

Kinetic Stable Delaunay Graphs*

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Abstract

We consider the problem of maintaining the Euclidean Delaunay triangulation DT of a set P of n moving points in the plane, along algebraic trajectories of constant description complexity. Since the best known upper bound on the number of topological changes in the full Delaunay triangulation is only nearly cubic, we seek to maintain a suitable portion of the diagram that is less volatile yet retains many useful properties of the full triangulation. We introduce the notion of a *stable Delaunay graph*, which is a dynamic subgraph of the Delaunay triangulation. The stable Delaunay graph (a) is easy to define, (b) experiences only a nearly quadratic number of discrete changes, (c) is robust under small changes of the norm, and (d) possesses certain useful properties for further applications.

The stable Delaunay graph (SDG in short) is defined in terms of a parameter $\alpha > 0$, and consists of Delaunay edges pq for which the (equal) angles at which p and q see the corresponding Voronoi edge e_{pq} are at least α . We show that (i) SDG always contains at least roughly one third of the Delaunay edges at any fixed time; (ii) it contains the β -skeleton of P , for $\beta = 1 + \Omega(\alpha^2)$; (iii) it is stable, in the sense that its edges survive for long periods of time, as long as the orientations of the segments connecting (nearby) points of P do not change by much; and (iv) stable Delaunay edges remain stable (with an appropriate redefinition of stability) if we replace the Euclidean norm by any sufficiently close norm.

In particular, if we approximate the Euclidean norm by a polygonal norm (with a regular k -gon as its unit ball, with $k = \Theta(1/\alpha)$), we can define and keep track of a Euclidean SDG by maintaining the full Delaunay triangulation of P under the polygonal norm (which is trivial to do, and which is known to involve only a nearly quadratic number of discrete changes).

We describe two kinetic data structures for maintaining SDG when the points of P move along pseudo-algebraic trajectories of constant description complexity. The first uses the polygonal norm approximation noted above, and the second is slightly more involved, but significantly reduces the dependence of its performance on α . Both structures use $O^*(n)$ storage and process $O^*(n^2)$ events during the motion, each in $O^*(1)$ time. (Here the $O^*(\cdot)$ notation hides multiplicative factors which are polynomial in $1/\alpha$ and polylogarithmic in n .)

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1 Introduction

Delaunay triangulations and Voronoi diagrams. Let P be a (finite) set of points in \mathbb{R}^2 . Let $\text{VD}(P)$ and $\text{DT}(P)$ denote the Voronoi diagram and Delaunay triangulation of P , respectively. For a point $p \in P$, let $\text{Vor}(p)$ denote the Voronoi cell of p . The Delaunay triangulation $\text{DT} = \text{DT}(P)$ consists of all triangles whose circumcircles do not contain points of P in their interior. Its edges form the *Delaunay graph*, which is the straight-edge dual graph of the Voronoi diagram of P . That is, pq is an edge of the Delaunay graph if and only if $\text{Vor}(p)$ and $\text{Vor}(q)$ share an edge, which we denote by e_{pq} . This is equivalent to the existence of a circle passing through p and q that does not contain any point of P in its interior—any circle centered at a point on e_{pq} and passing through p and q is such a circle. Delaunay triangulations and Voronoi diagrams are fundamental to much of computational geometry and its applications. See [5, 11] for a survey and a textbook on these structures.

In many applications of Delaunay/Voronoi methods (e.g., mesh generation and kinetic collision detection) the points are moving continuously, so these diagrams need to be efficiently updated as motion occurs. Even though the motion of the nodes is continuous, the combinatorial and topological structure of the Voronoi and Delaunay diagrams change only at discrete times when certain critical events occur. Their evolution under motion can be studied within the framework of *kinetic data structures* (KDS in short) of Basch *et al.* [6, 12, 13], a general methodology for designing efficient algorithms for maintaining such combinatorial attributes of mobile data.

For the purpose of kinetic maintenance, Delaunay triangulations are nice structures, because, as mentioned above, they admit local certifications associated with individual triangles. This makes it simple to maintain DT under point motion: an update is necessary only when one of these empty circumcircle conditions fails—this corresponds to cocircularities of certain subsets of four points.¹ Whenever such an event happens, a single edge flip easily restores Delaunayhood. Estimating the number of such events, however, has been elusive—the problem of bounding the number of combinatorial changes in DT for points moving along semi-algebraic trajectories of constant description complexity has been in the computational geometry lore for many years; see [10].

Let n be the number of moving points in P . We assume that each point moves along an algebraic trajectory of fixed degree or, more generally, along pseudo-algebraic trajectory of constant description complexity (see Section 2 for a more formal definition). Guibas *et al.* [14] showed a roughly cubic upper bound of $O(n^2 \lambda_s(n))$ on the number of discrete (also known as *topological*) changes in DT, where $\lambda_s(n)$ is the maximum length of an (n, s) -Davenport-Schinzel sequence [22], and s is a constant depending on the motions of the points. A substantial gap exists between this upper bound and the best known quadratic lower bound [22].

It is thus desirable to find approaches for maintaining a substantial portion of DT that *provably* experiences only a nearly quadratic number of discrete changes, that is reasonably easy to define and maintain, and that retains useful properties for further applications.

Polygonal distance functions. If the “unit ball” of our underlying norm is *polygonal* then things improve considerably. In more detail, let Q be a convex polygon with a constant number, k , of edges. It induces a *convex distance function*

$$d_Q(x, y) = \min\{\lambda \mid y \in x + \lambda Q\};$$

d_Q is a metric if Q is centrally symmetric with respect to the origin.

¹We assume that the motion of the points is sufficiently generic, so that no more than four points can become cocircular at any given time.

We can define the Q -Voronoi diagram of a set P of points in the plane in the usual way, as the partitioning of the plane into Voronoi cells, so that the cell $\text{Vor}^\diamond(p)$ of a point p is $\{x \in \mathbb{R}^2 \mid d_Q(x, p) = \min_{p' \in P} d_Q(x, p')\}$. Assuming that the points of P are in general position with respect to Q , these cells are nonempty, have pairwise disjoint interiors, and cover the plane.

As in the Euclidean case, the Q -Voronoi diagram of P has its dual representation, which we refer to as the Q -Delaunay triangulation $\text{DT}^\diamond(P) = \text{DT}^\diamond$. A triple of points in P define a triangle in DT^\diamond if and only if they lie on the boundary of some homothetic copy of Q that does not contain any point of P in its interior. Assuming that P is in general position, these Q -Delaunay triangles form a triangulation of a certain simply-connected polygonal region that is contained in the convex hull of P . Unlike the Euclidean case, it does not always coincide with the convex hull (see Figures 5 and 23 for examples). See Chew and Drysdale [8] and Leven and Sharir [19] for analysis of Voronoi and Delaunay diagrams of this kind.

For kinetic maintenance, polygonal Delaunay triangulations are “better” than Euclidean Delaunay triangulations because, as shown by Chew [7], when the points of P move (in the algebraic sense assumed above), the number of topological changes in the Q -Delaunay triangulation is only nearly quadratic in n . One of the major observations in this paper is that the *stable portions* of the Euclidean Delaunay triangulation and the Q -Delaunay triangulation are closely related.

Stable Delaunay edges. We introduce the notion of α -stable Delaunay edges, for a fixed parameter $\alpha > 0$, defined as follows. Let pq be a Delaunay edge under the Euclidean norm, and let $\triangle pqr^+$ and $\triangle pqr^-$ be the two Delaunay triangles incident to pq . Then pq is called α -stable if its opposite angles in these triangles satisfy $\angle pr^+q + \angle pr^-q < \pi - \alpha$. (The case where pq lies on the convex hull of P is treated as if one of r^+, r^- lies at infinity, so that the corresponding angle $\angle pr^+q$ or $\angle pr^-q$ is equal to 0.) An equivalent and more useful definition, in terms of the dual Voronoi diagram, is that pq is α -stable if the equal angles at which p and q see their common Voronoi edge e_{pq} are at least α . See Figure 1.

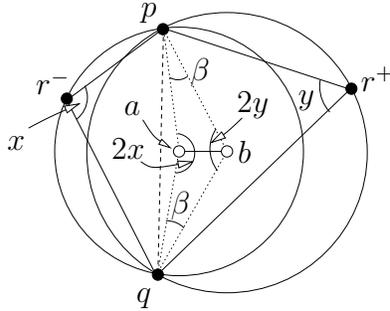


Figure 1: The points p and q see their common Voronoi edge ab at (equal) angles β . This is equivalent to the angle condition $x + y = \pi - \beta$ for the two adjacent Delaunay triangles.

A justification for calling such edges stable lies in the following observation: If a Delaunay edge pq is α -stable then it remains in DT during any continuous motion of P for which every angle $\angle prq$, for $r \in P \setminus \{p, q\}$, changes by at most $\alpha/2$. This is clear because at the time pq is α -stable we have $\angle pr^+q + \angle pr^-q < \pi - \alpha$ for any pair of points r^+, r^- lying on opposite sides of the line ℓ supporting pq , so, if each of these angles change by at most $\alpha/2$ we still have $\angle pr^+q + \angle pr^-q \leq \pi$, which is easily seen to imply that pq remains an edge of DT . (This argument also covers the cases when a point r crosses ℓ from side to side: Since each point, on either side of ℓ , sees pq at an angle of $\leq \pi - \alpha$, it follows that no point

can cross pq itself – the angle has to increase from $\pi - \alpha$ to π . Any other crossing of ℓ by a point r causes $\angle prq$ to decrease to 0, and even if it increases to $\alpha/2$ on the other side of ℓ , pq is still an edge of DT, as is easily checked.) Hence, as long as the “small angle change” condition holds, stable Delaunay edges remain a “long time” in the triangulation.

Informally speaking, the non-stable edges pq of DT are those for p and q are almost cocircular with their two common Delaunay neighbors r^+ , r^- , and hence are more likely to get flipped “soon”.

Overview of our results. Let $\alpha > 0$ be a fixed parameter. In this paper we show how to maintain a subgraph of the full Delaunay triangulation DT, which we call a $(c\alpha, \alpha)$ -stable Delaunay graph (SDG in short), so that (i) every edge of SDG is α -stable, and (ii) every $c\alpha$ -stable edge of DT belongs to SDG, where $c > 1$ is some (small) absolute constant. Note that SDG is not uniquely defined, even when c is fixed.

In Section 2, we introduce several useful definitions and show that the number of discrete changes in the SDGs that we consider is nearly quadratic. What this analysis also implies is that if the true bound for kinetic changes in a Delaunay triangulation is really close to cubic, then the overwhelming majority of these changes involve edges which never become stable and just flicker in and out of the diagram by cocircularity with their two Delaunay neighbors.

In Sections 3 and 4 we show that SDG can be maintained by a kinetic data structure that uses only near-linear storage (in the terminology of [6], it is *compact*), encounters only a nearly quadratic number of critical events (it is *efficient*), and processes each event in polylogarithmic time (it is *responsive*). For the second data structure, described in Section 4, can be slightly modified to ensure that each point appears at any time in only polylogarithmically many places in the structure (it then becomes *local*).

The scheme described in Section 3 is based on a useful and interesting “equivalence” connection between the (Euclidean) SDG and a suitably defined “stable” version of the Delaunay triangulation of P under the “polygonal” norm whose unit ball Q is a regular k -gon, for $k = \Theta(1/\alpha)$. As noted above, Voronoi and Delaunay structures under polygonal norms are particularly favorable for kinetic maintenance because of Chew’s result [7], showing that the number of topological changes in $DT^\circ(P)$ is $O^*(n^2k^4)$; here the $O^*(\cdot)$ notation hides a factor that depends sub-polynomially on both n and k . In other words, the scheme simply maintains the “polygonal” diagram $DT^\circ(P)$ in its entirety, and selects from it those edges that are also stable edges of the Euclidean diagram DT.

The major disadvantage of the solution in Section 3 is the rather high (proportional to $\Theta(1/\alpha^4)$) dependence on $1/\alpha$ ($\approx k$) of the bound on the number of topological changes. We do not know whether the upper bound $O^*(n^2k^4)$ on the number of topological changes in $DT^\circ(P)$ is nearly tight (in its dependence on k). To remedy this, we present in Section 4 an alternative scheme for maintaining stable (Euclidean) Delaunay edges. The scheme is reminiscent of the kinetic schemes used in [1] for maintaining closest pairs and nearest neighbors. It extracts $O^*(n)$ pairs of points of P that are candidates for being stable Delaunay edges. Each point $p \in P$ then runs $O(1/\alpha)$ *kinetic and dynamic tournaments* involving the other points in its candidate pairs. Roughly, these tournaments correspond to shooting $O(1/\alpha)$ rays from P in fixed directions and finding along each ray the nearest point equally distant from p and from some other candidate point q . We show that pq is a stable Delaunay edge if and only if q wins many (at least some constant number of) consecutive tournaments of p (or p wins many consecutive tournaments of q). A careful analysis shows that the number of events that this scheme processes (and the overall processing time) is only $O^*(n^2/\alpha^2)$.

Section 5 establishes several useful properties of stable Delaunay graphs. In particular, we show that at any given time the stable subgraph contains at least $\left[1 - \frac{3}{2(\pi/\alpha - 2)}\right] n$ Delaunay edges, i.e., at least about one third of the maximum possible number of edges. In addition, we show that at any moment the SDG contains the closest pair, the so-called β -skeleton of P , for $\beta = 1 + \Omega(\alpha^2)$ (see [4, 18]), and the *crust* of a

sufficiently densely sampled point set along a smooth curve (see [3, 4]). We also extend the connection in Section 3 to arbitrary distance functions d_Q whose unit ball Q is sufficiently close (in the Hausdorff sense) to the Euclidean one (i.e., the unit disk).

2 Preliminaries

Stable edges in Voronoi diagrams. Let $\{u_0, \dots, u_{k-1}\} \subset \mathbb{S}^1$ be a set of $k = \Theta(1/\alpha)$ equally spaced directions in \mathbb{R}^2 . For concreteness take $u_i = (\cos(2\pi i/k), -\sin(2\pi i/k))$, $0 \leq i < k$ (so our directions u_i go clockwise as i increases).² For a point $p \in P$ and a unit vector u let $u[p]$ denote the ray $\{p + \lambda u \mid \lambda \geq 0\}$ that emanates from p in direction u . For a pair of points $p, q \in P$ let b_{pq} denote the perpendicular bisector of p and q . If b_{pq} intersects $u_i[p]$, then the expression

$$\varphi_i[p, q] = \frac{\|q - p\|^2}{2\langle q - p, u_i \rangle} \quad (1)$$

is the distance between p and the intersection point of b_{pq} with $u_i[p]$. If b_{pq} does not intersect $u_i[p]$ we define $\varphi_i[p, q] = \infty$. The point q minimizes $\varphi_i[p, q']$, among all points q' for which $b_{pq'}$ intersects $u_i[p]$, if and only if the intersection between b_{pq} and $u_i[p]$ lies on the Voronoi edge e_{pq} . We call q the *neighbor of p in direction u_i* , and denote it by $N_i(p)$; see Figure 2.

The (*angular*) *extent* of a Voronoi edge e_{pq} of two points $p, q \in P$ is the angle at which it is seen from either p or q (these two angles are equal). For a given angle $\alpha \leq \pi$, e_{pq} is called α -*long* (resp., α -*short*) if the extent of e_{pq} is at least (resp., smaller than) α . We also say that $pq \in \text{DT}(P)$ is α -*long* (resp., α -*short*) if e_{pq} is α -*long* (resp., α -*short*). As noted in the introduction, these notions can also be defined (equivalently) in terms of the angles in the Delaunay triangulation: A Delaunay edge pq , which is not a hull edge, is α -long if and only if $\angle pr^+q + \angle pr^-q \leq \pi - \alpha$, where $\triangle pr^+q$ and $\triangle pr^-q$ are the two Delaunay triangles incident to pq . See Figure 1; hull edges are handled similarly, as discussed in the introduction.

Given parameters $\alpha' > \alpha > 0$, we seek to construct (and maintain under motion) an (α', α) -*stable Delaunay graph* (or *stable Delaunay graph*, for brevity, which we further abbreviate as SDG) of P , which is any subgraph G of $\text{DT}(P)$ with the following properties:

- (S1) Every α' -long edge of $\text{DT}(P)$ is an edge of G .
- (S2) Every edge of G is an α -long edge of $\text{DT}(P)$.

An (α', α) -stable Delaunay graph need not be unique. In what follows, α' will always be some fixed (and reasonably small) multiple of α .

Kinetic tournaments. Kinetic tournaments were first studied by Basch *et al.* [6], for kinetically maintaining the lowest point in a set P of n points moving on some vertical line, say the y -axis, so that their trajectories are algebraic of bounded degree, as above. These tournaments are a key ingredient in the data structures that we will develop for maintaining stable Delaunay graphs. Such a tournament is represented and maintained using the following variant of a heap. Let T be a minimum-height balanced binary tree, with the points stored at its leaves (in an arbitrary order). For an internal node $v \in T$, let P_v denote the set of points stored in the subtree rooted at v . At any specific time t , each internal node v stores the lowest

²The index arithmetic is modulo k , i.e., $u_i = u_{i+k}$.

point among the points in P_v at time t , which is called the *winner* at v . The winner at the root is the desired overall lowest point of P .

To maintain T we associate a certificate with each internal node v , which asserts which of the two winners, at the left child and at the right child of v , is the winner at v . This certificate remains valid as long as (i) the winners at the children of v do not change, and (ii) the order along the y -axis between these two “sub-winners” does not change. The actual certificate caters only to the second condition; the first will be taken care of recursively. Each certificate has an associated failure time, which is the next time when these two winners switch their order along the y -axis. We store all certificates in another heap, using the failure times as keys.³ This heap of certificates is called the *event queue*.

Processing an event is simple. When the two sub-winners p, q at some node v change their order, we compute the new failure time of the certificate at v (the first future time when p and q meet again), update the event queue accordingly, and propagate the new winner, say p , up the tree, revising the certificates at the ancestors of v , if needed.

If we assume that the trajectories of each pair of points intersect at most r times then the overall number of changes of winners, and therefore also the overall number of events, is at most $\sum_v |P(v)|\beta_r(|P(v)|) = O(n\beta_r(n) \log n)$. Here $\beta_r(n) = \lambda_r(n)/n$, and $\lambda_r(n)$ is the maximum length of a Davenport-Schinzel sequence of order r on n symbols; see [22].

This is larger by a logarithmic factor than the maximum possible number of times the lowest point along the y -axis can indeed change, since this latter number is bounded by the complexity of the lower envelope of the trajectories of the points in P (which, as noted above, records the changes in the winner at the root of T).

Agarwal *et al.* [1] show how to make such a tournament also *dynamic*, supporting insertions and deletions of points. They replace the balanced binary tree T by a *weight-balanced* ($BB(\alpha)$) *tree* [21] (and see also [20]). This allows us to insert a new point anywhere we wish in T , and to delete any point from T , in $O(\log n)$ time. Each such insertion or deletion may change $O(\log n)$ certificates, along the corresponding search path, and therefore updating the event queue takes $O(\log^2 n)$ time, including the time for the structural updates of (rotations in) T ; here n denotes the actual number of points in T , at the step where we perform the insertion or deletion. The analysis of [1] is summarized in Theorem 2.1.

Theorem 2.1 (Agarwal *et al.* [1]). *A sequence of m insertions and deletions into a kinetic tournament, whose maximum size at any time is n (assuming $m \geq n$), when implemented as a weight-balanced tree in the manner described above, generates at most $O(m\beta_{r+2}(n) \log n)$ events, with a total processing cost of $O(m\beta_{r+2}(n) \log^2 n)$. Here r is the maximum number of times a pair of points intersect, and $\beta_{r+2}(n) = \lambda_{r+2}(n)/n$. Processing an update or a tournament event takes $O(\log^2 n)$ worst-case time. A dynamic kinetic tournament on n elements can be constructed in $O(n)$ time.*

Remarks: (1) Theorem 2.1 subsumes the static case too, by inserting all the elements “at the beginning of time”, and then tracing the kinetic changes.

(ii) Note that the amortized cost of an update or of processing a tournament event is only $O(\log n)$ (as opposed to the $O(\log^2 n)$ worst-case cost).

Maintenance of an SDG. Let $P = \{p_1, \dots, p_n\}$ be a set of points moving in \mathbb{R}^2 . Let $p_i(t) = (x_i(t), y_i(t))$ denote the position of p_i at time t . We call the motion of P *algebraic* if each $x_i(t), y_i(t)$ is a polynomial function of t , and the *degree* of motion of P is the maximum degree of these polynomials. Throughout this paper we assume that the motion of P is algebraic and that its degree is bounded by a constant. In this

³Any “standard” heap that supports **insert**, **delete**, and **deletemin** in $O(\log n)$ time is good for our purpose.

subsection we present a simple technique for maintaining a $(2\alpha, \alpha)$ -stable Delaunay graph. Unfortunately this algorithm requires quadratic space. It is based on the following easy observation (see Figure 2), where k is an integer, and the unit vectors (directions) u_0, \dots, u_{k-1} are as defined earlier.

Lemma 2.2. *Let $\alpha = 2\pi/k$. (i) If the extent of e_{pq} is larger than 2α then there are two consecutive directions u_i, u_{i+1} , such that q is the neighbor of p in directions u_i and u_{i+1} . (ii) If there are two consecutive directions u_i, u_{i+1} , such that q is the neighbor of p in both directions u_i and u_{i+1} , then the extent of e_{pq} is at least α .*

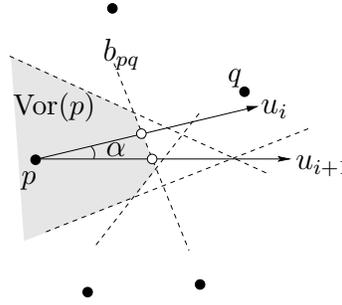


Figure 2: q is the neighbor of p in the directions u_i and u_{i+1} , so the Voronoi edge e_{pq} is α -long.

The algorithm maintains Delaunay edges pq such that there are two consecutive directions u_i and u_{i+1} along which q is the neighbor of p . For each point p and direction u_i we get a set of at most $n - 1$ piecewise continuous functions of time, $\varphi_i[p, q]$, one for each point $q \neq p$, as defined in (1). (Recall that $\varphi_i[p, q] = \infty$ when $u_i[p]$ does not intersect b_{pq} .) By assumption on the motion of P , for each p and q , the domain in which $\varphi_i[p, q](t)$ is defined consists of a constant number of intervals.

For each point p , and ray $u_i[p]$, consider each function $\varphi_i[p, q]$ as the trajectory of a point moving along the ray and corresponding to q . The algorithm maintains these points in a dynamic and kinetic tournament $K_i(p)$ (see Theorem 2.1) that keeps track of the minimum of $\{\varphi_i[p, q](t)\}_{q \neq p}$ over time. For each pair of points p and q such that q wins in two consecutive tournaments, $K_i(p)$ and $K_{i+1}(p)$, of p , it keeps the edge pq in the stable Delaunay graph. It is trivial to update this graph as a by-product of the updates of the various tournaments. The analysis of this data structure is straightforward using Theorem 2.1, and yields the following result.

Theorem 2.3. *Let P be a set of n moving points in \mathbb{R}^2 under algebraic motion of bounded degree, let k be an integer, and let $\alpha = 2\pi/k$. A $(2\alpha, \alpha)$ -stable Delaunay graph of P can be maintained using $O(kn^2)$ storage and processing $O(kn^2\beta_{r+2}(n)\log n)$ events, for a total cost of $O(kn^2\beta_{r+2}(n)\log^2 n)$ time. The processing of each event takes $O(\log^2 n)$ worst-case time. Here r is a constant that depends on the degree of motion of P .*

Later on, in Section 4, we will revise this approach and reduce the storage to nearly linear, by letting only a small number of points to participate in each tournament. The filtering procedure for the points makes the improved solution somewhat more involved.

3 An SDG Based on Polygonal Voronoi Diagrams

Let $Q = Q_k$ be a regular k -gon for some even $k = 2s$, circumscribed by the unit disk, and let $\alpha = \pi/s$ (this is the angle at which the center of Q sees an edge). Let $\text{VD}^\diamond(P)$ and $\text{DT}^\diamond(P)$ denote the Q -Voronoi diagram and the dual Q -Delaunay triangulation of P , respectively. In this section we show that the set of edges of $\text{VD}^\diamond(P)$ with sufficiently many *breakpoints* (see below for details) form a (β, β') -stable (Euclidean) Delaunay graph for appropriate multiples β, β' of α . Thus, by kinetically maintaining $\text{VD}^\diamond(P)$ (in its entirety), we shall get “for free” a KDS for keeping track of a stable portion of the Euclidean DT.

3.1 Properties of $\text{VD}^\diamond(P)$

We first review the properties of the (stationary) $\text{VD}^\diamond(P)$ and $\text{DT}^\diamond(P)$. Then we consider the kinetic version of these diagrams, as the points of P move, and review Chew’s proof [7] that the number of topological changes in these diagrams, over time, is only nearly quadratic in n . Finally, we present a straightforward kinetic data structure for maintaining $\text{DT}^\diamond(P)$ under motion that uses linear storage, and that processes a nearly quadratic number of events, each in $O(\log n)$ time. Although later on we will take Q to be a regular k -gon, the analysis in this subsection is more general, and we only assume here that Q is an arbitrary convex k -gon, lying in general position with respect to P .

Stationary Q -diagrams. The *bisector* b_{pq}^\diamond between two points p and q , with respect to $d_Q(\cdot, \cdot)$, is the locus of all placements of the center of any homothetic copy Q' of Q that touches p and q . Q' can be classified according to the pair of its edges, e_1 and e_2 , that touch p and q , respectively. If we slide Q' so that its center moves along b_{pq}^\diamond (and its size expands or shrinks to keep it touching p and q), and the contact edges, e_1 and e_2 , remain fixed, the center traces a straight segment. The bisector is a concatenation of $O(k)$ such segments. They meet at *breakpoints*, which are placements of the center of a copy Q' that touches p and q and one of the contact points is a vertex of Q ; see Figure 3. We call such a placement a *corner contact* at the appropriate point. Note that a corner contact where some vertex w of (a copy Q' of) Q touches p has the property that the center of Q' lies on the fixed ray emanating from p and parallel to the directed segment from w to the center of Q .

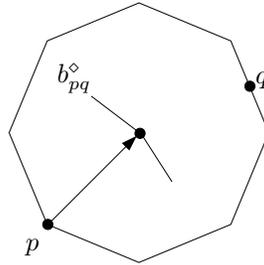


Figure 3: Each breakpoint on b_{pq}^\diamond corresponds to a corner contact of Q at one of the points p, q , so that ∂Q also touches the other point.

A useful property of bisectors and Delaunay edges, in the special case where Q is a regular k -gon, which will be used in the next subsection, is that the breakpoints along a bisector b_{pq}^\diamond alternate between corner contacts at p and corner contacts at q . Indeed, assuming general position, each point $w \in \partial Q$ determines a unique placement of Q where it touches p at w and also touches q , as is easily checked. A symmetric

property holds when we interchange p and q . Hence, as we slide the center of Q along the bisector b_{pq}^\diamond , the points of contact of ∂Q with p and q vary continuously and monotonically along ∂Q . Consider two consecutive corner contacts, Q', Q'' , of Q at p along b_{pq}^\diamond , and suppose to the contrary that the portion of b_{pq}^\diamond between them is a straight segment, meaning that, within this portion, ∂Q touches each of p, q at a fixed edge. Since the center of Q moves along the angle bisector of the lines supporting these edges (a property that is easily seen to hold for regular k -gons), it is easy to see that the distance between the two contact points of p , at the beginning and the end of this sliding, and the distance between the two contact points of q (measured, say, on the boundary of the standard placement of Q) are equal. However, this distance for p is the length of a full edge of ∂Q , because the motion starts and ends with p touching a vertex, and therefore the same holds for q , which is impossible (unless q also starts and ends at a vertex, which contradicts our general position assumption).

Another well known property of Q -bisectors and Voronoi edges, for arbitrary convex polygons in general position with respect to P , is that two bisectors $b_{pq_1}^\diamond, b_{pq_2}^\diamond$, can intersect at most once (again, assuming general position), so every Q -Voronoi edge e_{pq}^\diamond is connected. Equivalently, this asserts that there exists at most one homothetic placement of Q at which it touches p, q_1 , and q_2 . Indeed, since homothetic placements of Q behave like pseudo-disks (see, e.g., [17]), the boundaries of two distinct homothetic placements of Q intersect in at most two points, or, in degenerate position, in at most two connected segments. Clearly, in the former case the boundaries cannot both contain p, q_1 , and q_2 , and this also holds in the latter case because of our general position assumption.

Consider next an edge pq of $\text{DT}^\diamond(P)$. Its dual Voronoi edge e_{pq}^\diamond is a portion of the bisector b_{pq}^\diamond , and consists of those center placements along b_{pq}^\diamond for which the corresponding copy Q' has an *empty interior* (i.e., its interior is disjoint from P). Following the notation of Chew [7], we call pq a *corner edge* if e_{pq}^\diamond contains a breakpoint (i.e., a placement with a corner contact); otherwise it is a *non-corner edge*, and is therefore a straight segment.

Kinetic Q -diagrams. Consider next what happens to $\text{VD}^\diamond(P)$ and $\text{DT}^\diamond(P)$ as the points of P move continuously with time. In this case $\text{VD}^\diamond(P)$ changes continuously, but undergoes topological changes at certain critical times, called *events*. There are two kinds of events:

- (i) FLIP EVENT. A Voronoi edge e_{pq}^\diamond shrinks to a point, disappears, and is “flipped” into a newly emerging Voronoi edge $e_{p'q'}^\diamond$.
- (ii) CORNER EVENT. An endpoint of some Voronoi edge e_{pq}^\diamond becomes a breakpoint (a corner placement). Immediately after this time e_{pq}^\diamond either gains a new straight segment, or loses such a segment, that it had before the event.

Some comments are in order:

- (a) A flip event occurs when the four points p, q, p', q' become “cocircular”: there is an empty homothetic copy Q' of Q that touches all four points.
- (b) Only non-corner edges can participate in a flip event, as both the vanishing edge e_{pq}^\diamond and the newly emerging edge $e_{p'q'}^\diamond$ do not have breakpoints near the event.
- (c) A flip event entails a discrete change in the Delaunay triangulation, whereas a corner event does not. Still, for algorithmic purposes, we will keep track of both kinds of events.

We first bound the number of corner events.

Lemma 3.1. *Let P be a set of n points in \mathbb{R}^2 under algebraic motion of bounded degree, and let Q be a convex k -gon. The number of corner events in $\text{DT}^\diamond(P)$ is $O(k^2 n \lambda_r(n))$, where r is a constant that depends on the degree of motion of P .*

Proof. Fix a point p and a vertex w of Q , and consider all the corner events in which w touches p . As noted above, at any such event the center c of Q lies on a ray γ emanating from p at a fixed direction. (Since p is moving, γ is a moving ray, but its orientation remains fixed.) For each other point $q \in P \setminus \{p\}$, let $\varphi_\gamma^\diamond[p, q]$ denote the distance, at time t , from p along γ to the center of a copy of Q that touches p (at w) and q . The value $\min_q \varphi_\gamma^\diamond[p, q](t)$ represents the intersection of $\partial \text{Vor}^\diamond(p)$ with γ at time t , where $\text{Vor}^\diamond(p)$ is the Voronoi cell of p in $\text{VD}^\diamond(P)$. The point q that attains the minimum defines the Voronoi edge e_{pq}^\diamond (or vertex if the minimum is attained by more than one point q) of $\text{Vor}^\diamond(p)$ that γ intersects.

In other words, we have a collection of $n - 1$ partially defined functions $\varphi_\gamma^\diamond[p, q]$, and the breakpoints of their lower envelope represent the corner events that involve the contact of w with p . By our assumption on the motion of P , each function $\varphi_\gamma^\diamond[p, q]$ is piecewise algebraic, with $O(k)$ pieces. Each piece encodes a continuous contact of q with a specific edge of Q' , and has constant description complexity. Hence (see, e.g., [22, Corollary 1.6]) the complexity of the envelope is at most $O(k\lambda_r(n))$, for an appropriate constant r . Repeating the analysis for each point p and each vertex w of Q , the lemma follows. \square

Consider next flip events. As noted, each flip event involves a placement of an empty homothetic copy Q' of Q that touches simultaneously four points p_1, p_2, p_3, p_4 of P , in this counterclockwise order along $\partial Q'$, so that the Voronoi edge $e_{p_1 p_3}^\diamond$, which is a non-corner edge before the event, shrinks to a point and is replaced by the non-corner edge $e_{p_2 p_4}^\diamond$. Let e_i denote the edge of Q' that touches p_i , for $i = 1, 2, 3, 4$.

We fix the quadruple of edges e_1, e_2, e_3, e_4 , bound the number of flip events involving a quadruple contact with these edges, and sum the bound over all $O(k^4)$ choices of four edges of Q . For a fixed quadruple of edges e_1, e_2, e_3, e_4 , we replace Q by the convex hull Q_0 of these edges, and note that any flip event involving these edges is also a flip event for Q_0 . We therefore restrict our attention to Q_0 , which is a convex k_0 -gon, for some $k_0 \leq 8$.

We note that if (p, q) is a Delaunay edge representing a contact of some homothetic copy Q'_0 of Q_0 where p and q touch two *adjacent* edges of Q'_0 , then (p, q) must be a corner edge—shrinking Q'_0 towards the vertex common to the two edges, while it continues to touch p and q , will keep it empty, and eventually reach a placement where either p or q touches a corner of Q'_0 . The same (and actually simpler) argument applies to the case when p and q touch the same edge⁴ of Q_0 .

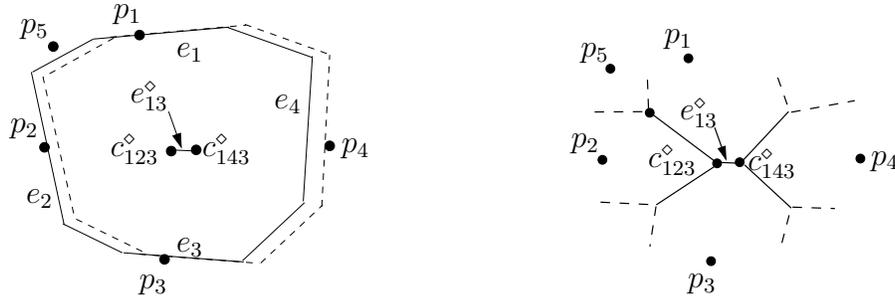


Figure 4: Left: The edge e_{13}^\diamond in the diagram $\text{VD}^\diamond(P)$ before disappearing. The endpoint c_{123}^\diamond (resp., c_{143}^\diamond) of e_{13}^\diamond corresponds to the homothetic copy of Q_0 whose edges e_1, e_2, e_3 (resp., e_1, e_4, e_3) are incident to the respective vertices p_1, p_2, p_3 (resp., p_1, p_4, p_3). Right: The tree of non-corner edges.

Consider the situation just before the critical event takes place, as depicted in Figure 4 (left). The Voronoi edge $e_{p_1 p_3}^\diamond$ (to simplify the notation, we write this edge as e_{13}^\diamond , and similarly for the other edges and

⁴In general position this does not occur, but it can happen at discrete time instances during the motion,

vertices in this analysis) is delimited by two Voronoi vertices, one, c_{123}^\diamond , being the center of a copy of Q_0 which touches p_1, p_2, p_3 at the respective edges e_1, e_2, e_3 , and the other, c_{143}^\diamond , being the center of a copy of Q_0 which touches p_1, p_4, p_3 at the respective edges e_1, e_4, e_3 . Consider the two other Voronoi edges e_{12}^\diamond and e_{23}^\diamond adjacent to c_{123}^\diamond , and the two Voronoi edges e_{14}^\diamond and e_{43}^\diamond adjacent to c_{143}^\diamond . Among them, consider only those which are non-corner edges; assume for simplicity that they all are. For specificity, consider the edge e_{12}^\diamond . As we move the center of Q_0 along that edge away from c_{123}^\diamond , Q_0 loses the contact with p_3 ; it shrinks on the side of $p_1 p_2$ which contains p_3 (and p_4 , already away from Q_0), and expands on the other side. Since this is a non-corner edge, its other endpoint is a placement where the (artificial) edge e_{12} of Q_0 between e_1 and e_2 touches another point p_5 . Now, however, since e_{12} is adjacent to both edges e_1, e_2 , the new Voronoi edges e_{15}^\diamond and e_{25}^\diamond are both corner edges.

Repeating this analysis to each of the other three Voronoi edges adjacent to e_{13}^\diamond , we get a tree of non-corner Voronoi edges, consisting of at most five edges, so that all the other Voronoi edges adjacent to its edges are corner edges. As long as no discrete change occurs at any of the surrounding corner edges, the tree can undergo only $O(1)$ discrete changes, because all its edges are defined by a total of $O(1)$ points of P . When a corner edge undergoes a discrete change, this can affect only $O(1)$ adjacent non-corner trees of the above kind. Hence, the number of changes in non-corner edges is proportional to the number of changes in corner edges, which, by Lemma 3.1 (applied to Q_0) is $O(n\lambda_r(n))$. Multiplying by the $O(k^4)$ choices of quadruples of edges of Q , we thus obtain:

Theorem 3.2. *Let P be a set of n moving points in \mathbb{R}^2 under algebraic motion of bounded degree, and let Q be a convex k -gon. The number of topological changes in $\text{VD}^\diamond(P)$ with respect to Q is $O(k^4 n \lambda_r(n))$, where r is a constant that depends on the degree of motion of P .*

Kinetic maintenance of $\text{VD}^\diamond(P)$ and $\text{DT}^\diamond(P)$. As already mentioned, it is a fairly trivial task to maintain $\text{DT}^\diamond(P)$ and $\text{VD}^\diamond(P)$ kinetically, as the points of P move. All we need to do is to assert the correctness of the present triangulation by a collection of local certificates, one for each edge of the diagram, where the certificate of an edge asserts that the two homothetic placements Q^-, Q^+ of Q that circumscribe the two respective adjacent Q -Delaunay triangles $\Delta pqr^-, \Delta pqr^+$, are such that Q^- does not contain r^+ and Q^+ does not contain r^- . The failure time of this certificate is the first time (if one exists) at which $p, q, r^-,$ and r^+ become Q -cocircular—they all lie on the boundary of a common homothetic copy of Q . Such an event corresponds to a flip event in $\text{DT}^\diamond(P)$. If pq is an edge of the periphery of $\text{DT}^\diamond(P)$, so that Δpqr^+ exists but Δpqr^- does not, then Q^- is a limiting wedge bounded by rays supporting two *consecutive* edges of (a copy of) Q , one passing through p and one through q (see Figure 5). The failure time of the corresponding certificate is the first time (if any) at which r^+ also lies on the boundary of that wedge.

We maintain the breakpoints using “sub-certificates”, each of which asserts that Q^- , say, touches each of p, q, r^- at respective specific edges (and similarly for Q^+). The failure time of this sub-certificate is the first failure time when one of p, q or r^- touches Q^- at a vertex. In this case we have a corner event—two of the adjacent Voronoi edges terminate at a corner placement. Note that the failure time of each sub-certificate can be computed in $O(1)$ time. Moreover, for a fixed collection of valid sub-certificates, the failure time of an initial certificate (asserting non-cocircularity) can also be computed in $O(1)$ time (provided that it fails before the failures of the corresponding sub-certificates), because we know the four edges of Q^- involved in the contacts.

We therefore maintain an event queue that stores and updates all the active failure times (there are only $O(n)$ of them at any given time—the bound is independent of k , because they correspond to actual DT edges. When a sub-certificate fails we do not change $\text{DT}^\diamond(P)$, but only update the corresponding Voronoi edge, by adding or removing a segment and a breakpoint, and by replacing the sub-certificate by a new one;

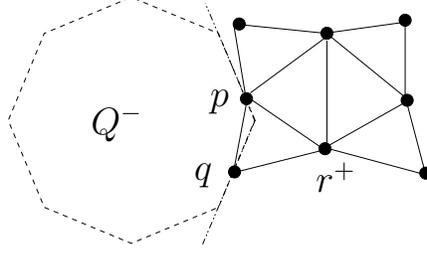


Figure 5: If r^- does not exist then Q^- is a limiting wedge bounded by rays supporting two consecutive edges of (a copy of) Q .

we also update the cocircularity certificate associated with the edge, because one of the contact edges has changed. When a cocircularity certificate fails we update $DT^\diamond(P)$ and construct $O(1)$ new sub-certificates and certificates. Altogether, each update of the diagram takes $O(\log n)$ time. We thus have

Theorem 3.3. *Let P be a set of n moving points in \mathbb{R}^2 under algebraic motion of bounded degree, and let Q be a convex k -gon. $DT^\diamond(P)$ and $VD^\diamond(P)$ can be maintained using $O(n)$ storage and $O(\log n)$ update time, so that $O(k^4 n \lambda_r(n))$ events are processed, where r is a constant that depends on the degree of motion of P .*

3.2 Stable Delaunay edges in $DT^\diamond(P)$

We now restrict Q to be a regular k -gon. Let v_0, \dots, v_{k-1} be the vertices of Q arranged in a clockwise direction, with v_0 the leftmost. We call a homothetic copy of Q whose vertex v_j touches a point p , a v_j -placement of Q at p . Let u_j be the direction of the vector that connects v_j with the center of Q , for each $0 \leq j < k$ (as in Section 2). See Figure 6 (left).

We follow the machinery in the proof of Lemma 3.1. That is, for any pair $p, q \in P$ let $\varphi_j^\diamond[p, q]$ denote the distance from p to the point $u_j[p] \cap b_{pq}^\diamond$; we put $\varphi_j^\diamond[p, q] = \infty$ if $u_j[p]$ does not intersect b_{pq}^\diamond . If $\varphi_j^\diamond[p, q] < \infty$ then the point $b_{pq}^\diamond \cap u_j[p]$ is the center of the v_j -placement Q' of Q at p that also touches q , and it is easy to see that there is a unique such point. The value $\varphi_j^\diamond[q, p]$ is equal to the circumradius of Q' . See Figure 6 (middle).

The neighbor $N_j^\diamond[p]$ of p in direction u_j is defined to be the point $q \in P \setminus \{p\}$ that minimizes $\varphi_j^\diamond[p, q]$. Clearly, for any $p, q \in P$ we have $N_j^\diamond[p] = q$ if and only if there is an empty v_j -placement Q' of Q at p so that q touches one of its edges.

Remark: Note that, in the Euclidean case, we have $\varphi_j[p, q] < \infty$ if and only if the angle between \overline{pq} and $u_j[p]$ is at most $\pi/2$. In contrast, $\varphi_j^\diamond[p, q] < \infty$ if and only if the angle between \overline{pq} and $u_j[p]$ is at most $\pi/2 - \pi/k = \pi/2 - \alpha/2$. Moreover, we have $\varphi_j[p, q] \leq \varphi_j^\diamond[p, q]$. Therefore, $\varphi_j^\diamond[p, q] < \infty$ always implies $\varphi_j[p, q] < \infty$, but not vice versa; see Figure 6 (right). Note also that in both the Euclidean and the polygonal cases, the respective quantities $N_j[p]$ and $N_j^\diamond[p]$ may be undefined.

Lemma 3.4. *Let $p, q \in P$ be a pair of points such that $N_j(p) = q$ for $h \geq 3$ consecutive indices, say $0 \leq j \leq h-1$. Then for each of these indices, except possibly for the first and the last one, we also have $N_j^\diamond[p] = q$.*

Proof. Let w_1 (resp., w_2) be the point at which the ray $u_0[p]$ (resp., $u_{h-1}[p]$) hits the edge e_{pq} in $VD(P)$. (By assumption, both points exist.) Let D_1 and D_2 be the disks centered at w_1 and w_2 , respectively, and

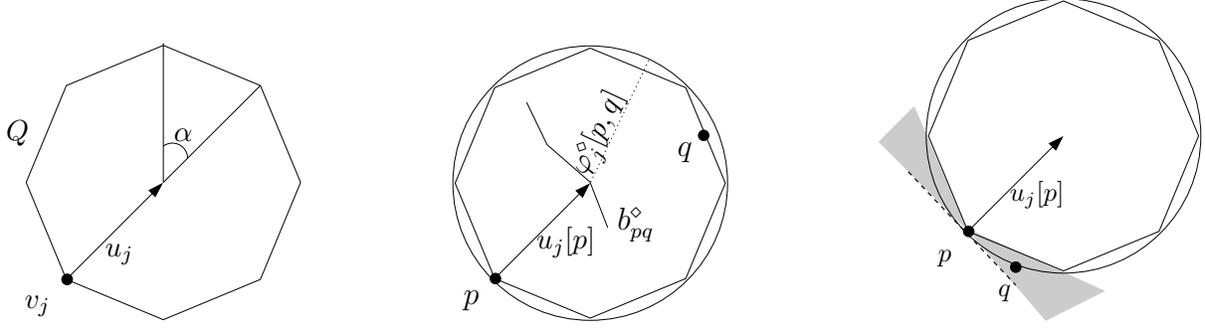


Figure 6: Left: u_j is the direction of the vector connecting vertex v_j to the center of Q . Middle: The function $\varphi_j^\circ[p, q]$ is equal to the radius of the circle that circumscribes the v_j -placement of Q at p that also touches q . Right: The case when $\varphi_j^\circ[p, q] = \infty$ while $\varphi_j[p, q] < \infty$. In this case q must lie in one of the shaded wedges.

touching p and q . By definition, neither of these disks contains a point of P in its interior. The angle between the tangents to D_1 and D_2 at p or at q (these angles are equal) is $\beta = (h - 1)\alpha$; see Figure 7 (left).

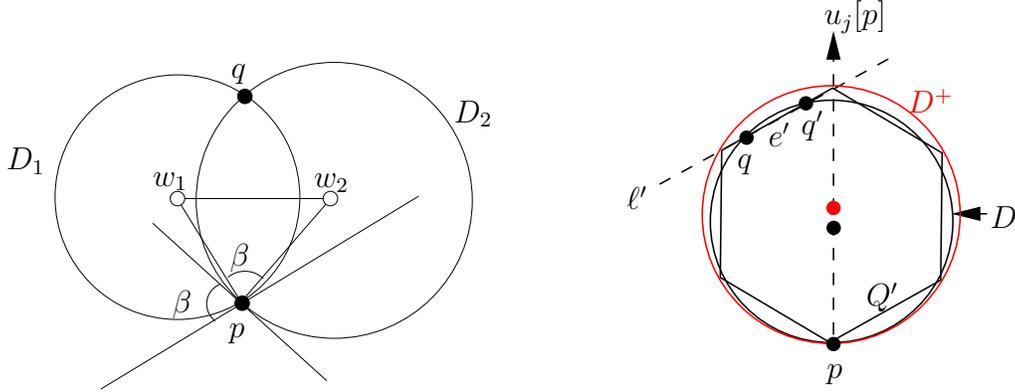


Figure 7: Left: The angle between the tangents to D_1 and D_2 at p (or at q) is equal to $\angle w_1 p w_2 = \beta = (h - 1)\alpha$. Right: The line ℓ' crosses D in a chord qq' which is fully contained in e' .

Fix an arbitrary index $1 \leq j \leq h - 2$, so $u_j[p]$ intersects e_{pq} and forms an angle of at least α with each of pw_1, pw_2 . Let Q' be the v_j -placement of Q at p that touches q . To see that such a placement exists, we note that, by the preceding remark, it suffices to show that the angle between \overline{pq} and $u_j[p]$ is at most $\pi/2 - \alpha/2$; that is, to rule out the case where q lies in one of the shaded wedges in Figure 6 (right). This case is indeed impossible, because then one of $u_{j-1}[p], u_{j+1}[p]$ would form an angle greater than $\pi/2$ with \overline{pq} , contradicting the assumption that both of these rays intersect the (Euclidean) b_{pq} .

We claim that $Q' \subset D_1 \cup D_2$. Establishing this property for every $1 \leq j \leq h - 2$ will complete the proof of the lemma. Let e' be the edge of Q' passing through q . See Figure 7 (right). Let D be the disk whose center lies on $u_j[p]$ and which passes through p and q , and let D^+ be the circumscribing disk of Q' . Since $q \in \partial D$ and is interior to D^+ , and since D and D^+ are centered on the same ray $u_j[q]$ and pass through p , it follows that $D \subset D^+$. The line ℓ' containing e' crosses D in a chord qq' that is fully contained in e' . The angle between the tangent to D at q and the chord qq' is equal to the angle at which p sees qq' . This is

smaller than the angle at which p sees e' , which in turn is equal to $\alpha/2$.

Arguing as in the analysis of D_1 and D_2 , the tangent to D at q forms an angle of at least α with each of the tangents to D_1, D_2 at q , and hence e' forms an angle of at least $\alpha/2$ with each of these tangents; see Figure 8 (left). The line ℓ' marks two chords q_1q, qq_2 within the respective disks D_1, D_2 . We claim that e' is fully contained in their union q_1q_2 . Indeed, the angle q_1pq is equal to the angle between ℓ' and the tangent to D_1 at q , so $\angle q_1pq \geq \alpha/2$. On the other hand, the angle at which p sees e' is $\alpha/2$, which is smaller. This, and the symmetric argument involving D_2 , are easily seen to imply the claim.

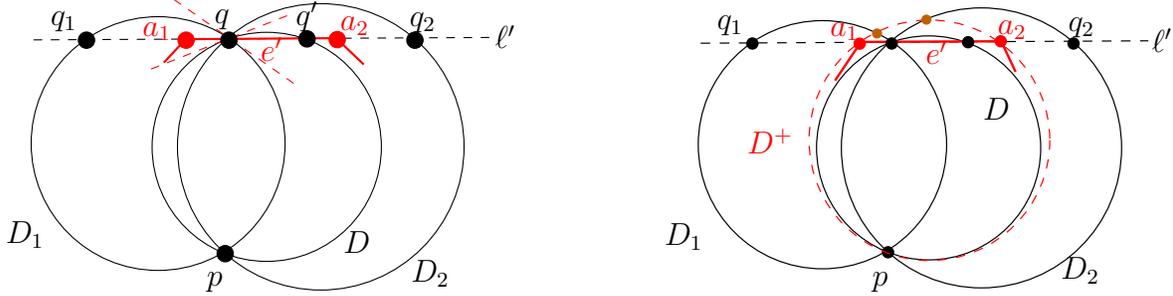


Figure 8: Left: The line ℓ' forms an angle of at least $\alpha/2$ with each of the tangents to D_1, D_2 at q . Right: The edge $e' = a_1a_2$ of Q' is fully contained in $D_1 \cup D_2$.

Now consider the circumscribing disk D^+ of Q' . Denote the endpoints of e' as a_1 and a_2 , where a_1 lies in q_1q and a_2 lies in qq_2 . Since the ray $\overline{pa_1}$ hits ∂D^+ before hitting D_1 , and the ray $\overline{pa_2}$ hits these circles in the reverse order, it follows that the second intersection of ∂D_1 and ∂D^+ (other than p) must lie on a ray from p which lies between the rays $\overline{pa_1}, \overline{pa_2}$ and thus crosses e' . See Figure 8 (right). Symmetrically, the second intersection point of ∂D_2 and ∂D^+ also lies on a ray which crosses e' .

It follows that the arc of ∂D^+ delimited by these intersections and containing p is fully contained in $D_1 \cup D_2$. Hence all the vertices of Q' (which lie on this arc) lie in $D_1 \cup D_2$. This, combined with the argument in the preceding paragraphs, is easily seen to imply that $Q' \subseteq D_1 \cup D_2$, so its interior does not contain points of P , which in turn implies that $N_j^\diamond[p] = q$. As noted, this completes the proof of the lemma. \square

Since Q -Voronoi edges are connected, Lemma 3.4 implies that e_{pq}^\diamond is “long”, in the sense that it contains at least $h - 2$ breakpoints that represent corner placements at p , interleaved (as promised in Section 3.1) with at least $h - 3$ corner placements at q . This property is easily seen to hold also under the weaker assumptions that: (i) for the first and the last indices $j = 0, h - 1$, the point $N_j[p]$ either is equal to q or is undefined, and (ii) for the rest of the indices j we have $N_j[p] = q$ and $\varphi_j^\diamond[p, q] < \infty$ (i.e., the v_j -placement of Q at p that touches q exists). In this relaxed setting, it is now possible that any of the two points w_1, w_2 lies at infinity, in which case the corresponding disk D_1 or D_2 degenerates into a halfplane. This stronger version of Lemma 3.4 is used in the proof of the converse Lemma 3.5, asserting that every edge e_{pq}^\diamond in $\text{VD}^\diamond(P)$ with sufficiently many breakpoints has a stable counterpart e_{pq} in $\text{VD}(P)$.

Lemma 3.5. *Let $p, q \in P$ be a pair of points such that $N_j^\diamond[p] = q$ for at least three consecutive indices $j \in \{0, \dots, k - 1\}$. Then for each of these indices, except possibly for the first and the last one, we have $N_j[p] = q$.*

Proof. Again, assume with no loss of generality that $N_j^\diamond[p] = q$ for $0 \leq j \leq h - 1$, with $h \geq 3$. Suppose to the contrary that, for some $1 \leq j \leq h - 2$, we have $N_j[p] \neq q$. Since $N_j^\diamond[p] = q$ by assumption, we

have $\varphi_j[p, q] \leq \varphi_j^\diamond[p, q] < \infty$, so there exists $r \in P$ for which $\varphi_j[p, r] < \varphi_j[p, q]$. Assume with no loss of generality that r lies to the left of the line from p to q . In this case $\varphi_{j-1}[p, r] < \varphi_{j-1}[p, q] < \infty$. Indeed, we have (i) $N_{j-1}^\diamond[p] = q$ by assumption, so $\varphi_{j-1}^\diamond[p, q] < \infty$, and (ii) $\varphi_{j-1}[p, q] \leq \varphi_{j-1}^\diamond[p, q]$. Moreover, because r lies to the left of the line from p to q , the orientation of b_{pr} lies counterclockwise to that of b_{pq} , implying that $\varphi_{j-1}[p, q] < \infty$. See Figure 9. Since $u_j[p]$ hits b_{pr} before hitting b_{pq} , any ray emanating from p counterclockwise to $u_j[p]$ must do the same, so we have $\varphi_{j-1}[p, r] < \varphi_{j-1}[p, q]$, as claimed. Similarly, we get that either $\varphi_{j-2}[p, r] < \varphi_{j-2}[p, q] < \infty$ or $\varphi_{j-2}[p, r] \leq \varphi_{j-2}[p, q] = \infty$ (where the latter can occur only for $j = 1$). Now applying (the extended version of) Lemma 3.4 to the point set $\{p, q, r\}$ and to the index set $\{j-2, j-1, j\}$, we get that $\varphi_{j-1}^\diamond[p, r] < \varphi_{j-1}^\diamond[p, q]$. But this contradicts the fact that $N_{j-1}^\diamond[p] = q$. \square

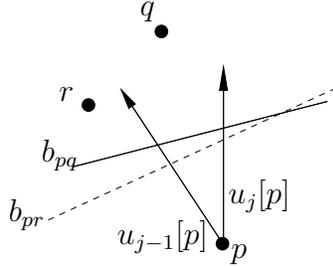


Figure 9: Proof of Lemma 3.5. If $N_j[p] \neq q$ because some r , lying to the left of the line from p to q , satisfies $\varphi_j[p, r] < \varphi_j[p, q]$. Since $\varphi_{j-1}[p, q] < \varphi_{j-1}^\diamond[p, q] < \infty$, we have $\varphi_{j-1}[p, r] < \varphi_{j-1}[p, q]$.

Maintaining an SDG using $\text{VD}^\diamond(P)$. Lemmas 3.4 and 3.5 together imply that an SDG can be maintained using the fairly straightforward kinetic algorithm for maintaining the whole $\text{VD}^\diamond(P)$, provided by Theorem 3.3. We use $\text{VD}^\diamond(P)$ to maintain the graph G on P , whose edges are all the pairs $(p, q) \in P \times P$ such that p and q define an edge e_{pq}^\diamond in $\text{VD}^\diamond(P)$ that contains at least seven breakpoints. As shown in Theorem 3.3, this can be done with $O(n)$ storage, $O(\log n)$ update time, and $O(k^4 n \lambda_r(n))$ updates (for an appropriate r). We claim that G is a $(6\alpha, \alpha)$ -SDG in the Euclidean norm.

Indeed, if two points $p, q \in P$ define a 6α -long edge e_{pq} in $\text{VD}(P)$ then this edge stabs at least six rays $u_j[p]$ emanating from p , and at least six rays $u_j[q]$ emanating from q . Thus, according to Lemma 3.4, $\text{VD}^\diamond(P)$ contains the edge e_{pq}^\diamond with at least four breakpoints corresponding to corner placements of Q at p that touch q , and at least four breakpoints corresponding to corner placements of Q at q that touch p . Therefore, e_{pq}^\diamond contains at least 8 breakpoints, so $(p, q) \in G$.

For the second part, if $p, q \in P$ define an edge e_{pq}^\diamond in $\text{VD}^\diamond(P)$ with at least 7 breakpoints then, by the interleaving property of breakpoints, we may assume, without loss of generality, that at least four of these breakpoints correspond to P -empty corner placements of Q at p that touch q . Thus, Lemma 3.5 implies that $\text{VD}(P)$ contains the edge e_{pq} , and that this edge is hit by at least two consecutive rays $u_j[p]$. But then, as observed in Lemma 2.2, the edge e_{pq} is α -long in $\text{VD}(P)$. We thus obtain the main result of this section.

Theorem 3.6. *Let P be a set of n moving points in \mathbb{R}^2 under algebraic motion of bounded degree, and let $\alpha \geq 0$ be a parameter. A $(6\alpha, \alpha)$ -stable Delaunay graph of P can be maintained by a KDS of linear size that processes $O(n \lambda_r(n) / \alpha^4)$ events, where r is a constant that depends on the degree of motion of P , and that updates the SDG at each event in $O(\log n)$ time.*

4 An Improved Data Structure

The data structure of Theorem 3.6 requires $O(n)$ storage but the best bound we have on the number of events it may encounter is $O^*(n^2/\alpha^4)$, which is much larger than the number of events encountered by the data structure of Theorem 2.3 (which, in terms of the dependence on α , is only $O^*(n^2/\alpha)$). In this section we present an alternative data structure that requires $O^*(n/\alpha^2)$ space and $O^*(n^2/\alpha^2)$ overall processing time. The structure processes each event in $O^*(1/\alpha)$ time and is also *local*, in the sense that each point is stored at only $O^*((1/\alpha)^2)$ places in the structure.

Notation. We use the directions u_i and the associated quantities $N_i[p]$ and $\varphi_i[p, q]$ defined in Section 2. We assume that k , the number of canonical directions, is even, and write, as in Section 2, $k = 2s$. We denote by C_i the cone (or wedge) with apex at the origin that is bounded by u_i and u_{i+1} . Note that C_i and $C_{i\pm s}$ are antipodal. As before, for a vector u , we denote by $u[x]$ the ray emanating from x in direction u . Similarly, for a cone C we denote by $C[x]$ the translation of C that places its apex at x . Let $0 \leq \beta \leq \pi/2$ be an angle. For a direction $u \in \mathbb{S}^1$ and for two points $p, q \in P$, we say that the edge $e_{pq} \in \text{VD}(P)$ is β -long around the ray $u[q]$ if p is the Voronoi neighbor of q in all directions in the range $[u - \beta, u + \beta]$, i.e., for all $v \in [u - \beta, u + \beta]$, the ray $v[q]$ intersects e_{pq} . The β -cone around $u[q]$ is the cone whose apex is q and each of its bounding rays makes an angle of β with $u[q]$.

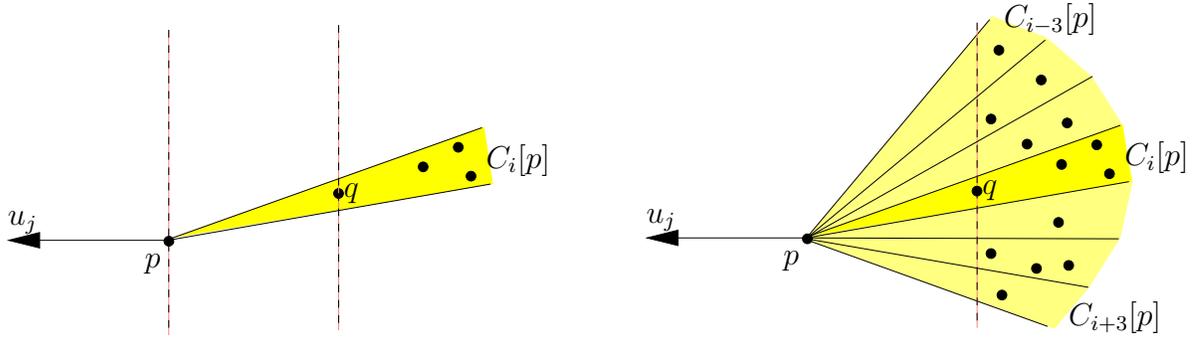


Figure 10: Left: q is j -extremal for p . Right: q is *strongly* j -extremal for p .

Definition (j -extremal points). (i) Let $p, q \in P$, let i be the index such that $q \in C_i[p]$, and let u_j be a direction such that $\langle u_j, x \rangle \leq 0$ for all $x \in C_i$. We say that q is j -extremal for p if $q = \arg \max\{\langle p', u_j \rangle \mid p' \in C_i[p] \cap P \setminus \{p\}\}$. That is, q is the nearest point to p in this cone, in the $(-u_j)$ -direction. Clearly, a point p has at most s j -extremal points, one for every admissible cone $C_i[p]$, for any fixed j . See Figure 10 (left).

(ii) For $0 \leq i < k$, let C'_i denote the extended cone that is the union of the seven consecutive cones C_{i-3}, \dots, C_{i+3} . Let $p, q \in P$, let i be the index such that $q \in C_i[p]$, and let u_j be a direction such that $\langle u_j, x \rangle \leq 0$ for all $x \in C'_i$ (such u_j 's exist if α is smaller than some appropriate constant). We say that the point $q \in P$ is *strongly* j -extremal for p if $q = \arg \max\{\langle p', u_j \rangle \mid p' \in C'_i[p] \cap P \setminus \{p\}\}$.

(iii) We say that a pair $(p, q) \in P \times P$ is (strongly) (j, ℓ) -extremal, for some $0 \leq j, \ell \leq k - 1$, if p is (strongly) ℓ -extremal for q and q is (strongly) j -extremal for p .

The stable Delaunay graph. We kinetically maintain a $(10\alpha, \alpha)$ -stable Delaunay graph, whose precise definition is given below, using a data-structure which is based on a collection of 2-dimensional orthogonal range trees similar to the ones used in [1].

Fix $0 \leq i < s$, and choose a “sheared” coordinate frame in which the rays u_i and u_{i+1} form the x - and y -axes, respectively. That is, in this coordinate frame, $q \in C_i[p]$ if and only if q lies in the upper-right quadrant anchored at p .

We define a 2-dimensional range tree \mathcal{T}_i consisting of a *primary* balanced binary search tree with the points of P stored at its leaves ordered by their x -coordinates, and of secondary trees, introduced below. Each internal node v of the primary tree of \mathcal{T}_i is associated with the *canonical subset* P_v of all points that are stored at the leaves of the subtree rooted at v . A point $p \in P_v$ is said to be *red* (resp., *blue*) in P_v if it is stored at the subtree rooted at the left (resp., right) child of v in \mathcal{T}_i . For each primary node v we maintain a secondary balanced binary search tree \mathcal{T}_i^v , whose leaves store the points of P_v ordered by their y -coordinates. We refer to a node w in a secondary tree \mathcal{T}_i^v as a *secondary node* w of \mathcal{T}_i .

Each node w of a secondary tree \mathcal{T}_i^v is associated with a canonical subset $P_w \subseteq P_v$ of points stored at the leaves of the subtree of \mathcal{T}_i^v rooted at w . We also associate with w the sets $R_w \subset P_w$ and $B_w \subset P_w$ of points residing in the *left* (resp., *right*) *subtree* of w and are red (resp., blue) in P_w . It is easy to verify that the sum of the sizes of the sets R_w and B_w over all secondary nodes of \mathcal{T}_i is $O(n \log^2 n)$.

For each secondary node $w \in \mathcal{T}_i$ and each $0 \leq j < k$ we maintain the points

$$\xi_{i,j}^R(w) = \arg \max_{p \in R_w} \langle p, u_j \rangle, \quad \xi_{i,j}^B(w) = \arg \max_{p \in B_w} \langle p, u_j \rangle,$$

provided that both R_w, B_w are not empty. See Figure 12. It is straightforward to show that if (p, q) is a (j, ℓ) -extremal pair, so that $q \in C_i[p]$, then there is a secondary node $w \in \mathcal{T}_i$ for which $q = \xi_{i,j}^B(w)$ and $p = \xi_{i,\ell}^R(w)$.

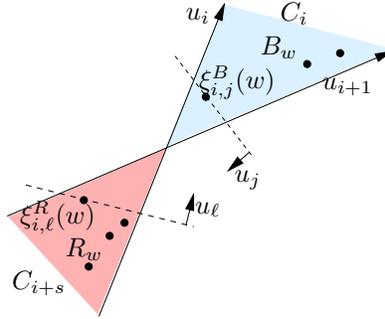


Figure 12: The points $\xi_{i,\ell}^R(w), \xi_{i,j}^B(w)$ for a secondary node w of \mathcal{T}_i .

For each $p \in P$ we construct a set $\mathcal{N}[p]$ containing all points $q \in P$ for which (p, q) is a (j, ℓ) -extremal pair, for some pair of indices $0 \leq j, \ell < k$. Specifically, for each $0 \leq i < s$, and each secondary node $w \in \mathcal{T}_i$ such that $p = \xi_{i,\ell}^R(w)$ for some $0 \leq \ell < k$, we include in $\mathcal{N}[p]$ all the points q such that $q = \xi_{i,j}^B(w)$ for some $0 \leq j < k$. Similarly, for each $0 \leq i < s$, and each secondary node $w \in \mathcal{T}_i$ such that $p = \xi_{i,\ell}^B(w)$ for some $0 \leq \ell < k$ we include in $\mathcal{N}[p]$ all the points q such that $q = \xi_{i,j}^R(w)$ for some $0 \leq j < k$. It is easy to verify that, for each (i, ℓ) -extremal pair (p, q) , for some $0 \leq j, \ell < k$, q is placed in $\mathcal{N}[p]$ by the preceding process. The converse, however, does not always hold, so in general $\{p\} \times \mathcal{N}[p]$ is a superset of the pairs that we want.

For each $0 \leq i < s$, each point $p \in P$ belongs to $O(\log^2 n)$ sets R_w and B_w , so the size of $\mathcal{N}[p]$ is bounded by $O(s^2 \log^2 n)$. Indeed, p may be coupled with up to $k = 2s$ neighbors at each of the $O(s \log^2 n)$ nodes containing it.

For each point $p \in P$ and $0 \leq \ell < k$ we maintain all points in $\mathcal{N}[p]$ in a kinetic and dynamic tournament $\mathcal{D}_\ell[p]$ whose winner q minimizes the directional distance $\varphi_\ell[p, q]$, as given in (1). That is, the winner in $\mathcal{D}_\ell[p]$ is $N_\ell[p]$ in the Voronoi diagram of $\{p\} \cup \mathcal{N}[p]$.

We are now ready to define the stable Delaunay graph G that we maintain. For each pair of points $p, q \in P$ we add the edge (p, q) to G if the following hold.

(G1) There is an index $0 \leq \ell < k$ such that q wins the 8 consecutive tournaments $\mathcal{D}_\ell[p], \dots, \mathcal{D}_{\ell+7}[p]$.

(G2) The point p is strongly $(\ell + 3)$ -extremal and strongly $(\ell + 4)$ -extremal for q .

The $(10\alpha, \alpha)$ -stability of G is implied by a combination of Theorems 4.3 and 4.4.

Theorem 4.3. *For every 10α -long edge $e_{pq} \in \text{VD}(P)$, the graph G contains the edge (p, q) .*

Proof. By Corollary 4.2 (i), there are j and ℓ such that (p, q) is a (j, ℓ) -extremal pair. By the preceding discussion this implies that q is in $\mathcal{N}[p]$. Now since e_{pq} is 10α -long there is an ℓ' such that $N_{\ell'}[p], \dots, N_{\ell'+7}[p] = q$ in $\text{VD}(P)$, and therefore also in the Voronoi diagram of $\{p\} \cup \mathcal{N}[p]$. So it follows that q indeed wins the tournaments $\mathcal{D}_{\ell'}[p], \dots, \mathcal{D}_{\ell'+7}[p]$.

By the proof of Corollary 4.2 (ii), p is strongly $(\ell' + 3)$ -extremal and strongly $(\ell' + 4)$ -extremal for q . \square

Theorem 4.4. *For every edge $(p, q) \in G$, the edge e_{pq} belongs to $\text{VD}(P)$ and is α -long there.*

Proof. Since $(p, q) \in G$ we know that q is in $\mathcal{N}[p]$ and wins the tournaments $\mathcal{D}_\ell[p], \mathcal{D}_{\ell+1}[p], \dots, \mathcal{D}_{\ell+7}[p]$, for some $0 \leq \ell < k$ and that the point p is strongly $(\ell + 3)$ -extremal and $(\ell + 4)$ -extremal for q . We prove that the rays $u_{\ell+3}[p]$ and $u_{\ell+4}[p]$ stab e_{pq} , from which the theorem follows.

Assume then that one of the rays $u_{\ell+3}[p], u_{\ell+4}[p]$ does not stab e_{pq} ; suppose it is the ray $u_{\ell+4}[p]$. (This includes the case when e_{pq} is not present at all in $\text{VD}(P)$.) By definition, this means that $r = N_{\ell+4}[p] \neq q$. We use Lemma 4.5, given shortly below, to show that q cannot win in at least one of the tournaments among $\mathcal{D}_\ell[p], \dots, \mathcal{D}_{\ell+7}[p]$ and thereby get a contradiction.

According to Lemma 4.5, there exists a point r such that $\varphi_{\ell+4}[p, r] < \varphi_{\ell+4}[p, q]$ and p is $(\ell + 4)$ -extremal for r . Let $x = u_{\ell+4}[p] \cap b_{pr}$ and let D be the circle which is centered at x , and passes through r and p ; see Figure 13.

We consider the case where r is to the right of the line from p to q ; the other case is treated symmetrically. In this case the intersection of b_{pr} and b_{pq} is to the left of the directed line from p to x . Let $0 \leq i \leq k - 1$ be the index for which $r \in C_i[p]$. If $i \leq s - 1$ then there is a secondary node w in the tree \mathcal{T}_i for which $p \in R_w$ and $r \in B_w$, and since p is $(\ell + 4)$ -extremal for r , $\xi_{i, \ell+4}^R(w)$ is equal to p . If $i > s$ then, symmetrically, we have a node $w \in \mathcal{T}_{i-s}$ such that $r \in R_w$ and $p \in B_w$ and $\xi_{i, \ell+4}^B(w)$ is equal to p . We assume that $i \leq s - 1$ in the sequel; the other case is treated in a fully symmetric manner.

Let $v[r]$ be the ray from r through x , for an appropriate direction $v \in \mathcal{S}^1$, and let u_j be the direction which lies counterclockwise to v and forms with it an angle of at least α and at most 2α . Put $r' = \xi_{i, j}^B(w)$, implying that $r' \in C_i[p]$ and $\langle r', u_j \rangle \geq \langle r, u_j \rangle$. In particular, r' belongs to $\mathcal{N}[p]$. If r' is inside D (and in particular if $r' = r$) then q cannot win the tournament $\mathcal{D}_{\ell+4}[p]$ which is the contradiction we are after. So we may assume that r' is outside D .

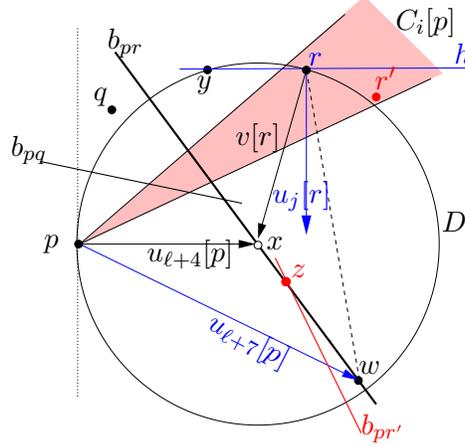


Figure 13: Proof of Theorem 4.4: the case when r is to the right of the line from p to q . The line h orthogonal to u_j through r intersects the circle D at a point y outside $C_i[p]$, which implies that r' is to the right of the line from p to r . Assuming $r \neq r'$, the point $z = b_{pr} \cap b_{pr'}$ is inside the cone bounded by $u_{\ell+4}[p]$ and $u_{\ell+7}[p]$. Hence, $u_{\ell+7}[p]$ hits $b_{pr'}$ before b_{pr} .

Let h be the line through r orthogonal to u_j . Clearly, h intersects D at two points, r and another point y (lying counterclockwise to r along ∂D , by the choice of u_j). Since $\angle rpy = \frac{1}{2}\angle rxy$, and $\angle rxy$ equal to twice the angle between v and u_j , $\angle rpy$ is at least α , so y is outside $C_i[p]$. By assumption, r' lies in the halfplane bounded by h and containing p . Since we assume that r' is not in D it must be to the right of the line from x to r . It follows that $b_{pr'}$ intersects b_{pr} at some point z to the right of the line from p to x ; see Figure 13.

We claim that z is inside the cone with apex p bounded by the rays $u_{\ell+4}[p]$ and $u_{\ell+7}[p]$. Indeed, suppose to the contrary that the claim is false. It follows that in the diagram $\text{VD}(\{r, r', p\})$ the edge e_{pr} is α -long around $u_j[r]$. Indeed, denote the intersection point of $u_{\ell+7}[p]$ and b_{pr} as w (see Figure 13). Then $\angle xrw = \angle xpw = 3\alpha$. Since the angle between $v[r]$ and $u_j[r]$ is between α and 2α , the claim follows. Now, according to Lemma 4.1, $\langle r, u_j \rangle \leq \langle r', u_j \rangle$, which contradicts the choice of r' . It follows that z is in the cone bounded by $u_{\ell+4}[p]$ and $u_{\ell+7}[p]$ and thus $u_{\ell+7}[p]$ hits $b_{pr'}$ before b_{pr} , and therefore also before b_{pq} . Hence, q cannot win $\mathcal{D}_{\ell+7}[p]$, and we get the final contradiction which completes the proof of the theorem. \square

Remark: We have not made any serious attempt to reduce the constants c appearing in the definitions of various $(c\alpha, \alpha)$ -SDGs that we maintain. We suspect, though, that they can be significantly reduced.

To complete the proof of Theorem 4.4, we provide the missing lemma.

Lemma 4.5. *Let $p, q \in P$ be a pair of points and $0 \leq \ell \leq k - 1$ an index, such that the point p is strongly ℓ -extremal for q but $N_\ell[p] \neq q$. Then there exists a point r such that $\varphi_\ell[p, r] < \varphi_\ell[p, q]$ and p is ℓ -extremal for r .*

Proof. Let $0 \leq i \leq k - 1$ be the index for which $q \in C_i[p]$ and let h be the line through p , orthogonal to u_ℓ . Assume without loss of generality that h is vertical and the ray $u_\ell[p]$ extends to the right of h .

Let a be the point at which $u_\ell[p]$ intersects the bisector b_{pq} , and let D be the disk centered at a whose boundary contains both p and q . Since $N_\ell[p] \neq q$, the interior of D must contain some other point $r \in P$; see Figure 14.

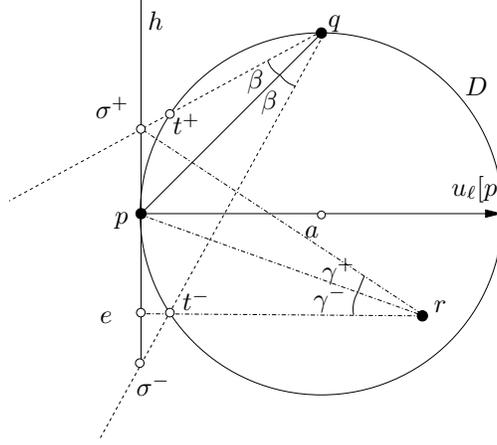


Figure 14: The proof of Lemma 4.5: The point p is strongly ℓ -extremal for q and ℓ -extremal for r .

Let $C[q]$ be the cone emanating from q such that each of its bounding rays makes an angle of $\beta = 3\alpha$ with the ray from q through p ; in particular $C[q]$ contains p . Let σ^+ (resp., σ^-) denote the upper (resp., lower) endpoint of the intersection of $C[q]$ and h . Since p is strongly ℓ -extremal for q , the interior of the triangle $\Delta\sigma^+q\sigma^-$ does not contain any points of P . Hence, r must be outside the triangle $\Delta\sigma^+q\sigma^-$. So either r is above $q\sigma^+$ (and inside D) or below $q\sigma^-$ (and inside D).

Assume, without loss of generality, that r is below $q\sigma^-$, as shown in Figure 14. (The case where r is above $q\sigma^+$ is fully symmetric.) Let t^+ and t^- denote the intersection points $q\sigma^+ \cap \partial D$ and $q\sigma^- \cap \partial D$, respectively. Let e be the point at which the ray from r through t^- intersects h . Then the intersection of the triangle $\Delta\sigma^+re$ and $\Delta\sigma^+q\sigma^-$ is empty. Among all the points of P in D we choose r so that its x -coordinate is the smallest. For this choice of r we also have that $\Delta\sigma^+re \setminus \Delta\sigma^+q\sigma^-$ is empty (since it is contained in D and lies to the left of r). In other words, $\Delta\sigma^+re$ is empty.

Let γ^+ (resp., γ^-) denote the angle $\angle pr\sigma^+$ (resp., $\angle prt^-$). It remains to show that $\gamma^+ \geq \frac{1}{3}\beta$ and $\gamma^- \geq \frac{1}{3}\beta$. This will imply that the cone $C_\nu[r]$ that contains p is fully contained in the cone bounded by the rays from r through σ^+ and t^- , so p is extreme in the u_ℓ -direction within $C_\nu[r]$, which is what the lemma asserts. Since r is inside D , it is clear that $\gamma^- \geq \angle pqt^- = \beta$. The angle γ^+ however may be smaller than β , but, as we next show, $\tan \gamma^+ \geq \frac{1}{3} \tan \beta$. Indeed, fix an angle θ and let $\Gamma(\theta)$ denote the circular arc which is the locus of all points z that are to the right of h and the angle $\angle pzs^+$ is θ . The endpoints of $\Gamma(\theta)$ are p and σ^+ , and its center a^* is on the (horizontal) bisector of ps^+ ; see Figure 15 (left).

Notice that $\Gamma(\theta)$ intersects ∂D at two points, one of which is p , which are symmetric with respect to the line through a and a^* . As θ decreases a^* moves to the right, and the intersection of $\Gamma(\theta)$ with ∂D rotates clockwise around ∂D . Consider the smallest θ such that $\Gamma(\theta)$ intersects D on or below qt^- . It follows that this intersection is at t^- . See Figure 15 (right).

This shows that for fixed p and q , the position of r in D below the line qt^- which minimizes γ^+ is at t^- . To complete the analysis, we look for the position of q that minimizes γ^+ when r is at t^- . Note that, as q moves along ∂D , the points t^+ and t^- do not change. As shown in Figure 16 (left), γ^+ decreases when q tends counterclockwise to t^+ . When q is at t^+ , $q\sigma^+$ is tangent to D . A simple calculation, illustrated in Figure 16 (right), shows that $\tan \gamma^+ = \frac{1}{3} \tan \beta$. By the inequality $\tan(3x) > 3 \tan x$, for x sufficiently small, it follows that $\gamma^+ > \frac{1}{3}\beta$, implying, as noted above, that the point p is ℓ -extremal for r . This completes the proof of the lemma. \square

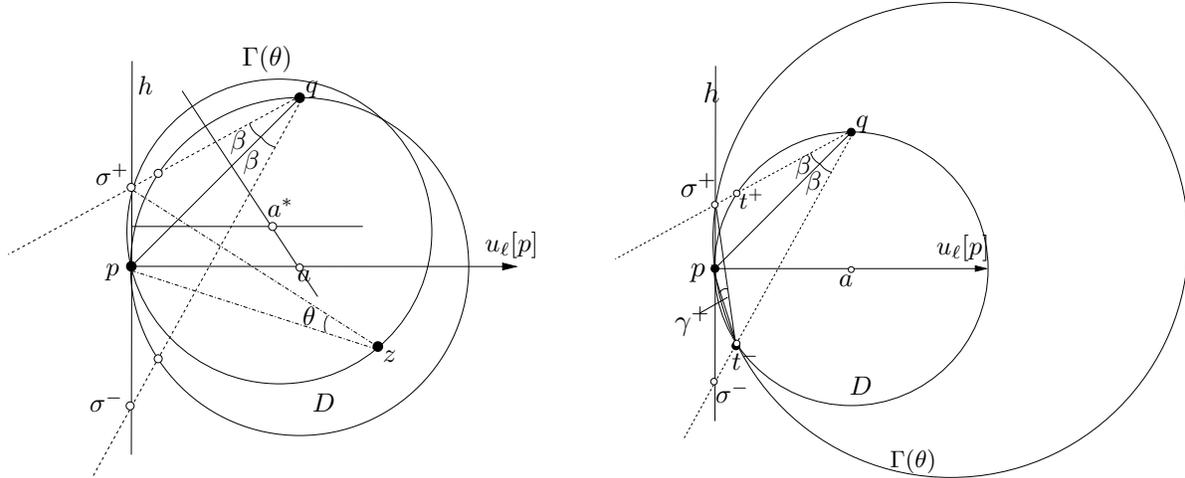


Figure 15: Left: The circular arc $\Gamma(\theta)$ is the locus of all points which are to the right of $p\sigma^+$ and see it at angle θ . Right: To minimize θ we increase the radius of $\Gamma(\theta)$ until one of its intersection points with D coincides with t^- .

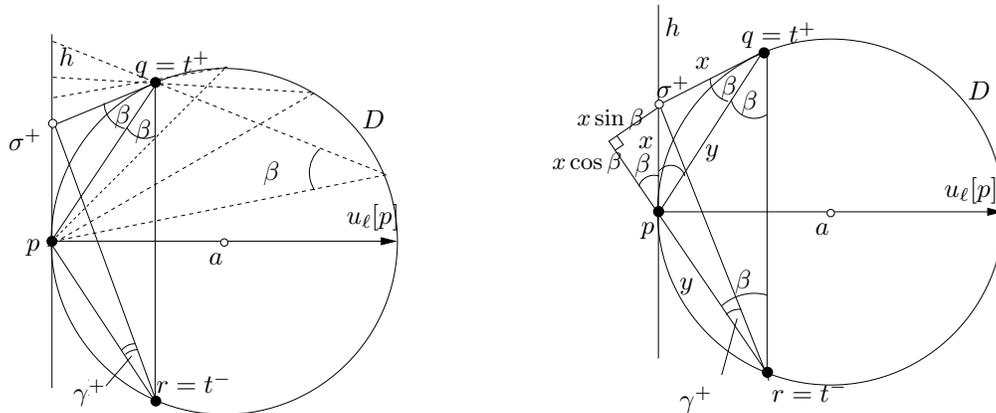


Figure 16: Left: γ^+ is minimized as q tends counterclockwise to t^+ . Right: Proving that $\tan \gamma^+ = \frac{1}{3} \tan \beta$ when $q = t^+$ and $r = t^-$. The triangles $\triangle q\sigma^+p$ and $\triangle pqr$ are isosceles and similar, and $y = 2x \cos \beta$. Thus $\tan \gamma^+ = \frac{x \sin \beta}{x \cos \beta + y} = \frac{1}{3} \tan \beta$.

In Section 4.1 we describe a naive algorithm for kinetic maintenance of G , which encounters a total of $O^*(k^4 n^2)$ events in the tournaments $\mathcal{D}_\ell[p]$. In Section 4.2 we consider a slightly more economical definition of the tournaments $\mathcal{D}_\ell[p]$, yielding a solution which processes only $O^*(k^2 n^2)$ events in $O^*(k^2 n^2)$ overall time.

4.1 Naive maintenance of G

As the points of P move, we need to update the SDG G , which, as we recall, contains those edges (p, q) such that q wins 8 consecutive tournaments $\mathcal{D}_\ell[p], \dots, \mathcal{D}_{\ell+7}[p]$ of p , and p is strongly $(\ell + 3)$ -extremal and $(\ell + 4)$ -extremal for q . We thus need to detect and process instances at which one of these conditions

changes. There are several events at which such a change can occur:

- (a) A change in the sets of neighbors $\mathcal{N}[p]$, for $p \neq P$.
- (b) A change in the status of being strongly ℓ -extremal for some pair (p, q) .
- (c) A change in the winner of some tournament $\mathcal{D}_\ell[p]$ (at which two existing members of $\mathcal{N}[p]$ attain the same minimum distance in the direction u_ℓ).

Note that each of the events (a)–(b) can arise only during a swap of two points in one of the s directions u_0, \dots, u_{s-1} or in one of the directions orthogonal to these vectors.

For each $0 \leq i \leq s-1$ we maintain two lists. The first list, L_i , stores the points of P ordered by their projections on a line in the u_i -direction, and the second list, K_i , stores the points ordered by their projections on a line orthogonal to the u_i -direction. We note that, as long as the order in each of the $2s$ lists K_i, L_i remains unchanged, the discrete structure of the range trees \mathcal{T}_i , and the auxiliary items $\xi_{i,\ell}^R(w), \xi_{i,j}^B(w)$, does not change either. More precisely, the structure of \mathcal{T}_i changes only when two consecutive elements in K_i or in K_{i+1} swap their order in the respective list; whereas the auxiliary items $\xi_{i,j}^R(w), \xi_{i,j}^B(w)$, stored at secondary nodes of \mathcal{T}_i , may also change when two consecutive points swap their order in the list L_j . There are $O(sn^2) = O(n^2)$ discrete events where consecutive points in K_i or L_i swap. We call these events K_i -swaps and L_i -swaps, respectively. Each such event happens when the line through a pair of points becomes orthogonal or parallel to u_i . We can maintain each list in linear space for a total of $O(sn)$ space for all lists. Processing a swap takes $O(\log n)$ time to replace a constant number of elements in the event queue (and more time to update the various structures, as discussed next).

The range trees \mathcal{T}_i . As just noted, the structure of \mathcal{T}_i changes either at a K_i -swap or at a K_{i+1} -swap. As described in [1, Section 4], we can update \mathcal{T}_i when such a swap occurs, including the various auxiliary data that it stores, in $O(s \log^2 n)$ time. (The factor s is due to the fact that we maintain $O(s)$ extreme points $\xi_{i,\ell}^B(w)$ and $\xi_{i,j}^R(w)$ in each secondary node w of \mathcal{T}_i , whereas in [1] only two points are maintained.)

In a similar manner, an L_j -swap of two points p, q may affect one of the items $\xi_{i,j}^B(w)$ and $\xi_{i,j}^R(w)$ stored at any secondary node w of any \mathcal{T}_i , for $0 \leq i \leq s-1$, such that both p, q belong to R_w or to B_w . Each \mathcal{T}_i has only $O(\log^2 n)$ such nodes, and the data structure of [1] allows us to update \mathcal{T}_i , when an L_j -swap occurs in $O(\log^2 n)$ time. Summing up over all $0 \leq i \leq s-1$, we get that the total update time of the range trees after an L_j -swap is $O(s \log^2 n)$. As follows from the analysis in [1, Section 4], the trees \mathcal{T}_i , for $0 \leq i \leq s-1$, require a total of $O(s^2 n \log n)$ storage (because of the $O(s)$ items $\xi_{i,\ell}^B(w), \xi_{i,j}^R(w)$ stored at each secondary node of each of the s trees).

The tournaments $\mathcal{D}_\ell[p]$. The kinetic tournament $\mathcal{D}_\ell[p]$, for $p \in P$ and $0 \leq \ell \leq k-1$ contains the points in the set $\mathcal{N}[p]$. Since $\mathcal{N}[p]$ varies both kinetically and dynamically and therefore the tournaments $\mathcal{D}_\ell[p]$ need to be maintained as kinetic and dynamic tournaments, in the manner reviewed in Section 2.

For $0 \leq i \leq s-1$, we define Π_i to be the set of pairs of points (p, q) , such that there exists a secondary node w in \mathcal{T}_i , and indices $0 \leq j, \ell \leq k-1$, for which $p = \xi_{i,\ell}^R(w)$ and $q = \xi_{i,j}^B(w)$. For a fixed i , a point p belongs to $O(s \log^2 n)$ pairs (p, q) in Π_i , for a total of $O(s^2 \log^2 n)$ pairs over all sets Π_i . It follows that the total size of all the sets Π_i is $O(s^2 n \log^2 n)$. Any secondary node of any tree \mathcal{T}_i , for $0 \leq i \leq s-1$, contributes at most $O(s^2)$ pairs to the respective set Π_i .

The set $\mathcal{N}[p]$ consists of all the points q such that there exists a set Π_i that contains the pair (p, q) or the pair (q, p) . So the total size of the sets $\mathcal{N}[p]$, over all points p , is $O(s^2 n \log^2 n)$. A set $\mathcal{N}[p]$ changes only when one of the sets Π_i changes, which can happen only as the result of a swap.

Specifically, when $\xi_{i,\ell}^R(w)$ changes for some $0 \leq i \leq s-1$ and $0 \leq \ell \leq k-1$, from a point p to a point p' , we make the following updates. (i) If $p \neq \xi_{i,\ell'}^R(w)$ for all $\ell' \neq \ell$ then for every $0 \leq j \leq k-1$ we delete the pair $(p, \xi_{i,j}^B(w))$ from Π_i . (ii) We add the pair $(p', \xi_{i,j}^B(w))$ to Π_i . We make analogous updates when one

of the values $\xi_{i,j}^B(w)$ changes. When a node w is created, deleted, or involved in a rotation, we update the pairs $(\xi_{i,\ell}^B(w), \xi_{i,j}^R(w))$ in Π_i for every ℓ and j . In such a case we say that node w is *changed*.

A change of $\xi_{i,\ell}^R(w)$ or $\xi_{i,j}^B(w)$ in an existing node w generates $O(s)$ changes in Π_i and thereby $O(s)$ changes to the sets $\mathcal{N}[p]$. Thus, it may generate $O(s^2)$ updates to the tournaments $\mathcal{D}_\ell[p]$. A change of a secondary node may generate $O(s^2)$ changes to the sets $\mathcal{N}[p]$ and thereby $O(s^3)$ updates to the tournaments $\mathcal{D}_\ell[p]$.

A point $\xi_{i,\ell}^R(w)$ or $\xi_{i,\ell}^B(w)$ changes during either a K_i , K_{i+1} , or L_ℓ -swap. Each L_ℓ -swap, for any ℓ , causes $O(s \log^2 n)$ points $\xi_{i,\ell}^R(w)$ or $\xi_{i,\ell}^B(w)$ to change (over the entire collection of trees), and therefore each swap causes $O(s^3 \log^2 n)$ updates to the tournaments $\mathcal{D}_\ell[p]$. The number of nodes which change in \mathcal{T}_i by a K_i or K_{i+1} -swap is $O(\log^2 n)$. Each such change causes $O(s^3)$ updates to the tournaments $\mathcal{D}_\ell[p]$. Therefore the total number of updates to tournaments due to changes of nodes is also $O(s^3 \log^2 n)$ per swap.

The number of swaps is $O(sn^2)$, so overall we get $O(s^4 n^2 \log^2 n)$ updates to the tournaments. The size of each individual tournament is $O(s^2 \log^2 n)$. By Theorem 2.1 these updates generate

$$O(s^4 n^2 \log^2 n \cdot \beta_{r+2}(s^2 \log^2 n) \log(s^2 \log^2 n)) = O(s^4 n^2 \beta_{r+2}(s \log n) \log^2 n \log(s \log n))$$

tournament events, which are processed in

$$O(s^4 n^2 \log^2 n \cdot \beta_{r+2}(s^2 \log^2 n) \log^2(s^2 \log^2 n)) = O(s^4 n^2 \cdot \beta_{r+2}(s \log n) \log^2 n \log^2(s \log n))$$

time. Processing each individual tournament event takes $O(\log^2 \log n + \log^2 s)$ time.

Since the size of each tournament is $O(s^2 \log^2 n)$ and there are $O(ns)$ tournaments, the total size of all tournaments is $O(s^3 n \log^2 n)$.

Testing whether p is strongly ℓ -extremal for the winner of $\mathcal{D}_\ell[p]$. For each $0 \leq i \leq s-1$, and for each pair $(p, q) \in \Pi_i$ we maintain those indices $0 \leq \ell \leq k-1$ (if there are any) for which p is strongly ℓ -extremal for q . Recall that each point p belongs to $O(s^2 \log^2 n)$ pairs in the sets Π_i .

We use the trees \mathcal{T}_j for $i-3 \leq j \leq i+3$ to find, for a query q , the point $\arg \max_{q' \in P \cap C'_i[q]} \langle q', u_\ell \rangle$, for each $0 \leq \ell \leq k-1$. The query time is $O(s \log^2 n)$. Using this information we easily determine, for a pair (p, q) , for which values of ℓ p is strongly ℓ -extremal for q .

As explained above, every swap changes $O(s^2 \log^2 n)$ pairs of the sets Π_i . When a new pair is added to a set Π_i we query the trees \mathcal{T}_j , $i-3 \leq j \leq i+3$, to find for which values of ℓ , p is strongly ℓ -extremal for q (and vice versa). This takes a total of $O(s^3 \log^4 n)$ time for each swap.

Furthermore, a point p can cease (or start) being strongly ℓ -extremal for q only during a swap which involves either p or q . So when we process a swap between p and some other point we recompute, for all pairs (p, x) and (x, p) in the current sets Π_i and for every $0 \leq \ell \leq k-1$, whether p is strongly ℓ -extremal for x , and whether x remains strongly ℓ -extremal for p . This adds an overhead of $O(s^3 \log^4 n)$ time at each swap.

The following theorem summarizes the results obtained so far in this section.

Theorem 4.6. *The SDG G can be maintained using a data structure which requires $O((n/\alpha^3) \log^2 n)$ space and encounters two types of events: swaps and tournament events.*

There are $O(n^2/\alpha)$ swaps, each processed in $O(\log^4 n/\alpha^3)$ time. There are

$$O((n^2/\alpha^4) \log^2 n \beta_{r+2}(\log n/\alpha) \log(\log n/\alpha))$$

tournament events which are processed in overall

$$O((n^2/\alpha^4) \log^2 n \beta_{r+2}(\log n/\alpha) \log^2(\log n/\alpha))$$

time. Processing each individual tournament event takes $O(\log^2 \log n + \log^2(1/\alpha))$ time.

4.2 An even faster data structure

We next reduce the overall time and space required to maintain G roughly by factors of s^2 and s , respectively (bringing the dependence on s of both bounds down to roughly s^2). We achieve that by restricting each tournament $\mathcal{D}_\ell[p]$ to contain a carefully chosen subset $\mathcal{N}_\ell[p] \subseteq \mathcal{N}[p]$ of size $O(s \log^2 n)$ (recall that the size of the entire set $\mathcal{N}[p]$ is $O(s^2 \log^2 n)$). The definition of $\mathcal{N}_\ell[p]$ is based on the following lemma. Its simple proof is given in Figure 17.

Lemma 4.7. *Let $p, q \in P$ and let i be the index for which $q \in C_i[p]$. Let $0 \leq \ell \leq k - 1$ be an index, and $v \in \mathbb{S}^1$ a direction such that the rays $u_\ell[p]$ and $v[q]$ intersect b_{pq} at the same point. Then v lies in one of the two consecutive cones $C_{\zeta(i,\ell)}, C_{\zeta(i,\ell)+1}$, where $\zeta(i,\ell) = 2i + s - \ell$.*

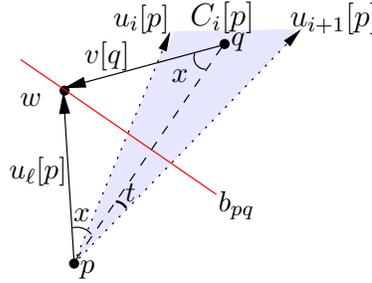


Figure 17: Proof of Lemma 4.7: We assume that $q \in C_i[p]$, and that the rays $u_\ell[p]$ and $v[q]$ hit b_{pq} at the same point w . Then the angle $x = \angle wpq = (i + 1 - \ell)\alpha - t$, for some $0 \leq t \leq \alpha$. The orientation of \overline{qp} is $(i + 1)\alpha - t + \pi = (i + s + 1)\alpha - t$. Hence, the orientation of v is $(i + s + 1)\alpha - t + x = (2i + s - \ell + 2)\alpha - 2t$. Thus, the direction v lies in the union of the two consecutive cones $C_{\zeta(i,\ell)}, C_{\zeta(i,\ell)+1}$, for $\zeta(i,\ell) = 2i + s - \ell$.

It follows that in Corollary 4.2, we can require that the indices $0 \leq j, \ell \leq k - 1$, for which (p, q) is a (strongly) (j, ℓ) -extremal pair, satisfy $\zeta(i, \ell) \leq j \leq \zeta(i, \ell) + 2$. Indeed, we may require that the vectors $u_j[q], u_\ell[p]$ hit b_{pq} at the respective points x and y for which the angle $\angle xpy = \angle xqy$ is at most α , which, in turn, happens only if u_j bounds one of the cones $C_{\zeta(i,\ell)}, C_{\zeta(i,\ell)+1}$.

For all $0 \leq i \leq s - 1$ and $0 \leq \ell \leq k - 1$ we define a set $\Pi_{i,\ell}$ which consists of all pairs (p, q) of points of P such that there exists a secondary node w in \mathcal{T}_i , and indices ℓ and $\zeta(i, \ell) \leq j \leq \zeta(i, \ell) + 2$, such that $p = \xi_{i,\ell}^B(w)$ and $q = \xi_{i,j}^R(w)$ or $p = \xi_{i,\ell}^R(w)$ and $q = \xi_{i,j}^B(w)$. We define the set $\mathcal{N}_\ell[p]$ to consist of all points q such that $(p, q) \in \Pi_{i,\ell}$. For a point p the set of points that participate in the *reduced* tournament $\mathcal{D}_\ell[p]$ is $\bigcup_{\ell'=\ell-3}^{\ell+3} \mathcal{N}_{\ell'}[p]$. (Note that this rule distributes a point $q \in \mathcal{N}_\ell[p]$ to only seven nearby tournaments. Nevertheless, when the edge pq is sufficiently long, q will belong to several consecutive neighborhoods $\mathcal{N}_\ell[p]$, and therefore will appear in more tournaments, in particular in at least eight consecutive tournaments at which it should win, according to the definition of our SDG.)

We claim that, with this redefinition of the tournaments $\mathcal{D}_\ell[p]$, Theorems 4.3 and 4.4 still hold. To verify that Theorem 4.3 holds one has to follow its (short) proof and notice that, by Lemma 4.7, the point q belongs to the eight reduced tournaments which it is supposed to win.

We next indicate the changes required in the proof of Theorem 4.4. We use the same notation as in the original proof of Theorem 4.4, and recall that it assumed by contradiction that, say, $N_{\ell+4}[p] \neq q$ even though q wins the tournaments $\mathcal{D}_\ell[p], \mathcal{D}_{\ell+1}[p], \dots, \mathcal{D}_{\ell+7}[p]$, and the point p is strongly $(\ell + 3)$ - and $(\ell + 4)$ -extremal for q . We use Lemma 4.5 to establish the existence of some point $r \in P$ such that $\varphi_{\ell+4}[p, r] < \varphi_{\ell+4}[p, q]$ and p is $(\ell + 4)$ -extremal for r . Let i be the index for which $r \in C_i[p]$, and let w be the secondary node in

\mathcal{T}_i for which $r \in B_w$ and $p \in R_w$. Note that $p = \xi_{i,\ell+4}^R(w)$. We next choose an index j such that the point $r' = \xi_{i,j}^B(w)$ either satisfies that $\varphi_{\ell+7}[p, r'] < \varphi_{\ell+7}[p, q]$ if r is to the right of the line from p to q , or that $\varphi_{\ell+1}[p, r'] < \varphi_{\ell+1}[p, q]$ if r is to the left of the line from p to q . To re-establish Theorem 4.4 it suffices to show that r' participates in the reduced tournament $\mathcal{D}_{\ell+7}[p]$ (resp., $\mathcal{D}_{\ell+1}[p]$) if r is to the right (resp., left) of the line from p to q .

It follows from the way we defined j in the original proof and from Lemma 4.7 that $\zeta(i, \ell + 4) - 2 \leq j \leq \zeta(i, \ell + 4) - 1$ (if r is to the right of the line from p to q) or $\zeta(i, \ell + 4) + 1 \leq j \leq \zeta(i, \ell + 4) + 2$ (if r is to the left of the line from p to q). So $r' \in \mathcal{N}_{\ell+4}[p]$ and therefore r' does participate in the reduced tournament $\mathcal{D}_{\ell+1}[p]$ or $\mathcal{D}_{\ell+7}[p]$. Indeed, the direction v used in that proof lies in one of the cones $C_{\zeta(i,\ell+4)}, C_{\zeta(i,\ell+4)+1}$. The direction u_j then forms an angle between α and 2α with v , which lies counterclockwise from v if r lies to the right of the line from p to q , or clockwise from v in the other case. This is easily seen to imply the two corresponding constraints on j ; see Figure 13.

We change our algorithm accordingly to maintain only the reduced tournaments.

Now every secondary node w of any range tree \mathcal{T}_i contributes only seven pairs to each set $\Pi_{i,\ell}$, for $0 \leq \ell \leq k - 1$, so the size of each such set is $O(n \log n)$. Since there are $O(s^2)$ sets $\Pi_{i,\ell}$, their total size is $O(s^2 n \log n)$. Each pair in each $\Pi_{i,\ell}$ contributes an item to a constant number of tournaments, so the total size of the tournaments is $O(s^2 n \log n)$. Each individual tournament $\mathcal{D}_\ell[p]$ is now of size $O(s \log^2 n)$, because p belongs to $O(\log^2 n)$ pairs in each set $\Pi_{i,\ell'}$ for $0 \leq i \leq s - 1$, $0 \leq \ell' \leq k - 1$, and $\mathcal{D}_\ell[p]$ inherits only those points q that come from pairs $(p, q) \in \Pi_{i,\ell'}$, for $0 \leq i \leq s - 1$ and $\ell - 3 \leq \ell' \leq \ell + 3$.

When $\xi_{i,\ell}^B[w]$ changes from p to p' for some $0 \leq i \leq s - 1$ and $0 \leq \ell \leq k - 1$, at most a constant number of pairs $(p, \xi_{i,j}^R(w))$ for $\zeta(i, \ell) \leq j \leq \zeta(i, \ell) + 2$ are deleted from $\Pi_{i,\ell}$, and a constant number of pairs $(p', \xi_{i,j}^R(w))$ for $\zeta(i, \ell) \leq j \leq \zeta(i, \ell) + 2$ are added to $\Pi_{i,\ell}$. Similar changes take place in $\Pi_{i,j}$ for those three indices j satisfying $\zeta(i, j) \leq \ell \leq \zeta(i, j) + 2$. When $\xi_{i,j}^R[w]$ changes from q to q' for some $0 \leq i \leq s - 1$ and $0 \leq j \leq k - 1$, at most a constant number of pairs $(\xi_{i,\ell}^B(w), q)$ are deleted from $\Pi_{i,j}$ for the indices ℓ satisfying $\zeta(i, j) \leq \ell \leq \zeta(i, j) + 2$, and a constant number of pairs $(\xi_{i,\ell}^B(w), q')$ are added for the same values of ℓ . Similarly, at most a constant number of pairs $(\xi_{i,\ell}^B(w), q)$ are deleted from $\Pi_{i,\ell}$ for the indices ℓ satisfying $\zeta(i, \ell) \leq j \leq \zeta(i, \ell) + 2$, and a constant number of pairs $(\xi_{i,\ell}^B(w), q')$ are added for the same values of ℓ .

A change of a secondary node w in the tree \mathcal{T}_i causes $O(s)$ pairs in the sets $\Pi_{i,\ell}$ to change.

Any K_i -swap changes $O(\log^2 n)$ nodes in \mathcal{T}_i and thereby causes $O(s \log^2 n)$ pairs in the sets $\Pi_{i,\ell}$ to change. Any L_j -swap changes $O(s \log^2 n)$ extremal points $\xi_{i,j}^R[w]$, $\xi_{i,j}^B[w]$ at secondary nodes w of the trees \mathcal{T}_i , and thereby causes $O(s \log^2 n)$ pairs in the sets $\Pi_{i,\ell}$ to change. Since each pair in $\Pi_{i,\ell}$ contributes an item to a constant number of tournaments it follows that $O(s \log^2 n)$ points are inserted to and deleted from the tournaments $\mathcal{D}_\ell[p]$ at each swap.

According to Theorem 2.1 the size of each tournament is $O(s \log^2 n)$ – the number of elements that it contains. So the total size of all tournaments is $O(s^2 n \log n)$. In total we get that there are $O(s^2 n^2 \log^2 n)$ updates to tournaments during swaps. These updates generate

$$O(s^2 n^2 \log^2 n \beta_{r+2}(s \log n) \log(s \log n))$$

tournament events that are processed in overall

$$O(s^2 n^2 \log^2 n \beta_{r+2}(s \log n) \log^2(s \log n))$$

time. Each individual tournament event is processed in $O(\log^2 \log n + \log^2 s)$ time and each swap can be processed in $O(s \log^2 n \log^2(s \log n))$ time.

In addition, for each pair $(p, q) \in \Pi_{i,\ell}$ we record whether p is strongly ℓ -extremal for q . We maintain this information using the trees \mathcal{T}_j , for $i - 3 \leq j \leq i + 3$, as described above, which allow for any $p, q \in P$ and $0 \leq \ell \leq k - 1$ to test, in $O(\log^2 n)$ time, if p is strongly ℓ -extremal for q . At each swap event we spend $O(s \log^4 n)$ extra time to compute for $O(s \log^2 n)$ pairs (p, q) which are added to the sets $\Pi_{i,\ell}$ whether p is strongly ℓ -extremal for q .

Consider a pair $(p, q) \in \Pi_{i,\ell}$. The point p may stop being strongly ℓ -extremal for q only during a swap which involves p or q . So, as before, at each swap we find the $O(s \log^2 n)$ pairs containing one of the points involved in the swap, and recompute, in $O(s \log^4 n)$ total time, for each such pair (p, q) , whether the strong extremal relation holds. We thus obtain the following summary result.

Theorem 4.8. *Let P be a set of n moving points in \mathbb{R}^2 under algebraic motion of bounded degree, and let $\alpha > 0$ be a sufficiently small parameter. A $(10\alpha, \alpha)$ -SDG of P can be maintained using a data structure that requires $O((n/\alpha^2) \log n)$ space and encounters two types of events: swap events and tournament events. There are $O(n^2/\alpha)$ swap events, each processed in $O(\log^4(n)/\alpha)$ time. There are*

$$O((n/\alpha)^2 \beta_{r+2}(\log(n)/\alpha) \log^2 n \log(\log(n)/\alpha))$$

tournament events, which are handled in a total of

$$O((n/\alpha)^2 \beta_{r+2}(\log(n)/\alpha) \log^2 n \log^2(\log(n)/\alpha))$$

processing time. The worst-case processing time of a tournament event is $O(\log^2(\log(n)/\alpha))$. The data structure is also local, in the sense that each point is stored, at any given time, at only $O(\log^2 n/\alpha^2)$ places in the structure.

Concerning locality, we note that a point participates in $O(s)$ projection tournaments at each of $O(s \log^2 n)$ tree nodes. If it wins in at least one of the projection tournaments at a node, it is fed to $O(s)$ directional tournaments. So it appears in $O(s^2 \log n)$ places.

Remarks: (1) Comparing this algorithm with the space-inefficient one of Section 2, we note that they both use the same kind of tournaments, but here much fewer pairs of points ($O^*(n/\alpha^2)$ instead of $O(n^2/\alpha)$) participate in the tournaments. The price we have to pay is that the test for an edge pq to be stable is more involved. Moreover, keeping track of the subset of pairs that participate in the tournaments requires additional work, which is facilitated by the range trees \mathcal{T}_j .

(2) To be fair, we note that our $O^*(\cdot)$ notation hides polylogarithmic factors in n . Hence, comparing the analysis in this section with Theorem 3.6, we gain when n is smaller than some threshold, which is exponential in $1/\alpha$.

5 Properties of SDG

We conclude the paper by establishing some of the properties of stable Delaunay graphs.

Near cocircularities do not show up in an SDG. Consider a critical event during the kinetic maintenance of the full Delaunay triangulation, in which four points a, b, c, d become cocircular, in this order, along their circumcircle, with this circle being empty. Just before the critical event, the Delaunay triangulation involved two triangles, say, abc, acd . The Voronoi edge e_{ac} shrinks to a point (namely, to the circumcenter of $abcd$ at the critical event), and, after the critical cocircularity, is replaced by the Voronoi edge e_{bd} , which expands from the circumcenter as time progresses.

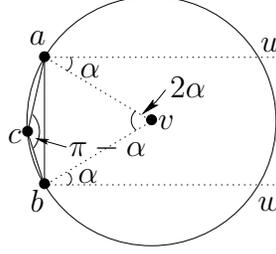


Figure 18: The near collinearity that corresponds to a Voronoi edge becoming α -short.

Our algorithm will detect the possibility of such an event before the criticality occurs, when e_{ac} becomes α -short (or even before this happens). It will then remove this edge from the stable subgraph, so the actual cocircularity will not be recorded. The new edge e_{bd} will then be detected by the algorithm only when it becomes sufficiently long (if this happens at all), and will then enter the stable Delaunay graph. In short, critical cocircularities do not arise *at all* in our scheme.

As noted in the introduction, a Delaunay edge ab (interior to the hull) is just about to become α -short or α -long when the sum of the opposite angles in its two adjacent Delaunay triangles is $\pi - \alpha$ (see Figure 1). This shows that changes in the stable Delaunay graph occur when the “cocircularity defect” of a nearly cocircular quadruple (i.e., the difference between π and the sum of opposite angles in the quadrilateral spanned by the quadruple) is between α and $c\alpha$, where c is the constant used in our definitions in Section 3 or Section 4. Note that a degenerate case of cocircularity is a collinearity on the convex hull. Such collinearities also do not show up in the stable Delaunay graph.⁵ A hull collinearity between three nodes a, b, c is detected before it happens, when (or before) the corresponding Voronoi edge becomes α -short, in which case the angle $\angle acb$, where c is the middle point of the (near-)collinearity becomes $\pi - \alpha$ (see Figure 18). Therefore a hull edge is removed from the SDG if the Delaunay triangle is almost collinear. The edge (or any new edge about to replace it) re-appears in the SDG when its corresponding Voronoi edge is long enough, as before.

SDGs are not too sparse. Consider the Voronoi cell $\text{Vor}(p)$ of a point p , and suppose that p has only one α -long edge e_{pq} . Since the angle at which p sees e_{pq} is at most π , the sum of the angles at which p sees the other edges is at least π , so $\text{Vor}(p)$ has at least π/α α -short edges. Let m_1 denote the number of points p with this property. Then the sum of their degrees in $\text{DT}(P)$ is at least $m_1(\pi/\alpha + 1)$. Similarly, if m_0 points do not have any α -long Voronoi edge, then the sum of their degrees is at least $2\pi m_0/\alpha$. Any other point at least two α -long Voronoi edges and its degree is at least 3 if it is an interior point, or at least 2 otherwise. So the number of α -long edges is at least (recall that each α -long edge is counted twice)

$$n - m_1 - m_0 + m_1/2 = n - (m_1 + 2m_0)/2. \quad (2)$$

Let h denote the number of hull vertices. Since the sum of the degrees is $6n - 2h - 6$, we get

$$3(n - h - m_1 - m_0) + 2h + m_1 \left(\frac{\pi}{\alpha} + 1 \right) + 2m_0 \frac{\pi}{\alpha} \leq 6n - 2h - 6,$$

⁵Even if they did show up, no real damage would be done, because the number of such collinearities is only $O^*(n^2)$; see, e.g., [22].

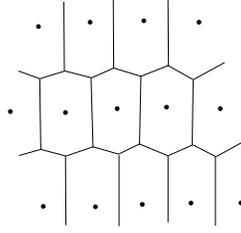


Figure 19: If the points of P lie on a sufficiently spaced shifted grid then the number of α -long edges in $\text{VD}(P)$ (the vertical ones) is close to n .

implying that

$$m_1 + 2m_0 \leq \frac{3n}{\pi/\alpha - 2}.$$

Plugging this inequality in (2), we conclude that the number of α -long edges is at least

$$n \left[1 - \frac{3}{2(\pi/\alpha - 2)} \right].$$

As α decreases, the number of edges in the SDG is always at least a quantity that gets closer to n . This is nearly tight, since there exist n -point sets for which the number of stable edges is only roughly n , see Figure 19.

Closest pairs, crusts, β -skeleta, and the SDG. Let $\beta \geq 1$, and let P be a set of n points in the plane. The β -skeleton of P is a graph on P that consists of all the edges pq such that the union of the two disks of radius $(\beta/2)d(p, q)$, touching p and q , does not contain any point of $P \setminus \{p, q\}$. See, e.g., [4, 18] for properties of the β -skeleton, and for its applications in surface reconstruction. We show that the edges of the β -skeleton are α -stable in $\text{DT}(P)$, provided $\beta \geq 1 + \Omega(\alpha^2)$. In Figure 20 we sketch a straightforward proof of the fact that the edges of the β -skeleton are α -stable in $\text{DT}(P)$, provided that $\beta \geq 1 + \Omega(\alpha^2)$.

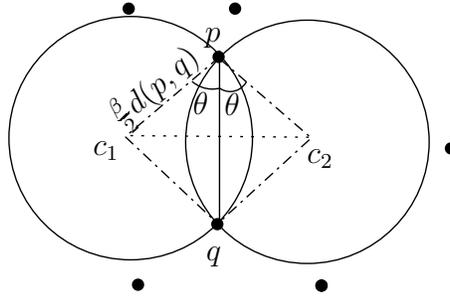


Figure 20: An edge pq of the β -skeleton of P (for $\beta > 1$). c_1 and c_2 are centers of the two P -empty disks of radius $(\beta/2)d(p, q)$ touching p and q . Clearly, each of p, q sees the Voronoi edge e_{pq} at an angle at least $2\theta = \angle c_1pq + \angle c_2pq$ (so it is 2θ -stable). We have $1/\beta = \cos \theta \approx 1 - \theta^2/2$ or $\beta = 1 + \Theta(\theta^2)$. That is, for $\beta \geq 1 + \Omega(\alpha^2)$ every edge of the β -skeleton is α -stable.

A similar argument shows that the stable Delaunay graph contains the closest pair in $P(t)$ as well as the crust of a set of points sampled sufficiently densely along a 1-dimensional curve (see [3, 4] for the definition

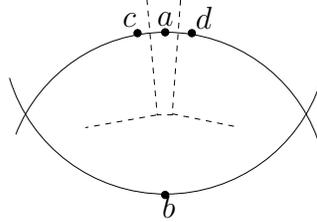


Figure 21: ab is an edge of the relative neighborhood graph but not of SDG.

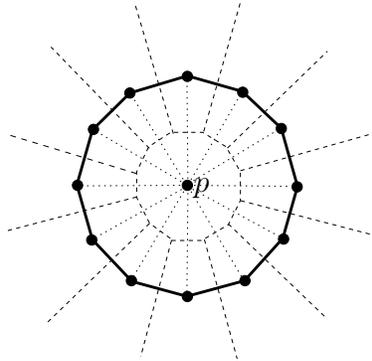


Figure 22: A wheel-like configuration that disconnects p in the stable Delaunay graph. The Voronoi diagram is drawn with dashed lines, the stable Delaunay edges are drawn as solid, and the remaining Delaunay edges as dotted edges. The points of the “wheel” need not be cocircular.

of crusts and their applications in surface reconstruction). We only sketch the argument for closest pairs: If (p, q) is a closest pair then $pq \in \text{DT}(P)$, and the two adjacent Delaunay triangles $\triangle pqr^+$, $\triangle pqr^-$ are such that their angles of r^+ , r^- are at most $\pi/3$ each, so e_{pq} is $(\pi/3)$ -long, ensuring that pq belongs to any stable subgraph for α sufficiently small; see [1] for more details. We omit the proof for crusts, which is fairly straightforward.

In contrast, stable Delaunay graphs need not contain all the edges of several other important subgraphs of the Delaunay triangulation, including the Euclidean minimum spanning tree, the Gabriel graph, the relative neighborhood graph, and the all-nearest-neighbors graph. An illustration for the relative neighborhood graph is given in Figure 21. As a matter of fact, the stable Delaunay graph need not even be connected, as is illustrated in Figure 22.

Completing SDG into a triangulation. As argued above, the Delaunay edges that are missing in the stable subgraph correspond to nearly cocircular quadruples of points, or to nearly collinear triples of points near the boundary of the convex hull. Arguably, these missing edges carry little information, because they may “flicker” in and out of the Delaunay triangulation even when the points move just slightly (so that all angles determined by the triples of points change only slightly). Nevertheless, in many applications it is desirable (or essential) to complete the stable subgraph into *some* triangulation, preferably one that is also stable in the combinatorial sense—it undergoes only nearly quadratically many topological changes.

By the analysis in Section 3 we can achieve part of this goal by maintaining the full Delaunay triangu-

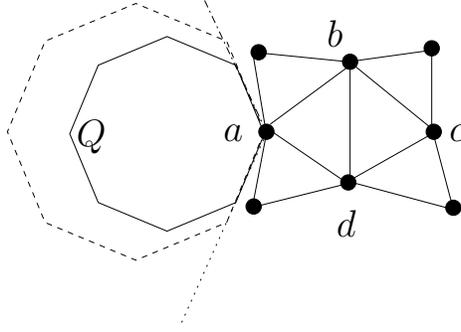


Figure 23: The triangulation $DT^\diamond(P)$ of an 8-point set P . The points a, b, c, d , which do not lie on the convex hull of P , still lie on the boundary of the union of the triangles of $DT^\diamond(P)$ because, for each of these points we can place an arbitrary large homothetic interior-empty copy of Q which touches that point.

lation $DT^\diamond(P)$ under the polygonal norm induced by the regular k -gon Q_k . This diagram experiences only a nearly quadratic number of topological changes, is easy to maintain, and contains all the stable Euclidean Delaunay edges, for an appropriate choice of $k \approx 1/\alpha$. Moreover, the union of its triangles is simply connected — it has no holes. Unfortunately, in general it is not a triangulation of the entire convex hull of P , as illustrated in Figure 23.

For the time being, we leave it as an open problem to come up with a simple and “stable” scheme for filling the gaps between the triangles of $DT^\diamond(P)$ and the edges of the convex hull. It might be possible to extend the kinetic triangulation scheme developed in [16], so as to kinetically maintain a triangulation of the “fringes” between $DT^\diamond(P)$ and the convex hull of P , which is simple to define, easy to maintain, and undergoes only nearly quadratically many topological changes.

Of course, if we only want to maintain a triangulation of P that experiences only a nearly quadratically many topological changes, then we can use the scheme in [16], or the earlier, somewhat more involved scheme in [2]. However, if we want to keep the triangulation “as Delaunay as possible”, we should include in it the stable portion of DT , and then the efficient completion of it, as mentioned above, becomes an issue, not yet resolved.

Nearly Euclidean norms and some of their properties. One way of interpreting the results of Section 3 is that the stability of Delaunay edges is preserved, in an appropriately defined sense, if we replace the Euclidean norm by the polygonal norm induced by the regular k -gon Q_k (for $k \approx 1/\alpha$). That is, stable edges in one Delaunay triangulation are also edges of the other triangulation, and are stable there too. Here we note that there is nothing special about Q_k : The same property holds if we replace the Euclidean norm by any sufficiently close norm (or convex distance function [8]).

Specifically, let Q be a closed convex set in the plane that is contained in the unit disk D_0 and contains the disk $D'_0 = (\cos \alpha)D_0$ that is concentric with D_0 and scaled by the factor $\cos \alpha$. This is equivalent to requiring that the Hausdorff distance $H(Q, D_0)$ between Q and D_0 be at most $1 - \cos \alpha \approx \alpha^2/2$. We define the center of Q to coincide with the common center of D_0 and D'_0 .

Q induces a convex distance function d_Q , defined by $d_Q(x, y) = \min\{\lambda \mid y \in x + \lambda Q\}$. Consider the Voronoi diagram $\text{Vor}^Q(P)$ of P induced by d_Q , and the corresponding Delaunay triangulation $DT^Q(P)$. We omit here the detailed analysis of the structure of these diagrams, which is similar to that for the norm induced by Q_k , as presented in Section 3. See also [7, 8] for more details. Call an edge e_{pq} of $\text{Vor}^Q(P)$

α -stable if the following property holds: Let u and v be the endpoints of e_{pq} , and let Q_u, Q_v be the two homothetic copies of Q that are centered at u, v , respectively, and touch p and q . Then we require that the angle between the supporting lines at p (for simplicity, assume that Q is smooth, and so has a unique supporting line at p (and at q); otherwise, the condition should hold for any pair of supporting lines at p or at q) to Q_u and Q_v is at least α , and that the same holds at q . In this case we refer to the edge pq of $\text{DT}^Q(P)$ as α -stable.

Note that Q_k -stability was (implicitly) defined in a different manner in Section 3, based on the number of breakpoints of the corresponding Voronoi edges. Nevertheless, it is easy to verify that the two definitions are essentially identical.

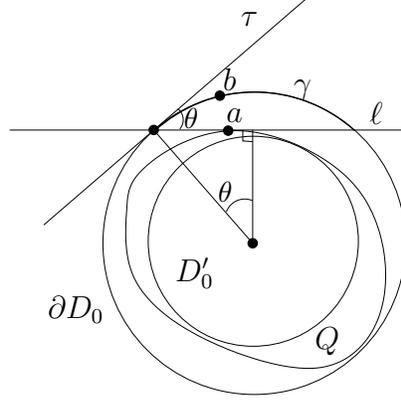


Figure 24: An Illustration for Claim 5.1.

A useful property of such a set Q is the following:

Claim 5.1. *Let a be a point on ∂Q and let ℓ be a supporting line to Q at a . Let b be the point on ∂D_0 closest to a (a and b lie on the same radius from the center o). Let γ be the arc of ∂D_0 , containing b , and bounded by the intersection points of ℓ with ∂D_0 . Then the angle between ℓ and the tangent, τ , to D_0 at any point along γ , is at most α .*

Proof. Denote this angle by θ . Clearly θ is maximized when τ is tangent to D_0 at an intersection of ℓ and ∂D_0 . See Figure 24. It is easy to verify that the distance from o to ℓ is $\cos \theta$. But this distance has to be at least $\cos \alpha$, or else ∂Q would have contained a point inside D'_0 , contrary to assumption. Hence we have $\cos \theta > \cos \alpha$, and thus $\theta < \alpha$, as claimed. \square

We need a few more properties:

Claim 5.2. *Let Q_1 and Q_2 be two homothetic copies of Q and let w be a point such that (i) w lies on ∂Q_1 and on ∂Q_2 , and (ii) w and the respective centers o_1, o_2 of Q_1, Q_2 are collinear. Then Q_1 and Q_2 are tangent to each other at w ; more precisely, they have a common supporting line at w , and, assuming ∂Q to be smooth, w is the only point of intersection of $\partial Q_1 \cap \partial Q_2$ (otherwise, $\partial Q_1 \cap \partial Q_2$ is a single connected arc containing w).*

Proof. Map each of Q_1, Q_2 back to the standard placement of Q , by translation and scaling, and note that both transformations map w to the same point w_0 on ∂Q . Let ℓ_0 be a supporting line of Q at w_0 , and let ℓ_1, ℓ_2 be the forward images of ℓ under the mappings of Q to Q_1 and to Q_2 , respectively. Clearly, ℓ_1 and ℓ_2

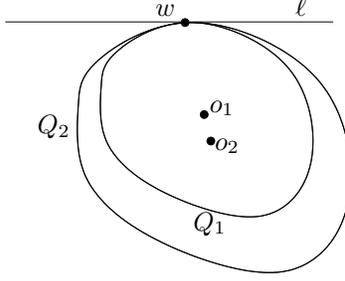


Figure 25: An Illustration for Claim 5.2.

coincide, and are a common supporting line of Q_1 and Q_2 at w . See Figure 25. The other asserted property follows immediately if ∂Q is smooth, and can easily be shown to hold in the non-smooth case too; we omit the routine argument. \square

Claim 5.3. *Let a and b be two points on ∂Q , and let ℓ_a and ℓ_b be supporting lines of Q at a and b , respectively. Then the difference between the angles that ℓ_a and ℓ_b form with ab is at most 2α .*

Proof. Denote the two angles in the claim by θ_a and θ_b , respectively. Let a' (resp., b') be the point on ∂D_0 nearest to (and co-radial with) a (resp., b). Let τ_1, τ_2 denote the respective tangents to D_0 at a' and at b' . Clearly, the respective angles θ_1, θ_2 between the chord $a'b'$ of D_0 and τ_1, τ_2 are equal. By Claim 5.1, we have $|\theta_1 - \theta_a| \leq \alpha$ and $|\theta_2 - \theta_b| \leq \alpha$, and the claim follows. \square

The connection between Euclidean stability and Q -stability. Let e_{pq} be a $t\alpha$ -long Voronoi edge of the Euclidean diagram, for $t \geq 9$, and let u, v denote its endpoints. Let D_u and D_v denote the disks centered respectively at u, v , whose boundaries pass through p and q , and let D be a disk whose boundary passes through p and q , so that $D \subset D_u \cup D_v$ and the angles between the tangents to D and to D_u and D_v at p (or at q) are at least $m\alpha$ each, where $m \geq 4$. (Recall that the angle between the tangents to D_u and D_v at u is at least $t\alpha \geq 9\alpha$.)

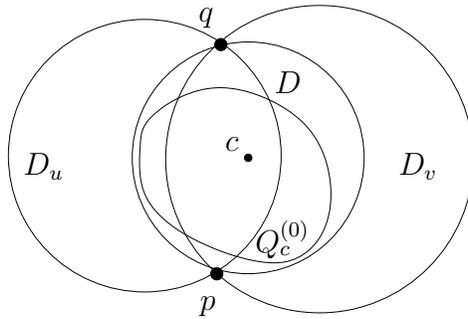


Figure 26: The homothetic copy $Q_c^{(0)}$.

Let c and ρ denote the center and radius of D , respectively. Note that c lies on e_{pq} “somewhere in the middle”, because of the angle condition assumed above. Let $Q_c^{(0)}$ denote the homothetic copy of Q centered

at c and scaled by ρ , so $Q_c^{(0)}$ is fully contained in D and thus also in $D_u \cup D_v$, implying that $Q_c^{(0)}$ is empty—it does not contain any point of P in its interior. (This scaling makes the unit circle D_0 bounding Q coincide with D .) See Figure 26.

Expand $Q_c^{(0)}$ about its center c until the first time it touches either p or q . Suppose, without loss of generality, that it touches p . Denote this placement of Q as Q_c . Let ℓ_p denote a supporting line of Q_c at p . We claim that the angle between ℓ_p and the tangent τ_p to D at p is at most α . Indeed, let ℓ_p^-, ℓ_p^+ denote the tangents from p to $Q_c^{(0)}$. By Claim 5.1, the angles that they form with the tangent τ_p to D at p are at most α each. As $Q_c^{(0)}$ is expanded to Q_c , these tangents rotate towards each other, one clockwise and one counterclockwise so when they coincide (at Q_0) the resulting supporting line ℓ_p lies inside the double wedge between them. Since τ_p also lies inside this double wedge, and forms an angle of at most α with each of them, it follows that ℓ_p must form an angle of at most α with τ_p , as claimed.

Since the angle between the tangent τ_p to D at p and the tangent τ_p^v to D_v at p is at least $m\alpha$ it follows that the angle between ℓ_p and τ_p^v is at least $(m-1)\alpha$. A similar argument shows that the angle between ℓ_p and the tangent τ_p^u to D_u at p is at least $(m-1)\alpha$.

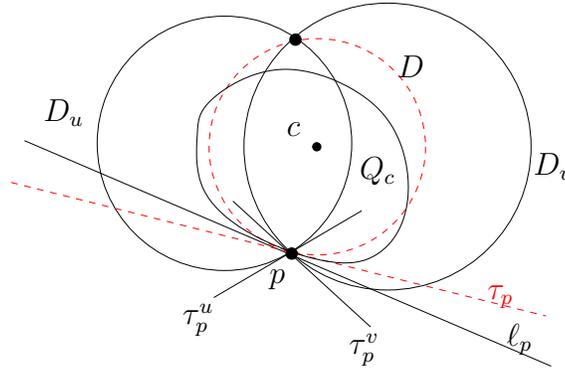


Figure 27: The homothetic copy Q_c .

Now expand Q_c by moving its center along the line passing through p and c , away from p , and scale it appropriately so that its boundary continues to pass through p , until it touches q too. Denote the center of the new placement as c' , and the placement itself as $Q_{c'}$. Let $D_{c'}$ be the corresponding homothetic copy of D_0 centered at c' and bounding $Q_{c'}$. See Figure 27.

We argue that $Q_{c'}$ is empty. By Claim 5.2, ℓ_p is also a supporting line of $Q_{c'}$ at p . Refer to Figure 29. We denote by x_p and y_p the intersections of the supporting line ℓ_p with $\partial D_{c'}$ and ∂D_v , respectively. We denote by z the intersection of $\partial D_{c'}$ and ∂D_v that lies on the same side of ℓ_p as q . The angle $\angle pzx_p$ is at most α since by Claim 5.1 the angle between ℓ_p and the tangent to $D_{c'}$ at x_p is at most α . On the other hand the angle $\angle pzy_p$ is at least $(m-1)\alpha$ since the angle between ℓ_p and τ_p^v at p is at least $(m-1)\alpha$. So it follows that the segment px_p is fully contained in D_v . Since the ray \overline{zp} meets ∂D_v (at p) before meeting $\partial D_{c'}$, and the ray $\overline{zx_p}$ meets $\partial D_{c'}$ (at x_p) before meeting ∂D_v , it follows that $\partial D_{c'}$ and ∂D_v intersect at a point on a ray between \overline{zp} and $\overline{zx_p}$.

Let ℓ_q denote a supporting line of $Q_{c'}$ at q . By Claim 5.3, the angles between pq and the lines ℓ_p, ℓ_q differ by at most 2α . Since each of the angles between ℓ_p and the two tangents τ_p^v and τ_p^u is at least $(m-1)\alpha$, it follows that each of the angles between ℓ_q and the two tangents τ_q^u and τ_q^v to D_u and D_v , respectively, at q , is at least $(m-3)\alpha$.

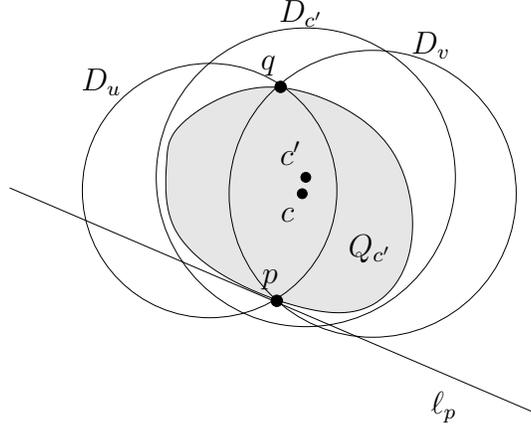


Figure 28: The homothetic copy $Q_{c'}$.

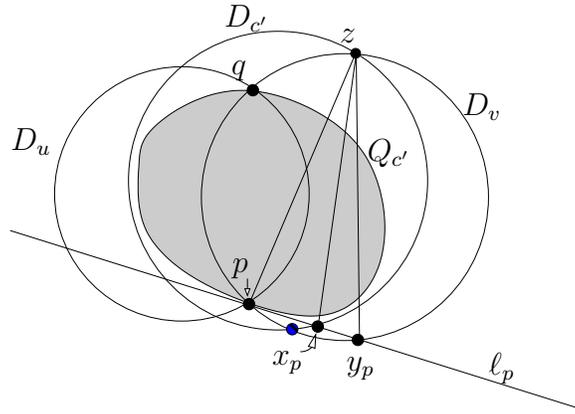


Figure 29: The segment px_p is fully contained in D_v . The circles $\partial D_{c'}$, ∂D_v intersect at a point on a ray emanating from z between zp and zx_p .

Refer now to Figure 30. We denote by z' the intersection of $D_{c'}$ and D_v distinct from z , and we denote by x_q, y_q the intersections between ℓ_q and $D_{c'}, D_v$, respectively. An argument analogous to the one given before shows that $\angle qz'x_q \leq \alpha$ while $\angle qz'y_q \geq (m-3)\alpha$. It follows that the segment qx_q is fully contained in D_v and we have an intersection between $\partial D_{c'}$ and ∂D_v on a ray emanating from z' between the ray from z' to q and the ray from z' to x_q .

Our argument about the position of the intersections between $D_{c'}$ and D_v implies that the entire section of $\partial D_{c'}$ between x_p and x_q is contained in D_v . Therefore the portion of $Q_{c'}$ to the right of the line through p and q (in the configuration depicted in the figures) is fully contained in D_v . A symmetric argument shows that the portion of $Q_{c'}$ to the left of the line through p and q is fully contained in D_u . Since $D_u \cup D_v$ is empty we conclude that $Q_{c'}$ is empty.

The emptiness of $Q_{c'}$ implies that p and q are neighbors in the Q -Voronoi diagram, and that c' lies on their common Q -Voronoi edge e_{pq}^Q .

We thus obtain the following theorem.

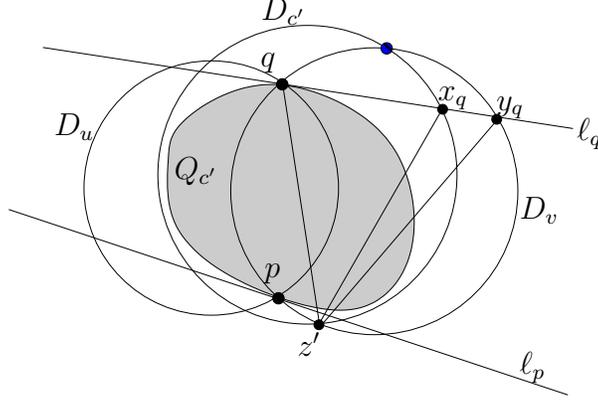


Figure 30: The segment qx_q is fully contained in D_v . The circles $\partial D_{c'}, \partial D_v$ intersect at a point on a ray emanating from z' between $z'q$ and $z'x_q$.

Theorem 5.4. *Let P , α , and Q be as above. Then (i) every 9α -stable edge of the Euclidean Delaunay triangulation is an α -stable edge of $\text{DT}^Q(P)$. (ii) Conversely, every 9α -stable edge of $\text{DT}^Q(P)$ is also an α -stable edge in the Euclidean norm.*

Note that parts (i) and (ii) are generalizations of Lemmas 3.4 and 3.5, respectively (with weaker constants).

Proof. Part (i) follows directly from the preceding analysis. Indeed, let pq be a $t\alpha$ -stable Delaunay edge, for $t \geq 9$, whose Voronoi counterpart has endpoints u and v . Let $Q_{c'}$ be the homothetic placement of Q , with center c' , that touches p and q . We have shown that $Q_{c'}$ has empty interior if the ray $\rho = \overline{pc'}$ lies between \overline{pu} and \overline{pv} and spans an angle of at least 4α with each of them. Assuming $t \geq 9$, such rays ρ form a cone of size $(t - 8)\alpha > \alpha$, which, in turn, gives the first part of the theorem.

Part (ii) follows from part (i) by repeating, almost verbatim, the proof of Lemma 3.5. □

There are many interesting open problems that arise here. One of the main problems is to extend the class of sets Q for which a near quadratic bound on the number of topological changes in $\text{DT}^Q(P)$, under algebraic motion of bounded degree of the points of P , can be established.

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