Polyharmonic Daubechies type wavelets in Image Processing and Astronomy, I

Ognyan Kounchev, Damyan Kalaglarsky

Abstract: *We introduce a new family of multivariate wavelets which are obtained by "polyharmonic subdivision". They generalize directly the original compactly supported Daubechies wavelets.*

Key words: *Wavelet Analysis, Daubechies wavelet, Image Processing.*

1 Introduction

We consider new **multivariate polyharmonic Daubechies type wavelets** which are called "polyharmonic subdivision wavelets". They have been recently introduced in the paper [5]. They are obtained by means of a procedure called "polyharmonic subdivision" which is a generalization of the classical one-dimensional subdivision scheme of Deslauriers-Dubuc [4] which is the original source for the first compactly supported wavelets of Daubechies in 1988, cf. [3]. This new family of polyharmonic wavelets is the second representative of the *Polyharmonic Wavelet Analysis* following the "polyspline wavelets" which have been introduced in the monograph [6].

An important feature of these newly-born wavelets is that they are a nice generalization of the one-dimensional wavelets of Daubechies: they form an orthonormal family, enjoy nice non-stationary "refinement operator" equations, and have compact filters. In addition to that they have elongated supports. Let us remind that a major drawback of the one-dimensional spline wavelets of Ch. Chui is that they do not have finite filters, and respectively, the polyspline wavelets of [6] do not have finite filters.

2 Construction of fundamental function Φ_m for exponential polynomials subdivision

The whole construction of the Daubechies type wavelets passes via the construction of the so-called *fundamental function of subdivision*, cf. [1]. In the present case we will work with **non-stationary** subdivision and we have a family of such functions Φ_m for all $m \in \mathbb{Z}$ which satisfy the **refinement equations** (two-scale relations) given by

$$\Phi_m(t) = \sum_{i\in\mathbb{Z}} a_i^{[m]} \Phi_{m+1}(2t-i) \qquad \text{for all } t \in \mathbb{R}. \tag{1}$$

We define the **non-stationary subdivision symbol** by putting

$$a^{[k]}(z) := \sum_{j\in\mathbb{Z}} a_j^{[k]} z^j. \tag{2}$$

We are interested in special subdivision processes arising through the solutions of Ordinary Differential Equations. We assume that we are given a number of frequencies $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_p$ and put for the frequency vector (with repetitions)

$$\Lambda = \{\lambda_1, \lambda_2, ..., \lambda_N\} \cup \{-\lambda_1, -\lambda_2, ..., -\lambda_N\}.$$

We consider the space of C^∞ solutions of the ODE

$$\prod_{j=1}^{p} \left(\frac{d^2}{dt^2} - \lambda_j^2 \right) f(t) = 0. \tag{3}$$

Let us recall a simple fact from ODEs: in the case of different λ_j's the space of all C^∞ solutions in (3) is spanned by the set $\left\{ e^{\lambda_j t} : j = 1, 2, ..., p \right\}$. In the case of s coinciding indices $\lambda_i = \lambda_{i+1} = ... = \lambda_{i+s-1}$ we have that the solution set contains the functions $\left\{ t^\ell e^{\lambda_i t} : \ell = 0, 1, ..., s \right\}$.

Let us proceed to the construction of the subdivision symbols. We put

$$x_j = e^{-\lambda_j / 2^{k+1}}.$$

We define the following Laurent polynomial

$$d(z) := d^{[k]}(z) := \prod_{j=1}^{N} \frac{(z + x_j)\left(z^{-1} + x_j\right)}{(1 + x_j)^2}$$

and

$$P(x) := P^{[k]}(x) := \prod_{j=1}^{N} \left(1 - \frac{4x_j}{(1 + x_j)^2} x \right). \tag{4}$$

They satisfy the equality

$$d\left(e^{i\omega}\right) = P\left(\sin^2 \frac{\omega}{2}\right) \qquad \text{for all } \omega \in \mathbb{R}; \tag{5}$$

cf. [7]. We will often drop the dependence on the upper index in d, a, P and the other functions and symbols.

An important step for construction of the subdivision coefficients $a_j^{[m]}$ is the application of the Bezout theorem:

Proposition 1 *There exists a unique polynomial Q with real coefficients of degree $N - 1$ such that*

$$P(x) Q(x) + P(1 - x) Q(1 - x) = 1$$

and

$$Q(x) > 0 \qquad \text{for } x \in (0, 1).$$

We define now the trigonometric polynomial $b(z) = b^{[k]}(z)$ by putting

$$b\left(e^{i\omega}\right) = Q\left(\sin^2 \frac{\omega}{2}\right).$$

We finally define the symmetric Laurent polynomial $a(z)$ by putting

$$a(z) := a^{[k]}(z) := 2d(z)\, b(z) \qquad \text{for } z \in \mathbb{C} \setminus \{0\}. \tag{6}$$

The following proposition is important for the application of the Riesz lemma to $a(z)$ and construction of the Wavelet Analysis, cf. [7], [5].

Proposition 2 *The polynomial $a(z)$ defined in (6) satisfies*

$$a(z)=\sum_{j=-2N+1}^{2N-1}a_jz^j$$

with $a_j=a_{-j}=\overline{a_j}$ and

$$a(z)\geq 0\qquad \text{for all } |z|=1.$$

The following fundamental result shows that the symbols $a(z)$ are the non-stationary subdivision symbols for symmetric set of frequencies Λ, cf. [5].

Theorem 3 *For every exponential polynomial, i.e. for every solution to the equation*

$$Lf(t):=\prod_{j=1}^{N}\left(\frac{d^2}{dt^2}-\lambda_j^2\right)f(t)=0 \tag{7}$$

we put

$$f_j^k=f\left(\frac{j}{2^k}\right).$$

Then f is reproduced by means of interpolatory subdivision, i.e.

$$f_{j'}^{k+1}=\sum_{j=-\infty}^{\infty}a_{j'-2j}^{[k]}f_j^k\qquad \text{for all } j'\in\mathbb{Z} \tag{8}$$

$$f_{2j}^{k+1}=f_j^k\qquad\qquad \text{for all } j\in\mathbb{Z},$$

For every $m\in\mathbb{Z}$ the fundamental function of subdivision $\Phi_m(t)$ is a continuous function obtained throught the subdivision process (8), where one starts from $f_j^0=\delta_j$ for $j\in\mathbb{Z}$ (here δ_j is the Kronecker symbol), i.e. we put $\Phi_m\left(\frac{j}{2^m}\right)=\delta_j$, and Φ_m satisfies the refinement equation (1).

Having in hand the functions Φ_m and their refinement symbols $a^{[m]}$ we may follow the usual scheme for construction of father and mother wavelets which has been used by Daubechies, cf. [3], [1]. The following fundamental result has been proved in [5].

Theorem 4 *There exists a polynomial $g(z)=\sum_{j\in\mathbb{Z}}g_jz^j$ such that it is the "square root" of $2a(z)$, i.e.*

$$a\left(e^{i\theta}\right)=\frac{1}{2}\left|g\left(e^{i\theta}\right)\right|^2 \tag{9}$$

For every $m\in\mathbb{Z}$ there exists a compactly supported function $\varphi_m(t)$ which satisfies the refinement equation

$$\varphi_m(t)=\sum_j g_j\varphi_{m+1}(2t-j), \tag{10}$$

and the family $\{\varphi_m(t-j)\}_{j\in\mathbb{Z}}$ is orthonormal. (These are the non-stationary father wavelets.) The functions

$$\psi_m(t)=\sum_{j\in\mathbb{Z}}(-1)^j g_{1-j}\varphi_{m+1}(2t-j) \tag{11}$$

are the mother wavelets; the family $\{\psi_m(t-j)\}_{j\in\mathbb{Z}}$ is orthonormal and the family $\{\psi_m(t-j)\}_{m,j\in\mathbb{Z}}$ forms an orthonormal basis of $L_2(\mathbb{R})$.

2.1 The polyharmonic case

For the polyharmonic subdivision we will work with very special ODEs defined by $L_\xi := \left(d^2/dt^2 - \xi^2\right)^N$ which are the Fourier transform of the polyharmonic operator Δ^N. For a fixed constant $\xi \geq 0$ we put

$$\Lambda := (-\xi, -\xi, ..., -\xi, \xi, \xi, ..., \xi) \in \mathbb{R}^{2N} \tag{12}$$

i.e. $\lambda_j = \xi$, for $j = 1, 2, ..., N$. Now for fixed $\xi \geq 0$ and $k \in \mathbb{Z}$ we define the polynomial

$$d(z) := d^{[k],\xi}(z) := d^{[k]}(z) := \frac{(z + x_0)^N \left(z^{-1} + x_0\right)^N}{(1 + x_0)^{2N}} \qquad \text{for } z \in \mathbb{C}; \tag{13}$$

here we put $x_0 := e^{-\xi/2^{k+1}}$. For the sake of simplicity we will very often drop the dependence on k and ξ. By (5) we have $d\left(e^{i\omega}\right) = P\left(\sin^2 \frac{\omega}{2}\right)$ where

$$P(x) = \left(1 - \frac{4x_0}{(1 + x_0)^2} x\right)^N = (1 - \eta x)^N, \tag{14}$$

and we have put

$$\eta = \eta^{[k],\xi} := \frac{4x_0}{(1 + x_0)^2} = \frac{2}{1 + \cosh\left(\xi/2^{k+1}\right)}.$$

Then following Proposition 1 we have to find the polynomial solution Q to the equation

$$P(x)\, Q(x) + Q(1 - x)\, P(1 - x) = 1$$

where Q has degree $\leq N - 1$.

Remark 5 *Let us recall that the polynomial Q in the classical case, cf. e.g. [1], p. 195, satisfies condition*

$$(1 - y)^N Q(y) + y^N Q(1 - y) = 1.$$

The lowest degree solution polynomial Q will be called Daubechies' polynomial and we put

$$R_N(x) := \sum_{j=0}^{N-1} \binom{N + j - 1}{j} y^j. \tag{15}$$

(Note that in [3] and [1] the notation used is P_N !)

It is **amazing** that it is possible to solve the problem in Proposition 1 explicitly.

Proposition 6 *Let $\Lambda = (-\xi, -\xi, ..., -\xi, \xi, \xi, ..., \xi) \in \mathbb{R}^{2N}$. Then for the corresponding polynomial $P(x) = (1 - \eta x)^N$, the polynomial Q of degree $N - 1$ defined by*

$$Q(x) = Q_N^{k;\xi}(x) = (2 - \eta)^{-N} \sum_{j=0}^{N-1} \binom{N + j - 1}{j} \frac{(1 - \eta(1 - x))^j}{(2 - \eta)^j} \tag{16}$$

solves the equation

$$P(x)\, Q(x) + P(1 - x)\, Q(1 - x) = 1. \tag{17}$$

Hence,

$$Q(x) = (2 - \eta)^{-N} R_N\left(\frac{1 - \eta(1 - x)}{2 - \eta}\right). \tag{18}$$

Hence, we find the trigonometric polynomial $b^{[k]}(z)$ by putting

$$b^{[k]}(z) := b^{[k],\xi}\left(e^{i\omega}\right) := Q^{[k],\xi}\left(\sin^2\frac{\omega}{2}\right) \tag{19}$$

where we recall the notations

$$x = \sin^2\frac{\omega}{2} = \frac{1-\cos\omega}{2} = \frac{1}{2} - \frac{z+z^{-1}}{4},$$

Finally, we obtain the subdivision symbol $a^{[k]}(z)$ by putting

$$a^{[k]}(z) := a^{[k],\xi}(z) := 2d^{[k],\xi}(z)\, b^{[k],\xi}(z). \tag{20}$$

Now by Theorem 4 we find the "square root" of the symbol $a^{[k]}(z)$. This means that we have to take separately the "square root" of the Laurent polynomials $d^{[k]}(z)$ and $b^{[k]}(z)$. The "square root" of $d^{[k]}(z)$ is obvious; taking the "square root" of $b^{[k]}(z)$ needs taking the "square root" of the polynomial Q.

3 Algorithm for finding the square root of the polynomials Q

For the algorithmic aspects of taking the "square root" of the polynomial Q it will be important to describe the polynomial Q through the zeros of the Daubechies' polynomial R_N in (15).

Proposition 7 *Let the zeros of the Daubechies' polynomial (15) be c_j^D, i.e.*

$$R_N(y) = \sum_{j=0}^{N-1}\binom{N+j-1}{j} y^j = \frac{(2N-2)!}{((N-1)!)^2}\prod_{j=1}^{N-1}\left(y - c_j^D\right).$$

Then the polynomial Q as determined by (16) is given by

$$Q(x) = (2-\eta)^{-2N+1}\,\eta^{N-1}\frac{(2N-2)!}{((N-1)!)^2}\prod_{j=1}^{N-1}(x - C_j),$$

where

$$C_j := \frac{c_j^D(2-\eta) + \eta - 1}{\eta}.$$

By formula (13) we have the representation

$$d^{[k]}(z) = \left|\frac{(z+x_0)^N}{(1+x_0)^N}\right|^2 \qquad \text{for } z = e^{i\omega},$$

hence, we take the trigonometric polynomial

$$M_1(z) := \frac{(z+x_0)^N}{(1+x_0)^N} \tag{21}$$

as its "square root", i.e. $d^{[k]}(z) = |M_1(z)|^2$ for $|z| = 1$. Further, we have to take care of the "square root" of the polynomial $b^{[k]}(z)$. Thus we have to find the polynomial M_2 of degree $\leq N-1$ such that

$$\left|M_2\left(e^{i\omega}\right)\right|^2 = \frac{1}{2}Q\left(\sin^2\frac{\omega}{2}\right), \tag{22}$$

which may be obtained by using the roots of the Daubechies polynomials.

Remark 8 *Let the polynomial Q have the zeros C_j as in Proposition 7, and let us put*

$$c_j = 1 - 2C_j.$$

We see that $Q\left(\sin^2 \frac{\omega}{2}\right) = \widetilde{Q}(\cos\omega)$ for some polynomial $\widetilde{Q}$ and c_j are the zeros of $\widetilde{Q}$. Hence, we may apply the algorithm for the Riesz representation of $\widetilde{Q}$, see e.g. [1], p. $197 - 198$.

Thus we obtain finally for every integer $m \geq 0$ and $\xi \in \mathbb{Z}^n$ the representation

$$a^{[m],|\xi|}(z) = \frac{1}{2}\left|M_1(z)\, M_2(z)\right|^2, \tag{23}$$

and the family of functions

$$M(z) := M^{[m]}(z) := M^{[m],\xi}(z) := M_1(z)\, M_2(z) \tag{24}$$

represents the refinement masks for the family of scaling functions (father wavelets) $\{\varphi_m(t)\}_{m\geq 0}$ for which the functions Φ_m are autocorrelation functions.

Remark 9 *Note that the above factorization has been found in the special case $\xi = 0$ by Daubechies in [3], p.* 266*; the coefficients of the "square root" polynomial for $N = 2..10$ are in table* 6.1 *in [3]. A detailed discussion of more efficient methods for choosing the proper polynomial $M_2(z)$ is available in Strang-Nguyen [8], p.* 157, *in chapter* 5.4 *on Spectral factorization. The factorization of the Daubechies' polynomial $R_N(y)$ is discussed in Burrus [2], on p.* 78 *and the Matlab program is* ***[hn,hin]=daub(N)*** *in Appendix C. They work with the zeros of the polynomial R_N and provide a number of manipulations for finding a more stable factorization.*

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ABOUT THE AUTHORS

Ognyan Kounchev, Prof., Dr., Institute of Mathematics and Informatics, Bulgarian Academy of Science, tel. $+359 - 2 - 9793851$; kounchev@gmx.de

Damyan Kalaglarsky, Institute of Astronomy, Bulgarian Academy of Science, tel. $+359 - 2 - 9793851$; damyan@skyarchive.org.