaces of functions with *stronger* structural properties than either $B_w \cap [h_0, w]$ (resp, $H_w \cap [h_0, w]$). In particular, per (ii) the greatest RE is in $H_w^u \cap [h_0, w]$, while the least RE be in either $H_w^l \cap [h_0, w]$ or $H_w^u \cap [h_0, w]$). As a matter of notation, the fixed point set Ah in the space B_w (for example) will be denoted by $\Psi_A^{B_w}$.

Theorem 8. *Under Assumptions 1, 2, and 3: (a) there exist a nonempty complete lattice of nontrivial RE in $B_w \cap [h_0, w]$ (resp, $H_w \cap [h_0, w]$), (b) the least RE $h_{\min} = \wedge \Psi_A^{B_w \cap [h_0,w]} = \wedge\ \Psi_A^{H_w \cap [h_0,w]}$ in $(B_w \cap [h_0, w], \leq)$ (resp, in $(H_w \cap [h_0, w], \leq)$) is an isotone lsc (i.e., $h_{\min} \in H_w^l \cap [h_0, w]$, while the greatest RE is $(B_w, \leq)$ (in $(H_w, \leq)$) is $h_{\max} = \vee \Psi_A^{B_w} = \vee \Psi_A^{H_w}$ is an isotone usc function (i.e., the greatest RE in H_w^u). Further, we can modify our iterations in (b) to compute a least usc isotone RE in $H_w^u \cap [h_0, w]$. Finally (d) all these extremal RE can be constructed by successive approximations.*

[20] This observation is important, as what we are showing in this theorem is that if one works in the space of functions B_w (resp, H_w), where the upper solution that define the space is w (which is a continuous isotone function), the *greatest* fixed point will always be in H_w^u (i.e., a space of functions with *stronger* structural properties that B_w (resp, H_w). This means if we want to compute RE in a *larger* subclass than H_w^u, we must to change the upper solution to the function m_b (resp, m_I) that is defined in the proof of Lemma 4, where m_b (resp, m_I) are only bounded (resp., only bounded and increasing), but always discontinuous.

Proof. (a) From Lemma 7, Ah transforms the complete lattice of bounded functions $[h_0, w] \subset B_w$ (resp, $[h_0, w] \subset H_w$). By Lemma 4, Ah is isotone. The result then follows from Tarski's theorem. (e.g., Tarski [55], Theorem 1).

(b): We first compute by successive approximation the least RE in H_w^l. By lemma 7, when restricted to the order interval $H_w^l \cap [h_0, w]$ (which is a complete lattice itself), we have $0 < h_0 = Ah_0$. As Ah is order continuous (e.g., Lemma 6), by the Tarski-Kantorvich Theorem (i.e., Dugundji and Granas [19], Theorem 4.2), we hve $\vee\{A^n h_0\} = h_{\min} = \wedge \Psi_A^{H_w^U}$ when $k > 0$ where $\Psi_A^{H_w^U}$ denotes the fixed points of Ah in $H_w^l \cap [h_0, w]$. That is, we have

$$h_{\min}(k) = \vee\{A^n h_0\}(k) = \lim_{n\to\infty} A^n h_0(k) = \sup\{A^n h_0(k\} = \wedge \Psi_A^{H_w^U}.$$

where $h_{\min}$ is lsc as it is the upper envelope of a family of elements of lsc functions, and hence in H_w^l. It is therefore the minimal bounded isotone and lsc RE in $H_w^l \cap [h_0, w]$. It is also the minimal RE in $H_w \cap [h_0, w]$ as $h_0 \in H_w$. It is a nontrivial RE as $h_{\min}(k) > 0$ when $k > 0$, so it satisfies the RE functional equation in 3.

Similarly, we can compute the maximal RE in $(B_w \cap [h_0, w], \leq)$ as the inf (pointwise limit) of a decreasing sequence beginning at w. That is, is:

$$h_{\max}(k) = \wedge\{A^n w\}(k) = \lim_{n\to\infty} A^n w(k) = \inf\{A^n w(k)\},$$

which implies that $h_{\max} \in H^u$ since it is the lower envelope of a family of elements of $(H_w^u \cap [h_0, w], \leq)$.

(c) We next compute the least RE in H_w^u. Following the same argument as in Theorem 8, it is only a matter of correcting $h_{\min}$ above at most at a countable number of points to obtain the minimal bounded isotone and *usc* RE. Specifically, the minimal RE in $(H_w^u, \leq)$ is the function $g_{\min} : X \to X$ defined as:

$$\begin{aligned} g_{\min}(k) &= \inf_{k'>k} \{\sup\{A^n h_0(k')\}\} \\ &= \inf_{k'>k} \{\vee\{A^n h_0\}(k')\} \ \forall\ k \in [0, k_{\max}) \end{aligned}$$

and $g_{\min}(k_{\max}) = \vee\{A^n h_0\}(k_{\max})$. Indeed, by construction $g_{\min} \in H_w^u$, $g_{\min}(k)$ and $h_{\min} = \vee\{A^n h_0\}(k)$ differ at most at the discontinuity points of $\vee\{A^n h_0\}(k)$, and $g_{\min}(., z)$ is the smallest usc function greater than $\vee\{A^n h_0\}(k)$. In addition, since $\vee\{A^n h_0\}$ is lsc, for any $k \in X$, $g_{\min}(k) = \lim_{k'\to k^+} \vee\{A^n h_0\}(k')$. For any $k \in [0, k_{\max})$, and for all $k' > k$, by definition of $h_{\min}(k')$=$q(k')$

$$-u'(w(k') - q(k')) + u'(r(q(k'))q(k'))r(q(k')$$

Both functions $u'(c)$ continuous (assumption 1) and r is continuous (assumption 2), taking limits when $k' \to k^+$ on both sides of the previous equality implies that:

$$-u'(w(k) - g_{\min}(k)) + u'(r(g_{\min}(k))g_{\min}(k))r(g_{\min}(k)) = 0$$

which proves that, $Ag_{\min}(k) = g_{\min}(k)$. The set of RE in $(H_w^u, \leq)$ is then simply the set of fixed points of A that are bounded, isotone, and usc.

(d) this is obvious by the constructions in parts (b) and (c). ■

We now prove a second existence theorem concerning the existence and computation of *non-trivial* least and greatest RE within the subclasses B_{m_b} and H_{m_I}. That is, in this theorem, we want to consider the existence of RE that are bounded, but not isotone (i.e., in subintervals of B_w but not in H_w), or bounded and isotone, but not not semicontinuous (i.e., in subintervals of H_w, but in neither H_w^u and H_w^l) . We can use the results of the previous Theorem to obtain least RE with these properties. To obtain greatest RE that are not in H_w^u, we must change the upper solutions w to a new function (i.e., either m_b or m_I defined in equation 6 and 7, respectively).

Theorem 9. *Under Assumptions 1-3, there exists a complete lattice of RE in $B_{m_b} \cap [h_0^b, m_b] \subset B_w$ (resp, $H_{m_I} \cap [h_0^I, m_I] \subset H_w$) where h_0^b and h_0^I are nontrivial lower solutions in B_w and H_w, respectively. Further, the least and greatest RE in each subclass can be computed by successive approximations.*

Finally, note that it is also easy to modify the usc function $h_{\max}$ at most at a countable number of points to construct the maximal bounded isotone and lsc RE. So by construction, least and greatest semicontinuous RE are not the same in general even in spaces where the lower (resp, upper solutions) are the same.

3.2. On Uniqueness of RE

Under the additional assumption of capital income monotonicity, we can sharpen our results. Specifically, we prove three things. First, we show in our context the existence of a *unique* Lipschitz continuous isotone h^*. A related result has been shown previously in Wang [57], Morand and Reffett [37] and McGovern, Morand, and Reffett [33]. Second, we prove this uniqueness result remains valid related to H_w, that is the space of bounded isotone functions. Finally, we prove this uniquness result *fails* in the space B_w (bounded functions).

Theorem 10. *Under Assumption 1-4, (i) there exists a unique bounded isotone RE h^* in H. Further, the corresponding (Markovian) equilibrium consumption policy, $w - h^*$ is also isotone, which implies that both h^* and $w - h^*$ are Lipschitz continuous. Further, this uniqueness result relative to the space B_{m_b}*

Proof. Under capital income monotonicity, for all $k \in X^*$ the following equation in y:

$$-u'(w(k) - x) + u'(r(x)x)r(x) =$$

has a unique solution. Notice, if $h^*(k)$ is RE in H_w (which exists by Theorem 8, for example) . Let $k_1 \geq k_2 > 0$. As $h^*(k)$ is increasing, under assumption A4,

$$-u'(w(k_2) - h^*(k_2)) + u'(r(h^*(k_1))h^*(k_1))r(h^*(k_1)) \leq 0$$

Therefore, it must be the case that $h^*(k_1)$ is such that $w(k_1) - h^*(k_1)$ is increasing in k. But as $h^*(k)$ is also increasing in k, and $w(k)$ is locally lipschitz of modulus $w'(k)$ near each k, $h^*(k)$ is an element of an equicontinuous collection with pointwise bound $w'(k)$. As $L = \sup_{k \in X} |w'(k)| < \infty$, this implies $h^*(k)$ is Lipschitz of modulus L on $(0, k_{\max}]$. By a standard Lipschitz extension argument, as $w(0) = 0 = h^*(0)$, $h^*(0)$ is a Lipschitz extension of $h^*(k)$ for $k > 0$, and we can wlog have $h^*(k)$ Lipschitz of modulus L on all of $[0, k_{\max}] = X$. So this proves the first part of the theorem.

Finally, following Wang [57], for each $k \in X^*$, for h^* increasing, we have both $w(k) - h^*(k)$ and $h^*(k)$ increasing in k. Further, as $w(k)$ is strictly increasing in k, $u(c)$ strictly concave, this implies both are strictly increasing in k for h^* increasing. Also, as the set of fixed points in any spaces of increasing functions is a complete lattice, for any fixed point h^* that is increasing, $\exists$ another fixed point h^{**} such that $h^{**} = t(k)h^*(k) \leq h^*$ for $0 < t(k) \leq 1$ for all $k \in X^*$ with $0 < t(k) < 1$ for at least 1 $k \in X^*$. If these fixed points aAll these facts together imply that as Z defined below

$$Z(h^*, k, h^*) = -u'(w(k) - h^*(k)) + u'(r(h^*(k))h^*(k))r(h^*(k))$$

is strictly falling in $h \in H_w$ at each fixed k, h^* when $h^{**} < h^*$ at any $k \in X^*$,there cannot be two solutions to this functional equation at each k. That is, $h^*(k)$ is a unique lipschitz RE, and that is true for any other candidate RE $h \in H_w$ with $h \neq h^*$. So this proves the second part of the theorem. ■

Finally, we stress two important facts: (i) capital income isotonicity is not necessary for uniqueness of RE (as shown by the following example shows), and

(ii) the uniqueness of RE even under capital income monotonicity only holds relative to spaces of *isotone* RE (i.e., this uniqueness result is not robust to RE many subsets of B_w).

We first show that capital income is not necessary for uniqueness by example.

Example 11. *Consider the utility function:*

$$\ln(c_t) + \ln(c_{t+1}),$$

in which case the maximization problem of an agent is:

$$\max_{x \in [0, w(s)]} \left\{ \ln(w(s) - x) + \int_Z \ln(r(h(s))x) \right\},$$

and the associated first order condition is

$$(w(s) - x) = x,$$

so that the unique continuous RE is the function $h^*(k) = .5w(k)$. Notice, under log utility, h disappears from the first order condition, so solving for the unique continuous RE in this special case is very simple.

We remark, that in this example, this points the our need to also rule out log utility (to obtain multiple RE across various spaces of functions). In the log utility case, one can obtain closed-form solutions to the functional equations, which great simplifies the existence of RE problem,

Now, we conclude with a corollary to Theorem 9, which explicitly computes an RE in B_w where Theorem 10 fails

Corollary 12. *In the space $(B_{m_b}, \leq)$, the successive approximations $\inf_n A^n(m_b) \to h^b_{\max}$*

Proof. Follows from Theorem 9 ■

We note that in the above corollary we cannot prove $h^b_{\max}$ is increasing in k, (an, hence, our uniqueness result fails). That is, if we choose $k_1 \geq k_2$, as $h^b_{\max}(k)$ is not increasing in k, $r(h^b_{\max}(k))h^b_{\max}(k)$ is not necessarily increasing. Hence, $(w - h^b_{\max})(k)$ is not increasing in k, and uniqueness argument in Theorem 10 fails (as at a fixed point $x^*(k; h^b_{\max}(k)) = h^b_{\max}(k)$, there can be multiple roots of the equation $Z(x^*(k, h^b_{\max}(k), k, h^b_{\max}(k)) = 0$.

3.3. Stable Iterations and Computable Equilibrium Comparative Statics

We finally consider ordered perturbations of the primitive data of our OLG economies, and show how our monotone methods can be used to study the question of existence of tractible computable selections. To do this, we first impose a slight modification of assumption 1:

Assumption 1$'$: $u(c)$ satisfies A1 plus (i) $u(c)$ is such that $u'(r \cdot x)r$ is increasing in r, and (ii) $u(c_1) + \beta u(c_1)$, for $\beta > 0$.

Assumption 1$'$ would be met, for example, for models with power utility. With this in mind, we now introduce partial orders on the primitive data of the economy. Let $\mathbf{F}$ be the space of production functions satisfying Assumption A2. Define the *gradient order* on $\mathbf{F}$ as follows: for $f_1 \in \mathbf{F}$, and $f_2 \in \mathbf{F}(K)$, with $f_1(0,1,0,1) = f_2(0,1,0,1)$, we say $f_1 \geq_F f_2$ if for all $K > 0$, $f_1(k,1,K,1) - f_2(k,1,K,1)$ is increasing in k. Notice, this is actually introduces a partial order on $\mathbf{F}(K)$ for each $K > 0$. We then have the following computable equilibrium comparative statics result:

Theorem 13. *Let $Ah(k; f, \beta)$ be the operator on $B^* = B_{m_b} \cap [h^b_\wedge, m_b]$ (resp, $H^* = H_{m_I} \cap [h^b_\wedge, m_b]$, $H^{u*} = H^u_w \cap [h^b_\wedge, w]$, $H^{l*} = H^l_w \cap [h^b_\wedge, w]$). Let $\Psi^{B^*}_A(f,\beta)$ (resp, $\Psi^{H^*}_A(f,\beta), \Psi^{H^{u*}}_A(f,\beta), \Psi^{H^{l*}}_A(f,\beta)$) be the set of RE in B^* (resp, H^*, H^{u*}, H^{l*}). Then, under A1'-A3, $\Psi^{B^*}_A(f,\beta)$ (resp, $\Psi^{H^*}_A(f,\beta), \Psi^{H^{u*}}_A(f,\beta), \Psi^{H^{l*}}_A(f,\beta)$) are each nonempty complete lattices, with least and greatest RE in each class increasing selection. . Further, the iterations from least and greatest elements of each space B^* (resp, H^*, H^{u*}, H^{l*}) converge in order to these increasing selections.*

Proof. That $\Psi^{B^*}_A(f,\beta)$ (resp, $\Psi^{H^*}_A(f,\beta), \Psi^{H^{u*}}_A(f,\beta), \Psi^{H^{l*}}_A(f,\beta)$) be the set of RE in B^* (resp, H^*, H^{u*}, H^{l*}) are nonempty complete lattices for each (f,β) follows from Tarski's theorem. Using the definition of $Ah(k; f, \beta)$ in $Z(x,k,h)$, one can easily verify $Ah(k; f, \beta)$ is increasing in both (f,β) (where the partial order on $\mathbf{F}$ is the gradient order). The fact that that least and greatest elements of $\Psi^{B^*}_A(f,\beta)$ (resp, $\Psi^{H^*}_A(f,\beta), \Psi^{H^{u*}}_A(f,\beta), \Psi^{H^{l*}}_A(f,\beta)$) are increasing selections follows, therefore, from Veinott's fixed point comparative statics theorem. The computability result follows from the order continuity of $Ah(k; f, \beta)$ and the Tarski-Kantorovich theorem. ■

4. Economies with Infinitely-Lived Agents and Progressive Taxes

We now consider infinite horizon economies with progressive taxes (i.e., exactly the economy studied in Coleman [10][12]. It is a special case of the economies studied in Mirman, Morand, and Reffett [35]. For these economies, we begin by constructing a recursive representation of a typical household's decision problem that we shall repeatedly appeal to throughout the paper. We again seek recursive equilibrium on a minimal state space. In this case, a household enters the period with individual capital stock k facing prices in the econmy generated by an aggregate capital stock K, with all future capital stocks calculate using a fixed law of motion on that aggregate capital stock

$$K' = h(K)$$

with initial states $(k_0, K_0) \in X= \mathbf{K}\times\mathbf{K} \subset \mathbf{R}^2_+$ given. Let this beginning period state variable, therefore, be denoted by $s = (k, K) \in X$. Let $\mathbf{B}(X)$ denote the space of bounded functions endowed with (i) the topology of uniform convergence, and (ii) the pointwise partial order, and $\mathbf{B}^f(\mathbf{X})$ be a subset of $\mathbf{B}(\mathbf{X})$ that consist of all the socially feasible aggregate laws of motion; i.e.,

$$\mathbf{B}^f(X) = \{h(s)|0 \leq h(x) \leq r(K)k + w(K)\} \subset \mathbf{B}(X)$$

Notice, for an equilibrium trajectory, we shall ahve $k = K$, so $r(x)x+w(x) = f(x)$, so $s_D = (k, k) \in D = \{s \in X|s = (k, k), k \in \mathbf{K}\}$. Notice, D is an diagonal subspace of X. Endow $\mathbf{B}^f(X)$ with its relative topology and relative partial order. The collection $\mathbf{B}^f(X)$ is a complete sublattice in $\mathbf{B}(X)$.

For simplicity, we assume household's own the firms, and rent the factors of production to those firms in competitive markets. Using the definitions of r and w, and appealing to zero profits under constant returns to scale in Assumption 2, the household income process can be written equivalently as either y^1 or y^2 in the following expression:

$$\begin{aligned} y^1(k, K) &= (1 - \tau(K))\{f(K) + (k - K)r(K)\} + J(K) && (8) \\ &= (1 - \tau(K))\{r(K)k + w(K)\} + J(K) \\ &= y^2(k, K) && (9) \end{aligned}$$

where $y^i : \mathbf{K} \times \mathbf{K}_{++} \to \mathbb{R}_+$ for $i = 1, 2$.[21] For simplicity, lets write the budget correspondence just using $y^1 = y$, so household's budget correspondence can be written as:

$$\Psi(k,K) = \{c, k' | c + k' \leq y(k,K), c \geq 0, k' \geq 0\} \tag{10}$$

In equilibrium, where $k = K$, as the government's budget constraint is imposed, we require $\tau f = \tau(rK + w) = J$. Therefore, the household's income processes can be written, respectively, as

$$\begin{aligned} y(k,k) &= f(k) \\ &= y^2(k,k) = rk + w \end{aligned}$$

Under Assumptions 1, 2, and 5, as f, r, w, τ and J are each at least locally Lipschitz, so the household's feasible correspondence $\Psi^i(k,K)$ is locally Lipschitz continuous on $\mathbf{K} \times \mathbf{K}$ when $K > 0$.[22]

Let $\mathbf{K}^* = \mathbf{K} \backslash 0$, and $X^* = \mathbf{K} \times \mathbf{K}^*$. To construct a recursive representation of the household's decision problem, for a household entering a period in state $s = (k,K) \in X^*$ in a candidate recursive equilibrium $h \in \mathbf{B}^f(X)$, when $h > 0$, we can construct a unique value function $V^* : \mathbf{K} \times \mathbf{K}^* \times \mathbf{B}^f(\mathbf{X})$ that satisfies the following parameterized Bellman's equation:

$$V^*(k,K;h) = \sup_{x \in \Psi(k,K)} \{u(y(k,K) - x) + \beta V^*(x, h(K)\} \tag{11}$$

where the household's feasible correspondence is simply $\Psi(k,K) = [0, y(k,K)]$. Under Assumptions 1, 2 and 5, appealing to a standard argument in the literature, it can be shown that the unique real-valued solution to this Bellman equation is a function $V^*(k,K,h)$ in the set $W = \{ V : \mathbf{K} \times \mathbf{K}^* \times \mathbf{B}^f \to \mathbf{R}$, v (i) isotone in k, each (K,h), (ii) strictly concave (hence, continuous) in k for each $(K,h)\}$. Further, additionally, by the Mirman-Zilcha lemma, $V^*(k,K;h)$ also (iii) has a envelope theorem in $V_1^*(k,K;h) = u'(y^i_{x^*})r(1-\tau)$, where (iv) the optimal policy $x^* = x^*(k,K;h)$ is single valued and continuous in its first argument. Allowing for unbounded returns above in our setting is not a problem. (See Morand and Reffett [40] for power utility, or Morand, Reffett and Wang [42], more generally).

[21] It will become clear in a moment why keeping track of the two different equivalent expressions for household's income process is useful. As a convention, unless we mention y^2, we will use $y = y^1$ as the HH income process.

[22] See Rockafellar [52] for a discussion of Lipschitizian properties of correspondences.

[23] Notice, if in addition, $h(K)$ is continuous, $x^*(k,K;h)$ is also continuous in K. Therefore, in any recursive equilibrium $h^* \in \mathbf{B}^f$, when $k = K$, h^* must be such that conditions (i)-(iv) hold for equilibrium value function $V^*(k,k;h^*)$ and unique optimal solution $x^*(k,K;h^*)$. Defining the mapping $y_{x^*} = y - x^*$, we can construct a necessary and sufficient first order characterization of the unique optimal solution $x^* = x^*(k,K,h)$ as:

$$u'(y_{x*}) - \beta u'(y_{x^*})(x^*, h(K))\ r(h(K))(1 - \tau(h(K)) = 0 \tag{12}$$

With these properties of $V^*(k,k;h^*)$ and $x^*(k,K;h)$ now clear, we now ready define an recursive equilibrium for our economies as follows:[24]

Definition 14. *A recursive equilibrium is any function $h^*(k,k) \in \mathbf{B}^f(X)$, such that (i) positivity: $h^*(k,k) > 0$ and $(f - h^*)(k,k) > 0$ when $k \in \mathbf{K}^*$, (ii) RE functional equation: $h^*(k,k) = x^*(k,k;h^*(k,k))$, and $h^*(k,k) = 0$, else, and (iii) Necessary Structural Properties for HH optimization in RE: when $k > 0$, given a law of motion $h^*(k,K) \in \mathbf{B}^f$, when $k = K$, (a) Strict concavity, individual states: there is a dynamic program $V^*(k,k,h^*(k,k))$ that solves (11), strictly concave in its first argument, that satisfies the Bellman equation (11) with associated unique optimal solution $x^*(k,k;h^*(k,k))$; (b) Envelope theorem holds: $V^*(k,k,h^*(k,k))$ has an envelope theorem $V_1^*(k,k;h^*) = u'(y_{x*})r(K)(1-\tau(K))$ at $x^*(k,k;h^*(k,k))$, and (c) Euler equation is satisified: $x^*(k,k;h^*(k,k))$ can be characterized by the necessary and sufficient Euler equation (12).*

We need to make a remark at this point. In this definition above, we emphasize the requirements that **any** recursive equilibrium must satisfy. In particular,

[23] That is, the key step to dealing with the unbounded below case is to first solve the Euler equation abstractly, and then use well-known methods for solving dynamic programming problems with unbounded returns in a candidate to show there exists, and has a unique value function evaluated at a (positive) equilibrium solution that satisfies a necessary and sufficient Euler equation. This can be done by a modification of the local contraction arguments in Martins-Da-Rocha and Vailakis [31] in our deterministic model. See Reffett [48] for a discussion.

[24] The requirement of interiority of consumption implied in our definition is a natural requirement for a recursive equilibrium. For example, it is needed to show that the candidate recursive equilibrium decision rule $a^*(k,k,h^*(k,k)) = h^*(k,k)$ induces sequential equilibrium with a price system in an appropriate dual space to the (infinite) commodity space. Although this is not the focus of this paper, proofs for our economies can be built from the recent results presented in Morand, Reffett and Wang [42] for this economy.

$h^*(k)$ is a RE iff conditions (i), (ii), and (iii.a-iii.c) hold. The key conditions that will be a problem to check for correspondence based recursive methods (e.g., Phelan and Stacchetti [46], Kubler and Schmedders [26]. Miao and Santos [34] and Feng, et. al. [21]) will becontinuity requirements that any RE must satisfy along the *diagonal* of a function $x^*(k, K, h^*(k, k)) = h^*(k, k)$ in its *first* argument. More on this in the next section for the regressive tax case. It also bears mentioning that for any correspondence-based continuations method that does not work in function spaces (i.e., all of the methods in the existing literature), the requirement of continuity in the individual state variables along the diagonal is demanding. For example, it implies that for $G^*(k, k)$ the equilibrium correspondence generated by the "APS" type operator, one must guarantee the existence of a selection, say $g^*(k, k) \in G^*(k, k)$, that is *continuous* in its first argument (hence, guaranteeing the resulting Generalized Markov equilibrium decision rule for investment $k' = x^*(k, k; g^*(k, k)) \in X^*(k, k, g^*(k, k))$ is a continuous selection in its first argument). This condition is very difficult to check even in very simple one dimensional problems as we shall argue in Section 5 of the paper, as the typical equilibrium correspondence $G^*(k, k)$ is simply a nonempty upper semicontinuous correspondence in its arguments. More in this in the next section.

We first consider function-based continuation methods for our economies under Assumptions 1, 2, and 3(i) and 3(iii) (that is, nonoptimal Cass growth with a state-contingent progressive tax, and lump-sum transfers.) For these economies, we prove two new results. First, we extend the uniqueness result obtained in Coleman [12] for his policy-iteration procedure to a much larger class of domains (namely, a class of bounded functions with bounded consumption functions, not necessarily even monotone). We then construct a second new fixed point procedure that is not policy iteration, but admits a complete lattice of (locally Lipschitz) continuous fixed points, where at least its least fixed point (for equilibrium investment) is a recursive equilibrium. This recursive equilibrium cannot be guaranteed to be in Coleman's fixed point set obtained using policy iteration.

4.1. Some Useful Complete Lattices

Any discussion of solution methods for functional equations begins with a discussion function spaces that serve as domains for fixed points of operators used to solve the equations. At this stage, we define a number of function spaces that we use in the paper. We begin with subsets of the bounded socially feasible decision rules $\mathbf{B}^f(X)$. For the moment, let $s = (s_1, s_2) \in X$ for the moment, where $s_1 = k$, and $s_2 = K$.[25] To guarantee our solutions are recursive equilibrium satisfy conditions (iii.a)-(iii.c) along their diagonal (where $s_1 = s_2$), it is important keep track of individual vs. aggregate states separately. Therefore, partition the components of the state space $x \in \mathbf{X}$ as $x = (s_1, s_2) \in \mathbf{X}_1 \times \mathbf{X}_2 = \mathbf{X} \subset \mathbf{R}^2_+$**,** where s_1 can be viewed as a individuals holds of capital, while s_2 is the aggregate state of the economy's per capita capital stock, $\mathbf{X}_1 = \mathbf{X}_2 = \mathbf{K}$. Consider a subset of $\mathbf{B}^f(X)$ consisting of a set of upper semicontinuous, monotone functions on **X:**

$$\begin{aligned} \mathbf{UCS}(\mathrm{X}) = \{h(x) \in \mathbf{B}^f| \quad & h(x) \text{ monotone increasing (isotone)} \\ & \text{and upper semicontinuous in } x\} \end{aligned}$$

For an element $h \in \mathbf{USC}(X)$, if we interpret a typical element $h(x)$ as an candidate equilibrium investment decision rule, notice the implied equilibrium consumption $c^*(x) = y(x) - h^*(x)$ is lower semicontinuous in x (and not necessarily isotone).

An important subset of $\mathbf{USC}(X)$ occurs when consumption decisions rules $c = y - h$ are also isotone on X, namely, the space:

$$\begin{aligned} \mathbf{C}(X) &= \{h(x) \in \mathbf{USC}(X) | h(x) \text{ continuous, s.t.} \\ & \quad\; y(x) - h(x) \text{ is isotone in } x\} \end{aligned}$$

The space $\mathbf{C}(X)$ is the domain for policy iteration methods studied in Coleman [10][12] (as well as many subsequent related papers on policy iteration methods based upon equilibrium versions of the household's Euler equations). Therefore, for $h \in \mathbf{C}$, as both $h(x)$ and $c(x) = y(x) - h(x)$ are isotone in x, $h(x)$ and its implied consumption $c(x)$ are necessarily continuous (as with both $c(x)$ and $h(x)$ are increasing, hence, locally Lipschitzian with modulus $y'(x) = f'(x)$ near $x \in \mathbf{X}$, $x > 0$). Give the subcollections $\mathbf{USC}(X)$ and $\mathbf{C}(X)$ their relative topologies and partial orders to the space $\mathbf{B}^f(X)$.[26]

[25] We introduce this new notation just to make clear we are dealing with individual states $x_1 = k$ vs. aggregate states $x_2 = K$ separately in each argument. We will really be interested in the case where $k = K$, so $x_D \in D$ which is the diagonal of X, but the structural properties of our spaces that we define will be often asymmetric with respect to the components of x.

[26] In our subsequent discussion, when the context for $\mathbf{X}$ is obvious, we shall refer to these spaces spaces $\mathbf{B}^f$, **USC,** and **C,** respectively, where the domains of the functions defining each space is understood.

In our first lemma, mention the order completeness properties of subsets of $\mathbf{B}^f$ under pointwise partial orders:

Lemma 15. *$\boldsymbol{B}^f$ is a complete lattice; (ii)* $\mathbf{USC}$ *is subcomplete in $\boldsymbol{B}^f$, (iii) $\boldsymbol{C}$ is subcomplete* $\mathbf{USC}(X)$.

Proof. To see the completeness claims, let $B \subset \mathbf{B}^f$. As the pointwise inf and sup of B satisfies the pointwise bounds, i.e., $0\leq \inf_x B \leq m$, and $0\leq \sup_x B \leq m$, we have $\wedge B \in \mathbf{B}^f$ and $\vee B \in \mathbf{B}^f$. Hence, $\mathbf{B}^f$ is a complete lattice.

Further, as monotonicity in x (resp, equicontinuity at x) are preserved also under pointwise sup and inf operations in X, if $B \subset \mathbf{C}$, $\wedge B \in \mathbf{C}$ and $\vee B \in \mathbf{C}$. Hence, $\mathbf{C}$ is a complete lattice.

Finally, if $B \subset$**USC,** then the pointwise inf of any arbitrary B is upper semi-continuous (e.g., Aliprantis and Border [5], lemma 2.41)). As $\vee\mathbf{USC}=y(x)$ is continuous, by the characterization of a complete lattice in Davey and Priestley ([16], Theorem 2.31), $\mathbf{USC}$ is complete lattice.

Finally, noting obvious sublattice and subchain inclusions, the subcompleteness and subchain completeness claims in the lemma follow. ■

We next consider subsets of $\mathbf{B}^f(X)$ where the restrictions on $h \in \mathbf{B}^f(X)$ are stated in terms of their implied properties on the inverse of marginal utility of consumption. To do this, we first construct an analog to space of bounded feasible decision rules $\mathbf{B}^f(X)$ in terms of inverse marginal utilities. We can let the inverse marginal utility implied for any element $h \in \mathbf{B}^f(X)$ be denoted by:

$$
\begin{aligned}
m_h(x) &= \frac{1}{u'(y(x)-h(x))}, \text{for } h \in \mathbf{B}^f(X),\ u'(y-h)>0 \\
&= 0,\ \text{else}
\end{aligned}
$$

Under Assumption 1, the function $m_h(x)$ is well-defined. Recalling Assumptions 1, 2, and 3(i) and 3(iii), it is known that for our economies, there exists a maximal sustainable capital stock, say $k^u > 0$. Therefore, we can define the maximal sustainable inverse marginal utility of consumption as $m_0^u = \frac{1}{u'(y(k^u,k^u))}$. Noticing $m^u(0,0) = 0$,define the set of socially feasible inverse marginal utilities is given as follows:

$$\mathbf{M}^f(X) = \{m | 0 \leq m \leq m^u(k,k)\}$$

Notice, as promised, the space $\mathbf{M}^f(X)$ is simply a restatement of the space $\mathbf{B}^f(X)$ in the previous section. That is, we have $h \in \mathbf{B}^f$ iff $m = \frac{1}{u'(h)} \in \mathbf{M}^f$.

One important subset of $\mathbf{M}^f(X)$ occurs when continuous $m \in \mathbf{M}^f(X)$, and is an element of the following subcollection:

$$\begin{aligned}\mathbf{M}^A(X) = \{&h(x) \in \mathbf{B}^f(X) | h(x) \text{ continuous}, 0 \leq m_h \leq m_0^u, \text{ s. t.} \\ &0 \leq |m_h(x') - m_h(x)| \leq \frac{1}{u''(y(k^u, k^u))}\}\end{aligned}$$

For an investment function $h(x)$ to be consistent with the inverse marginal utility level $m(x) \in \mathbf{M}^A(X)$, we only require the equilibrium decision rules to have an implied consumption function that has an implied variation of its inverse marginal utility an element of a collection of functions that each exhibit (uniform) equicontinuity near each x bounded above by m'_f. Therefore, for $h \in \mathbf{M}^A$, as $u(c)$ is C^2, $c(x) = y - h$ is also locally Lipschitz continuous in x. Therefore, as y is also locally Lipschitz under A2 and A5, $h(x)$ is locally Lipschitz (as Lipschitz structure is closed under scaler multiplication and addition).

Finally, consider the following subset of $\mathbf{M}^f(X)$ that also prove useful in our subsequent arguments:

$$\mathbf{M}(X) = \{m \in \mathbf{M}^f(X) | m \text{ s.t. } \frac{R_\tau(k)}{m(k,k)} \text{ strictly decreasing } k \text{ for } k > 0\} \tag{13}$$

The subset $\mathbf{M}(X) \subset \mathbf{M}^f(X)$ is closely related to the domain of functions studied on Coleman [12] for his uniqueness argument (the difference being that for $m(x) \in \mathbf{M}(X)$, we do not required $m(x)$ to be continuous).

Endow $\mathbf{M}^f(X)$ with the pointwise partial order, and give the subsets $\mathbf{M}^d$ and $\mathbf{M}$ each their relative partial orders and topologies. First, note, that $\mathbf{M}$ is not order closed; hence, is not a suitable domain for existence arguments via order theoretic fixed point methods (e.g,, Tarski's theorem or its variants). $\mathbf{M}(X)$ will prove very useful for uniqueness arguments. In the Lemma 16, we discuss the order completeness properties of the remaining function spaces $\mathbf{M}^f$ and $\mathbf{M}^A$**:**

Lemma 16. *(i) $\mathbf{M}^f(X)$ is a complete lattice; (ii) $\mathbf{M}^A(X)$ is chain complete.*

Proof. Proof: That $\mathbf{M}^f(X)$ is a complete lattice follows directly from $\mathbf{B}^f(X)$ a complete lattice (noting, the one-to-one lattice morphism defined in (**??**) between elements of $\mathbf{M}^f(X)$ and $\mathbf{B}^f(X)$.

Further, as $\mathbf{M}^A$ is equicontinuous and pointwise compact, it is a compact subset of the space of continuous functions on $\mathbf{X}$ in the topology of uniform convergence. Hence, in the pointwise partial order, by a theorem in Amann (e.g., Amann ([4], Theorem 10), $\mathbf{M}^A(X)$ is chain complete. ■

In the subsequent discussion, when the context is clear, for all function spaces, we will delete the reference to the state-space X (e.g., $\mathbf{C}(X)$ is denoted as **C).** We are now ready to discuss function-based approaches to equilibrium in our economies with a progressive tax.

4.2. A New Uniqueness Result

The first function-based method we consider is the policy iteration method proposed in Coleman [10] [12]).[27] Coleman's procedure can be defined as follows: for $h \in \mathbf{C}$, rewrite the equilibrium version of the household's Euler equation in (12) as the mapping $Z_A : \mathbf{K} \times \mathbf{X} \times \mathbf{C} \to \mathbf{R}$:

$$Z_A(x, k, k, h) = u'(x) - \beta u'(y_h(y_x))R_\tau(y_x) \tag{14}$$

where, the function $R_\tau(K) = r(K) \cdot (1 - \tau(K))$ denotes the distorted return on capital, and, $y_h = y - h$. [28] Then, a nonlinear operator $A(h)(k, k)$ can be constructed implicitly using Z_A as follows:

$$\begin{aligned} A(h)(k,k) &= x^* \text{ s.t. } Z_A(x^*(k,k,h), k, k, h) = 0,\ k > 0,\ h > 0, \forall k \\ &= 0 \text{ else.} \end{aligned}$$

It is important to remember the operator equation $A(h)(k, k) = h$ is only an abstract operator equation (with solutions that are not necessarily recursive equilibrium). Therefore, to make its fixed points of $A(h)$ recursive equilibrium, further argument is typically required.

[27] We shall referred to this procedure as the "Coleman's procedure". See also Bizer and Judd [9]. This procedure has been studied in a number of other papers. See Mirman, Morand, and Reffett ([35], Section 4) for a detailed set of references.

[28] We should note, we consider the mapping Z_A (and all similar mappings in the paper) to be a real-valued function. We are careful to only use Z_A to define our operators when it is real-valued. This makes the need for the extended reals unnecessary.

The properties of iterative methods based upon the Coleman procedure have been studied extensively in the literature. For the sake of completeness, we summarize what is known about the solutions to the operator equation $A(h) = h$ in $\mathbf{C}$:[29]

Proposition 17. *(Coleman [10][12] and Mirman, Morand, and Reffett [35]). Let Ψ_A^C be the set of fixed points associated with $A(h)(k,k)$. Then, under Assumptions 1-3(i) and 3(iii), $\Psi_A^C = \{0, h^*(k,k)\} \subset \mathbf{C}$, with $h^*(k,k) = \vee\Psi_A > 0$ when $k > 0$. Further, the iterations $\lim_n A^n(f) = \sup_n A^n(f) \to h^*$ (where the convergence is both in topology and order, respectively). Finally, $h^*(k,k)$ is C^1 when $k > 0$.*

We now extend the uniqueness result for the policy iteration methods in Proposition 17 to a more general setting. Following Coleman [12], we construct a second operator whose fixed points can be shown to be isomorphic to those of $A(h)(x)$ using the domain $\mathbf{M}(X)$. To do this, define the function $H(m)$ implicitly by:

$$u'(H(m)) = \frac{1}{m} \text{ for } m > 0, \text{ 0 elsewhere.}$$

The function $H(m) = c(m)$ is the consumption level required to obtain the inverse marginal utility level of $\frac{1}{m}$ when $m > 0$. Under Assumption 1, as $u'(c)$ is strictly decreasing, for each $m > 0$, the mapping H is well-defined, bounded, strictly increasing, and it has the following important boundary properties: (i) $\lim_{m\to 0} H(m) = 0$, and (ii) $\lim_{m\to f} H(m) = f = m^u$.

Using the function H, for each $m \in \mathbf{M}$, next consider the mapping $\hat{Z}_A$:$\mathbf{K}\times\mathbf{K}\times\mathbf{K} \times \mathbf{M}$ based, again, upon an on equation (12) as follows:

$$\hat{Z}_A(x,k,k,m) = -\frac{1}{x} + \beta\frac{R_\tau(y - H(x))}{m(y(k,k) - H(x), y - H(x))}$$

Define a new nonlinear operator $\widehat{A}m(k,k)$ implicitly in $\hat{Z}_A(x,k,k,m)$ as follows:

$$\widehat{A}m(k,k) = \{x^*(k,k;\tilde{m}) \mid \widehat{Z}_A(x,k,k,m) = 0 \text{ for } \tilde{m} > 0, \text{ 0 elsewhere}\}.$$

[29] Although the focus here is on economies with bounded state spaces, Morand and Reffett [40] extend Coleman [10][12] to the case of unbounded state spaces, and power utility. Further generalizations are also available. See Morand, Reffett and Wang [42]

In Lemma 18, we show the operator $\widehat{A}m(k,k)$ is well-defined, transforms the space $\mathbf{M}$ into itself, and has strong geometric properties when restricted to the domain $\mathbf{M}$.

Lemma 18. *Under Assumptions 1, 2, 3(i) and 3(iii), $\widehat{A}m(k,k)$ is well-defined in $\mathbf{M}(X)$, with $\hat{A}(m)(k,k) \in \mathbf{M}$, and $H(\widehat{A}m)(k,k) \in \mathbf{C}(X)$. Finally, $\widehat{A}m(k,k) \in \mathbf{M}(X)$, $\widehat{A}m(k,k)$ isotone, pseudo concave, and k_0-monotone.*

Proof. Proof: As $\hat{Z}_A$ is (i) strictly increasing in x, each (k,k,m), $m>0$, $\widehat{A}m(k,k)$ is well-defined. For $k_1 \geq k_2 > 0$, the second term in $\hat{Z}_A$ falls. Therefore, for such (m,k), have $\hat{A}(m)(k_1,k_1) \geq \hat{A}(m)(k_2,k_2)$. Noting the definition of $\widehat{A}m(k,k)$, under Assumption 2 and 3(i), $\hat{A}(m)(k,k)$ is isotone in (k,k). Therefore, $\hat{A}(m)(k,k) \in \mathbf{M}$.

To see $\hat{A}(m)(k,k)$ is such that $H(\hat{A}(m))(k,k) \in \mathbf{C}$, simply note that as $\hat{A}(m)(k,k)$ is increasing in k, when $k_1 \geq k_2 > 0$, $m>0$, then as the first term of $\hat{Z}_A$ must fall, $\hat{A}(m)(k,k)$ must be such that the second term of $\hat{Z}_A$ falls; hence, as $m \in \mathbf{M}$, $\hat{A}(m)(k,k)$ is such that $y(k,k) - H(\hat{A}(m))(k,k)$ is increasing in k. As $H(\hat{A}(m))(k,k)$ is also increasing in k, $\hat{A}(m)(k,k)$ is such that $H(\hat{A}(m))(k,k) \in \mathbf{C}$.

Let $m' \geq m, m > 0$ and $k > 0$. As $\hat{Z}_A$ is strictly decreasing in m, for such (m,k), $\hat{A}(m')(k,k) \geq \hat{A}(m)(k,k)$. Again, noting the definition of $\widehat{A}m(k,k)$ elsewhere, we have $\hat{A}(m)(k,k)$ isotone on $\mathbf{M}$.

Finally, that $\hat{A}(m)(k,k)$ is pseudo-concave and k_0-monotone follows from Coleman ([12], Lemma 3 and 4, respectively). ■

We now are ready to prove the following important result in this section. That is, we extend of the main uniqueness theorem in Coleman [12] to a much larger set of functions:[30]

Theorem 19. *Let h^* be the unique positive fixed point in Proposition 17. Then, under Assumptions 1, 2, 3(i) and 3(iii), the set of fixed points of $\widehat{A}m(k,k)$ is $\Psi_{\hat{A}}^{M}$ $=\{0,\ m^*\} \subset \mathbf{M}(X)$, with $m^* > 0$ when $k > 0$. The iterations $\inf_n \widehat{A}^n(m^u) \rightarrow m^*$ where the convergence is in order and topology, where $H(m^*)(k,k) = h^* \in \mathbf{C}(X)$.*

[30] Also, it is important to note that Coleman [10] and others have studied economies with stochastic Markov shocks defined in a discrete shock space. Therefore, obviously, our new uniqueness result extends with a trivial modification to such economies.

Proof. First, consider $m \in \mathbf{M(}X\mathbf{)}$, $m > 0$, $k > 0$. Under Assumptions 1, 2, and 3(i), as $\widehat{Z_A}$ is strictly in x, $x \in \mathbf{K} \in \mathbf{R}$. Therefore, $\widehat{Z_A}$ (x, k, k, m) is upper-semicontinuous from the left, and lower semicontinuous from the right in x. By Assumptions 1 and 2, we have additionally $\lim_{x\to 0} \widehat{Z_A} = +\infty$ and $\lim_{x\to f} \widehat{Z_A} = -\infty$. Hence, at all such points (k, m), by Guillerme's coincidence theorem (Guillerme [24], Theorem 3), there exists a root $\widehat{A}m(k, k) = x^*(k, k; \tilde{m})$ such that $\widehat{Z_A}$ $(x^*, k, k, m) = 0$. Further, as $\widehat{Z_A}$ is strictly increasing in x, $\widehat{A}m(k, k) = x^*(k, k; \tilde{m})$ is unique. Noting $\widehat{A}m(k, k) = 0$ else, $\widehat{A}m(k, k)$ is well-defined in $\mathbf{M}(X)\mathbf{.}$

Next, note that the minimal fixed point of $\hat{A}(m)(k, k)$ is by definition 0. To establish the only other fixed point of $\widehat{\ Am}(k, k)$ is $m^*(k, k)$ with $m^* > 0$ when $k > 0$, using the definition of m, we have:

$$H(m_0(k, k)) = \frac{1}{u'(c_0(k, k))}.$$

By the definition of the operator $\hat{A}(m)(k, k)$ in equation (**??**), we have

$$\frac{1}{\widehat{A}m_0(k, k)} = \beta \left\{ \frac{R_\tau(y^1(k, k) - H(\widehat{A}m_0(k, k))}{m_0(y^1(k, k) - H(\widehat{A}m_0(k, k))} \right\}$$

or, equivalently (from the definition of c_0):

$$\frac{1}{\widehat{A}m_0(K, z)} = \beta\{R_\tau(y(k, k) - H(\widehat{A}m_0(k, k))) \\ \cdot u'(c_0(f(k, k) - H(\widehat{A}m_0(k, k))\}.$$

Therefore, by construction, Ac_0 satisfies:

$$\begin{aligned} u'((Ac_0)(k, k) \;&=\; \beta\{R_\tau(y^1(k, k) - Ac_0(k, k)) \\ &\quad \cdot u'(c_0(y(k, k) - Ac_0(k, k))\}. \end{aligned}$$

By the uniqueness of $\widehat{A}m_0$, it must be that $1/\widehat{A}m_0 = u'(Ac_0)$ (or, equivalently, $H(\widehat{A}m_0) = Ac_0)$). By induction, for all $n = 1, 2, ...,$ $A^n c_0 = H(\widehat{A}^n m_0)$. Hence, a fixed point of $\widehat{\ Am}(k, k)$ corresponds with a fixed point of $A(h)(k, k)$.

Next, we prove a fixed point of $A(h)(k, k)$ corresponds to fixed point of $\widehat{A}m(k, k)$. To see this, consider an x such that $Ax = x$, and define $z = 1/u'(x)$ (or, equivalently $H(z) = x$). By definition, x satisfies:

$$u'(x(k,k)) = \beta\{R_\tau(y(k,k) - x(k,k))$$
$$\cdot u'(x(y(k,k) - x(k,k)\} \text{ for all } (k,k).$$

Substituting the definition of y into this expression, we have:

$$\frac{1}{y} = \beta \frac{R_\tau(y(k,k) - H(z(k,k))}{z(y(k,k) - H(z(k,k))},$$

hence, $z(k,k)$ is a fixed point of $\widehat{A}$. Therefore, $h^*(k,k) \in \Psi_A \Leftrightarrow m^* \in \Psi_{\hat{A}}$.

As $\widehat{A}m(k,k)$ is pseudo-concave and k_0-monotone, it has at most two fixed points, one non-zero (e.g., Coleman ([12], Theorem 5). Therefore, the fixed point set of $\hat{A}(m)(k,k)$ is $\Psi_{\hat{A}}^M = \{0, m^*\}$, with $m^* = \frac{1}{u'(c^*(k,k)))}$ for $c^*(k,k) > 0$ when $k > 0$. Hence, $H(m^*(k,k)) = c^*(k,k) \in \mathbf{C}$.

Finally, to show uniform convergence of the iterations $\inf_n \widehat{A}^n(m^u) = \lim_n \widehat{\mathbf{A}}^{\mathbf{n}}(m^u)$ $\to m^*$, first note that for any $m \in \mathbf{M}$, as $A(m)(k,k)$ is increasing in k, we have the following inequality when $k_1 \geq k_2 > 0$:

$$\begin{aligned} &- \frac{1}{\hat{A}(m)(k_1,k_1)} + \\ &\beta \frac{R_\tau(y(k_2,k_2 - H(\hat{A}(m)(k_2,k_2)))}{m(y(k_2,k_2) - H(\hat{A}(m)(k_2,k_2)), y(k_2,k_2 - H(\hat{A}(m)(k_2,k_2)))} \\ &\geq 0 \end{aligned}$$

Hence, $\hat{A}(m)(k,k)$ is such that $y(k,k) - H(\hat{A}(m)(k,k)$ is increasing in k. Hence, $\hat{A}(m)(k,k)$ implies $H(\hat{A}(m)) \in \mathbf{C}$ as $f-H(\hat{A}(m))$ increasing in k. Hence, each element of a $\{\hat{A}^n(m^u)(k,k)$ equicontinuous set such that $H(\hat{A}(m))$ and $f - H(\hat{A}(m))$ has maximal variation $f'(k)$ at each $k > 0$. Therefore, by Dini's theorem, for each $k > 0$, as the limiting function m^* is continuous, and $\lim_n \widehat{\mathbf{A}}^{\mathbf{n}}(m^u) \to m^*$ uniformly when $k > 0$. Noting the definition of $\hat{A}(m)$ when $k = 0$, this convergence is uniform on X. Finally, convergence in order is implied by the fact that pointwise and uniform convergence coincide in $\mathbf{C}$, and $\{\hat{A}^n(m^u)(k,k)\}$ forms a subchain in $\mathbf{C}$ with sup and inf operations for $\{\hat{A}^n(m^u)(k,k)\}$ equal to pointwise/uniform limits. ■

Theorem 19 extends the uniqueness result in Coleman [12] In particular, elements $m \in \mathbf{M}(\mathbf{X})$ do not require monotonicity of either investment, nor continuity of either consumption or investment. All that is required is that consumption be monotone (jointly) in (k, k) continuous in its first argument, bound in its second argument.

4.3. A New Method and More Continuous RE

Our uniqueness result in Theorem 19 pertains to the standard operator that has been studied in the literature.[31] We now show compute RE in the exact same space $\mathbf{C}$ (where our new uniqueness result in Theorem 19 holds), but using a completely different procedure. What will be interesting is this method will *not* allow use to check the geometric conditions needed in Theorem 19, hence we will not be able to rule out additional RE. The new method is a simple value iteration procedure, that produces a *decreasing* operator that is continuous in the space $\mathbf{C}$. As $\mathbf{C}$ is a nonempty, compact, and convex set, existence of some RE will be guaranteed by Schauder's theorem. Further, its fixed point set will be an antichain. [32]

The method works directly with the household's dynamic program. We modify the household problem from before as follows: for a household entering a period in state $s = (k, K) \in X^*$ in a candidate recursive equilibrium $h \in \mathbf{C}(X)$, when $h > 0$, construct the unique value function $V^* : \mathbf{K} \times \mathbf{K}^* \times \mathbf{C}(\mathbf{X})$ that satisfies the following parameterized Bellman's equation:

$$V^*(k, K; h) = \sup_{x \in \Psi(k,K)} \{u(y(k, K) - x) + \beta V^*(x, h(K))\} \tag{15}$$

where the household's feasible correspondence is again simply $\Psi(k, K) = [0, y(k, K)]$. Again, using $y_{x^*} = y - x^*$, we can construct a necessary and sufficient first order characterization of the unique optimal solution $x^* = x^*(k, K, h)$ is:

$$u'(y_{x*}) - \beta u'(y_{x^*})(x^*, h(K))\ r(h(K))(1 - \tau(h(K)) = 0 \tag{16}$$

Now, define a new operator

$$\begin{aligned} A^*(h)(k) &= x^*(k, k; h) \text{ for } k > 0,\ h \in \mathbf{C},\ h > 0 \\ &= y \text{ else.} \end{aligned}$$

That is, define an operator this exactly the HH's best response map to the aggregate law of motion $h \in \mathbf{C}$ when h and k are not zero (and zero, else). We now have the following Lemma:

[31] For example, in addition to Coleman [10][11][12], also this operator is used in Greenwood and Huffman [23], Datta, Mirman, and Reffett [14], Morand and Reffett [40], among others.

[32] i.e., not two fixed points will be ordered.

Lemma 20. *$A^*(h)(k)$ is continuous and antitone on $\boldsymbol{C}(X)$.*

Proof. To be completed. Basically, apply Bonsall-Nadler theorem to household's dynamic program to get pointwise continuity of policies in $h(k)$. Then, use equicontinuity of $\mathbf{C}(X)$ to get uniform convergence on $X^* = (0, \bar{k}]$. Noting definition of $A(h)(k)$ elsewhere, by equicontinuity, prove $A(h)(k)$ is continuous at 0. ■

Let $\Psi_{A^*}^{\mathbf{C}}$ be the fixed point set of $A^*(h)(k)$. We now verify the existence of a RE using $A^*(h)(k)$

Theorem 21. *The set of fixed points for $A^*(h)(k)$ is nonempty, compact, antichain (hence, chain complete). Further, each fixed point of $A^*(h)(k)$ is a RE in $\boldsymbol{C}(X)$*

Proof. As $\mathbf{C}$ is a nonempty, compact, and convex set, and $A^*(h)(k)$ is continuous by Lemma 20, the fact that $\Psi_{A^*}^{\mathbf{C}}$ is nonempty and compact follow from Schauder's theorem. That $\Psi_{A^*}^{\mathbf{C}}$ forms an antichain follows the fact that $A^*(h)$ is decreasing in a complete lattice $\mathbf{C}$ (e.g., Dacic [13]), and that $\Psi_{A^*}^{\mathbf{C}}$ is chain complete follows from Amann ([4], Theorem 10).

Now, whenever $h = 0, A^*(h)(k)$ is y, and as $h_n \to y$, $r(h_n) \to r(h)$, so $A^*(h)(k) > 0$ (hence, there are no trivial fixed points of $A^*(h)(k)$). Therefore, all the elements of $\Psi_{A^*}^{\mathbf{C}}$ are actually RE. ■

We conclude with a remark about the fixed points of $A^*(h)$ and our uniqueness result in Theorem 19. Using the Euler equation, we have

$$Z(A^*(h), k, h) = u'(y_{A^*(h)}) - \beta u'(y_{A^*(h)})(A^*, h(K))\ r(h(K))(1 - \tau(h(K)) = 0$$

Therefore without further restrictions on F, we cannot checking the standard pseudo-concavity condition; that is, we do not have $A^*(th)(k) > tA^*(h)$ for all $t > 0, h > 0$. Additional, it can be verified we do not sufficient convexity conditions to apply related arguments for unique decreasing operators.

4.4. More Continuous RE

Now we propose a second fixed point procedure that verifies even more continuous RE that lie outside the realm of our uniqueness result. This new result is not inconsistent with Theorem 19, as we prove relative existence outside the domain of bounded functions where the uniqueness argument of Theorem 19 holds (namely, **M**). As our new operator does not have the requisite concavity and monotonicity properties needed to guarantee the existence of unique fixed points in its domain, we cannot expect RE to be unique. What is interesting is this new procedure also will allow us to build additional RE that are discontinuous (essentially just bounded) in K, for each k.

The new procedure is a "two-step" monotone map method. To define the method, fix a pair of functions $h \in \mathbf{USC}$ and $\hat{h} = \hat{h}(K) \in \mathbf{B}^f$, and consider the mapping $Z_B(x, k, k, h, \hat{h})$:

$$\begin{aligned} Z_B(x, k, k, h, \hat{h}) = & -u'(y^2(k, k) - x) \\ & + \beta u'(r(h(k, k))x + w(x) - \hat{h}(\hat{h}, \hat{h})) \cdot R_\tau(x) \end{aligned} \tag{17}$$

where, in the definition of Z_B, we have used the fact that that $y^2(k, k) = r(k)k + w(k) = f(k)$. For fixed $\hat{h} \in \mathbf{B}^f$, define a "first-step" nonlinear operator $B^1(h)(k, k; \hat{h})$ in the space **USC** discussed in Section 3 in Lemma 15 implicitly in the household's equilibrium Euler equation as follows:

$$\begin{aligned} B^1(h)(k, k; \hat{h}) = f & \text{ if } \nexists \text{ an } x \text{ st } r(h(K))x + w(x) - \hat{h}(\hat{h}(K), \hat{h}(K)) > 0, \\ & \text{when } k > 0, h(k) > 0,\ \hat{h} \in \mathbf{B}^f \\ & = x^*| \text{ for } Z_B(x^*, k, k, h, \hat{h}) = 0, \text{ else, when } k > 0, h(k) > 0,\ \hat{h} \in \mathbf{B}^f \\ & = 0\ , \text{ elsewhere.} \end{aligned}$$

The following lemma characterizes the properties of the "first step" operator $B^1(h)(k, k, \hat{h}) = B^1(h; \hat{h})$:

Lemma 22. *For $\hat{h} = \hat{h}(K) \in \mathbf{B}^f$, for each $h \in \mathbf{USC}$, $B^1(h; \hat{h}) \in \mathbf{USC}$. Further, $B^1(h; \hat{h})$ jointly increasing on in $(h, \hat{h}(K)) \in \mathbf{USC} \times \mathbf{B}^f$, for each $(k, k) \in X = \mathbf{K} \times \mathbf{K}$.*

Proof: Under Assumptions 1 and 2, Z_B is strictly decreasing in $x \in \mathbf{K} \subset \mathbf{R}$ with $Z_B \to \infty$ for $x \to 0$ and $Z_B \to -\infty$ as $x \to m^2$. Noting the definition of $B^1(h;\hat{h})$ elsewhere, we conclude $B^1(h;\hat{h})$ is well-defined. Further, noting that Z_B is upper-semicontinuous and increasing in (k,k) (hence, right continuous in k), and Z_B is continuous in x, the root $x^*(k,k,h,\hat{h})$ is right continuous and monotone (hence, upper semicontinuous) in k Finally, for each (k,k), $k>0$, we have $Z_B(x,k,k,h_1,\hat{h}_1\ )\geq Z_B(x,k,k,h_2,\hat{h}_2\ )$ when $(h_1,\hat{h}_1)\geq(h_2,\hat{h}_2)$. Therefore, $B^1(h_1,\hat{h}_1)\geq B^1(h_2,\hat{h}_2)$ for such k. Noting the definition of $B^1(h,\hat{h})$, elsewhere, $B^1(h,\hat{h})$ is jointly increasing in $(h,\hat{h})$ on $\mathbf{USC}\times\mathbf{B}^f$ for each (k,k).$\blacksquare$

Let $\Psi_{B^1}^{USC}(\hat{h}) \subset \mathbf{USC}$ be the fixed point correspondence for the operator $B^1(h;\hat{h})$ at $\hat{h} \in \mathbf{B}^f$. We now prove an important lemma concerning the fixed point set of the "first-step" of our modified policy iteration procedure[33]:

Lemma 23. *The fixed point set $\Psi_{B^1}^{USC} : \mathbf{B}^f \to 2^{\mathbf{USC}}\backslash\varnothing$ is a nonempty complete lattice-valued correspondence, with $\Psi_{B^1}^{USC}(\hat{h})$ ascending in Veinott strong set order on $\boldsymbol{B}^f$. Further, for each fixed point $h^* \in \Psi_{B^1}(\hat{h})$, $h^* \in \mathbf{C}(X)$. Additionally, for fixed $\hat{h} \in \mathbf{B}^f$, the iterations $\lim_n B^{1^n}(0;\hat{h}) \to \inf_n B^{1n}(0;\hat{h}) = h^*(\hat{h}) = \wedge\Psi_{B^1}^{USC}(\hat{h}) \in \mathbf{C}$, such that $h^*(\hat{h})>0$ and $f(k)-h^*(\hat{h})>0$ when $k>0$, where the convergence is uniform. Finally, the selection $B^2_\wedge(\hat{h}) = \wedge\Psi_{B^1}^{USC}(\hat{h})$ is an increasing selection of $\Psi_{B^1}^{USC}(\hat{h})$.*

Proof. Proof: As $B^1(h;\hat{h}) \in \mathbf{USC}(X)$, isotone in h, each $(k,k,\hat{h})$, and $\mathbf{USC(}X\mathbf{)}$ as complete lattice, by Tarski's theorem, $\Psi_{B^1}^{USC}(\hat{h})$ is a nonempty complete lattice. Further, that the fixed point correspondence $\Psi_{B^1}^{USC}(\hat{h})$ is ascending in Veinott's strong set order follows from a theorem in Veinott ([58], Theorem 14, Chapter 4).[34]

[33] We should be very clear: Coleman [10][12] and Mirman, Morand, and Reffett [35] are explicit when noting precisely how to interpret their uniqueness results; what is new, here, is (i) a new method for computing recursive equilibrium outside the side of function for which they claim uniqueness, (ii) an argument that uniqueness results, at best, only can be claimed relative to operators, not sets of recursive equilibrium.

[34] See also Topkis ([56], Theorem 2.5.2).

Let $h^*(\hat{h}) \in \Psi_{B^1}^{USC}(\hat{h})$. By the definition of $B^1(h;\hat{h})$, when $k > 0$,

$$\begin{aligned} h^*(\hat{h}) &= B^1(h^*;\hat{h}) = \\ & \inf\{x^*(k,k,h^*,\hat{h}), f) \end{aligned}$$

where $Z_B(x^*(k,k,h,\hat{h}),k,k,h,\hat{h}) = 0$. Therefore, when $k_1 \geq k_2 > 0$, using the notation $h_k^*(\hat{h}) = h^*(k,k;\hat{h})$, and $x_k^*(\hat{h}) = x^*(k,k,h^*,\hat{h})$, the following:

$$\begin{aligned} & u'(y^2(k,k) - x_k^*(\hat{h})) - \\ & \qquad \beta u'(r(h_{k_1}^*(\hat{h})h_{k_1}^*(\hat{h}) + w(h_{k_1}^*(\hat{h})) - \hat{h}(\hat{h},\hat{h}) \cdot R_\tau(h_{k_1}^*(\hat{h})) \geq \end{aligned}$$

$$\begin{aligned} & u'(y^2(k,k) - h_{k_2}^*(\hat{h})) - \\ & \qquad \beta u'(f(h_{k_2}^*(\hat{h})) - \hat{h}(\hat{h},\hat{h}) \cdot R_\tau(h_{k_2}^*(\hat{h}) \\ & \qquad\qquad = 0 \end{aligned}$$

as $r(h^*)h^* + w(h^*) = f(h^*)$ by the definition of income process y^2, and $h_k^*(\hat{h})$ is increasing in k.

Therefore, if $B^1(h^*(\hat{h});\hat{h}) = x_k^*(\hat{h})$, we have $B^1(h^*(\hat{h});\hat{h})$ such that $u'(y^2(k,k) - h_k^*(\hat{h}))$ is decreasing in k. Therefore, $y^2(k,k) - h_k^*(\hat{h})$ must be increasing in k. Hence, $B^1(h^*(\hat{h});\hat{h}) \in \mathbf{C}$. Further, if $k > 0$, then $B^1(h^*(\hat{h});\hat{h}) {=} f$, hence we trivially have $y^2 - B^1(h^*(\hat{h});\hat{h})$ increasing in k. Therefore, when $k > 0$, for $h^* \in$**USC**$(X)$$\Rightarrow$ $B^1(h^*(\hat{h});\hat{h}) \in \mathbf{C}(X)$. Elsewhere, $B^1(h^*(\hat{h});\hat{h}) = 0 \in \mathbf{C}$. Therefore, for all $k \in \mathbf{K}$, $h_k^*(\hat{h}) \in \mathbf{USC} \Rightarrow$ $h_k^*(\hat{h}) = B^1(h^*(\hat{h});\hat{h}) \in \mathbf{C}(X)$. Finally, as the subset $\mathbf{C}(X)$ is compact in $\mathbf{USC}(X)$),we have any fixed point $h_k^*(\hat{h}) \in \mathbf{C}$.

To conclude the proof, as for each $\hat{h} = \hat{h}\ (K,K)$, the iterations $\lim_n B^{1^n}(0;\hat{h}) \to h_k^*(\hat{h}) = \wedge\Psi_{B^1}^{USC}(\hat{h})$ form a monotone sequence of continuous function with the limit h^* continuous**,** hence, by Dini's theorem, the convergence is uniform. ■

We next construct a new operator from the fixed point correspondence of our "first-step" operator based upon the selection $h_k^*(\hat{h}(K,K))$ when $k = K$ as follows

$$\begin{aligned} B^2(h)(k,k) &= h_k^*(\hat{h}(k)),\ h \in \mathbf{M}^A, h < f, k > 0 \\ &= f,\ h = f, k > 0 \\ &= 0,\ \text{else} \end{aligned}$$

In Theorem 24, we now show the existence of additional recursive equilibrium in $\mathbf{M}^A(X)$. Further, we show that when $B^2(h)(k,k)$ is restricted to $\mathbf{C}$, $B^2(h)(k,k)$ has a (unique) positive fixed point in $\mathbf{C}$ (hence, by Theorem 19, Coleman's policy iteration procedure is robust to *alternative fixed point procedures.*

Let $\Psi_{B^2}^{M^A}$ be the set of fixed points of $B^2(h)(x)$. Further, define the set of functions:

$$\mathbf{B}_*^f(X) = \{h| \text{ for fixed } \hat{h}(k) \in \mathbf{B}^f,\ h_k^*(\hat{h}(k)) \in \mathbf{M}^A\}$$

Notice the elements of $h_k^*(\hat{h}(k)) \in \mathbf{B}^*(\mathbf{K})$ are *not* continuous on $\mathbf{K}$, as $\hat{h} \in \mathbf{B}^f$. It can easily be verified that $\mathbf{B}^*(X)$ is chain complete. We have the following:

Theorem 24. *The operator $B^2 : \mathbf{M}^A \to \mathbf{M}^A$ is isotone on $\boldsymbol{M}^A$. Therefore, its fixed points $\Psi_{B^2}^{M^A}$ form a chain complete set, with $\wedge\Psi_{B^2}^{M^A} \in \mathbf{M}^A$ a continuous recursive equilibrium, with $\wedge\Psi_{B^2}^{M^A} \notin \mathbf{C}$. Further, when the operator $B^2(h)(k,k)$ is restricted to $\mathbf{C}$, $\Psi_{B^2}^{M^A} = \{0, h^*\} \subset \mathbf{C}$. Finally, for $\hat{h} \in \mathbf{B}^f$, $B^2 : \mathbf{B}^* \to \mathbf{B}^*$ and is isotone, so the set of fixed points $\Psi_{B^2}^{\mathbf{B}^*}$ is chain complete, and $\wedge\Psi_{B^2}^{\mathbf{B}^*}$ a bounded RE.*

Proof. Proof: Let $h \in \mathbf{M}^A$, with $h < f$, $k > 0$. Let $k_1 \geq k_2 > 0$. As $h \in \mathbf{M}^A$, and B^2 is increasing in k, we have

$$\begin{aligned}|\beta u'(f(B^2(k_1,k_1) - h(h(k_1,k_1),h(k_1,k_1)) \cdot R_\tau(B^2(k_1,k_1) - \\ \beta u'(f(B^2(k_2,k_2) - h(h(k_2,k_2),h(k_2,k_2)) \cdot R_\tau(B^2(k_2,k_2)| \geq 0\end{aligned}$$

Hence, $B^2(k,k)$ must be such that

$$|u'(y^2(k_1,k_1) - B^2(k_1,k_1)) - u'(y^2(k_2,k_2) - B^2(k_2,k_2))|$$

Using the definition of the inverse marginal utility in (4.2), using $c^m = y^1 - B^2$, defining m_{B^3} to be the implied inverse marginal utility at c^m, this implies

$$|m_{B^2}(k_1,k_1) - m_{B^2}(k_2,k_2)| \leq \frac{1}{u''(m^u)}$$

Noting the definition of B^2 elsewhere, $B^2(k,k) \in \mathbf{M}^A$.

To see $B^2(h)$ is isotone, note that when $h' \geq h$, the second term of Z_B rises (noting that the least fixed of $B^1(h;\hat{h})$, $B^2_{\vee}(\hat{h})$, rises by Veinott's fixed point comparatives statics result, e.g., Topkis ([56], Theorem 2.5.2). Therefore, noting the definition of $B^2(h)$ elsewhere, $B^2(h)$ is isotone in $\mathbf{M}^A$.

As $\mathbf{M}^A$ is chain complete, by Markowsky's fixed point theorem ([30], Theorem 9), the fixed point set for B^2, $\Psi^{M^A}_{B^2}$, is chain complete.

To complete the proof of the first part of the Theorem, appealing to the Inada conditions in Assumptions 1 and 2, the greatest fixed point has $\wedge\Psi^{M^A}_{B^2} > 0$ when $m > 0$, and $k > 0$. By the local Lipschitz continuity $\wedge\Psi^{M^A}_{B^2}$ near each k, the implied fixed points for consumption and investment at $\wedge\Psi^{M^A}_{B^2}$ near each k are Locally Lipschitz continuous. Under Assumptions 1, 2, 3(i), all the primitive data that defines Z_B is locally Lipschitz continuous. Further as $m \in \mathbf{M}^A$, m is locally Lipschitz. As it is known in our setting, local Lipschitz structure is closed under composition,[35] we have Z_{B^2} locally Lipschitz in k at $\vee\Psi^{M^A}_{B^2}$. As under Assumption 1, for any element of Clarke partial $\partial_m Z$, the element does not vanish (as, for example, the Clarke gradient of the first term does not vanish), by Clarke's implicit function theorem, as $\wedge\Psi^{M^A}_{B^3} = B^3(h^*) = B^2_{\wedge}(h^*, h^*)(k,k) = h^*_k(\hat{h})$ is just the root of Z_{B^2}, when $\wedge\Psi^{M^A}_{B^2} > 0$, $k > 0$, $\wedge\Psi^{M^A}_{B^2})(k,k)$ is locally Lipschitz near any such k. Noting the definition of $\wedge\Psi^{M^A}_{B^2}$ elsewhere, $\wedge\Psi^{M^A}_{B^2}(k,k)$ is continuous.

Finally, the continuity of equilibrium decision rules guarantee we evaluated the pair of conditions(11) and (12) in the definition of a recursive equilibrium, and verify they are satisfied with $x^*(k,k;\wedge\Psi_{B^2})$ being the optimal solution at $V^*(k,k,\wedge\Psi^{M^A}_{B^2})$.

To see that when $h \in \mathbf{C}$, $\wedge\,\Psi^{M^A}_{B^2} = h^* \in \mathbf{C}$, notice first that for such h, $B^2(h)$ in Z_{B^2} in

$$\begin{aligned}&u'(y^2(k_1,k_1) - B^2(k_1,k_1))\\&-\beta u'(f(B^3(h) - h(h(k_1,k_1),h(k_1,k_1)) \cdot R_\tau(B^2(k_1,k_1)\end{aligned}$$

is now increasing in (k,k) such that $f - B^2$ is increasing in (k,k).Therefore, $B^2 \in \mathbf{C}$, additionally. Therefore, noting that in Lemma 15, $\mathbf{C}$ is a complete lattice, as B^2 is isotone, the fixed point set $\Psi_{B^3} \subset \mathbf{C}$. Therefore, $\wedge\Psi^{M^A}_{B^2} \in \mathbf{C}$. It can easily be verified that appealing to obvious modifications of the arguments in Theorem 19, the fixed points of $B^2(h)$ for $h \in \mathbf{C}$ can be related one-to-one with

[35] If $f : I_1 \to I_2$, and $g : I_2 \to \mathbf{R}$, f and g Lipschitz any I_1 and I_2 compact in $(0, k^u]$, then $g \circ f$ is Lipschitz on I_1.

the fixed points of $\hat{A}(m)(k,k)$ defined in theorem in $\mathbf{M}$. Hence, $\Psi_{B^2}^{M^A} = \{0, h^*\}$ as in Proposition 17

To complete the proof, simply note that $B^2(h)$ on $\mathbf{B}^*$ self map follows from the fact that for each $\hat{h}$, $h_k^*(\hat{h}) \in \mathbf{M}^A$. Isotonicity follows from the fixed point comparative statics result in Theorem 23. Therefore, as $\mathbf{B}^*$ is chain complete, by Markowsky's theorem, $\Psi_{B^2}^{\mathbf{B}^*}$ is chain complete. Further, it can easily be verified $B^2(h)$ is order continuous on $\mathbf{B}^*$, and $\exists$ a lower bound $h^b \in \mathbf{B}^f$, $h^b \notin \mathbf{C}$ sufficiently close to 0, such that $h^b \leq B^2(h^b)$. Therefore, the the Tarski-Kantorovich theorem, the iterations $\sup_n B^{2,n}(h^b)(k,k) \to \wedge\Psi_{B^2}^{\mathbf{B}^*}$. As $B^2(h)$ corresponds with the root of first order condition for the HH in equilibrium, given the Inada condition, we must have $\wedge\Psi_{B^2}^{\mathbf{B}^*}(k,k) > 0$ when $k > 0$, so $\wedge\Psi_{B^2}^{\mathbf{B}^*}$ is bounded RE, continuous only in its first argument. ■

5. Economies with Infinitely-Lived Agents and Regressive Taxes

We finally consider the existence of recursive equilibrium in the example studied in Santos [54] for case of regressive income taxes. For this case, we will again develop a two-step modification of Coleman's policy iteration procedure to establish the existence of both an very narrow set of RE (i.e., (locally Lipschitz) continuous isotone recursive equilibrium), and an very large set for RE (i.e., bounded RE, locally Lipschitz continuous in k, for each $K = k$). So in the latter case, the RE has essentially *no* structural properties in K along the path $k = K$.The of this section is to make a very simple point about RE in nonoptimal economies: in an RE, there are basically *no* restrictions places on solutions to RE functional equation in "big K". This is because of the "k-K" structure of the RE functional equation. This allows us to (in effect) solve the RE functional equation "on sections" of the state space X, with the first step verifying the required properties for an RE properties of an RE in "little k" (holding "big K" constant), and in the second step, just making sure the solution is consistent with making the Euler equation hold given for the RE function for all "big K". This latter step (as we shall see) places very little restrictions on the solutions.

5.1. Two Step Monotone Map Methods

To see how this two step procedure works, lets first compute locally Lipschitz continuous RE.[36] To do this, consider the following modification of Coleman's procedure: for $h \in \mathbf{C}(X)$, and $\hat{h}(K,K) \in \mathbf{C}(X)$, define the mapping $Z_{A^S}(x,k,k,h,\hat{h})$ as follows

$$Z_{A^S}(x,k,k,h,h) = u'(y^1(k,k) - x) - \beta u'(y_h^1)(x,x))r(x)(1-\tau(\hat{h}(K,K)))`$$

where, under Assumption 3(ii), the income tax $\tau(K)$ is now assumed to be decreasing and Lipschitzian in its argument. Fixing both h and $\hat{h}$, we can define a nonlinear operator $A^S(h)(k,k;\hat{h}) = A^S(h,\hat{h})$ implicitly in $Z_{A^S}(x,k,k,\hat{h}(K))$ at $\hat{h} = \hat{h}(K,K)$ as follows:

$$\begin{aligned} A^S(h,\hat{h}) = x^* \text{ s.t. } & Z_{A^S}(x^*(k,k,h,\hat{h}(K)),k,k,h,\hat{h}) = 0, \\ & h > 0,\ k > 0,\ \text{all } K \\ & = 0 \text{ else.} \end{aligned}$$

For fixed $(h,\hat{h}) \in \mathbf{C} \times \mathbf{C}$, we first prove some basic properties of the operator $A^S(h;\hat{h})$:

Lemma 25. *For $(h,\hat{h}) \in \mathbf{C} \times \mathbf{C}$, at $\hat{h} = \hat{h}(K,K)$, $A^S(h;\hat{h}) \in \mathbf{C}$. Further, when $k > 0$, $k = K$, $A^s(h)(k,k,\hat{h}(k,k))$ is locally Lipschitz.*

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[36] Note, Santos [54] claims nonexistence of continuous RE in the class of models we study presently. We show this is not not the case. Actually, what Santo's verifies (correctly) is there are not continuous RE on a state space of k (i.e., a one dimensional state space). We verify continuous RE exist on $x = (k,k)$, which is not the same (so in principle, our claim does not contradict the claim in Santos).

But a few remarks. First, in addition to continuity of RE, we are able to verify *isotone* RE for investment in this example. That contradicts the numerical claims from the computations reported in Feng, et. al. [21] that at some unstable steady state for this example, there is a spiral sink. Also, it should be noted that the Grobman-Hartman theorem does not apply in this problem (as RE dynamics are not necessarily smooth, only locally Lipschitz). Hence, some other method has to be used to verify the stability claims in Santos [54] . See Reffett [48] for discussion.

Proof. Proof: For $k = 0$, and $(h, \hat{h}) \in \mathbf{C} \times \mathbf{C}$,the operator is well-define. Fix $\hat{h}(K) \in \mathbf{C}$, $k > 0$. For such points, the fact that the operator $A^S(h)(k, k; \hat{h}(K))$ is well-defined, isotone, and continuous in the uniform topology in h follows from Coleman ([10], Proposition 4). Further, if $\mathbf{K}^* = (0, k^u]$, under Assumptions 1, 2, 3(ii)-(iii), noting local Lipschitz continuity is pointwise closed under composition in this context, for $(h, \hat{h}) \in \mathbf{C} \times \mathbf{C}$, the mapping $Z_A(x, k, k, h, \hat{h})$ is (x, k) jointly locally Lipschitz on $\mathbf{K}^* \times \mathbf{K}^*$. Further, under Assumptions 1 and 2, noting $y^1(k, k) = f(k)$, as each partial Clarke gradient for Z_A in x does not vanish, i.e., the Clarke generalized gradient $\partial_x Z_A$ is of full rank, by Clarke's Implicit Function Theorem, $x^*(k, k, h, \hat{h}(k, k)), k, k, h, \hat{h}(k, k)) = A^S(h)(k, k; \hat{h}(k))$ is locally Lipschitz near each $k > 0$.[37] ■

We now study the monotonicity of the operator $A^S(h)(k, k; \hat{h}(k))$. In particular, we prove the mapping $A^S(h)(k, k; \hat{h}(K))$ is jointly isotone on $\mathbf{C}\times\mathbf{C}$:

Lemma 26. *$A^S(h)(k, k; \hat{h}(k))$ increasing jointly in $(h, \hat{h})$.*

Proof. Proof: When $h > 0$, $k > 0$, as $Z_{A^S}(x, k, k, h, \hat{h})$ is strictly increasing in x, and decreasing jointly in $(h, \hat{h}) \in \mathbf{C} \times \mathbf{C}$, $A^S(h)(k, k, \hat{h})$ is jointly isotone for such $(h, \hat{h})$, when $k > 0$. Noting the definition of $A^S(h)(k, k; \hat{h}(K))$ elsewhere, $A^S(h)(k, k; \hat{h}(K))$ is jointly isotone on $\mathbf{C} \times \mathbf{C}$. ■

Let $\Psi^C_{A^S}\ (\hat{h})(k, k)$ be the set of fixed points of $A^S(h)(k, k; \hat{h}(K))$ at $\hat{h}(K) \in \mathbf{C}(X)$ when $k = K$. We have the following result:

Lemma 27. *$\Psi^C_{A^S}\ (\hat{h})(k, k)$ is a nonempty complete lattice for each $\hat{h} \in \mathbf{C}(X)$. Furthermore, $B^S(\hat{h}) = \wedge_{\hat{h}} \Psi^C_{A^S}(\hat{h})(k, k)$ is an increasing selection.*

[37] By a theorem in Matoušková (e.g, [32], Theorem 2.4), as $A^S(h)(k, k, \hat{h}(k, k))$ is Lipschitz on each closed $I \subset (0, k^u]$, there exists a Lipschitz extension of A^S onto $[0, k^u]$ with the same Lipschitz module. As we really only need this property in equilibrium, we defer this issue to the existence of equilibrium result in Theorem 25 below.

Proof. As $A^S(h)(k,k;\hat{h}(K))$ is isotone in h on $\mathbf{C}$, each $\in \mathbf{C}$, the fixed point correspondence $\Psi^C_{A^S}\ (\hat{h})(k,k)$ is a nonempty complete lattice follows by Tarski's theorem ([55], Theorem 1). By Veinott's fixed point comparative statics result in the appendix (e.g., Veinott [58], Theorem 14, Chapter 4), $\Psi^C_{A^S}\ (\hat{h})(k,k)$ is strong set order isotone jointly in the parameter $\hat{h}(k,k)$ with all isotone selections forming a complete lattice. That $\wedge_{\hat{h}}\Psi^C_{A^S}\ (\hat{h})(k,k)$ is an increasing selection follows from the fact that as $\Psi^C_{A^S}\ (\hat{h})(k,k)$ is also complete lattice-valued that is ascending in Veinott's strong set order, hence, $\wedge_{\hat{h}}\Psi^C_{A^S}\ (\hat{h})(k,k)$ is well-defined an is the least increasing selection for investment. ■

Let $\Psi^C_{B^S}$ be the fixed point set of the operator $B^S : \mathbf{C} \to \mathbf{C}$ define in Lemma 27.We now prove the main theorem of this section, namely, that our modification of Coleman's policy iteration algorithm converges to a greatest continuous recursive equilibrium:

Theorem 28. *Under Assumptions 1, 2, and 3(ii)-(iii), there exists complete lattice $\Psi^C_{B^S}$ of fixed points of $B^S(h)$ in $\mathbf{C}(\mathbf{X})$. Furthermore, the iterations $\lim_n B^{Sn}(0) \to h^* = \wedge\Psi^C_{B^S}$. Finally, h^* a recursive equilibrium that is continuous, locally Lipschitz when $k > 0$, and convergence of $\lim_n \wedge B^{Sn}(f) \to h^*$is uniform on $\mathbf{X}$.*

Proof. Proof: As $B^S(h)$ is isotone in $\mathbf{C}$, $\mathbf{C}$ a complete lattice, the set Ψ_{B^S} is a nonempty complete lattice by Tarski's theorem. Further, by the isotonicity of $B(h)$, the iterations from the maximal element of $\mathbf{C}$, namely $\wedge\mathbf{C}$ =0 is the minimal investment (or, maximal consumption associated with a 1 period economy). Then, $\lim_n B^{Sn}(0)\}_n \to h^* = \wedge\Psi_{B^S}$.

As the limiting fixed point $A^S(h^*(k,k),k,k,h^*(k,k))$ is defined implicitly in Z_{A^S}, when $k > 0$, from Lemma 25, the fixed point $h^*(k,k) = A^S(h^*(k,k),k,k,h^*(k,k))$ is locally Lipschitz when $k > 0$.

Finally, for $k \in [0,k^u]$, let $d_1(k) = 0$, and $d_2(k) = f$. Notice, $d_1 \leq h^* \leq d_2$ for all $k \in (0,k^u]$. Hence, this holds for all closed subsets $I \subset (0,k^u]$. Under Assumption 2, both d_1 and d_2 are pointwise bounded, such that $\exists c > 0$, such that for all k_1, k_2, $d_1 - d_2 \leq c|k_1 - k_2|$ for all $(k_1,k_2) \in\ [0,k^u] \times [0,k^u]$. Therefore, by a theorem in Matoušková ([32], Theorem 2.2), $h^*(k,k)$ is a continuous extension onto $[0,k^U]$, with $h^*(k,k)$ at $k = 0$. Further, as $\{B^{Sn}(0)\}_n$ is an increasing sequence of continuous functions converging pointwise to a continuous function $h^* = \wedge\Psi_{B^S}$.

Hence, by Dini theorem, this convergence is uniform (also, see Amann [3], Theorem 6.1). ■

A few remarks. First, in the above result, there is no claim of uniqueness of locally Lipschitz RE. Actually, a simple inspection of the RE functional equation parameterized with our two step method reveals there is very little geometric structure for the *second* step (i.e., the first step does generate unique RE for each $\hat{h}$, but because of the implicit nature of the second step operator in the RE functional equation, aside from monotonicity, very little else can be established.

Second, the result does indeed verify the existence of continuous RE in the Santos example. Also, it is easily verified that the RE investment decision rule is isotone in (k, k).

Finally, it important to note that Santos ([54], Section 3.2) never establishes the existence recursive equilibrium in his example; rather, he only considers the nonexistence of continuous recursive equilibrium on a state space of **K.** As we shall show in the next section, discontinuous solutions to the equilibrium Euler equation of the household's can be constructed. They are not RE. Further, It bears mentioning that in Santo's proof, to character the local properties of a continuous RE near a particular unstable steady state, it appears he calculates eigenvalues needed for this characterization applying the Grobmann-Hartmann theorem. Indeed, it is precisely this application that obtains the necessary contradiction with the continuity of RE on an open manifold near this "spiral sink". It is important to remember that the Grobmann-Hartman theorem in his context would assume the local RE decision rule is a *smooth* dynamical systems mapping in C^1-manifold.

5.2. Existing Correspondence-based GME Methods

We finally consider a correspondence-based Generalized Markov equilibrium methods for constructing recursive equilibrium in enlarged state spaces. Our main focus in this section will be on economies with a state contingent regressive taxes (as this case has been studied extensively using correspondence-based continuation methods in the work of Santos [54], Miao and Santos [34] and Feng, et. al. [21]). In the Miao-Santos correspondence-based continuation method, instead of parameterizing the continuation structure of the economy in the previous section with functions, we use correspondences, say $G(x) \in \mathbf{G(X)}$, where $\mathbf{G(X)}$ is a complete lattice of correspondences under the set inclusion partial order. We will define a mapping in $\mathbf{G}$ essentially as follows: (i) given an element $G(x) \in \mathbf{G}$, we can solve the first order conditions for all solutions to the household's Euler equation that are consistent with this implied continuation structure for the economy; then, (ii) we use these solutions to define mapping, say $T(G)(x) \in \mathbf{G}(\mathbf{X})$, that returns the updated values of these continuation variables today. We then compute fixed points of this "set to set" mapping.

It will turn out that for the regressive tax case, the critical complication for the correspondence-based approach is the fact that under Assumptions 1 and 3(ii), the function

$$\Phi(x, k) = \frac{u'(y^1(k,k) - x)}{(1-\tau)(x)}$$

does not exhibit any particular pattern of monotone comparative statics in the pair (x, k). Hence, for each (k, k), the set of solutions to a modified version of our equation Z_{A^S} in equation (5.1) continuing to an element $g \in G(k', k')$ will be correspondence $T(G)(x)$ that is simply nonempty, upper semicontinuous correspondence and preserves compactness. Unfortunately, such a correspondence will not admit continuous selection in its first argument, in general, and, hence generalized Markovian decision rules, say $k' = x^*(k, k, g^*(k, k))$, cannot be guaranteed to be optimal solutions to a strictly concave dynamic programming problem for the household in its individual state k in equation (11). This will be core issue with generating situation where the Miao-Santos procedure fails guarantee the existence recursive equilibrium selections without additional arguments.

5.2.1. The Miao-Santos Procedure

To define the Miao-Santos operator $T(G)(x)$, consider for a subset $\mathbf{D} \subset \mathbf{R}$, the collection of subsets $\mathbf{D}' = 2^{\mathbf{D}}$, where $\mathbf{D}$ is compact. It is known that the pair $(\mathbf{D}', \geq)$ is a continuous lattice (hence, a complete lattice) under the set inclusion partial order. Endow $(\mathbf{D}', \geq)$ with the Hausdorff metric. Under this metric, $\mathbf{D}'$ is also a complete metric space. Define the following set of correspondences $\mathbf{G} \subset \mathbf{D}'$, defined as follows:

$$\begin{gathered}\mathbf{G} = \mathbf{G}(\mathbf{X}) = \{G(x) | G : \mathbf{X} \to \mathbf{D}',\ G(x) \subset G^u(x) = \mathbf{D}\ \forall\ x = (k, k) \in \mathbf{X}, \\ G(x) \text{ a nonempty, compact-valued, and upper-semicontinuous in } x\}\end{gathered}$$

When defining $\mathbf{G}$, we require a top element, say G^u. When applying the Miao-Santos procedure to our economies with Inada conditions, this element can often be challenging to construct without prior knowledge of the actual recursive equilibrium. For the moment, we shall assume that for our economy with a regressive tax, such a greatest element $G^u = \mathbf{D}$ can be specified for all (k, k) such that $k > 0$. [38].

Consider an operator $T(G)(k,k) \subset \mathbf{G}$ mapping in spaces of correspondences defined implicitly in the temporary equilibrium version of the household's Euler equation (12) as follows:[39]

$$\begin{aligned} T(G)(k,k) &= \{g' | \exists x^* \text{ st } \frac{u'(y^1 - x^*)}{(1-\tau)(x^*)} - \beta g,\ g \in G(k', k') \in \mathbf{G} \\ k' &= x^*,\ g' = u'(y^1 - x^*) f'(k),\ x^*(k,k,g) \in X^*(k,k;g),\ y^1 = y^1(k,k)\} \end{aligned} \tag{18}$$

where $x^* = x^*(k,k,g) \in X^*(k,k;g)$ is an implied selection for investment decision in equilibrium defined on the expanded state space that includes the "shadow value" of household capital holdings in equilibrium, namely g. That is, when $g \in G(k', k')$ is an equilibrium envelope in a recursive equilibrium tomorrow, today's decision rules for investment in equilibrium will be selections from $X^*(k,k;g)$. Therefore, a continuation g will induce an auxiliary state variable in the decision rules for equilibrium policies $k' = x^*(k,k,g)$. This is precisely the "generalized Markov equilibrium" structure that is studied in the literature (e.g., see additionally for example Phelan and Stacchetti [46] and Kubler and Schmedders [26]).

As shall be mentioned in a moment (e.g., in Proposition 29 below), it is known that the operator $T(G)(x)$ maps the space of correspondences $\mathbf{G}$ into itself, and is isotone under the set inclusion order. Hence, $T(G)(x)$ has a complete lattice of fixed points by Tarski's theorem (ordered under set inclusion). Further, under Assumptions 1 and 3(ii) on the Lipschitz structure of u' and τ, as the mapping Z_T defined as

$$Z_T(x,k,k,g) = \frac{u'(y^1 - x)}{(1-\tau)(x)} - \beta g$$

is jointly continuous in all its arguments, by a standard argument, $T(G)(k,k)$ is (pointwise) Hausdorff continuous. This implies $T(G)(k,k)$ is order continuous on

[38] It is not clear often how to do this (for example, in our economy). In Miao and Santos [34] and Feng et. al. [21], the avoid this question by endowing agents with an interior income point. For even simple models like ours, no such element exists.

[39] The idea here is this is a typical sequential equilibrium Euler eqiuation at any date (say date 0), where the continuation envelope in date 1 g is treated as a state variable, and "todays" decision rule on investment now depends on an enlarged state space (k,k,g). The existence of this envelope follows from the fact that the household's sequential decision problem is smooth in k_0, for the sequence of aggregate capital stock $\{K_t\}$ from date 0.

G.[40] Therefore, by the Tarski-Kantorovich theorem (e.g, Dugundji and Granas [19], Theorem 4.2), $T(G)(k,k)$ will have a greatest fixed point in the down-set $\{G|G \in \mathbf{G},\, G \leq G^u = \mathbf{D}\}$ under set inclusion that is computed as

$$\inf_n T^n(G^u)(k,k) \to G^*(k,k)$$

Finally, given "self-generation" arguments first discussed in Abreu, Pearce, and Stacchetti (e.g., [1][2]), it turns out that our interest is only on this greatest fixed point $G^*(x)$.

Given the existence of such an upper bound $G^u = \mathbf{D} \in \mathbf{G}$, Miao and Santos [34] and Reffett [48] have proven a number of results concerning iterative methods based upon the correspondence-based operator $T(G)(k,k)$. We state the key facts in the next proposition proved in Miao and Santos [34]:

Proposition 29. *(Miao and Santos [34]; Feng et. al. [21]). For each* $(k,k) \in \mathbf{X}$, $k > 0$, $T(G(k,k)) \in \mathbf{G}$ *is isotone on* $\mathbf{G}$ *under set inclusion. Further, if* $\exists\; G^u \in \mathbf{G}$ *such that* $G^u(k,k) \leq T(G^u)(k,k)$ *under set inclusion, for all* (k,k), *then, the iterations* $\lim_n T^n(G^u)(k,k) \to \inf_n T^n(G^u)(k,k)G^*(k,k)$ *(where the* inf *is with respect to set inclusion), and* $G^*(k,k)$=G^* *the greatest fixed point of* $T(G)(k,k)$, *where the convergence both in the Hausdorff.*

Therefore, the question of existence of recursive equilibrium is now reduced to guaranteeing the existence of selection $g^*(k,k) \in G^*(x)$ such that the generalized Markov equilibrium decision rule $k' = x^*(k,k,g^*) \in X^*(k,k,g^*)$ is a recursive equilibrium per our definition in Section 2.

A few remarks on Proposition 29. First, and most importantly, Proposition 29 is *not* sufficient to establish the existence of a recursive equilibrium. That is, although $T(G)(k,k)$ maps $\mathbf{G}$ into itself, $G^* \in \mathbf{G}$ does not guarantee the existence of selections $g(k,k) \in G^*(k,k)$, such that the implied decision rule for investment from date 0, (namely the sequence individual states $\{k_t\}_{t=0}^{\infty}$ generated recursively as $k_{t+1} = x^*(k_t, \{K_t\}, g(k_t, \{K_t\}))$ from k_0 when $k_t = K_t$ satisfies the following the necessary and sufficient conditions for the existence of an optimal solution to

[40] See Reffett [48] for discussion.

the households sequential optimization problem in a sequential equilibrium from individual state $k_0 > 0$:

$$V_0^*(k_0, k_0; g(k_0, k_0)) = \sup_{\{\hat{x}_t\}_{t=0}^{\infty} \in \Psi_i(k_0, \{k_t\}_{t=0}^{\infty})} \sum \beta^t u(y^1(k,k) - \hat{x}_t)$$

where $\Psi_0(k_0\ \{K_t\})$ is the lifetime budget constraint for the household with prices $\{r_t\}$ and $\{w_t\}$ generated by $\{K_t\}$ when $k_t = K_t$, with optimal solution $\hat{x}_t^*(k_t, \{K_t\})$ the households sequential equilibrium solution for investment corresponding with the Generalized Markov equilibrium $x_t^*(k, \{K_t\}, g(k_t, \{K_t\}) \in X_t^*(k_t, K_t, g(k_t, K_t))$ when $k_t = K_t$. The problem is the nonexistence of a selection $g(k,k) \in G^*(k,k)$ that is continuous its first argument (e.g., Aubin and Frankowska [6], Example, p. 358), which implies that Proposition 29 does not verify of a sequential equilibrium with household decision rule that is generates a smooth value function in its first argument.

5.3. A New Correspondence-Based Method in Function Spaces

We can now provide a very simple fix for this situation. To do this, restrict our interval operator $\hat{T}(I^G)(x)$ to a smaller domain, say subintervals of the set $\bar{\mathbf{C}} \subset \mathbf{B}^f$. Further, define this restriction of $\hat{T}(I^G)(x)$ by using the mapping $B^S(\hat{h}) = \vee_{\hat{h}} \Psi_{A^S}(\hat{h})(k,k)$ defined Lemma 27. To see the details, let the interval powerdomain of $\mathbf{C}$ be defined by:

$$\mathfrak{I}(\mathbf{C}) = \{I^C | I^C = [h_1, h_2], h_1 \in \mathbf{C}, h_2 \in \mathbf{C}\} \cup \varnothing$$

Modifying the definition of g_h, replacing $\bar{\mathbf{B}}^f$ in the definition of $\mathbf{G}^B$, and restricting the set of continuation envelopes to be those defined using the operator $B^s(h)$ defined in $\mathbf{C}$, we can define an "APS" method valued in function spaces. That is, using $B^S(0)$, recompute the candidate continuation envelopes $\mathbf{G}^C$ as follows:

$$g_h^e(k,k) = g_h(k,k) \text{ for } h \in \bar{\mathbf{C}}$$

with

$$\bar{\mathbf{C}} = \{h \in \mathbf{C} |\ B^s(0) \le 0 \le h^u < f\}$$

Then, define $\mathbf{G}^C$ to be the collection of continuation envelopes:

$$\mathbf{G}^C = \{g_h^e(k,k) | g_{h^u}^e \le g_h^e \le g_{B^S(0)}^e,\ h \in \bar{\mathbf{C}}\}$$

We now study the operator $\hat{T}(I)(x)$ define in equation (**??**) in the smaller interval power domain $\mathfrak{I}(\mathbf{G}^C)$ given by

$$\mathfrak{I}(\mathbf{G}^C) = \{I^C | I^C = [g_1, g_2],\ g_1 \leq g_2, g_1 \in \mathbf{G}^C, g_2 \in \mathbf{G}^C\} \cup \varnothing$$

Letting $I_u^C = \vee \mathbf{G}^C$, denoting the restriction $\hat{T}(I^B)(x)$ to $\mathfrak{I}(\mathbf{G}^C)$ by $\hat{T}_e(I)(x)$, we now prove a stronger version of Proposition 29 relative to interval mapping $\hat{T}_e(I^C)$. Further, we can give a very simple explicit operator on the set of fixed points of $\hat{T}_e(I)(x)$, say $\Psi^C_{\hat{T}_e}$, that computes a jointly continuous selection $g^*(k,k) \in G^*(k,k)$, such that $g^*(k,k)$ induces an recursive equilibrium decision rule $a^*_{g^*}(k,k,g^*(k,k))$ that is a generalized Markov equilibrium in the sense of Miao and Santos [34].

Theorem 30. *For every n, and all $x \in \mathbf{X}$, the orbits for the operators $\hat{T}_e(I)(x)$, $\hat{T}(I)(x)$, and $T(G)(x)$ are ordered, respectively, under reverse set inclusion from top elements I_u^C, I_u^B, and G^u, as follows: $\hat{T}_e^n(I_u^C)(x) \geq\ \hat{T}^n(I_c^B)(x) \geq\ T^n(G^u)(x)$. Further, $\lim_n \hat{T}_e(I_u^C) = \sup_n \hat{T}_e(I_u^C) \to \wedge\Psi_{\ \hat{T}_e} = [0, g^*] \geq \Psi_{T^B} \geq G^*(k,k)$, where the* sup *is taken with respect to the reverse set inclusion order. Finally, $g^*(x)$ corresponds to a recursive equilibrium $h^* = \wedge\Psi_{B^S}$ in Theorem 28.*

Proof. Proof: First, for $n = 1$, that we have

$$\hat{T}_e(I_u^C)(x) \geq\ \hat{T}(I_u^B)(x) \geq\ T(G^u)(x)$$

follows from $I_u^C \geq I_u^B \geq G^u$, and the fact that $\hat{T}_e^n(I_u^C)(x)$, by definition, consists of selections in $\bar{\mathbf{C}}$ from $\hat{T}^n(I_u^B)(x)$, and $\hat{T}^n(I_u^B)(x)$ consists of selections $\bar{\mathbf{B}}^f$ in $T^n(G^u)(x)$. As reverse set inclusion is a closed on the powersets $\mathbf{D}'$, this comparative dynamics result per iterations is preserved in the limit: i.e,

$$\lim_n \hat{T}_e^n(I)(x) \geq\ \lim_n \hat{T}^n(I)(x) \geq\ \lim_n T(G^u)(x)$$

Finally, as by construction, $\hat{T}_e^n(I_u^C)(x) = [0, \frac{r(f-B^s(0))}{m(f-B^s(0))}]$, we have $g^*(k,k) = \frac{r(f-\wedge\Psi_{B^S})}{m(f-\wedge\Psi_{B^S})}$. ■

6. Appendix: Definitions and Results

To keep the paper self-contained, many definitions needed in the paper are now provided.

6.1. Spaces

An arbitrary set $(P, \geq)$ is *partially ordered set (or Poset)* if P is equipped with an order relation $\geq: P \times P \to P$ that is reflexive, antisymmetric and transitive. If every element of a poset P is comparable, then P is *chain.* If P is a chain and countable, P is a *countable chain***.** The space P^{op} shall denote the poset P equipped with its dual partial order $\geq^{op}$. An *upper (*respectively, *lower) bound* for a set $B \subset P$ is an element x^u(respectively, x^l) $\in P$ such that for any other element $x \in B$, $x \leq x^u$ (respectively, $x^l \leq x$) for all $x \in B$. If there is a point x^u (respectively, x^l) such that x^u is the least element in the subset of upper bounds of $B \subset X$ (respectively, the greatest element in the subset of lower bounds of $B \subset P$), we say x^u (respectively, x^l) is the *supremum* (respectively, *infimum)* of B. Clearly if the supremum or infimum of a set P exists, it must be unique.

We say a set L is a *lattice* if for any two elements, say x and x' in L, L is closed under the operation of infimum (denoted by $x \wedge x'$), and supremum (denoted $x \vee x'$).The former is referred to as "the meet" of the two points, while the latter is "the join". A subset L_1 of L is a *sublattice* of L if it contains the sup and the inf (with respect to L) of any pair of points in L_1. A lattice is *complete* if any $L_1 \subset L$, upper bound (denoted $\vee L_1$) and a greatest lower bound (denoted $\wedge L_1$) are both in L. If this completeness property only holds for countable subsets L_c, the lattice is $\sigma-$*complete.* In a poset P, if every subchain $C \subset P$ is complete, then P is referred to as a *chain complete poset (or equivalent, a complete partially ordered set* or *CPO).* A set C is *countable* if it is either finite or there is a bijection from the natural numbers onto C. If every chain $C \subset P$ is countable and complete, then P is referred to as a *countably chain complete poset.* An *order interval* is defined to be $[a, b] = [a) \cap (b]$, $a \leq b$.

6.2. Mappings in Posets

Let $(P_1, \geq_{P_1})$ and $(P_2, \geq_{P_2})$ be Posets. A function (or, equivalently, operator) $f : P_1 \to P_2$ is *isotone* (or *order-preserving)* if $f(x') \geq_{P_2} f(x)$, when $x' \geq_{P_1} x$, for $x, x' \in P_1$. A function $f(x)$ is *antitone* (or *order-reversing*) if $f(x) \geq_{p_1} f(x')$ when $x' \geq_{P_1} x$, for x, $x' \in P_1$. A function that is isotone or antitone is *monotone.* If P_1 and P_2 be Posets, X a set, $g : X \to P_2$ a function, we say a function $g(x)$ admits an *isotone decomposition* $f(p_1, p_2)$ if there exists a function $f : P_1 \times P_1 \to P_2$ such that $f(p_1, p_2)$ is isotone on $P_1 \times P_1$. If X and Y are three sets, $f : X \times X \to Y$, the *diagonal* of a function $f(x, y)$ is a function $g = f(x, x)$.

Finally, a sequence $\{h_n \to h\}$ in H is *order convergent* if there exists two monotonic sequences of elements from H, one decreasing $\{h_{\downarrow n}\}$, and one increasing $\{h_{\uparrow n}\}$, such that $h = \inf h_{\downarrow n} = \sup h_{\uparrow n}$ and $h_{\uparrow n} \leq h_n \leq h_{\downarrow n}$. A necessary and sufficient condition for an increasing sequence $h_n \to h$ to be order convergent is $h = \sup h_n$. An operator Ah is *order continuous* on H if for all countable chains $C' = \{h_n\}$, $\vee A(C') = A(\vee C')$.

For a set X, define by 2^X the powersets of X, and $\mathbf{L}(X)$ the nonempty sublattices of L, and L_1 and L_2 be two arbitrary sublattices. Let R_{X_2} be an order relation on 2^X. We say a correspondence $F : P \longrightarrow 2^{X_2}$is *ascending* in the relation R_{X_2} from a poset $(P, \geq)$ to 2^{X_2} if $F(x')R_{X_2}F(x)$, when $x' \geq x$. If this set relation R_{X_2} induces a partial order on the a subclass of the powersets 2^{P_2}, say $P(X_2)$, and if $F(x) : P \longrightarrow P(X_2)$, we refer to $F(x)$ is a *isotone correspondence*. Dually, we can define a *descending* and *antitone* correspondence.

In this paper, we shall focus primarily on a few order relations on the powersets 2^X of a set X. For an arbitrary set X, the *Set inclusion Partial order* $\geq_{SI}$is the following: $A \geq_{si} B$ if $B \subset A$. Set inclusion induces a continuous lattice structure on 2^X with $\wedge = \cap$, $\vee = \cup$. If $X = L$ is additionally a lattice, define $\mathbf{L}(X) = \{L_1 \subset X|$ L_1 a nonempty sublattice$\}$, and let $L_1, L_2 \in \mathbf{L}(X)$. Then, define *Veinott's Strong Set Order* $\geq_s$ on $\mathbf{L}(X)$: $L_1 \geq_s L_2$, if for any $a \in L_1$, $b \in L_2$, $a \wedge b \in L_2$ and $a \vee b \in L_1$. Veinott's strong set order will be used extensively in this paper.

Finally, let $F : X \to 2^X \backslash \varnothing$ be a non-empty valued correspondence for each $x \in X$. The correspondence F is said to have a *fixed point* if there exists an x^* such that $x^* \in F(x^*)$. If this correspondence is actually a function, say $f(x)$, then a fixed point x^* has $x^* = f(x^*)$. A correspondence $F : X \times T \to 2^X \backslash \varnothing$ is referred to as *parameterized correspondence*. For $F(x, t)$, denote the fixed point set at $t \in T$ by $\Psi_F^X(t) : T \to 2^X$. A fixed point $x^* \in \Psi_F^X(t)$ is *minimal (resp, maximal)* in X if there does not exist another fixed point, say $y^* \in \Psi_F^X(t)$, such that $y^* \leq x^*$ (resp, $x^* \leq y^*$). If a fixed point is either minimal or maximal, we say it is *extremal.* If $x_L^*(t) = \wedge \Psi_F^X(t)$ exists relative to the order structure in X (resp, $x_G^*(t) = \vee \Psi_F^X(t)$ relative to X), then x^* is the *least fixed point* (resp, *greatest fixed point*) of F relative to X.

6.3. Some Useful Fixed Point Theorems

In our constructions, we shall apply various versions of Tarski's fixed point theorem. We begin with an interesting version of Tarski's fixed point theorem due to Veinott. Let X be a complete lattice, 2^X the powersets of X, T a partially ordered set, and $F : X \times T \to \mathbf{L}(X)$ be a parameterized correspondence, where

$\mathbf{L}(X) \subset 2^X$is the collection of nonempty sublattices of X endowed with Veinott's strong set order. The set $(\mathbf{L}(X), \geq)$ is the largest partially ordered set in 2^X.[41] Fixing $t \in T$, let $\Psi_F^X(t)$ be the set of fixed points of $F(x,t)$ in X. We have the following version of Tarski's theorem. [42]

Proposition 31. *Veinott [?][58]. Let X be a complete lattice, $F(x,t) \in \mathbf{L}(X)$ is nonempty, subcomplete-valued correspondence that is jointly strong set order ascending. Then (i) $\Psi_F^X(t)$ is a nonempty complete lattice, (ii) $\Psi_F^X(t)$ is strong set order ascending, and (iii) $\vee\Psi_F^X(t)$ and $\wedge\Psi_F^X(t)$ are isotone selections.*

Tarski's original theorem (Tarski [55], Theorem 1) occurs as a special case of Proposition 31 where $F(x,t) = f(x)$, and $f : X \to X$ is a operator (i.e., a function).

There are many useful constructive versions of Proposition 31 in the literature for the special case that the parameterized correspondence $F(x,t)$ is a parameterized operator $f : X \times T \to X$. In the first case, we assume for each $t \in T$, the partial map $f_t(x) : X \to X$ is order continuous. For this case, we have the following version of Tarski-Kantorvich-Markowsky theorem (e.g., Dugundji and Granas [19], Theorem 4.2, and Markowsky [30], Theorem 9).

Proposition 32. *Let $(X, \geq)$ be a CPO, $f : X \times T \to X$ isotone, $\exists$ a $x_L \in X$ such that $x_L \leq f(x_L, t)$, and $\Psi_f^X(t) : T \to 2^X$ a fixed point correspondence for f, and for all $t \in T$,Then, (i) $\Psi_F^X(t) \subset \{x \in X | x_L \leq x\}$ is nonempty CPO. Further, if (X,$\geq$) is countably chain complete, and $f(x,t)$ is order continuous on X, each $t \in T$, the iterations $\sup_n f^n(x_L) = \mu(t) \in \Psi_f^X(t)$, where $\mu(t)$ $=\inf_{x \in \{x \in X | x \geq x_L\}} \Psi_f^X(t)$.*

An important special case of the above corollary occurs when was add additional structure to both $(X, \geq)$ and $f(x,t)$. We state the theorem in the context we shall apply it in the paper, namely when $(X, \geq)$ is a compact order interval in

[41] $L_c(X)$ is not generally a lattice (e.g., for X not distributive). If X is completely distributive complete (i.e., $X \subset \mathbf{R}^\mathbf{I}$, where $\mathbf{I}$ is any arbitrary set, $\mathbf{R}^\mathbf{I}$ given the product order, and X=[0,1]$^\mathbf{I}$ with its relative partial order), then $L(X)$ is a complete lattice (actually, completely distributive complete lattce, hence, a continuous lattice).

[42] See Topkis ([56], Theorem 2.5.2) for proof.

the space of bounded real-valued continuous functions $C(Y)$ defined on compact $Y \subset \mathbf{R}^n$endowed with (i) the topology of uniform convergence, and (ii) pointwise partial order. This version of the theorem is proven in Amann ([3], Theorem 6.1). The existence result holds in more general topological spaces, while the continuous fixed point comparative statics result holds in more general ordered metric spaces.

Proposition 33. *In Corollary 32, if additional, (a) $(X, \geq)$ is a compact order interval in $C(Y)$, (b) $f(x;t)$ continuous in x, each $t\in T$, then, (i) the iterations* $\sup_n f^n(x_L;t) = \lim_n f(x_L;t) = \mu(t) \in \Psi_F^X(t)$, *and* $\mu(t) = \inf_{x\in\{x\in X|x\geq x_L\}} \Psi_F^X(t)$. *Further, if, additionally, (c) $T \subset C(Y)$, (d) $f(x,t)$ is jointly continuous, (e) and relative to the set $X_1 = \{x \in X|x \geq x_L\} \subset X$, for all $x_0 \in X_1$,* $\lim_n f^n(x_0;t) \to \mu(t) \in \Psi_f^X(t)$, *then (ii) $\mu(t)$ is continuous on T.*

Proposition 33 is a particularly useful result for many of our arguments. The Tarski-Kantorvich theorem tells us that the upper envelope (resp, dually, lower envelope) from some least point in X (resp, any greatest point in X), order converges to least (resp, greatest) fixed points in X. We can make this convergence in topology by introducing the Scott topology (an order topology where the bases of the order topology is form from directed sets in X). This topology is not always easily related to approximate solution methods. On the other hand, in Proposition 33, the topology is standard in the numerical approximation literature (i.e., uniform approximation). So the constructive result is particularly appealing in this context.

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