MODIFICATION OF AHO AND ULLMAN'S CORRECTNESS PROOF OF WARSHALL'S ALGORITHM

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Aho and Ullman's correctness proof of Warshall's algorithm (ref. 1, p. 48) is obscure and at one point incorrect. A modified version of this proof is presented.

Algorithm 0.2

Minimum cost of paths through a graph.

Input. An n x n cost matrix $C = [c_{ij}]$ with $1 \leq i,j \leq n$ and $c_{ij} \geq 0$ for all i and j.

Output. An n x n matrix $M = [m_{ij}]$ where m_{ij} represents the lowest cost of any path from node i to node j.

Comment. The nodes of the graph are assumed to be numbered 1,2,...,n. The costs c_{ij} represent the direct costs associated with the arc from node i to node j. The variables m_{ij} represent the minimum cost from node i to node j which has so far been found. The algorithm successively considers paths from i to j through each node k.

Method.

1. set $m_{ij} = c_{ij}$ for all i and j such that $1 \leq i,j \leq n$
2. set k = 1.
3. For all i and j, if $m_{ij} > m_{ik} + m_{kj}$, set m_{ij} to $m_{ik} + m_{kj}$.
4. If $k < n$, increase k by 1 and go to step (3). If k = n, halt.

Discussion of Algorithm

The heart of Algorithm 0.2 is step 3 in which it is determined whether the current minimum cost of going from node i to node j is greater than the current minimum cost of going from node i to node k and then from node k to node j.

On completion of step 3 with $k = \ell$, the value of each m_{ij} will be the minimum cost of a path from i to j through intermediate nodes in the set $1,2,\ldots,\ell$. This value will be denoted by m_{ij}^{ℓ}. The correctness proof is essentially an inductive proof of the assertion that, at the end of step 3 with $k = \ell$, m_{ij} has the value m_{ij}^{ℓ} for all i and j. From this assertion it follows that when step 3 is completed with $k = n$, the value of m_{ij} is the minimum cost path from i to j through intermediate nodes in the set 1,2 .. ..,n, and therefore the minimal cost of any path from i to j.

Notation: the notation

$$c_{iv_2} + c_{v_2v_3} + \ldots + c_{v_{m-1}j}$$

denotes the cost of the path from i to j through intermediate vertices $v_2, v_3, \ldots, v_{m-1}$. The minimum cost of a path from i to j is clearly the minimum of all expressions of the form $c_{iv_2} + c_{v_2v_3} + \ldots + c_{v_{m-1}j}$.

Theorem 0.5 (Correctness of the Warshall algorithm)

When algorithm 0.2 terminates, m_{ij} is the smallest value expressible as $c_{iv_2} + c_{v_2v_3} + \ldots + c_{v_{m-1}j}$ for all i and j.

<u>Proof</u>. The theorem will be proved by induction on the value of k at the end of step 3 of the algorithm, using the following inductive hypothesis:

<u>Inductive hypothesis</u>: After step 3 is executed with $k = \ell$, m_{ij} has the smallest value expressible as a sum of the form

$$c_{iv_2} + c_{v_2v_3} + \ldots + c_{v_{m-1}j},$$

where none of $v_2, v_3, \ldots, v_{m-1}$ is greater than ℓ. This value will be denoted by m_{ij}^{ℓ}.

<u>Basis</u>: When $\ell = 0$ the value of m_{ij} is $m_{ij}^{0} = c_{ij}$, which is the minimal cost of all paths from i to j passing through no intermediate nodes.

<u>Inductive step:</u> Assume that after step 3 with $k = \ell-1$ we have $m_{ij} = m_{ij}^{\ell-1}$ and prove that after step 3 with $k = \ell$, $m_{ij} = m_{ij}^{\ell}$.

The following two cases may be distinguished.

a) There is a path from i to j with minimal cost m_{ij}^{ℓ} which does not contain the vertex ℓ.

In this case $m_{ij}^{\ell} = m_{ij}^{\ell-1}$, and, since the current values of m_{ij} are $m_{ij}^{\ell-1}$ by the inductive hypothesis, the algorithm correctly determines that $m_{i\ell}^{\ell-1} + m_{\ell j}^{\ell-1} \geq m_{ij}^{\ell}$, and that the value of m_{ij} is not to be changed on this iteration.

Note that $m_{\ell j}^{\ell} = m_{\ell j}^{\ell-1}$ for all j since the path from ℓ to j through an intermediate vertex ℓ would contain a cycle, and cycles may always be removed without increasing the cost of the path (because $c_{ij} \geq 0$ for all i and j). Thus there is always a path with minimal $m_{\ell j}^{\ell}$ not containing an intermediate vertex ℓ. Similarly, there is always a path from i to ℓ with minimal cost $m_{i\ell}^{\ell}$ not containing an intermediate vertex ℓ. Thus when $k = \ell$, $m_{i\ell}$ and $m_{\ell j}$ will fall under case (a) for all i and j.

(b) All paths with minimal cost m_{ij}^{ℓ} pass through the intermediate vertex ℓ. In this case there is a path with cost m_{ij}^{ℓ} passing through a sequence of intermediate vertices $v_2, v_3, \ldots, v_{m-1}$ such that exactly one of the intermediate vertices is ℓ and all other vertices are less than ℓ . Let v_p be the vertex ℓ. Then the cost of this path may be expressed as follows:

$$c_{iv_2} + \ldots + c_{v_{p-1}\ell} + c_{\ell v_{p+1}} + \ldots + c_{v_{m-1}j}.$$

By the inductive hypothesis,

$$c_{iv_2} + \ldots + c_{v_{p-1}\ell} = m_{i\ell}^{\ell-1} \text{ and } c_{\ell v_{p+1}} + \ldots + c_{v_{m-1}j} = m_{\ell j}^{\ell-1}$$

are the current values of $m_{i\ell}$ and $m_{\ell j}$. Step 3 of the algorithm will correctly determine that $m_{i\ell}^{\ell-1} + m_{\ell j}^{\ell-1} < m_{ij}^{\ell-1}$ and will replace $m_{ij}^{\ell-1}$ by $m_{i\ell}^{\ell-1} + m_{\ell j}^{\ell-1} = m_{ij}^{\ell}$. Thus, at the end of step 3 m_{ij} will have the value m_{ij}^{ℓ} for all values of i and j.

1. Aho, A.V., and Ullman, J.D., Theory of Parsing, Translation and Compiling Volume 1, Prentice-Hall, 1972.

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by Lawrence Yelowitz (CSR 119)

ABSTRACT

The resolution principle of mechanical theorem proving is described in the context of the propositional calculus. Semantic resolution and a refinement of it are introduced as complete restrictions of general resolution. A novel matrix notation is utilized, leading to a systematic exposition of these ideas.

APPLICATION OF FUZZY LOGIC TO THE DETECTION OF STATIC HAZARDS IN COMBINATIONAL SWITCHING SYSTEMS
by Abraham Kandel (CSR 122)

ABSTRACT

In this paper, the fuzzy set (1) is viewed as a multi-valued logic with a continuum of truth values in the interval [0,1]. The concept of static hazard in combinational switching systems is related to fuzzy logic and various properties of this relation are established. The paper derives the necessary and sufficient conditions for a fuzzy function to adequately describe the steady-state and static hazard behavior of a combinational system, by extending the ternary method (2) and using the resolution principle of mechanical theorem proving.