# Triangular Decomposition of Semi-algebraic Systems 

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#### Abstract

Regular chains and triangular decompositions are fundamental and well-developed tools for describing the complex solutions of polynomial systems. This paper proposes adaptations of these tools focusing on solutions of the real analogue: semi-algebraic systems.

We show that any such system can be decomposed into finitely many regular semi-algebraic systems. We propose two specifications of such a decomposition and present corresponding algorithms. Under some assumptions, one type of decomposition can be computed in singly exponential time w.r.t. the number of variables. We implement our algorithms and the experimental results illustrate their effectiveness.


## Categories and Subject Descriptors

I.1.2 [Symbolic and Algebraic Manipulation]: Algo-rithms-Algebraic algorithms, Analysis of algorithms

## General Terms

Algorithms, Experimentation, Theory

## Keywords

regular semi-algebraic system, regular chain, triangular decomposition, border polynomial, fingerprint polynomial set

## 1. INTRODUCTION

Regular chains, the output of triangular decompositions of systems of polynomial equations, enjoy remarkable properties. Size estimates play in their favor [12] and permit the design of modular [13] and fast [17] methods for computing triangular decompositions. These features stimulate the development of algorithms and software for solving polynomial systems via triangular decompositions.

For the fundamental case of semi-algebraic systems with rational number coefficients, to which this paper is devoted, we observe that several algorithms for studying the real solutions of such systems take advantage of the structure of a

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regular chain. Some are specialized to isolating the real solutions of systems with finitely many complex solutions 23, 10 3. Other algorithms deal with parametric polynomial systems via real root classification (RRC) [25] or with arbitrary systems via cylindrical algebraic decompositions (CAD) [9.

In this paper, we introduce the notion of a regular semialgebraic system, which in broad terms is the "real" counterpart of the notion of a regular chain. Then we define two notions of a decomposition of a semi-algebraic system: one that we call lazy triangular decomposition, where the analysis of components of strictly smaller dimension is deferred, and one that we call full triangular decomposition where all cases are worked out. These decompositions are obtained by combining triangular decompositions of algebraic sets over the complex field with a special Quantifier Elimination (QE) method based on RRC techniques.
Regular semi-algebraic system. Let $T$ be a regular chain of $\mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$ for some ordering of the variables $\mathbf{x}=x_{1}, \ldots, x_{n}$. Let $\mathbf{u}=u_{1}, \ldots, u_{d}$ and $\mathbf{y}=y_{1}, \ldots, y_{n-d}$ designate respectively the variables of $\mathbf{x}$ that are free and algebraic w.r.t. $T$. Let $P \subset \mathbb{Q}[\mathbf{x}]$ be finite such that each polynomial in $P$ is regular w.r.t. the saturated ideal of $T$. Define $P_{>}:=\{p>$ $0 \mid p \in P\}$. Let $\mathcal{Q}$ be a quantifier-free formula of $\mathbb{Q}[\mathbf{x}]$ involving only the variables of $\mathbf{u}$. We say that $R:=\left[\mathcal{Q}, T, P_{>}\right]$ is a regular semi-algebraic system if:
(i) $\mathcal{Q}$ defines a non-empty open semi-algebraic set $S$ in $\mathbb{R}^{d}$,
(ii) the regular system $[T, P]$ specializes well at every point $u$ of $S$ (see Section 2 for this notion),
(iii) at each point $u$ of $S$, the specialized system $\left[T(u), P(u)_{>}\right]$ has at least one real zero.
The zero set of $R$, denoted by $Z_{\mathbb{R}}(R)$, is defined as the set of points $(u, y) \in \mathbb{R}^{d} \times \mathbb{R}^{n-d}$ such that $\mathcal{Q}(u)$ is true and $t(u, y)=0, p(u, y)>0$, for all $t \in T$ and all $p \in P$.

Triangular decomposition of a semi-algebraic system. In Section 3 we show that the zero set of any semi-algebraic system $\mathfrak{S}$ can be decomposed as a finite union (possibly empty) of zero sets of regular semi-algebraic systems. We call such a decomposition a full triangular decomposition (or simply triangular decomposition when clear from context) of $\mathfrak{S}$, and denote by RealTriangularize an algorithm to compute it. The proof of our statement relies on triangular decompositions in the sense of Lazard (see Section 2 for this notion) for which it is not known whether or not they can be computed in singly exponential time w.r.t. the number of variables. Meanwhile, we are hoping to obtain an algorithm for decomposing semi-algebraic systems (certainly under some genericity assumptions) that would fit in that complexity class. Moreover, we observe that, in practice, full triangular
decompositions are not always necessary and that providing information about the components of maximum dimension is often sufficient. These theoretical and practical motivations lead us to a weaker notion of a decomposition of a semi-algebraic system.
Lazy triangular decomposition of a semi-algebraic system. Let $\mathfrak{S}=\left[F, N_{\geq}, P_{>}, H_{\neq}\right]$(see Section 3 for this notation) be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$ and $Z_{\mathbb{R}}(\mathfrak{S}) \subseteq \mathbb{R}^{n}$ be its zero set. Denote by $d$ the dimension of the constructible set $\left\{x \in \mathbb{C}^{n} \mid f(x)=0, g(x) \neq 0\right.$, for all $\left.f \in F, g \in P \cup H\right\}$. A finite set of regular semi-algebraic systems $R_{i}, i=1 \cdots t$ is called a lazy triangular decomposition of $\mathfrak{S}$ if

- $\cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right) \subseteq Z_{\mathbb{R}}(\mathfrak{S})$ holds, and
- there exists $G \subset \mathbb{Q}[\mathbf{x}]$ such that the real-zero set $Z_{\mathbb{R}}(G)$ $\subset \mathbb{R}^{n}$ contains $Z_{\mathbb{R}}(\mathfrak{S}) \backslash\left(\cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right)\right)$ and the complexzero set $V(G) \subset \mathbb{C}^{n}$ either is empty or has dimension less than $d$.
We denote by LazyRealTriangularize an algorithm computing such a decomposition. In the implementation presented hereafter, LazyRealTriangularize outputs additional information in order to continue the computations and obtain a full triangular decomposition, if needed. This additional information appears in the form of unevaluated function calls, explaining the usage of the adjective lazy in this type of decompositions.

Complexity results for lazy triangular decomposition. In Section 4 we provide a running time estimate for computing a lazy triangular decomposition of the semi-algebraic system $\mathfrak{S}$ when $\mathfrak{S}$ has no inequations nor inequalities, (that is, when $N_{\geq}=P_{>}=H_{\neq}=\emptyset$ holds) and when $F$ generates a strongly equidimensional ideal of dimension $d$. We show that one can compute such a decomposition in time singly exponential w.r.t. $n$. Our estimates are not sharp and are just meant to reach a singly exponential bound. We rely on the work of J. Renagar [20] for QE. In Sections 5 and 6 we turn our attention to algorithms that are more suitable for implementation even though they rely on sub-algorithms with a doubly exponential running time w.r.t. $d$.
A special case of quantifier elimination. By means of triangular decomposition of algebraic sets over $\mathbb{C}$, triangular decomposition of semi-algebraic systems (both full and lazy) reduces to a special case of QE. In Section 5 we implement this latter step via the concept of a fingerprint polynomial set, which is inspired by that of a discrimination polynomial set used for RRC in 25, 24.

Implementation and experimental results. In Section 6 we describe the algorithms that we have implemented for computing triangular decompositions (both full and lazy) of semi-algebraic systems. Our Maple code is written on top of the RegularChains library. We provide experimental data for two groups of well-known problems. In the first group, each input semi-algebraic system consists of equations only while the second group is a collection of QE problems. To illustrate the difficulty of our test problems, and only for this purpose, we provide timings obtained with other well-known polynomial system solvers which are based on algorithms whose running time estimates are comparable to ours. For this first group we use the Maple command Groebner:Basis for computing lexicographical Gröbner bases. For the second group we use a general purpose QE software: QEPCAD B (in its non-interactive mode) [5]. Our experimental results show that our LazyRealTriangularize code can solve most of
our test problems and that it can solve more problems than the package it is compared to.

We conclude this introduction by computing a triangular decomposition of a particular semi-algebraic system taken from [6]. Consider the following question: when does $p(z)=$ $z^{3}+a z+b$ have a non-real root $x+i y$ satisfying $x y<1$ ? This problem can be expressed as $(\exists x)(\exists y)[f=g=0 \wedge y \neq$ $0 \wedge x y-1<0$ ], where $f=\operatorname{Re}(p(x+i y))=x^{3}-3 x y^{2}+a x+b$ and $g=\operatorname{Im}(p(x+i y)) / y=3 x^{2}-y^{2}+a$.

We call our LazyRealTriangularize command on the semialgebraic system $f=0, g=0, y \neq 0, x y-1<0$ with the variable order $y>x>b>a$. Its first step is to call the Triangularize command of the RegularChains library on the algebraic system $f=g=0$. We obtain one squarefree regular chain $T=\left[t_{1}, t_{2}\right]$, where $t_{1}=g$ and $t_{2}=8 x^{3}+2 a x-b$, satisfying $V(f, g)=V(T)$. The second step of LazyRealTriangularize is to check whether the polynomials defining inequalities and inequations are regular w.r.t. the saturated ideal of $T$, which is the case here. The third step is to compute the so called border polynomial set (see Section (2) which is $B=\left[h_{1}, h_{2}\right]$ with $h_{1}=4 a^{3}+27 b^{2}$ and $h_{2}=-4 a^{3} b^{2}-27 b^{4}+16 a^{4}+512 a^{2}+4096$. One can check that the regular system $[T,\{y, x y-1\}]$ specializes well outside of the hypersurface $h_{1} h_{2}=0$. The fourth step is to compute the fingerprint polynomial set which yields the quantifierfree formula $\mathcal{Q}=h_{1}>0$ telling us that $[\mathcal{Q}, T, 1-x y>0]$ is a regular semi-algebraic system. After performing these four steps, (based on Algorithm 50ction (6) the function call

$$
\text { LazyRealTriangularize }([f, g, y \neq 0, x y-1<0],[y, x, b, a])
$$

in our implementation returns the following:

$$
\begin{cases}{\left[\left\{t_{1}=0, t_{2}=0,1-x y>0, h_{1}>0\right\}\right]} & h_{1} h_{2} \neq 0 \\ \text { \%LazyRealTriangularize }\left(\left[t_{1}=0, t_{2}=0, f=0,\right.\right. & \\ \left.\left.h_{1}=0,1-x y>0, y \neq 0\right],[y, x, b, a]\right) & h_{1}=0 \\ \text { \%LazyRealTriangularize }\left(\left[t_{1}=0, t_{2}=0, f=0,\right.\right. & \\ \left.\left.h_{2}=0,1-x y>0, y \neq 0\right],[y, x, b, a]\right) & h_{2}=0\end{cases}
$$

The above output shows that $\{[\mathcal{Q}, T, 1-x y>0]\}$ forms a lazy triangular decomposition of the input semi-algebraic system. Moreover, together with the output of the recursive calls, one obtains a full triangular decomposition. Note that the cases of the two recursive calls correspond to $h_{1}=0$ and $h_{2}=0$. Since our LazyRealTriangularize uses the MAPLE piecewise structure for formatting its output, one simply needs to evaluate the recursive calls with the value command, yielding the same result as directly calling RealTriangularize

$$
\begin{cases}{\left[\left\{t_{1}=0, t_{2}=0,1-x y>0, h_{1}>0\right\}\right]} & h_{1} h_{2} \neq 0 \\ {[]} & h_{1}=0 \\ \begin{cases}\left(\left[\left\{t_{3}=0, t_{4}=0,\right.\right.\right. \\ \left.\left.\left.h_{2}=0,1-x y>0\right\}\right]\right) & h_{3} \neq 0 \\ {[]} & h_{3}=0\end{cases} & h_{2}=0\end{cases}
$$

where $t_{3}=x y+1, t_{4}=2 a^{3} x-a^{2} b+32 a x-48 b+18 x b^{2}$, $h_{3}=\left(a^{2}+48\right)\left(a^{2}+16\right)\left(a^{2}+12\right)$.

From this output, after some simplification, one could obtain the equivalent quantifier-free formula, $4 a^{3}+27 b^{2}>0$, of the original QE problem.

## 2. TRIANGULAR DECOMPOSITION OF ALGEBRAIC SETS

We review in the section the basic notions related to regular chains and triangular decompositions of algebraic sets. Throughout this paper, let $\mathbf{k}$ be a field of characteristic 0 and $\mathbf{K}$ be its algebraic closure. Let $\mathbf{k}[\mathbf{x}]$ be the polynomial ring over $\mathbf{k}$ and with ordered variables $\mathbf{x}=x_{1}<\cdots<x_{n}$. Let $p, q \in \mathbf{k}[\mathbf{x}]$ be polynomials. Assume that $p \notin \mathbf{k}$. Then denote by $\operatorname{mvar}(p), \operatorname{init}(p)$, and $\operatorname{mdeg}(p)$ respectively the greatest variable appearing in $p$ (called the main variable of $p$ ), the leading coefficient of $p$ w.r.t. $\operatorname{mvar}(p)$ (called the initial of $p$ ), and the degree of $p$ w.r.t. $\operatorname{mvar}(p)$ (called the main degree of $p$ ); denote by $\operatorname{der}(p)$ the derivative of $p$ w.r.t. $\operatorname{mvar}(p)$; denote by $\operatorname{discrim}(p)$ the discriminant of $p$ w.r.t. $\operatorname{mvar}(p)$.
Triangular sets. Let $T \subset \mathbf{k}[\mathbf{x}]$ be a triangular set, that is, a set of non-constant polynomials with pairwise distinct main variables. Denote by $\operatorname{mvar}(T)$ the set of main variables of the polynomials in $T$. A variable $v$ in $\mathbf{x}$ is called algebraic w.r.t. $T$ if $v \in \operatorname{mvar}(T)$, otherwise it is said free w.r.t. $T$. If no confusion is possible, we shall always denote by $\mathbf{u}=$ $u_{1}, \ldots, u_{d}$ and $\mathbf{y}=y_{1}, \ldots, y_{m}$ respectively the free and the main variables of $T$. Let $h_{T}$ be the product of the initials of the polynomials in $T$. We denote by sat $(T)$ the saturated ideal of $T$ : if $T$ is the empty triangular set, then $\operatorname{sat}(T)$ is defined as the trivial ideal $\langle 0\rangle$, otherwise it is the ideal $\langle T\rangle: h_{T}^{\infty}$. The quasi-component $W(T)$ of $T$ is defined as $V(T) \backslash V\left(h_{T}\right)$. Denote $\overline{W(T)}=V(\operatorname{sat}(T))$ as the Zariski closure of $W(T)$.
Iterated resultant. Let $p$ and $q$ be two polynomials of $\mathbf{k}[\mathbf{x}]$. Assume $q$ is non-constant and let $v=\operatorname{mvar}(q)$. We define $\operatorname{res}(p, q, v)$ as follows: if $v$ does not appear in $p$, then $\operatorname{res}(p, q, v):=p$; otherwise $\operatorname{res}(p, q, v)$ is the resultant of $p$ and $q$ w.r.t. $v$. Let $T$ be a triangular set of $\mathbf{k}[\mathbf{x}]$. We define $\operatorname{res}(p, T)$ by induction: if $T$ is empty, then $\operatorname{res}(p, T)=p$; otherwise let $v$ be the greatest variable appearing in $T$, then $\operatorname{res}(p, T)=\operatorname{res}\left(\operatorname{res}\left(p, T_{v}, v\right), T_{<v}\right)$, where $T_{v}$ and $T_{<v}$ denote respectively the polynomials of $T$ with main variables equal to and less than $v$.
Regular chain. A triangular set $T \subset \mathbf{k}[\mathbf{x}]$ is a regular chain if: either $T$ is empty; or (letting $t$ be the polynomial in $T$ with maximum main variable), $T \backslash\{t\}$ is a regular chain, and the initial of $t$ is regular w.r.t. $\operatorname{sat}(T \backslash\{t\})$. The empty regular chain is denoted by $\varnothing$. Let $H \subset \mathbf{k}[\mathbf{x}]$. The pair [ $T, H$ ] is a regular system if each polynomial in $H$ is regular modulo sat $(T)$. A regular chain $T$ or a regular system $[T, H]$, is squarefree if for all $t \in T$, the $\operatorname{der}(t)$ is regular w.r.t. $\operatorname{sat}(T)$.
Triangular decomposition. Let $F \subset \mathbf{k}[\mathbf{x}]$. Regular chains $T_{1}, \ldots, T_{e}$ of $\mathbf{k}[\mathbf{x}]$ form a triangular decomposition of $V(F)$ if either: $V(F)=\cup_{i=1}^{e} \overline{W\left(T_{i}\right)}$ (Kalkbrener's sense) or $V(F)=$ $\cup_{i=1}^{e} W\left(T_{i}\right)$ (Lazard's sense). In this paper, we denote by Triangularize an algorithm, such as the one of [18], computing a triangular decomposition of the former kind.
Regularization. Let $p \in \mathbf{k}[\mathbf{x}]$. Let $T$ be a regular chain of $\mathbf{k}[\mathbf{x}]$. Denote by $\operatorname{Regularize}(p, T)$ an operation which computes a set of regular chains $\left\{T_{1}, \ldots, T_{e}\right\}$ such that (1) for each $i, 1 \leq i \leq e$, either $p \in \operatorname{sat}\left(T_{i}\right)$ or $p$ is regular w.r.t. $\operatorname{sat}\left(T_{i}\right)$; (2) we have $\overline{W(T)}=\overline{W\left(T_{1}\right)} \cup \cdots \cup \overline{W\left(T_{e}\right)}, \operatorname{mvar}(T)=$ $\operatorname{mvar}\left(T_{i}\right)$ and $\operatorname{sat}(T) \subseteq \operatorname{sat}\left(T_{i}\right)$ for $1 \leq i \leq e$.
Good specialization [8. Consider a squarefree regular system $[T, H]$ of $\mathbf{k}[\mathbf{u}, \mathbf{y}]$. Recall that $\mathbf{y}$ and $\mathbf{u}=u_{1}, \ldots, u_{d}$ stand respectively for $\operatorname{mvar}(T)$ and $\mathbf{x} \backslash \mathbf{y}$. Let $z=\left(z_{1}, \ldots, z_{d}\right)$ be
a point of $\mathbf{K}^{d}$. We say that $[T, H]$ specializes well at $z$ if: $(i)$ none of the initials of the polynomials in $T$ vanishes modulo the ideal $\left\langle z_{1}-u_{1}, \ldots, z_{d}-u_{d}\right\rangle ;(i i)$ the image of $[T, H]$ modulo $\left\langle z_{1}-u_{1}, \ldots, z_{d}-u_{d}\right\rangle$ is a squarefree regular system. Border polynomial [25]. Let $[T, H]$ be a squarefree regular system of $\mathbf{k}[\mathbf{u}, \mathbf{y}]$. Let $b p$ be the primitive and square free part of the product of all $\operatorname{res}(\operatorname{der}(t), T)$ and all $\operatorname{res}(h, T)$ for $h \in H$ and $t \in T$. We call $b p$ the border polynomial of $[T, H]$ and denote by $\operatorname{BorderPolynomial}(T, H)$ an algorithm to compute it. We call the set of irreducible factors of $b p$ the border polynomial set of $[T, H]$. Denote by BorderPolynomialSet $(T, H)$ an algorithm to compute it. Proposition 1 follows from the specialization property of subresultants and states a fundamental property of border polynomials.

Proposition 1. The system $[T, H]$ specializes well at $u \in$ $\mathbf{K}^{d}$ if and only if the border polynomial $b p(u) \neq 0$.

## 3. TRIANGULAR DECOMPOSITION OF SEMI-ALGEBRAIC SYSTEMS

In this section, we prove that any semi-algebraic system can be decomposed into finitely many regular semi-algebraic systems. This latter notion was defined in the introduction.
Semi-algebraic system. Consider four finite polynomial subsets $F=\left\{f_{1}, f_{2}, \cdots, f_{s}\right\}, N=\left\{n_{1}, n_{2}, \cdots, n_{t}\right\}, P=$ $\left\{p_{1}, p_{2}, \cdots, p_{r}\right\}$, and $H=\left\{h_{1}, h_{2}, \cdots, h_{\ell}\right\}$ of $\mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$. Let $N_{\geq}$denote the set of non-negative inequalities $\left\{n_{1} \geq\right.$ $\left.0, \ldots, n_{t} \geq 0\right\}$. Let $P_{>}$denote the set of positive inequalities $\left\{p_{1}>0, \ldots, p_{r}>0\right\}$. Let $H_{\neq}$denote the set of inequations $\left\{h_{1} \neq 0, \ldots, h_{\ell} \neq 0\right\}$. We will denote by $\left[F, P_{>}\right.$] the basic semi-algebraic system $\left\{f_{1}=0, \ldots, f_{s}=0, p_{1}>\right.$ $\left.0, \ldots, p_{r}>0\right\}$. We denote by $\mathfrak{S}=\left[F, N_{\geq}, P_{>}, H_{\neq}\right]$the semi-algebraic system (SAS) which is the conjunction of the following conditions: $f_{1}=0, \ldots, f_{s}=0, n_{1} \geq 0, \ldots, n_{t} \geq 0$, $p_{1}>0, \ldots, p_{r}>0$ and $h_{1} \neq 0, \ldots, h_{\ell} \neq 0$.
Notations on zero sets. In this paper, we use " $Z$ " to denote the zero set of a polynomial system, involving equations and inequations, in $\mathbb{C}^{n}$ and " $Z_{\mathbb{R}}$ " to denote the zero set of a semialgebraic system in $\mathbb{R}^{n}$.
Pre-regular semi-algebraic system. Let $[T, P]$ be a squarefree regular system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Let $b p$ be the border polynomial of $[T, P]$. Let $B \subset \mathbb{Q}[\mathbf{u}]$ be a polynomial set such that $b p$ divides the product of polynomials in $B$. We call the triple $\left[B_{\neq}, T, P_{>}\right]$a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Its zero set, written as $Z_{\mathbb{R}}\left(B_{\neq}, T, P_{>}\right)$, is the set $(u, y) \in \mathbb{R}^{n}$ such that $b(u) \neq 0, t(u, y)=0, p(u, y)>0$, for all $b \in B$, $t \in T, p \in P$. Lemma 1 and Lemma 2 are fundamental properties of pre-regular semi-algebraic systems.

Lemma 1. Let $\mathfrak{S}$ be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Then there exists finitely many pre-regular semi-algebraic systems $\left[B_{i \neq}, T_{i}, P_{i>}\right], i=1 \cdots e$, s.t. $Z_{\mathbb{R}}(\mathfrak{S})=\cup_{i=1}^{e} Z_{\mathbb{R}}\left(B_{i \neq}, T_{i}, P_{i>}\right)$.

Proof. The semi-algebraic system $\mathfrak{S}$ decomposes into basic semi-algebraic systems, by rewriting inequality of type $n \geq 0$ as: $n>0 \vee n=0$. Let $\left[F, P_{>}\right]$be one of those basic semi-algebraic systems. If $F$ is empty, then the triple [ $\left.P, \varnothing, P_{>}\right]$, is a pre-regular semi-algebraic system. If $F$ is not empty, by Proposition 1 and the specifications of Triangularize and Regularize, one can compute finitely many squarefree regular systems $\left[T_{i}, H\right]$ such that $V(F) \cap Z\left(P_{\neq}\right)=$ $\cup_{i=1}^{e}\left(V\left(T_{i}\right) \cap Z\left(B_{i \neq}\right)\right)$ holds and where $B_{i}$ is the border polynomial set of the regular system $\left[T_{i}, H\right]$. Hence, we have
$Z_{\mathbb{R}}\left(F, P_{>}\right)=\cup_{i=1}^{e} Z_{\mathbb{R}}\left(B_{i \neq}, T_{i}, P_{>}\right)$, where each $\left[B_{i \neq}, T_{i}, P_{>}\right]$ is a pre-regular semi-algebraic system.

Lemma 2. Let $\left[B_{\neq}, T, P_{>}\right]$be a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Let $h$ be the product of polynomials in $B$. The complement of the hypersurface $h=0$ in $\mathbb{R}^{d}$ consists of finitely many open cells of dimension d. Let $C$ be one of them. Then for all $\alpha \in C$, the number of real zeros of $\left[T(\alpha), P_{>}(\alpha)\right]$ is the same.

Proof. From Proposition 1 and recursive use of Theorem 1 in 11] on the delineability of a polynomial.

Lemma 3. Let $\left[B_{\neq}, T, P_{>}\right]$be a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. One can decide whether its zero set is empty or not. If it is not empty, then one can compute a regular semi-algebraic system $\left[\mathcal{Q}, T, P_{>}\right]$whose zero set in $\mathbb{R}^{n}$ is the same as that of $\left[B_{\neq}, T, P_{>}\right]$.

Proof. If $T=\varnothing$, we can always test whether the zero set of $\left[B_{\neq}, P_{>}\right]$is empty or not, for instance using CAD. If it is empty, we are done. Otherwise, defining $\mathcal{Q}=B_{\neq} \wedge P_{>}$, the triple $\left[\mathcal{Q}, T, P_{>}\right]$is a regular semi-algebraic system. If $T$ is not empty, we solve the quantifier elimination problem $\exists \mathbf{y}(B(\mathbf{u}) \neq 0, T(\mathbf{u}, \mathbf{y})=0, P(\mathbf{u}, \mathbf{y})>0)$ and let $\mathcal{Q}$ be the resulting formula. If $\mathcal{Q}$ is false, we are done. Otherwise, by Lemma 2, above each connected component of $B(\mathbf{u}) \neq 0$, the number of real zeros of the system $\left[B_{\neq}, T, P_{>}\right]$is constant. Then, the zero set defined by $\mathcal{Q}$ is the union of the connected components of $B(\mathbf{u}) \neq 0$ above which $\left[B_{\neq}, T, P_{>}\right.$] possesses at least one solution. Thus, $\mathcal{Q}$ defines a nonempty open set of $\mathbb{R}^{d}$ and $\left[\mathcal{Q}, T, P_{>}\right]$is a regular semi-algebraic system.

Theorem 1. Let $\mathfrak{S}$ be a semi-algebraic system of $\mathbb{Q}[\mathbf{x}]$. Then one can compute a (full) triangular decomposition of $\mathfrak{S}$, that is, as defined in the introduction, finitely many regular semi-algebraic systems such that the union of their zero sets is the zero set of $\mathfrak{S}$.

Proof. It follows from Lemma 1 and 3

## 4. COMPLEXITY RESULTS

We start this section by stating complexity estimates for basic operations on multivariate polynomials.
Complexity of basic polynomial operations. Let $p, q \in$ $\mathbb{Q}[\mathbf{x}]$ be polynomials with respective total degrees $\delta_{p}, \delta_{q}$, and let $x \in \mathbf{x}$. Let $\hbar_{p}, \hbar_{q}, \hbar_{p q}$ and $\hbar_{r}$ be the height (that is, the bit size of the maximum absolute value of the numerator or denominator of a coefficient) of $p, q$, the product $p q$ and the resultant $\operatorname{res}(p, q, x)$, respectively. In [14, it is proved that $\operatorname{gcd}(p, q)$ can be computed within $O\left(n^{2 \delta+1} \hbar^{3}\right)$ bit operations where $\delta=\max \left(\delta_{p}, \delta_{q}\right)$ and $\hbar=\max \left(\hbar_{p}, \hbar_{q}\right)$. It is easy to establish that $\hbar_{p q}$ and $\hbar_{r}$ are respectively upper bounded by $\hbar_{p}+\hbar_{q}+n \log \left(\min \left(\delta_{p}, \delta_{q}\right)+1\right)$ and $\delta_{q} \hbar_{p}+\delta_{p} \hbar_{q}+n \delta_{q} \log \left(\delta_{p}+\right.$ 1) $+n \delta_{p} \log \left(\delta_{q}+1\right)+\log \left(\left(\delta_{p}+\delta_{q}\right)!\right)$. Finally, let $M$ be a $k \times k$ matrix over $\mathbb{Q}[\mathbf{x}]$. Let $\delta$ (resp. $\hbar$ ) be the maximum total degree (resp. height) of a polynomial coefficient of $M$. Then $\operatorname{det}(M)$ can be computed within $O\left(k^{2 n+5}(\delta+1)^{2 n} \hbar^{2}\right)$ bit operations, see [15.

We turn now to the main subject of this section, that is, complexity estimates for a lazy triangular decomposition of a polynomial system under some genericity assumptions. Let $F \subset \mathbb{Q}[\mathbf{x}]$. A lazy triangular decomposition (as defined in the Introduction) of the semi-algebraic system $\mathfrak{S}=[F, \emptyset, \emptyset, \emptyset]$,

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Algorithm 1: LazyRealTriangularize(S)
    Input: a semi-algebraic system \(\mathfrak{S}=[F, \emptyset, \emptyset, \emptyset]\)
    Output: a lazy triangular decomposition of \(\mathfrak{S}\)
    \(\mathfrak{T}:=\) Triangularize \((F)\)
    for \(T_{i} \in \mathfrak{T}\) do
        \(b p_{i}:=\operatorname{BorderPolynomial}\left(T_{i}, \emptyset\right)\)
        solve \(\exists \mathbf{y}\left(b p_{i}(\mathbf{u}) \neq 0, T_{i}(\mathbf{u}, \mathbf{y})=0\right)\), and let \(\mathcal{Q}_{i}\) be the
        resulting quantifier-free formula
        if \(\mathcal{Q}_{i} \neq\) false then output \(\left[\mathcal{Q}_{i}, T_{i}, \emptyset\right]\)
```

which only involves equations, is obtained by the above algorithm.
Proof of Algorithm 1 The termination of the algorithm is obvious. Let us prove its correctness. Let $R_{i}=\left[\mathcal{Q}_{i}, T_{i}, \emptyset\right]$, for $i=1 \cdots t$ be the output of Algorithm $\square$ and let $T_{j}$ for $j=t+1 \cdots s$ be the regular chains such that $\mathcal{Q}_{j}=$ false. By Lemma 3, each $R_{i}$ is a regular semi-algebraic system. For $i=1 \cdots s$, define $F_{i}=\operatorname{sat}\left(T_{i}\right)$. Then we have $V(F)=$ $\cup_{i=1}^{s} V\left(F_{i}\right)$, where each $F_{i}$ is equidimensional. For each $i=$ $1 \cdots s$, by Proposition 1 we have

$$
V\left(F_{i}\right) \backslash V\left(b p_{i}\right)=V\left(T_{i}\right) \backslash V\left(b p_{i}\right)
$$

Moreover, we have

$$
V\left(F_{i}\right)=\left(V\left(F_{i}\right) \backslash V\left(b p_{i}\right)\right) \cup V\left(F_{i} \cup\left\{b p_{i}\right\}\right) .
$$

Hence,

$$
Z_{\mathbb{R}}\left(R_{i}\right)=Z_{\mathbb{R}}\left(T_{i}\right) \backslash Z_{\mathbb{R}}\left(b p_{i}\right) \subseteq Z_{\mathbb{R}}\left(F_{i}\right) \subseteq Z_{\mathbb{R}}(F)
$$

holds. In addition, since $b p_{i}$ is regular modulo $F_{i}$, we have

$$
\begin{aligned}
Z_{\mathbb{R}}(F) \backslash \cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right) & =\cup_{i=1}^{s} Z_{\mathbb{R}}\left(F_{i}\right) \backslash \cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right) \\
& \subseteq \cup_{i=1} Z_{\mathbb{R}}\left(F_{i}\right) \backslash\left(Z_{\mathbb{R}}\left(T_{i}\right) \backslash Z_{\mathbb{R}}\left(b p_{i}\right)\right) \\
& \subset \cup_{i=1}^{s} Z_{\mathbb{T}}\left(F_{i} \cup\left\{b \eta_{i}\right\}\right) .
\end{aligned}
$$

and $\operatorname{dim}\left(\cup_{i=1}^{s} V\left(F_{i} \cup\left\{b p_{i}\right\}\right)\right)<\operatorname{dim}(V(F))$. So the $R_{i}$, for $i=1 \cdots t$, form a lazy triangular decomposition of $\mathfrak{S}$.

In this section, under some genericity assumptions for $F$, we establish running time estimates for Algorithm 11 see Proposition 3. This is achieved through:
(1) Proposition 2 giving running time and output size estimates for a Kalkbrener triangular decomposition of an algebraic set, and
(2) Theorem 2 giving running time and output size estimates for a border polynomial computation.
Our assumptions for these results are the following:
$\left(\mathbf{H}_{\mathbf{0}}\right) V(F)$ is equidimensional of dimension $d$,
$\left(\mathbf{H}_{1}\right) x_{1}, \ldots, x_{d}$ are algebraically independent modulo each associated prime ideal of the ideal generated by $F$ in $\mathbb{Q}[\mathbf{x}]$,
$\left(\mathbf{H}_{2}\right) F$ consists of $m:=n-d$ polynomials, $f_{1}, \ldots, f_{m}$.
Hypotheses $\left(\mathbf{H}_{\mathbf{0}}\right)$ and $\left(\mathbf{H}_{\mathbf{1}}\right)$ are equivalent to the existence of regular chains $T_{1}, \ldots, T_{e}$ of $\mathbb{Q}\left[x_{1}, \ldots, x_{n}\right]$ such that $x_{1}, \ldots, x_{d}$ are free w.r.t. each of $T_{1}, \ldots, T_{e}$ and such that we have $V(F)=\overline{W\left(T_{1}\right)} \cup \cdots \cup \overline{W\left(T_{e}\right)}$.

Denote by $\delta, \hbar$ respectively the maximum total degree and height of $f_{1}, \ldots, f_{m}$. In her PhD Thesis [22, Á. Szántó describes an algorithm which computes a Kalkbrener triangular decomposition, $T_{1}, \ldots, T_{e}$, of $V(F)$. Under Hypotheses $\left(\mathbf{H}_{\mathbf{0}}\right)$ to $\left(\mathbf{H}_{\mathbf{2}}\right)$, this algorithm runs in time $m^{O(1)}\left(\delta^{O\left(n^{2}\right)}\right)^{d+1}$ counting operations in $\mathbb{Q}$, while the total degrees of the polynomials in the output are bounded by $n \delta^{O\left(m^{2}\right)}$. In addition,
$T_{1}, \ldots, T_{e}$ are square free, strongly normalized 18 and reduced [1].

From $T_{1}, \ldots, T_{e}$, we obtain regular chains $E_{1}, \ldots, E_{e}$ forming another Kalkbrener triangular decomposition of $V(F)$, as follows. Let $i=1 \cdots e$ and $j=(d+1) \cdots n$. Let $t_{i, j}$ be the polynomial of $T_{i}$ with $x_{j}$ as main variable. Let $e_{i, j}$ be the primitive part of $t_{i, j}$ regarded as a polynomial in $\mathbb{Q}\left[x_{1}, \ldots, x_{d}\right]\left[x_{d+1}, \ldots, x_{n}\right]$. Define $E_{i}=\left\{e_{i, d+1}, \ldots, e_{i, n}\right\}$. According to the complexity results for polynomial operations stated at the beginning of this section, this transformation can be done within $\delta^{O\left(m^{4}\right) O(n)}$ operations in $\mathbb{Q}$.

Dividing $e_{i, j}$ by its initial we obtain a monic polynomial $d_{i, j}$ of $\mathbb{Q}\left(x_{1}, \ldots, x_{d}\right)\left[x_{d+1}, \ldots, x_{n}\right]$. Denote by $D_{i}$ the regular chain $\left\{d_{i, d+1}, \ldots, d_{i, n}\right\}$. Observe that $D_{i}$ is the reduced lexicographic Gröbner basis of the radical ideal it generates in $\mathbb{Q}\left(x_{1}, \ldots, x_{d}\right)\left[x_{d+1}, \ldots, x_{n}\right]$. So Theorem 1 in 12 applies to each regular chain $D_{i}$. For each polynomial $\dot{d}_{i, j}$, this theorem provides height and total degree estimates expressed as functions of the degree [7] and the height [19, 16] of the algebraic set $\overline{W\left(D_{i}\right)}$. Note that the degree and height of $\overline{W\left(D_{i}\right)}$ are upper bounded by those of $V(F)$. Write $d_{i, j}=\Sigma_{\mu} \frac{\alpha_{\mu}}{\beta_{\mu}} \mu$ where each $\mu \in \mathbb{Q}\left[x_{d+1}, \ldots, x_{n}\right]$ is a monomial and $\alpha_{\mu}, \beta_{\mu}$ are in $\mathbb{Q}\left[x_{1}, \ldots, x_{d}\right]$ such that $\operatorname{gcd}\left(\alpha_{\mu}, \beta_{\mu}\right)=1$ holds. Let $\gamma$ be the lcm of the $\beta_{\mu}$ 's. Then for $\gamma$ and each $\alpha_{\mu}$ :

- the total degree is bounded by $2 \delta^{2 m}$ and,
- the height by $O\left(\delta^{2 m}(m \hbar+d m \log (\delta)+n \log (n))\right)$.

Multiplying $d_{i, j}$ by $\gamma$ brings $e_{i, j}$ back. We deduce the height and total degree estimates for each $e_{i, j}$ below.

Proposition 2. The Kalkbrener triangular decomposition $E_{1}, \ldots, E_{e}$ of $V(F)$ can be computed in $\delta^{O\left(m^{4}\right) O(n)}$ operations in $\mathbb{Q}$. In addition, every polynomial $e_{i, j}$ has total degree upper bounded by $4 \delta^{2 m}+\delta^{m}$, and has height upper bounded by $O\left(\delta^{2 m}(m \hbar+d m \log (\delta)+n \log (n))\right)$.

Next we estimate the running time and output size for computing the border polynomial of a regular system.

Theorem 2. Let $R=[T, P]$ be a squarefree regular system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$, with $m=\# T$ and $\ell=\# P$. Let bp be the border polynomial of $R$. Denote by $\delta_{R}, \hbar_{R}$ respectively the maximum total degree and height of a polynomial in $R$. Then the total degree of bp is upper bounded by $(\ell+m) 2^{m-1} \delta_{R}{ }^{m}$, and bp can be computed within $(n \ell+n m)^{O(n)}\left(2 \delta_{R}\right)^{O(n) O(m)} \hbar_{R}{ }^{3}$ bit operations.

Proof. Define $G:=P \cup\{\operatorname{der}(t) \mid t \in T\}$. We need to compute the $\ell+m$ iterated resultants $\operatorname{res}(g, T)$, for all $g \in G$. Let $g \in G$. Observe that the total degree and height of $g$ are bounded by $\delta_{R}$ and $\hbar_{R}+\log \left(\delta_{R}\right)$ respectively. Define $r_{m+1}:=$ $g, \ldots, r_{i}:=\operatorname{res}\left(t_{i}, r_{i+1}, y_{i}\right), \ldots, r_{1}:=\operatorname{res}\left(t_{1}, r_{2}, y_{1}\right)$. Let $i \in$ $\{1, \ldots, m\}$. Denote by $\delta_{i}$ and $\hbar_{i}$ the total degree and height of $r_{i}$, respectively. Using the complexity estimates stated at the beginning of this section, we have $\delta_{i} \leq 2^{m-i+1} \delta_{R}{ }^{m-i+2}$ and $\hbar_{i} \leq 2 \delta_{i+1}\left(\hbar_{i+1}+n \log \left(\delta_{i+1}+1\right)\right)$. Therefore, we have $\hbar_{i} \leq\left(2 \delta_{R}\right)^{O\left(m^{2}\right)} n^{O(m)} \hbar_{R}$. From these size estimates, one can deduce that each resultant $r_{i}$ (thus the iterated resultants) can be computed within $\left(2 \delta_{R}\right)^{O(m n)+O\left(m^{2}\right)} n^{O(m)} \hbar_{R}{ }^{2}$ bit operations, by the complexity of computing a determinant stated at the beginning of this section.

Hence, the product of all iterated resultants has total degree and height bounded by $(\ell+m) 2^{m-1} \delta_{R}^{m}$ and $(\ell+$ $m)\left(2 \delta_{R}\right)^{O\left(m^{2}\right)} n^{O(m)} \hbar_{R}$, respectively. Thus, the primitive
and squarefree part of this product can be computed within $(n \ell+n m)^{O(n)}\left(2 \delta_{R}\right)^{O(n) O(m)} \hbar_{R}{ }^{3}$ bit operations, based on the complexity of a polynomial gcd computation stated at the beginning of this section.

Proposition 3. From the Kalkbrener triangular decomposition $E_{1}, \ldots, E_{e}$ of Proposition 圆 a lazy triangular decomposition of $f_{1}=\cdots=f_{m}=0$ can be computed in $\left(\delta^{n^{2}} n 4^{n}\right)^{O\left(n^{2}\right)} \hbar^{O(1)}$ bit operations. Thus, a lazy triangular decomposition of this system is computed from the input polynomials in singly exponential time w.r.t. $n$, counting operations in $\mathbb{Q}$.

Proof. For each $i \in\{1 \cdots e\}$, let $b p_{i}$ be the border polynomial of $\left[E_{i}, \emptyset\right]$ and let $\hbar_{R_{i}}$ (resp. $\delta_{R_{i}}$ ) be the height (resp. the total degree) bound of the polynomials in the pre-regular semi-algebraic system $R_{i}=\left[\left\{b p_{i}\right\}_{\neq}, E_{i}, \emptyset\right]$. According to Algorithm [1, the remaining task is to solve the QE problem $\exists \mathbf{y}\left(b p_{i}(\mathbf{u}) \neq 0, E_{i}(\mathbf{u}, \mathbf{y})=0\right)$ for each $i \in\{1 \cdots e\}$, which can be solved within $\left((m+1) \delta_{R_{i}}\right)^{O(d m)} \hbar_{R_{i}}^{O(1)}$ bit operations, based on the results of [20]. The conclusion follows from the size estimates in Proposition 22 and Theorem 2.

## 5. QUANTIFIER ELIMINATION BY RRC

In the last two sections, we saw that in order to compute a triangular decomposition of a semi-algebraic system, a key step is to solve the following quantifier elimination problem:

$$
\begin{equation*}
\exists \mathbf{y}(B(\mathbf{u}) \neq 0, T(\mathbf{u}, \mathbf{y})=0, P(\mathbf{u}, \mathbf{y})>0) \tag{1}
\end{equation*}
$$

where $\left[B_{\neq}, T, P_{>}\right]$is a pre-regular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. This problem is an instance of the so-called real root classification (RRC) [27]. In this section, we show how to solve this problem when $B$ is what we call a fingerprint polynomial set.
Fingerprint polynomial set. Let $R:=\left[B_{\neq}, T, P_{>}\right]$be a preregular semi-algebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. Let $D \subset \mathbb{Q}[\mathbf{u}]$. Let $d p$ be the product of all polynomials in $D$. We call $D$ a fingerprint polynomial set (FPS) of $R$ if:
(i) for all $\alpha \in \mathbb{R}^{d}$, for all $b \in B$ we have:

$$
d p(\alpha) \neq 0 \Longrightarrow b(\alpha) \neq 0
$$

(ii) for all $\alpha, \beta \in \mathbb{R}^{d}$ with $\alpha \neq \beta$ and $d p(\alpha) \neq 0, d p(\beta) \neq 0$, if the signs of $p(\alpha)$ and $p(\beta)$ are the same for all $p \in D$, then $R(\alpha)$ has real solutions if and only if $R(\beta)$ does.
Hereafter, we present a method to construct an FPS based on projection operators of CAD.
Open projection operator [21, 4]. Hereafter in this section, let $\mathbf{u}=u_{1}<\cdots<u_{d}$ be ordered variables. Let $p \in \mathbb{Q}[\mathbf{u}]$ be non-constant. Denote by factor $(p)$ the set of the non-constant irreducible factors of $p$. For $A \subset \mathbb{Q}[\mathbf{u}]$, define $\operatorname{factor}(A)=\cup_{p \in A}$ factor $(p)$. Let $C_{d}$ (resp. $C_{0}$ ) be the set of the polynomials in factor $(p)$ with main variable equal to (resp. less than) $u_{d}$. The open projection operator (oproj) w.r.t. variable $u_{d}$ maps $p$ to a set of polynomials of $\mathbb{Q}\left[u_{1}, \ldots, u_{d-1}\right]$ defined below:

$$
\begin{array}{r}
\left.\operatorname{oproj}\left(p, u_{d}\right):=C_{0} \cup \bigcup_{f, g \in C_{d}, f \neq g} \text { factor(res }\left(f, g, u_{d}\right)\right) \\
\cup \bigcup_{f \in C_{d}} \operatorname{factor}\left(\operatorname{init}\left(f, u_{d}\right) \cdot \operatorname{discrim}\left(f, u_{d}\right)\right) .
\end{array}
$$

Then, we define $\operatorname{oproj}\left(A, u_{d}\right):=\operatorname{oproj}\left(\Pi_{p \in A} p, u_{d}\right)$.
Augmentation. Let $A \subset \mathbb{Q}[\mathbf{u}]$ and $x \in\left\{u_{1}, \ldots, u_{d}\right\}$. Denote by $\operatorname{der}(A, x)$ the derivative closure of $A$ w.r.t. $x$, that is, $\operatorname{der}(A, x):=\cup_{p \in A}\left\{\operatorname{der}^{(i)}(p, x) \mid 0 \leq i<\operatorname{deg}(p, x)\right\}$. The
open augmented projected factors of $A$ is denoted by oaf $(A)$ and defined as follows. Let $k$ be the smallest positive integer such that $A \subset \mathbb{Q}\left[u_{1}, \ldots, u_{k}\right]$ holds. Denote by $C$ the set factor $\left(\operatorname{der}\left(A, u_{k}\right)\right)$; we have

- if $k=1$, then $\operatorname{oaf}(A):=C$;
- if $k>1$, then $\operatorname{oaf}(A):=C \cup \operatorname{oaf}\left(\operatorname{oproj}\left(C, u_{k}\right)\right)$.

Theorem 3. Let $A \subset \mathbb{Q}[\mathbf{u}]$ be finite and let $\sigma$ be a map from $\operatorname{oaf}(A)$ to the set of signs $\{-1,+1\}$. Then the set $S_{d}:=$ $\cap_{p \in \operatorname{oaf}(A)}\left\{u \in \mathbb{R}^{d} \mid p(u) \sigma(p)>0\right\}$ is either empty or a connected open set in $\mathbb{R}^{d}$.

Proof. By induction on $d$. When $d=1$, the conclusion follows from Thom's Lemma [2]. Assume $d>1$. If $d$ is not the smallest positive integer $k$ such that $A \subset \mathbb{Q}\left[u_{1}, \ldots, u_{k}\right]$ holds, then $S_{d}$ can be written $S_{d-1} \times \mathbb{R}$ and the conclusion follows by induction. Otherwise, write oaf $(A)$ as $C \cup E$, where $C=\operatorname{factor}\left(\operatorname{der}\left(A, u_{d}\right)\right)$ and $E=\operatorname{oaf}\left(\operatorname{oproj}\left(C, u_{d}\right)\right)$. We have: $E \subset \mathbb{Q}\left[u_{1}, \cdots, u_{d-1}\right]$. Denote by $M$ the set $\cap_{p \in E}\{u \in$ $\left.\mathbb{R}^{d-1} \mid p(u) \sigma(p)>0\right\}$. If $M$ is empty then so is $S_{d}$ and the conclusion is clear. From now on assume $M$ not empty. Then, by induction hypothesis, $M$ is a connected open set in $\mathbb{R}^{d-1}$. By the definition of the operator oproj, the product of the polynomials in $C$ is delineable over $M$ w.r.t. $u_{d}$. Moreover, $C$ is derivative closed (may be empty) w.r.t. $u_{d}$. Therefore $\cap_{p \in \operatorname{oaf}(A)}\left\{u \in \mathbb{R}^{d} \mid p(u) \sigma(p)>0\right\} \subset M \times \mathbb{R}$ is either empty or a connected open set by Thom's Lemma.

Theorem 4. Let $R:=\left[B_{\neq}, T, P_{>}\right]$be a pre-regular semialgebraic system of $\mathbb{Q}[\mathbf{u}, \mathbf{y}]$. The polynomial set $\operatorname{oaf}(B)$ is a fingerprint polynomial set of $R$.

Proof. Recall that the border polynomial $b p$ of $[T, P]$ divides the product of the polynomials in $B$. We have factor $(B)$ $\subseteq \operatorname{oaf}(B)$. So oaf $(B)$ satisfies $(i)$ in the definition of FPS. Let us prove (ii). Let $d p$ be the product of the polynomials in $\operatorname{oaf}(B)$. Let $\alpha, \beta \in \mathbb{R}^{d}$ such that both $d p(\alpha) \neq 0$, $d p(\beta) \neq 0$ hold and the signs of $p(\alpha)$ and $p(\beta)$ are equal for all $p \in \operatorname{oaf}(B)$. Then, by Theorem 3, $\alpha$ and $\beta$ belong to the same connected component of $d p(\mathbf{u}) \neq 0$, and thus to the same connected component of $B(\mathbf{u}) \neq 0$. Therefore the number of real solutions of $R(\alpha)$ and that of $R(\beta)$ are the same by Lemma 2.

From now on, let us assume that the set $B$ in the preregular semi-algebraic system $R=\left[B_{\neq}, T, P_{>}\right]$is an FPS of $R$. We solve the quantifier elimination problem (1) in three steps: $\left(s_{1}\right)$ compute at least one sample point in each connected component of the semi-algebraic set defined by $B(\mathbf{u}) \neq 0 ;\left(s_{2}\right)$ for each sample point $\alpha$ such that the specialized system $R(\alpha)$ possesses real solutions, compute the sign of $b(\alpha)$ for each $b \in B ;\left(s_{3}\right)$ generate the corresponding quantifier-free formulas.

In practice, when the set $B$ is not an FPS, one adds some polynomials from oaf $(B)$, using a heuristic procedure (for instance one by one) until Property (ii) of the definition of an FPS is satisfied. This strategy is implemented in Algorithm3 of Section 6

## 6. IMPLEMENTATION

In this section, we present algorithms for LazyRealTriangularize and RealTriangularize that we have implemented on top of the RegularChains library in Maple. We also provide experimental results for test problems which are available at www.orcca.on.ca/~cchen/issac10.txt.

```
Algorithm 2: GeneratePreRegularSas(S)
    Input: a semi-algebraic system \(\mathfrak{S}=\left[F, N_{\geq}, P_{>}, H_{\neq}\right]\)
    Output: a set of pre-regular semi-algebraic systems
    [ \(B_{i \neq}, T_{i}, P_{i>}\) ], \(i=1 \ldots e\), such that
    \(Z_{\mathbb{R}}(\mathfrak{S})=\cup_{i=1}^{e} Z_{\mathbb{R}}\left(B_{i \neq}, T_{i}, P_{i>}\right)\)
                                    \(\cup_{i=1}^{e} Z_{\mathbb{R}}\left(\operatorname{sat}\left(T_{i}\right) \cup\left\{\Pi_{b \in B_{i}} b\right\}, N_{\geq}, P_{>}, H_{\neq}\right)\).
    \(\mathfrak{T}:=\) Triangularize \((F) ; \mathfrak{T}^{\prime}:=\emptyset\)
    for \(p \in P \cup H\) do
        for \(T \in \mathfrak{T}\) do
            for \(C \in \operatorname{Regularize}(p, T)\) do
                if \(p \notin \operatorname{sat}(C)\) then \(\mathfrak{T}^{\prime}:=\mathfrak{T}^{\prime} \cup\{C\}\)
        \(\mathfrak{T}:=\mathfrak{T}^{\prime} ; \mathfrak{T}^{\prime}:=\emptyset\)
    \(\mathfrak{T}:=\{[T, \emptyset] \mid T \in \mathfrak{T}\} ; \mathfrak{T}^{\prime}:=\emptyset\)
    for \(p \in N\) do
        for \(\left[T, N^{\prime}\right] \in \mathfrak{T}\) do
            for \(C \in \operatorname{Regularize}(p, T)\) do
                if \(p \in \operatorname{sat}(C)\) then
                    \(\mathfrak{T}^{\prime}:=\mathfrak{T}^{\prime} \cup\left\{\left[C, N^{\prime}\right]\right\}\)
                else
                    \(\mathfrak{T}^{\prime}:=\mathfrak{T}^{\prime} \cup\left\{\left[C, N^{\prime} \cup\{p\}\right]\right\}\)
        \(\mathfrak{T}:=\mathfrak{T}^{\prime} ; \mathfrak{T}^{\prime}:=\emptyset\)
    \(\mathfrak{T}:=\left\{\left[T, N^{\prime}, P, H\right] \mid\left[T, N^{\prime}\right] \in \mathfrak{T}\right\}\)
    for \(\left[T, N^{\prime}, P, H\right] \in \mathfrak{T}\) do
        \(B:=\) BorderPolynomialSet \(\left(T, N^{\prime} \cup P \cup H\right)\)
        output \(\left[B, T, N^{\prime} \cup P\right]\)
```

```
Algorithm 3: GenerateRegularSas( \(B, T, P\) )
    Input: \(\mathfrak{S}=\left[B_{\neq}, T, P_{>}\right]\), a pre-regular semi-algebraic
            system of \(\mathbb{Q}[\mathbf{u}, \mathbf{y}]\), where \(\mathbf{u}=u_{1}, \ldots, u_{d}\) and
            \(\mathbf{y}=y_{1}, \ldots, y_{n-d}\).
    Output: A pair \((D, \mathcal{R})\) satisfying:
    (1) \(D \subset \mathbb{Q}[\mathbf{u}]\) such that factor \((B) \subseteq D\);
    (2) \(\mathcal{R}\) is a finite set of regular semi-algebraic systems,
    s.t. \(\cup_{R \in \mathcal{R}} Z_{\mathbb{R}}(R)=Z_{\mathbb{R}}\left(D_{\neq}, T, P_{>}\right)\).
    \(D:=\operatorname{factor}(B \backslash \mathbb{Q})\)
    if \(d=0\) then
        if RealRootCounting \((T, P)=0\) then
            return \((D, \emptyset)\)
        else
            return \((D,\{[\) true \(, T, P]\})\)
    while true do
        \(S:=\) SamplePoints \((D, d) ; G_{0}:=\emptyset ; G_{1}:=\emptyset\)
        for \(s \in S\) do
            if RealRootCounting \((T(s), P(s))=0\) then
                    \(G_{0}:=G_{0} \cup\{\) GenerateFormula \((D, s)\}\)
                else
                    \(\left\lfloor G_{1}:=G_{1} \cup\{\operatorname{GenerateFormula}(D, s)\}\right.\)
        if \(G_{0} \cap G_{1}=\emptyset\) then
            \(\mathcal{Q}:=\) Disjunction \(\left(G_{1}\right)\)
            if \(\mathcal{Q}=\) false then return \((D, \emptyset)\)
            else return \((D,\{[\mathcal{Q}, T, P]\})\)
        else
            select a subset \(D^{\prime} \subseteq \operatorname{oaf}(B) \backslash D\) by some
                heuristic method
                \(D:=D \cup D^{\prime}\)
```

Basic subroutines. For a zero-dimensional squarefree regular system $[T, P]$, RealRootCounting $(T, P)$ [23] returns the number of real zeros of $\left[T, P_{>}\right]$. For $A \subset \mathbb{Q}\left[u_{1}, \ldots, u_{d}\right]$ and a point $s$ of $\mathbb{Q}^{d}$ such that $p(s) \neq 0$ for all $p \in A$, GenerateFormula $(A, s)$ computes a formula $\wedge_{p \in A}\left(p \sigma_{p, s}>0\right)$, where $\sigma_{p, s}$ is defined as +1 if $p(s)>0$ and -1 otherwise. For a set of formulas $G$, Disjunction $(G)$ computes a logic formula $\Phi$ equivalent to the disjunction of the formulas in $G$.
Proof of Algorithm [2] Its termination is obvious. Let us prove its correctness. By the specification of Triangularize and Regularize, at line 16, we have

$$
Z\left(F, P_{\neq} \cup H_{\neq}\right)=\cup_{\left[T, N^{\prime}, P, H\right] \in \mathfrak{T}} Z\left(\operatorname{sat}(T), P_{\neq} \cup H_{\neq}\right)
$$

Write $\cup_{\left[T, N^{\prime}, P, H\right] \in \mathfrak{T}}$ as $\cup_{T}$. Then we deduce that

$$
Z_{\mathbb{R}}\left(F, N_{\geq}, P_{>}, H_{\neq}\right)=\cup_{T} Z_{\mathbb{R}}\left(\operatorname{sat}(T), N_{\geq}, P_{>}, H_{\neq}\right) .
$$

For each $\left[T, N^{\prime}, P, H\right]$, at line 19, we generate a pre-regular semi-algebraic system $\left[B_{\neq}, T, N_{>}^{\prime} \cup P_{>}\right]$. By Proposition 1 . we have

$$
\begin{aligned}
& Z_{\mathbb{R}}\left(\operatorname{sat}(T), N_{\geq}, P_{>}, H_{\neq}\right)= \\
& Z_{\mathbb{R}}\left(B_{\neq}, T, N_{>}^{\prime} \cup P_{>}\right) \cup Z_{\mathbb{R}}\left(\operatorname{sat}(T) \cup\left\{\Pi_{b \in B} b\right\}, N_{\geq}, P_{>}, H_{\neq}\right),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
Z_{\mathbb{R}}(\mathfrak{S})= & \cup_{T} Z_{\mathbb{R}}\left(B_{\neq}, T, N_{>}^{\prime} \cup P_{>}\right) \\
& \cup_{T} Z_{\mathbb{R}}\left(\operatorname{sat}(T) \cup\left\{\Pi_{b \in B} b\right\}, N_{\geq}, P_{>}, H_{\neq}\right) .
\end{aligned}
$$

So Algorithm 2 satisfies its specification.

```
Algorithm 4: SamplePoints \((A, k)\)
    Input: \(A \subset \mathbb{Q}\left[x_{1}, \ldots, x_{k}\right]\) is a finite set of non-zero
            polynomials
    Output: A finite subset of \(\mathbb{Q}^{k}\) contained in
    \(\left(\Pi_{p \in A} p\right) \neq 0\) and having a non-empty intersection with
    each connected component of \(\left(\Pi_{p \in A} p\right) \neq 0\).
    if \(k=1\) then
        return one rational point from each connected
        component of \(\Pi_{p \in A} p \neq 0\)
3 else
        \(A_{k}:=\left\{p \in A \mid \operatorname{mvar}(p)=x_{k}\right\} ; A^{\prime}:=\operatorname{oproj}\left(A, x_{k}\right)\)
        for \(s \in \operatorname{SamplePoints}\left(A^{\prime}, k-1\right)\) do
                Collect in a set \(S\) one rational point from each
                connected component of \(\Pi_{p \in A_{k}} p\left(s, x_{k}\right) \neq 0\);
                for \(\alpha \in S\) do output ( \(s, \alpha\) )
```

```
Algorithm 5: LazyRealTriangularize(S)
    Input: a semi-algebraic system \(\mathfrak{S}=\left[F, N_{\geq}, P_{>}, H_{\neq}\right]\)
    Output: a lazy triangular decomposition of \(\mathfrak{S}\)
    \(\mathfrak{T}:=\) GeneratePreRegularSas \((F, N, P, H)\)
    for \(\left[B, T, P^{\prime}\right] \in \mathfrak{T}\) do
        \((D, \mathcal{R})=\operatorname{GenerateRegularSas}\left(B, T, P^{\prime}\right)\)
        if \(\mathcal{R} \neq \emptyset\) then output \(\mathcal{R}\)
```

Proof of Algorithms 3 and 4 By the definition of oproj, Algorithm 4 terminates and satisfies its specification. By Theorem [4] oaf $(B)$ is an FPS. Thus, by the definition of an FPS, Algorithm 3 terminates and satisfies its specification.
Proof of Algorithm 5. Its termination is obvious. Let us prove the algorithm is correct. Let $R_{i}, i=1 \cdots t$ be the

```
Algorithm 6: RealTriangularize(S)
    Input: a semi-algebraic system \(\mathfrak{S}=\left[F, N_{\geq}, P_{>}, H_{\neq}\right]\)
    Output: a triangular decomposition of \(\mathfrak{S}\)
    \(\mathfrak{T}:=\) GeneratePreRegularSas \((F, N, P, H)\)
    for \(\left[B, T, P^{\prime}\right] \in \mathfrak{T}\) do
        \((D, \mathcal{R})=\) GenerateRegularSas \(\left(B, T, P^{\prime}\right)\)
        if \(\mathcal{R} \neq \emptyset\) then output \(\mathcal{R}\)
        for \(p \in D\) do
            output RealTriangularize \((F \cup\{p\}, N, P, H)\)
```

output. By the specification of each sub-algorithm, each $R_{i}$ is a regular semi-algebraic system and we have:

$$
\cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right) \subseteq Z_{\mathbb{R}}(\mathfrak{S})
$$

Next we show that there exists an ideal $\mathcal{I} \subseteq \mathbb{Q}[\mathbf{x}]$, whose dimension is less than $\operatorname{dim}\left(Z\left(F, P_{\neq} \cup H_{\neq}\right)\right)$and such that $Z_{\mathbb{R}}(\mathfrak{S}) \backslash \cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right) \subseteq Z_{\mathbb{R}}(\mathcal{I})$ holds.

At line 1, by the specification of Algorithm 2 we have

$$
\begin{aligned}
Z_{\mathbb{R}}(\mathfrak{S})= & \cup_{T} Z_{\mathbb{R}}\left(B_{\neq}, T, P_{>}^{\prime}\right) \\
& \cup_{T} Z_{\mathbb{R}}\left(\operatorname{sat}(T) \cup\left\{\Pi_{b \in B} b\right\}, N_{\geq}, P_{>}, H_{\neq}\right) .
\end{aligned}
$$

At line 3, by the specification of Algorithm 3 for each $B$, we compute a set $D$ such that factor $(B) \subseteq D$ and

$$
\cup_{T} Z_{\mathbb{R}}\left(D_{\neq}, T, P_{>}^{\prime}\right)=\cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right)
$$

both hold. Combining the two relations together, we have

$$
\begin{aligned}
Z_{\mathbb{R}}(\mathfrak{S})= & \cup_{T} Z_{\mathbb{R}}\left(R_{i}\right) \\
& \cup_{T} Z_{\mathbb{R}}\left(\operatorname{sat}(T) \cup\left\{\Pi_{p \in D} p\right\}, N_{\geq}, P_{>}, H_{\neq}\right) .
\end{aligned}
$$

Therefore, the following relations hold

$$
\begin{aligned}
& Z_{\mathbb{R}}(\mathfrak{S}) \backslash \cup_{i=1}^{t} Z_{\mathbb{R}}\left(R_{i}\right) \\
\subseteq & \cup_{T} Z_{\mathbb{R}}\left(\operatorname{sat}(T) \cup\left\{\Pi_{p \in D} p\right\}, N_{\geq}, P_{>}, H_{\neq}\right) \\
\subseteq & Z_{\mathbb{R}}\left(\cap_{T}\left(\operatorname{sat}(T) \cup\left\{\Pi_{p \in D} p\right\}\right)\right) .
\end{aligned}
$$

Define

$$
\mathcal{I}=\cap_{T}\left(\operatorname{sat}(T) \cup\left\{\Pi_{p \in D} p\right\}\right) .
$$

Since each $p \in D$ is regular modulo sat $(T)$, we have

$$
\operatorname{dim}(\mathcal{I})<\operatorname{dim}\left(\cap_{T} \operatorname{sat}(T)\right) \leq \operatorname{dim}\left(Z\left(F, P_{\neq} \cup H_{\neq}\right)\right)
$$

So all $R_{i}$ form a lazy triangular decomposition of $\mathfrak{S}$.
Proof of Algorithm 6] For its termination, it is sufficient to prove that there are only finitely many recursive calls to RealTriangularize. Indeed, if $[F, N, P, H]$ is the input of a call to RealTriangularize then each of the immediate recursive calls takes $[F \cup\{p\}, N, P, H]$ as input, where $p$ belongs to the set $D$ of some pre-regular semi-algebraic system $\left[D_{\neq}, T, P_{>}\right]$. Since $p$ is regular (and non-zero) modulo sat $(T)$ we have:

$$
\langle F\rangle \subsetneq\langle F \cup\{p\}\rangle .
$$

Therefore, the algorithm terminates by the ascending chain condition on ideals of $\mathbb{Q}[\mathbf{x}]$. The correctness of Algorithm 6 follows from the specifications of the sub-algorithms.
Table 1. Table 1 summarizes the notations used in Tables 2 and 3. Tables 2 and 3 demonstrate benchmarks running in Maple $14 \beta 1$, using an Intel Core 2 Quad CPU ( 2.40 GHz ) with 3.0 GB memory. The timings are in seconds and the time-out is 1 hour.

Table 2. The systems in this group involve equations only. We report the running times for a triangular decomposition

Table 1 Notations for Tables 2 and 3

| symbol | meaning |
| :--- | :--- |
| $\# \mathrm{e}$ | number of equations in the input system |
| $\# \mathrm{v}$ | number of variables in the input equations |
| d | maximum total degree of an input equation |
| G | Groebner:-Basis (plex order) in MAPLE |
| T | Triangularize in RegularChains library of MAPLE |
| LR | LazyRealTriangularize implemented in MAPLE |
| R | RealTriangularize implemented in MAPLE |
| Q | QEPCAD B |
| $>1 h$ | computation does not complete within 1 hour |
| FAIL | QEPCAD B failed due to prime list exhausted |

Table 2 Timings for varieties

| system | $\# \mathrm{v} / \# \mathrm{e} / \mathrm{d}$ | G | T | LR |
| :---: | :---: | :---: | :---: | :---: |
| Hairer-2-BGK | $13 / 11 / 4$ | 25 | 1.924 | 2.396 |
| Collins-jsc02 | $5 / 4 / 3$ | 876 | 0.296 | 0.820 |
| Leykin-1 | $8 / 6 / 4$ | 103 | 3.684 | 3.924 |
| 8-3-config-Li | $12 / 7 / 2$ | 109 | 5.440 | 6.360 |
| Lichtblau | $3 / 2 / 11$ | 126 | 1.548 | 11 |
| Cinquin-3-3 | $4 / 3 / 4$ | 64 | 0.744 | 2.016 |
| Cinquin-3-4 | $4 / 3 / 5$ | $>1 h$ | 10 | 22 |
| DonatiTraverso-rev | $4 / 3 / 8$ | 154 | 7.100 | 7.548 |
| Cheaters-homotopy-1 | $7 / 3 / 7$ | 3527 | 174 | $>1 h$ |
| hereman-8.8 | $8 / 6 / 6$ | $>1 h$ | 33 | 62 |
| L | $12 / 4 / 3$ | $>1 h$ | 0.468 | 0.676 |
| dgp6 | $17 / 19 / 2$ | 27 | 60 | 63 |
| dgp29 | $5 / 4 / 15$ | 84 | 0.008 | 0.016 |

of the input algebraic variety and a lazy triangular decomposition of the corresponding real variety. These illustrate the good performance of our tool.
Table 3. The examples in this table are quantifier elimination problems and most of them involve both equations and inequalities. We provide the timings for computing a lazy and a full triangular decomposition of the corresponding semi-algebraic system and the timings for solving the quantifier elimination problem via Qepcad B [5] (in noninteractive mode). Computations complete with our tool on more examples than with QEPCAD b.
Remark. The output of our tools is a set of regular semialgebraic systems, which is different than that of Qepcad b. We note also that our tool is more effective for systems with more equations than inequalities.

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Table 3 Timings for semi-algebraic systems

| system | $\# \mathrm{v} / \# \mathrm{e} / \mathrm{d}$ | T | LR | R | Q |
| :---: | :---: | :---: | :---: | :---: | :---: |
| BM05-1 | $4 / 2 / 3$ | 0.008 | 0.208 | 0.568 | 86 |
| BM05-2 | $4 / 2 / 4$ | 0.040 | 2.284 | $>1 h$ | FAIL |
| Solotareff-4b | $5 / 4 / 3$ | 0.640 | 2.248 | 924 | $>1 h$ |
| Solotareff-4a | $5 / 4 / 3$ | 0.424 | 1.228 | 8.216 | FAIL |
| putnam | $6 / 4 / 2$ | 0.044 | 0.108 | 0.948 | $>1 h$ |
| MPV89 | $6 / 3 / 4$ | 0.016 | 0.496 | 2.544 | $>1 h$ |
| IBVP | $8 / 5 / 2$ | 0.272 | 0.560 | 12 | $>1 h$ |
| Lafferriere37 | $3 / 3 / 4$ | 0.056 | 0.184 | 0.180 | 10 |
| Xia | $6 / 3 / 4$ | 0.164 | 191 | 739 | $>1 h$ |
| SEIT | $11 / 4 / 3$ | 0.400 | $>1 h$ | $>1 h$ | $>1 h$ |
| p3p-isosceles | $7 / 3 / 3$ | 1.348 | $>1 h$ | $>1 h$ | $>1 h$ |
| p3p | $8 / 3 / 3$ | 210 | $>1 h$ | $>1 h$ | FAIL |
| Ellipse | $6 / 1 / 3$ | 0.012 | $>1 h$ | $>1 h$ | $>1 h$ |

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